STRUCTURED PSEUDOSPECTRA FOR SMALL PERTURBATIONS

MICHAEL KAROW *

Abstract. In this paper we study the shape and growth of structured pseudospectra for small matrix perturbations of the form $A \rightsquigarrow A_\Delta = A + B\Delta C$, $\Delta \in \mathbf{\Delta}$, $\|\Delta\| \leq \delta$. It is shown that the properly scaled pseudospectra components converge to non-trivial limit sets as δ tends to 0. We discuss the relationship of these limit sets with μ-values and structured eigenvalue condition numbers for multiple eigenvalues.

Key words. eigenvalues, perturbations, spectral value sets, μ-values, condition numbers

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1. Introduction. A structured pseudospectrum (also called spectral value set) is the set of eigenvalues of all matrices which are obtained from a given matrix $A \in \mathbb{C}^{n\times n}$ by adding perturbations of a certain type [16, 18, 10, 20, 22, 35, 39]. The norm of the perturbations is bounded by a prescribed constant $\delta > 0$. The present paper deals with the shape and growth of structured pseudospectra for small perturbations [1, 2, 3, 21]. It is shown that, after scaling with a suitable power of δ, the connected components of pseudospecta converge to non-trivial limit sets as δ tends to zero. The limit sets reflect the mobility of the spectrum of A under small and structured perturbations. We demonstrate how the boundaries of these sets can be calculated using μ-values. Furthermore, we discuss the relationship of the limit sets with structured condition numbers of *multiple* eigenvalues. The latter have been defined and investigated in [27]. For structured condition numbers of simple eigenvalues see e.g. [5, 6, 7, 8, 14, 23, 24, 28, 35, 36]. Finally, we apply our results to the case of real perturbations of real matrices.

This paper is organized as follows. In Section 2 we introduce notation and recall basic facts about structured pseudospectra. Section 3 deals with the definition of structured condition numbers for multiple eigenvalues and their connection to pseudospectra. In Section 4 we state our main result on the convergence of the pseudospectra components. Examples are discussed in Section 5. Some technical proofs including the proof of the main result are given in the appendix. The notation is mainly adopted from the text book [16].

2. Notation and basic facts. The symbols $\mathbb{R}, \mathbb{C}$ stand for the sets of real and complex numbers respectively. By $\mathbb{K}^{n\times m}$ we denote the set of n by m matrices with entries in $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Furthermore, $\mathbb{K}^n = \mathbb{K}^{n\times 1}$ is the set of column vectors of length n. By $\bar{A}, A^\top, A^*, \Re A, \Im A$ we denote the conjugate, the transpose, the conjugate transpose, the real and the imaginary part of $A \in \mathbb{C}^{n\times m}$, respectively. If A is square then $\sigma(A)$, $\rho(A)$ and $\varrho(A)$ denote its spectrum, its resolvent set and its spectral radius, $\rho(A) = \mathbb{C} \setminus \sigma(A)$, $\varrho(A) = \max\{|s|;\ s \in \sigma(A)\}$. The n by n identity matrix is written I_n. The closed disk of radius r about $\lambda \in \mathbb{C}$ is denoted by $\mathcal{D}_\lambda(r) = \{s \in \mathbb{C}; |s - \lambda| \leq r\}$. The boundary and the topological closure of $\mathcal{S} \subset \mathbb{C}^{m\times n}$ are written $\partial\mathcal{S}$ and $cl(\mathcal{S})$.

We define $L_{n,l,q}$ to be the set of triples of matrices (A, B, C) with $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times l}, C \in \mathbb{C}^{q\times n}$. Throughout this paper the symbol $\mathbf{\Delta}$ stands for a non-empty closed cone in $\mathbb{C}^{l\times q}$, i.e. $\emptyset \neq \mathbf{\Delta} \subseteq \mathbb{C}^{l\times q}$, $cl(\mathbf{\Delta}) = \mathbf{\Delta}$, and $\Delta \in \mathbf{\Delta}$ implies $t\Delta \in \mathbf{\Delta}$ for all $t \geq 0$. Furthermore, $\|\cdot\|$ denotes a norm on $\mathbb{C}^{l\times q}$.

*Matheon, Technische Universität Berlin, D-10623 Berlin, Germany, (karow@math.TU-Berlin.de).

Given any triple $(A,B,C)\in L_{n,l,q}$ we consider perturbations of A of the form

$$A \rightsquigarrow A_\Delta = A + B\Delta C, \quad \Delta \in \mathbf{\Delta}. \tag{2.1}$$

Definition 2.1. *[16, 17] The structured pseudospectrum (also called spectral value set) of the triple $(A,B,C)\in L_{n,l,q}$ with respect to $(\mathbf{\Delta}, \|\cdot\|)$ and the perturbation level $\delta > 0$ is the following subset of the complex plane.*

$$\sigma_{\mathbf{\Delta}}(A,B,C;\delta) := \{\, s \in \mathbb{C};\ \ s \in \sigma(A+B\Delta C) \text{ for some } \Delta \in \mathbf{\Delta} \text{ with } \|\Delta\| < \delta \,\}. \tag{2.2}$$

The set $\sigma_{\mathbf{\Delta}}(A,B,C;\delta)$ is the union of all the spectra of the perturbed matrices A_Δ where $\Delta \in \mathbf{\Delta}$, $\|\Delta\| < \delta$. Note that in this definition, introduced by Hinrichsen and Pritchard [15, 16], the norm of the perturbations is bounded by a strikt inequality. The sets obtained in this way are bounded but not compact. However, our main result in Section 4 deals with convergence with respect to the Hausdorff metric. To this end we need compact sets. Hence, in the sequel we work with the topological closure $\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta)$ of $\sigma_{\mathbf{\Delta}}(A,B,C;\delta)$. We always have

$$\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta) = \{\, s \in \mathbb{C};\ \ s \in \sigma(A+B\Delta C) \text{ for some } \Delta \in \mathbf{\Delta} \text{ with } \|\Delta\| \le \delta \,\}. \tag{2.3}$$

This follows from Claim (c) of Proposition A.2 in the appendix.

The μ-value of $M\in\mathbb{C}^{q\times l}$ with respect to $(\mathbf{\Delta}, \|\cdot\|)$ is defined as [16, 41]

$$\mu_{\mathbf{\Delta}}(M) := \left[\, \inf\{ \|\Delta\|\,;\ \Delta \in \mathbf{\Delta},\ 1 \in \sigma(\Delta\, M)\,\}\,\right]^{-1}. \tag{2.4}$$

If the set $\{\Delta \in \mathbf{\Delta};\ 1 \in \sigma(\Delta M)\}$ is empty we define $\mu_{\mathbf{\Delta}}(M) = 0$. The proposition below specifies the relationship between spectral value sets and μ-values.

Proposition 2.2. *Let $(A,B,C)\in L_{n,l,q}$ and $G(s) = C(sI_n - A)^{-1}B$, $s \in \rho(A)$. Then for any $\delta > 0$,*

$$\begin{aligned}\sigma_{\mathbf{\Delta}}(A,B,C;\delta) &= \sigma(A) \cup \{ s \in \rho(A);\ 1 \in \sigma(\Delta\, G(s)) \textit{ for some } \Delta \in \mathbf{\Delta} \textit{ with } \|\Delta\| < \delta \,\} \\ &= \sigma(A) \cup \{ s \in \rho(A);\ \mu_{\mathbf{\Delta}}(G(s)) > \delta^{-1} \}\end{aligned}$$

$$\begin{aligned}\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta) &= \sigma(A) \cup \{ s \in \rho(A);\ 1 \in \sigma(\Delta\, G(s)) \textit{ for some } \Delta \in \mathbf{\Delta} \textit{ with } \|\Delta\| \le \delta \,\} \\ &= \sigma(A) \cup \{ s \in \rho(A);\ \mu_{\mathbf{\Delta}}(G(s)) \ge \delta^{-1} \}\end{aligned}$$

Proof. These identities are immediate from the definition of $\mu_{\mathbf{\Delta}}$ and the following chain of equivalences which holds for all $s \in \rho(A)$ and all $\Delta \in \mathbb{C}^{l\times q}$, (see also [16, Lemma 5.2.7]).

$$\begin{aligned} s\in\sigma(A + B\Delta C) \Leftrightarrow\ & 0 = \det(sI_n - (A + B\Delta C)) \\ & \quad = \det(sI_n - A)\,\det(I_n - (sI_n - A)^{-1}B\Delta C) \\ \Leftrightarrow\ & 1 \in \sigma((sI_n - A)^{-1}B\Delta C) \\ \Leftrightarrow\ & 1 \in \sigma(\Delta G(s)). \end{aligned}$$

In the last step we used the fact that the nonzero eigenvalues of a product of two matrices are independent of the order of the factors. □

Much work has been done in order to find estimates and computable formulae for μ-values with respect to several perturbation classes $\mathbf{\Delta}$ and norms [4, 11, 16, 17, 23, 30, 32, 34, 35, 40]. We only mention the following basic results which are necessary for the understanding of this paper.

(i) If $\boldsymbol{\Delta}$ is invariant under complex multiplication (i.e. $\Delta \in \boldsymbol{\Delta}$ implies $t\Delta \in \boldsymbol{\Delta}$ for all $t \in \mathbb{C}$) and $\|\cdot\|$ is an arbitrary norm on $\mathbb{C}^{l\times q}$ then for all $M \in \mathbb{C}^{q\times l}$ [16, Lemma 4.4.7],

$$\mu_{\boldsymbol{\Delta}}(M) = \max_{\substack{\Delta \in \boldsymbol{\Delta} \\ \|\Delta\| = 1}} \varrho(\Delta\, M). \tag{2.5}$$

(i) If the underlying norm is the spectral norm then for $M \in \mathbb{C}^{q\times l}$ [4, 16, 32],

$$\begin{aligned} \mu_{\mathbb{C}^{l\times q}}(M) &= \|M\| \, = \sigma_1(M) \\ \mu_{\mathbb{R}^{l\times q}}(M) &= \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \Re M & -\gamma\,\Im M \\ \gamma^{-1}\,\Im M & \Re M \end{bmatrix}\right), \end{aligned} \tag{2.6}$$

where $\sigma_1(\cdot), \sigma_2(\cdot)$ denote the largest and the second largest singular value respectively. For a scalar multiple of a real matrix $R \in \mathbb{R}^{q\times l}$ we have [16, Example 4.4.45]

$$\mu_{\mathbb{R}^{l\times q}}(e^{i\phi}R) = \begin{cases} \sigma_1(R) & \text{if } \phi \in \{0, \pi\}, \\ \sqrt{\sigma_1(R)\,\sigma_2(R)} & \text{if } 0 < \phi < 2\pi,\ \phi \neq \pi. \end{cases} \tag{2.7}$$

3. Condition numbers. In this section we introduce condition numbers of simple and multiple eigenvalues with respect to structured perturbations. Furthermore we establish their relationship to pseudospectra. Let us first recall the definition of condition numbers for functions between normed vector spaces.

Definition 3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Let U be a subset of X and let $x_0 \in U$ be an accumulation point of U. The Hölder condition number of order $\gamma > 0$ of a function $f : U \to Y$ at the point x_0 is defined by*

$$\kappa_\gamma(f, x_0) := \overline{\lim_{x\to x_0}} \frac{\|f(x) - f(x_0)\|_Y}{\|x - x_0\|_X^\gamma},$$

where $\overline{\lim}$ denotes the limit superior.

Note that $\kappa_\gamma(f, x_0) \in [0, \infty]$ is well defined for all $\gamma > 0$. However, there is at most one order $\gamma > 0$ such that $0 \neq \kappa_\gamma(f, x_0) \neq \infty$, since these inequalities imply that $\kappa_{\widetilde{\gamma}}(f, x_0) = 0$ for $\widetilde{\gamma} < \gamma$ and $\kappa_{\widetilde{\gamma}}(f, x_0) = \infty$ for $\widetilde{\gamma} > \gamma$. If f is discontinuous at x_0 then $\kappa_\gamma(f, x_0) = \infty$ for all $\gamma > 0$. We remark that our terminology here differs slightly from that in [10, Definiton 4.1] where the quantity $\kappa_\gamma(f, x_0)$ is called *asymptotic* Hölder condition number.

Lemma 3.1. *The condition number satisfies*

$$\kappa_\gamma(f, x_0) = \overline{\lim_{\delta\searrow 0}}\ \delta^{-\gamma} F(\delta), \qquad \textit{where} \quad F(\delta) = \sup_{\substack{\|x - x_0\|_X \leq \delta \\ x\in U}} \|f(x) - f(x_0)\|_Y.$$

Proof. For every $\delta, \epsilon > 0$ there exists an $x_\epsilon \in U$ such that $\|x_\epsilon - x_0\|_X \leq \delta$ and $F(\delta) \leq (1+\epsilon)\|f(x_\epsilon) - f(x_0)\|_Y$, whence $\frac{F(\delta)}{\delta^\gamma} \leq (1+\epsilon)\frac{\|f(x_\epsilon) - f(x_0)\|_Y}{\|x_\epsilon - x_0\|_X^\gamma}$. This yields

$$\overline{\lim}_{\delta\searrow 0} \frac{F(\delta)}{\delta^\gamma} \leq \kappa_\gamma(f, x_0). \tag{3.1}$$

By definition of F we have $\|f(x)-f(x_0)\|_Y \leq F(\|x-x_0\|_X)$ for every $x\in U$. Hence

$$\kappa_\gamma(f,x_0) \leq \overline{\lim_{x\to x_0}} \frac{F(\|x-x_0\|_X)}{\|x-x_0\|_X^\gamma} \leq \overline{\lim_{\delta\searrow 0}} \frac{F(\delta)}{\delta^\gamma}. \tag{3.2}$$

The inequalities (3.1) and (3.2) imply the Lemma. □

We now define the condition numbers of eigenvalues. In contrast to the approach in [27, 29] our definition is not based on the Puisseux expansion for perturbed eigenvalues. See however the remark after Theorem 4.1.

Definition 3.2. *For $\lambda\in\mathbb{C}$, $\widetilde{A}\in\mathbb{C}^{n\times n}$ and $m\leq n$ let*

$$d_m(\lambda,\widetilde{A}) := \min\{\,\delta\geq 0;\ \textit{the disk } \mathcal{D}_\lambda(\delta) \textit{ contains at least } m \textit{ eigenvalues of } \widetilde{A}\ \}.$$

In particular for $m=1$, $d_1(\lambda,\widetilde{A}) = \min_{\nu\in\sigma(\widetilde{A})}|\nu-\lambda|$.

Let $(A,B,C)\in L_{n,l,q}$ and let λ be an eigenvalue of A of algebraic multiplicity m. Then the structured Hölder condition number of order $\gamma >$ of the eigenvalue λ is defined as

$$\mathrm{cond}^\gamma_{\mathbf{\Delta}}(A,B,C,\lambda) := \lim_{\delta\searrow 0} \sup_{\substack{\Delta\in\mathbf{\Delta}\\ \|\Delta\|\leq\delta}} \frac{d_m(\lambda,A+B\Delta C)}{\|\Delta\|^\gamma}.$$

Note that $\mathrm{cond}^\gamma_{\mathbf{\Delta}}(A,B,C,\lambda)$ equals the condition number $\kappa_\gamma(f,0)$ of the function $f:\mathbf{\Delta}\to\mathbb{R}$, $f(\Delta)=d_m(\lambda,A+B\Delta C)$. Lemma 3.1 yields

$$\mathrm{cond}^\gamma_{\mathbf{\Delta}}(A,B,C,\lambda) = \overline{\lim_{\delta\searrow 0}}\ \delta^{-\gamma} \sup_{\substack{\Delta\in\mathbf{\Delta}\\ \|\Delta\|\leq\delta}} d_m(\lambda,A+B\Delta C) \tag{3.3}$$

Next, we relate eigenvalue condition numbers to spectral value sets.

Notation 3.3. From now on $\mathcal{C}_\lambda(\delta)$ denotes the connected component of $\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta)$ that contains the eigenvalue λ of A.

Proposition 3.4. *The structured condition number of $\lambda\in\sigma(A)$ satisfies*

$$\mathrm{cond}^\gamma_{\mathbf{\Delta}}(A,B,C,\lambda) = \overline{\lim_{\delta\searrow 0}}\ \delta^{-\gamma} \sup_{s\in\mathcal{C}_\lambda(\delta)} |s-\lambda|.$$

Proof. We set $\mathcal{S}(\delta) := \{A+B\Delta C;\ \ \Delta\in\mathbf{\Delta},\ \|\Delta\|\leq\delta\,\}$. Then $\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta) = \cup_{\widetilde{A}\in S(\delta)}\sigma(\widetilde{A})$. Since $\mathbf{\Delta}$ is a cone the set $\mathcal{S}(\delta)$ is connected. Hence, it follows from claim (d) of Proposition A.2 that each connected component of $\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta)$ contains at least one eigenvalue of A. Thus $\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta) = \cup_{\lambda\in\sigma(A)}\mathcal{C}_\lambda(\delta)$. Let $\epsilon>0$ be such that the disks $\mathcal{D}_\lambda(\epsilon)$, $\lambda\in\sigma(A)$, are pairwise disjoint. If δ is small enough then $\sigma^c_{\mathbf{\Delta}}(A,B,C;\delta)\subset\cup_{\lambda\in\sigma(A)}\mathcal{D}_\lambda(\epsilon)$. This follows from the continuity of eigenvalues (see Proposition A.2 (a)). Consequently, we have for the connected components that $\mathcal{C}_\lambda(\delta)\subseteq\mathcal{D}_\lambda(\epsilon)$. In particular $\mathcal{C}_\lambda(\delta)$ contains no eigenvalue of A different from λ. Let m_λ denote the algebraic multiplicity of λ. It follows from Claim (d) of Proposition A.2 that for each $\widetilde{A}\in S(\delta)$ the set $\mathcal{C}_\lambda(\delta)$ contains precisely m_λ eigenvalues of $\widetilde{A}$ counting algebraic multiplicities (see also [13]). This yields

$$\sup_{\substack{\Delta\in\mathbf{\Delta}\\ \|\Delta\|\leq\delta}} d_{m_\lambda}(\lambda,A+B\Delta C) = \sup_{s\in\mathcal{C}_\lambda(\delta)} |s-\lambda|.$$

Thus, the proposition is a consequence of (3.3). □

4. Main results. Let $\mathcal{X}_\lambda = \ker(A - \lambda I_n)^n$ denote the generalized eigenspaces of $A \in \mathbb{C}^{n\times n}$. Let $P_\lambda \in \mathbb{C}^{n\times n}$, $\lambda \in \sigma(A)$, be the projectors of direct decomposition $\mathbb{C}^n = \bigoplus_{\lambda\in\sigma(A)} \mathcal{X}_\lambda$, i.e. $P_\lambda^2 = P_\lambda$, $\operatorname{range}(P_\lambda) = \mathcal{X}_\lambda$, $\ker(P_\lambda) = \bigoplus_{\lambda\neq\nu\in\sigma(A)} \mathcal{X}_\nu$. Furthermore let $N_\lambda = (A - \lambda I_n)P_\lambda$. Then the matrices N_λ are nilpotent and the spectral representation of A is given by

$$A = \sum_{\lambda\in\sigma(A)} (\, \lambda P_\lambda + N_\lambda \,). \tag{4.1}$$

Let i_λ be the index of nilpotency of N_λ, i.e. $i_\lambda = \min\{\ \ell \geq 0;\ N_\lambda^\ell = 0\ \}$. Then i_λ is the size of the largest Jordan block associated with the eigenvalue λ in the Jordan canonical form of A. If $i_\lambda = 1$ (i.e. $N_\lambda = 0$) then λ is called a semi-simple (non-defective) eigenvalue of A. For any $s \in \rho(A)$ we have

$$(s\, I_n - A)^{-1} = \sum_{\lambda\in\sigma(A)} \left(\frac{P_\lambda}{s-\lambda} + \sum_{\ell=2}^{i_\lambda} \frac{N_\lambda^{\ell-1}}{(s-\lambda)^\ell} \right), \tag{4.2}$$

see e.g. [16, Lemma 4.2.21]. Let B, C be such that $(A, B, C) \in L_{n,l,q}$. Then for $G(s) = C(sI_n - A)^{-1}B$ and $s \in \rho(A)$,

$$G(s) = \sum_{\lambda\in\sigma(A)} \left(\frac{CP_\lambda B}{s-\lambda} + \sum_{\ell=2}^{\ell_\lambda} \frac{CN_\lambda^{\ell-1}B}{(s-\lambda)^\ell} \right), \tag{4.3}$$

where

$$\ell_\lambda := \begin{cases} 1 & \text{if } CN_\lambda^{\ell-1}B = 0 \text{ for all } \ell \geq 2, \\ \max\{\ \ell \geq 2;\ CN_\lambda^{\ell-1}B \neq 0\ \} & \text{otherwise.} \end{cases} \tag{4.4}$$

Obviously $\ell_\lambda \leq i_\lambda$. If $l = q = n$ and the matrices B, C are nonsingular then $\ell_\lambda = i_\lambda$ for all $\lambda \in \sigma(A)$. We denote the leading coefficients in (4.3) by

$$\Gamma_\lambda := \begin{cases} CP_\lambda B & \text{if } \ell_\lambda = 1, \\ CN_\lambda^{\ell_\lambda - 1}B & \text{otherwise.} \end{cases} \tag{4.5}$$

Note that $\Gamma_\lambda = 0$ if and only if $\ell_\lambda = 1$ and $CP_\lambda B = 0$. Next, we introduce the sets

$$\mathcal{L}_\lambda := \{\, z \in \mathbb{C};\ \ z^{\ell_\lambda} \in \sigma(\Delta\,\Gamma_\lambda) \text{ for some } \Delta \in \boldsymbol{\Delta} \text{ with } \|\Delta\| \leq 1 \,\}. \tag{4.6}$$

In words, $\mathcal{L}_\lambda$ is the set of roots of order ℓ_λ of all eigenvalues of the matrix products $\Delta\,\Gamma_\lambda$, where $\Delta \in \boldsymbol{\Delta}$ with $\|\Delta\| \leq 1$.

The theorem below is the main result of this paper. It provides information about the shape and growth of the connected components $\mathcal{C}_\lambda(\delta)$ for small δ. In order not to disturb the flow of exposition we give the technical proof of the theorem in the appendix.

Theorem 4.1. *Let $(A, B, C) \in L_{n,l,q}$ and $\lambda \in \sigma(A)$. Then*

$$\lim_{\delta\searrow 0} \frac{\mathcal{C}_\lambda(\delta) - \lambda}{\delta^{1/\ell_\lambda}} = \mathcal{L}_\lambda, \tag{4.7}$$

where $\mathcal{L}_\lambda$ is given by (4.6) and $\mathcal{C}_\lambda(\delta)$ denotes the connected component of the structured pseudospectrum $\sigma^{\mathrm{c}}_{\boldsymbol{\Delta}}(A, B, C, \delta)$ which contains λ. The limit is taken with respect to the Hausdorff distance of non-empty compact subsets of $\mathbb{C}$.

More explicitely, (4.7) states that to each $\epsilon > 0$ there exists an $\delta_0 > 0$ such that for all positive $\delta \leq \delta_0$,

$$(1) \quad \mathcal{C}_\lambda(\delta) \subset \lambda + \delta^{1/\ell_\lambda}\,\mathcal{U}_\epsilon(\mathcal{L}_\lambda), \qquad (2) \quad \lambda + \delta^{1/\ell_\lambda}\mathcal{L}_\lambda \subset \mathcal{U}_{(\epsilon\,\delta^{1/\ell_\lambda})}(\mathcal{C}_\lambda(\delta)),$$

where $\mathcal{U}_\epsilon(\mathcal{M}) = \{\, z \in \mathbb{C};\ |z-s| < \epsilon$ for some $s \in \mathcal{M}\,\}$ is an ϵ-neighborhood of $\mathcal{M} \subset \mathbb{C}$.

REMARK 4.2. The elements of $\mathcal{L}_\lambda$ are the coefficients in the first terms of the Puisseux epansions of λ with respect to the 1-parameter perturbations $A \rightsquigarrow A_\delta = A + \delta\, B\Delta_0 C,\ \Delta_0 \in \boldsymbol{\Delta},\ \|\Delta_0\| = 1$. See [29] for details. However, Theorem 4.1 is not an immediate consequence of this fact.

We continue with some statements about the limit sets $\mathcal{L}_\lambda$. The next proposition gives a characterization via μ-values. For $\phi \in \mathbb{R}$ let

$$r_\lambda(\phi) := [\, \mu_{\boldsymbol{\Delta}}(\, e^{-i\ell_\lambda\phi}\,\Gamma_\lambda\,)\,]^{1/\ell_\lambda}.$$

PROPOSITION 4.3. *We always have* $\mathcal{L}_\lambda = \{\, r\, e^{i\phi};\ \ \phi\in[0,2\pi),\ 0 \leq r \leq r_\lambda(\phi)\,\}$.

Proof. Obviously, $0 \in \mathcal{L}_\lambda$. For $r > 0$ and $\phi \in \mathbb{R}$ the following chain of equivalences holds.

$$\begin{aligned} r\, e^{i\phi} \in \mathcal{L}_\lambda \Leftrightarrow\ & (r\, e^{i\phi})^{\ell_\lambda} \in \sigma(\Delta\,\Gamma_\lambda) \text{ for some } \Delta \in \boldsymbol{\Delta} \text{ with } \|\Delta\| \leq 1 \\ \Leftrightarrow\ & 1 \in \sigma(\,(r^{-\ell_\lambda}\Delta)\,(e^{-i\ell_\lambda\phi}\Gamma_\lambda)\,) \text{ for some } \Delta \in \boldsymbol{\Delta} \text{ with } \|\Delta\| \leq 1 \\ \Leftrightarrow\ & 1 \in \sigma(\,\Delta\,(e^{-i\ell_\lambda\phi}\Gamma_\lambda)\,) \text{ for some } \Delta \in \boldsymbol{\Delta} \text{ with } \|\Delta\| \leq r^{-\ell_\lambda} \\ \Leftrightarrow\ & \mu_{\boldsymbol{\Delta}}(e^{-i\ell_\lambda\phi}\Gamma_\lambda) \geq r^{\ell_\lambda}. \end{aligned}$$

□

In the following, $R_\lambda \geq 0$ denotes the radius of the smallest disk about 0 that contains the set $\mathcal{L}_\lambda$, i.e.

$$R_\lambda = \sup_{\phi\in[0,2\pi]} r_\lambda(\phi).$$

PROPOSITION 4.4.

(i) If $\boldsymbol{\Delta} = -\boldsymbol{\Delta}$ *then* $r_\lambda(\phi + \pi/\ell_\lambda) = r_\lambda(\phi)$ *for all* $\phi \in \mathbb{R}$. *Thus* $e^{i\pi/\ell_\lambda}\mathcal{L}_\lambda = \mathcal{L}_\lambda$.

(ii) If $\mathbb{C}\boldsymbol{\Delta} = \boldsymbol{\Delta}$ *(i.e.* $\boldsymbol{\Delta}$ *is invariant under complex multiplication) then*

$$r_\lambda(\phi) = \text{const} = R_\lambda = \begin{cases} \mu_{\boldsymbol{\Delta}}(CP_\lambda B) & \text{if } \ell_\lambda = 1, \\ [\,\mu_{\boldsymbol{\Delta}}(CN_\lambda^{\ell_\lambda-1}B)\,]^{1/\ell_\lambda} & \text{otherwise.} \end{cases}$$

Hence, in this case the limit sets are closed disks, $\mathcal{L}_\lambda = \mathcal{D}_0(R_\lambda)$.

(iii) We have $R_\lambda = 0$ *(i.e.* $\mathcal{L}_\lambda = \{0\}$*) if and only if* $\Delta\Gamma_\lambda$ *is nilpotent for all* $\Delta \in \boldsymbol{\Delta}$.

Proof. Obvious. □

The next proposition gives an alternative representation of $\mathcal{L}_\lambda$, which may be useful for its computation.

PROPOSITION 4.5. *Suppose* Γ_λ *has the factorization* $\Gamma_\lambda = XY^*$ *with* $X \in \mathbb{C}^{q\times r}, Y \in \mathbb{C}^{l\times r}$. *Then*

$$\mathcal{L}_\lambda = \{\, z \in \mathbb{C};\ \ z^{\ell_\lambda} \in \sigma(Y^*\Delta X) \textit{ for some } \Delta \in \boldsymbol{\Delta} \textit{ with } \|\Delta\| \leq 1\,\}. \tag{4.8}$$

In particular, if $\operatorname{rank}(\Gamma_\lambda) = 1$ *and* $\Gamma_\lambda = xy^*,\ x \in \mathbb{C}^q, y \in \mathbb{C}^l$, *then*

$$\mathcal{L}_\lambda = \{\, z \in \mathbb{C};\ \ z^{\ell_\lambda} = y^*\Delta x \textit{ for some } \Delta \in \boldsymbol{\Delta} \textit{ with } \|\Delta\| \leq 1\,\}. \tag{4.9}$$

Proof. The matrices $\Delta\Gamma_\lambda = (\Delta X)Y^*$ and $Y^*(\Delta X)$ have the same nonzero eigenvalues. $\square$

The sets (4.9) have been investigated in [24]. It has been shown there that these sets are ellipses in many important cases. The next proposition connects the limit sets to eigenvalue condition numbers.

PROPOSITION 4.6. *The structured condition number of λ to the order $1/\ell_\lambda$ satisfies*

$$\begin{aligned}
\operatorname{cond}_{\mathbf{\Delta}}^{1/\ell_\lambda}(A,B,C,\lambda) &= R_\lambda \\
&= \max_{\substack{\Delta\in\mathbf{\Delta}\\ \|\Delta\|=1}} \left[\varrho(\Delta\Gamma_\lambda)\right]^{1/\ell_\lambda} \\
&= \max_{\substack{\Delta\in\mathbf{\Delta}\\ \|\Delta\|=1}} \left[\varrho(Y^*\Delta X)\right]^{1/\ell_\lambda} \qquad \text{if } \Gamma_\lambda = XY^*.
\end{aligned}$$

Proof. The first identity follows from Proposition (3.4) and Theorem 4.1. The second and the third are consequences of (4.6) and (4.8), respectively. $\square$

REMARK 4.7. As already mentioned in Proposition 4.4 we may have $R_\lambda = 0$. In this case there may be an order $\gamma_0 \neq 1/\ell_\lambda$ such that $\operatorname{cond}_{\mathbf{\Delta}}^{\gamma_0}(A,B,C,\lambda) \notin \{0,\infty\}$. For examples see the introduction of [27]. The order γ_0 can be found via Newton diagrams, see [29]. It is an open question whether $\lim_{\delta\searrow 0} \frac{\mathcal{C}(\delta)-\lambda}{\delta^{\gamma_0}}$ exists with respect to the Hausdorff metric.

5. Examples. We now give some examples that illustrate the results of the former section. In doing so we concentrate on real perturbations, i.e. $\mathbf{\Delta} = \mathbb{R}^{l\times q}$. Throughout this section the underlying norm is the spectral norm. The figures below have been generated using Proposition 4.3 and the formulae (2.6), (2.7).

EXAMPLE 5.1. Let $0 \neq M \in \mathbb{C}^{n\times n}$ be a nilpotent matrix and let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ be a non-real number. We set

$$A = \begin{bmatrix} \Re(\lambda I_n + M) & -\Im(\lambda I_n + M) \\ \Im(\lambda I_n + M) & \Re(\lambda I_n + M)\end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \qquad S = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & I_n \\ -i\,I_n & i\,I_n\end{bmatrix}.$$

Then $S^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & i\,I_n \\ I_n & -i\,I_n\end{bmatrix}$ and $A = S\begin{bmatrix}\lambda I_n + M & 0 \\ 0 & \overline{\lambda I_n + M}\end{bmatrix} S^{-1}$. Thus A has eigenvalues $\lambda, \overline{\lambda}$. The projector onto the generalized eigenspace associated with λ is

$$P_\lambda = S\begin{bmatrix} I_n & 0 \\ 0 & 0\end{bmatrix} S^{-1} = \frac{1}{2}\begin{bmatrix} I_n & i\,I_n \\ -i\,I_n & I_n\end{bmatrix}.$$

The powers of the eigennilpotent $N_\lambda = (A - \lambda I_n)P_\lambda$ satisfy

$$N_\lambda^\ell = \frac{1}{2}\begin{bmatrix} M^\ell & i\,M^\ell \\ -i\,M^\ell & M^\ell\end{bmatrix}, \qquad \ell = 1,2,\ldots. \tag{5.1}$$

We now consider perturbations of A of the form

$$A \rightsquigarrow A_\Delta = \begin{bmatrix} \Re(\lambda I_n + M) + \Delta & -\Im(\lambda I_n + M) \\ \Im(\lambda I_n + M) & \Re(\lambda I_n + M)\end{bmatrix} = A + B\Delta C,$$

where

$$B = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathbb{R}^{2n\times n}, \qquad C = \begin{bmatrix} I_n & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2n}$$

and $\Delta \in \mathbb{R}^{n\times n}$. The relation (5.1) yields that $CN_\lambda^\ell B = M^\ell$. Hence, in this example the number ℓ_λ defined in (4.4) equals the index of nilpotency of M. Furthermore, $\Gamma_\lambda = M^{\ell_\lambda - 1}$. The associated limit sets are

$$\begin{aligned} \mathcal{L}_\lambda &= \{\, z \in \mathbb{C};\ \ z^{\ell_\lambda} \in \sigma(\Delta\, M^{\ell_\lambda-1}) \text{ for some } \Delta \in \mathbb{R}^{n\times n} \text{ with } \|\Delta\| \le 1 \,\} \\ &= \{\, r\, e^{i\phi} \ \mid \phi \in [0, 2\pi),\ 0 \le r \le r_\lambda(\phi) \,\}, \end{aligned}$$

where

$$r_\lambda(\phi) = [\, \mu_{\mathbb{R}^{n\times n}}(e^{-i\,\ell_\lambda\,\phi} M^{\ell_\lambda-1}) \,]^{1/\ell_\lambda}.$$

Figure 5.1 shows the limit sets $\mathcal{L}_\lambda$ for $M = M_{jk}$, $j = 1, 2$, $k = 1, 2, 3$, where

$$M_{1k} = \begin{bmatrix} 0 & Z_k \\ 0 & 0 \end{bmatrix}, \qquad M_{2k} = \begin{bmatrix} 0 & Z_k & 0 \\ 0 & 0 & Z_k \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$Z_1 = \begin{bmatrix} 1-2i & 2-3i \\ -i & 4-3i \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} 4-5i & 1-i \\ 3-3i & -i \end{bmatrix}, \qquad Z_3 = \begin{bmatrix} 3-2i & 2 \\ 1-2i & 5-i \end{bmatrix}.$$

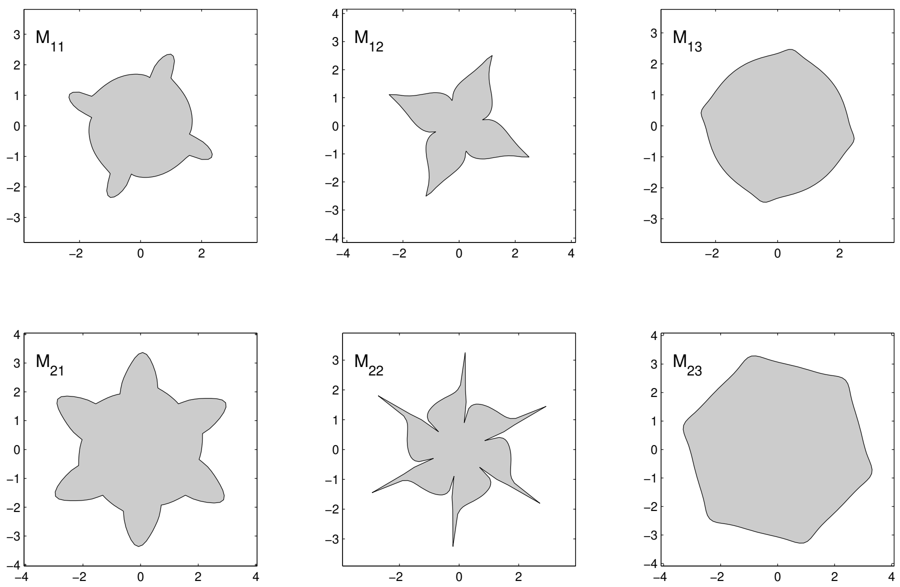

FIG. 5.1. *The limit sets $\mathcal{L}_\lambda$ of the matrices in Example 5.1 .*

EXAMPLE 5.2. Next, we consider real perturbations of a semi-simple eigenvalue $\lambda \in \sigma(A)$ with associated eigenprojector P_λ. We assume that $B = C = I_n$, so our matrix perturbations are of the form

$$A \rightsquigarrow A+\Delta, \qquad \Delta \in \mathbb{R}^{n\times n} \tag{5.2}$$

Since $N_\lambda = 0$ by assumption, we have $\ell_\lambda = 1$ and $\Gamma_\lambda = P_\lambda$. Thus, the associated limit sets are

$$\begin{aligned}\mathcal{L}_\lambda &= \{\, z\in\mathbb{C};\;\; z\in\sigma(\Delta\, P_\lambda) \text{ for some } \Delta\in\mathbb{R}^{n\times n} \text{ with } \|\Delta\|\le 1\,\}\\ &= \{\; r\, e^{i\phi} \;\mid \phi\in[0,2\pi),\; 0\le r\le r_\lambda(\phi)\;\},\end{aligned}$$

where $r_\lambda(\phi) = \mu_{\mathbb{R}^{n\times n}}(e^{-i\,\phi}P_\lambda)$.

The upper row of Figure 5.2 shows the limit sets $\mathcal{L}_\lambda$ for the projectors $P_\lambda = P_{1k}$, $k=1,2,3$, where $P_{1k} = d_k^{-1} X_k Y_k^*$ and

$$X_1 = \begin{bmatrix} -3i & -1+i & 2-3i & -2i\end{bmatrix}^\top,$$

$$Y_1^* = \begin{bmatrix} -4+6i & 18-24i & 6-12i & 6-6i\end{bmatrix},$$

$$X_2 = \begin{bmatrix} -2-i & -1-2i & 2 & -2+3i\\ -1+3i & 0 & 1-2i & i\end{bmatrix}^\top,$$

$$Y_2^* = \begin{bmatrix} -4-2i & 8-4i & -4-2i & 4+2i\\ -20+14i & -8-4i & -36+14i & -12+2i\end{bmatrix},$$

$$X_3 = \begin{bmatrix} -2-3i & -i & -3-4i & -i & -2-3i & -1\\ -1+i & 2 & -2 & -1 & 4+i & 3+i\end{bmatrix}^\top,$$

$$Y_3^* = \begin{bmatrix} -36+296i & -60+280i & 12-112i & -480i & -80i & 36-216i\\ -4+36i & -140+60i & -52-12i & 240-120i & 40 & 84-36i\end{bmatrix},$$

$$d_1 = -12, \qquad d_2 = -32, \qquad d_3 = 240i.$$

The projectors P_{1k} satisfy $P_{1k}\overline{P_{1k}} = 0$. Hence, they are the eigenprojectors to the eigenvalue $\lambda\in\mathbb{C}\setminus\mathbb{R}$ of any real matrix A with spectral representation

$$A = \lambda\, P_{1k} + \bar\lambda\,\overline{P_{1k}} + \sum_{\nu\in\sigma(A)\setminus\{\lambda,\bar\lambda\}} (\nu\, P_\nu + N_\nu).$$

The lower row of Figure 5.2 shows the limit sets for the real projectors

$$P_{21} = \begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}, \qquad P_{22} = \frac{1}{2}\begin{bmatrix} 1 & 1/2\\ 2 & 1\end{bmatrix}, \qquad P_{23} = diag(P_{21}, P_{22}).$$

The depicted limit sets for these projectors can be easily computed using formula (2.7): We have $\sigma_1(P_{23}) = \sigma_1(P_{22}) = 2$, $\sigma_2(P_{23}) = \sigma_1(P_{21}) = 1$, $\sigma_2(P_{21}) = \sigma_1(P_{22}) = 0$ and

$$r_\lambda(\phi) = \mu_{\mathbb{R}^{n\times n}}(e^{-i\,\phi}P_{2k}) = \begin{cases} \sigma_1(P_{2k}) & \text{if } \phi\in\{0,\pi\},\\ \sqrt{\sigma_1(P_{2k})\,\sigma_2(P_{2k})} & \text{if } 0<\phi<2\pi,\ \phi\neq\pi.\end{cases}$$

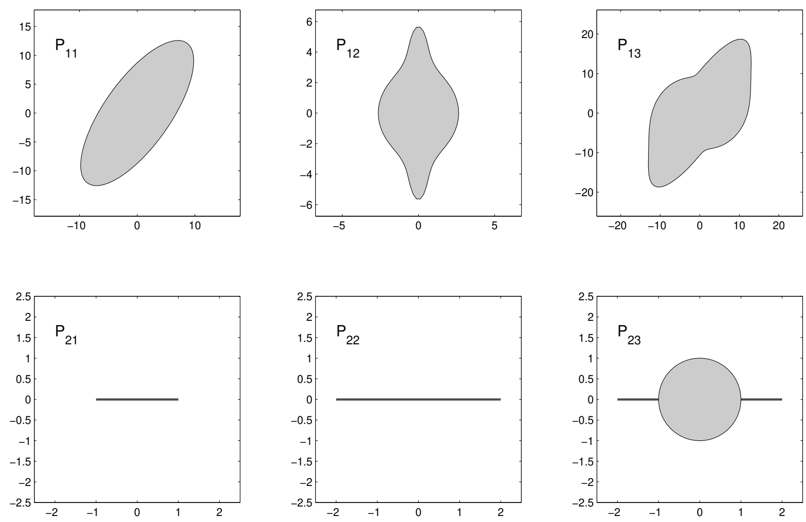

FIG. 5.2. *Limit sets of the projectors in Example 5.2*

EXAMPLE 5.3. Suppose the matrix $A \in \mathbb{R}^{n\times n}$ has the spectral representation

$$A = (\lambda P_\lambda + N_\lambda) + \sum_{\nu\in\sigma(A)\setminus\{\lambda\}} (\nu P_\nu + N_\nu),$$

where $\lambda, P_\lambda, N_\lambda$ are real and $N_\lambda \neq 0$. Then the limit set $\mathcal{L}_\lambda$ with respect to perturbations of the form (5.2) is given by

$$\begin{aligned}\mathcal{L}_\lambda &= \{\, z\in\mathbb{C};\;\; z\in\sigma(\Delta\, N_\lambda^{\ell_\lambda-1}) \text{ for some } \Delta\in\mathbb{R}^{n\times n} \text{ with } \|\Delta\|\le 1\,\}\\ &= \{\, r\,e^{i\phi};\;\; \phi\in[0,2\pi),\; 0\le r\le r_\lambda(\phi)\,\},\end{aligned}$$

where ℓ_λ is the index of nilpotency of N_λ and

$$r_\lambda(\phi) = \mu_{\mathbb{R}^{n\times n}}(e^{-i\,\ell_\lambda\phi} N_\lambda^{\ell_\lambda-1}) = \begin{cases}\sigma_1(N_\lambda^{\ell_\lambda-1}) & \text{if } \phi\in\{0,\pi\},\\ \sqrt{\sigma_1(N_\lambda^{\ell_\lambda-1})\,\sigma_2(N_\lambda^{\ell_\lambda-1})} & \text{if } 0<\phi<2\pi,\ \phi\neq\pi.\end{cases}$$

Figure 5.3 shows the limit sets $\mathcal{L}_\lambda$ for the cases $N_\lambda = N_1, N_2, N_3$, where

$$N_1 = \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\qquad N_2 = \begin{bmatrix}0&2&0\\0&0&2\\0&0&0\end{bmatrix},\qquad N_3 = diag(N_1, N_2).$$

Appendix A. In this section we derive some facts related to the continuity of eigenvalues and prove our main result Theorem 4.1.

Zero sets of holomorphic functions. Let (X,d) be a metric space, and let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X. The Hausdorff distance of

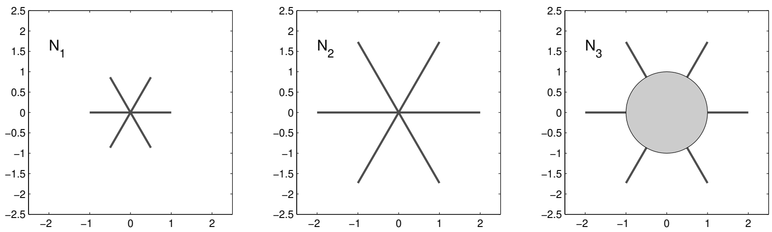

FIG. 5.3. *Limit sets for the nilpotent matrices in Example 5.3.*

$\mathcal{S}, \widetilde{\mathcal{S}} \in \mathcal{K}(X)$ is defined by

$$d_H(\mathcal{S}, \widetilde{\mathcal{S}}) \;:=\; \max\left\{ \max_{x\in\mathcal{S}} \min_{\widetilde{x}\in\widetilde{\mathcal{S}}} d(x,\widetilde{x}),\; \max_{\widetilde{x}\in\widetilde{\mathcal{S}}} \min_{x\in\mathcal{S}} d(\widetilde{x},x) \right\}.$$

Recall that $d_H(\cdot,\cdot)$ is a metric on $\mathcal{K}(X)$. In the following the Hausdorff distance of two sets in $\mathcal{K}(\mathbb{C})$ is induced by the metric $d_{\mathbb{C}}(z,\widetilde{z}) = |z-\widetilde{z}|$, $z,\widetilde{z}\in\mathbb{C}$. The lemma below yields the basis for the proof of Theorem 4.1 as well as for the statements concerning the continuity of eigenvalues that have been used in Sections 2 and 3.

LEMMA A.1. *Let (X,d) be a metric space, and let Ω be a non-empty open subset of $\mathbb{C}$. Let $f : X\times\Omega \to \mathbb{C}$ be a continuous map such that for any $x \in X$, the function $f(x,\cdot) : \Omega \to \mathbb{C}$ is holomorphic and non-constant. For any subset $\mathcal{S}$ of X let*

$$\mathcal{Z}(\mathcal{S}) := \{\, z \in \mathbb{C};\;\; f(x,z) = 0 \text{ for some } x \in \mathcal{S} \,\}.$$

Then the following holds.

(*a*) *Let $z \in \Omega$ be a zero of multiplicity m of the function $f(x,\cdot)$, $x \in X$. Then there is an $\epsilon_0 > 0$ such that the disk $\mathcal{D}_z(\epsilon_0)$ contains no zero of $f(x,\cdot)$ different from z. To each $\epsilon \in (0,\epsilon_0]$ there exists a $\delta > 0$ such that for all $\widetilde{x} \in X$ satisfying $d(\widetilde{x},x) \le \delta$ the disk $\mathcal{D}_z(\epsilon)$ contains precisely m zeros of $f(\widetilde{x},\cdot)$, counting multiplicities.*

(*b*) *Let $\mathcal{S}$ be a subset of X such that $cl(\mathcal{S})$ is compact. Then $cl(\mathcal{Z}(\mathcal{S})) = \mathcal{Z}(cl(\mathcal{S}))$, where $cl(\cdot)$ denotes the topological closure.*

(*c*) *Assume that each $f(x,\cdot)$ has at least one zero. Assume further that to any bounded subset $\mathcal{S}$ of X there exists a compact subset K of Ω such that $\mathcal{Z}(\mathcal{S}) \subseteq K$. Then the map*

$$\mathcal{Z} : \mathcal{K}(X) \to \mathcal{K}(\mathbb{C}), \qquad \mathcal{S} \overset{\mathcal{Z}}{\longmapsto} \mathcal{Z}(\mathcal{S})$$

is well defined and continuous.

(*d*) *Suppose all $f(x,\cdot)$ have the same (finite) number of zeros, counting multiplicites. Let $\mathcal{S}$ be a connected subset of X and let $\mathcal{C}$ be a connected component of $\mathcal{Z}(\mathcal{S})$. Suppose there is an $x \in \mathcal{S}$ such that precisely m zeros of $f(x,\cdot)$ are contained in $\mathcal{C}$, counting multiplicities. Then for any $\widetilde{x} \in \mathcal{S}$, precisely m zeros of $f(\widetilde{x},\cdot)$ are contained in $\mathcal{C}$, counting multiplicities.*

Proof. (a). By elementary function theory the zeros of the non-constant holomorphic function $f(x,\cdot)$ are isolated points. Hence, if $\epsilon > 0$ is small enough then $f(x,\zeta) \neq 0$ for all $\zeta \in \mathcal{D}_z(\epsilon)\setminus\{z\}$. Since f is continuous and $\partial\mathcal{D}_z(\epsilon)$ is compact, there is a $\delta > 0$ such that $d(x, \widetilde{x}) \leq \delta$ implies $|f(\widetilde{x}, \zeta) - f(x, \zeta)| < |f(x, \zeta)|$ for all $\zeta \in \partial\mathcal{D}_z(\epsilon)$. Then, by Rouche's theorem $f(\widetilde{x}, \cdot)$ and $f(x, \cdot)$ have the same number of zeros in $\mathcal{D}_z(\epsilon)$.

Below we will use the following corollary of (a).

(i) Let $f(x, z) = 0$. Then to each $\epsilon > 0$ there exists a $\delta(x, z, \epsilon) > 0$ such that $f(\widetilde{x}, \cdot)$ has a zero in $\mathcal{D}_z(\epsilon)$ whenever $d(x, \widetilde{x}) \leq \delta(x, z, \epsilon)$.

(b). Let $z \in \mathcal{Z}(cl(\mathcal{S}))$, i.e. $f(x, z) = 0$ for some $x \in cl(\mathcal{S})$. Let $\epsilon > 0$ and let $\widetilde{x} \in \mathcal{S}$ with $d(x, \widetilde{x}) < \delta(x, z, \epsilon)$. Then $f(\widetilde{x}, \cdot)$ has a zero in $\mathcal{D}_z(\epsilon)$. This yields $\mathcal{Z}(cl(\mathcal{S})) \subseteq cl(\mathcal{Z}(\mathcal{S}))$.

Let $z \in cl(\mathcal{Z}(\mathcal{S}))$. Then there are sequences $x_j \in \mathcal{S}$, $z_j \in \Omega$ with $\lim z_j = z$ and $f(x_j, z_j) = 0$. Since $cl(\mathcal{S})$ is compact a subsequence x_{j_k} of x_j converges to some $x \in cl(\mathcal{S})$. By continuity we have $f(x, z) = 0$. This yields $cl(\mathcal{Z}(\mathcal{S})) \subseteq \mathcal{Z}(cl(\mathcal{S}))$.

(c). A compact subset of a metric space is closed and bounded. Hence, if $\mathcal{S} \subseteq X$ is compact then (b) yields that $\mathcal{Z}(\mathcal{S})$ is closed. By assumption, $\mathcal{Z}(\mathcal{S})$ is contained in a compact set $K \subset \Omega$. Thus, $\mathcal{Z}(\mathcal{S})$ is compact. Thus we have shown that the function $\mathcal{S} \mapsto \mathcal{Z}(\mathcal{S})$ maps compact sets to compact sets.

In the following $\mathcal{S}_\epsilon$ and $\mathcal{Z}_\epsilon(\mathcal{S})$ denote closed ϵ-neighborhoods of $\mathcal{S} \subseteq X$ and $\mathcal{Z}(\mathcal{S})$, i.e.

$$\mathcal{S}_\epsilon = \{\widetilde{x} \in X;\ \ d(x, \widetilde{x}) \leq \epsilon \text{ for some } x \in \mathcal{S}\,\},$$
$$\mathcal{Z}_\epsilon(\mathcal{S}) = \{\widetilde{z} \in \Omega;\ \ |z - \widetilde{z}| \leq \epsilon \text{ for some } z \in \mathcal{Z}(\mathcal{S})\,\}.$$

The continuity of the function $\mathcal{S} \mapsto \mathcal{Z}(\mathcal{S})$, $\mathcal{S} \in \mathcal{K}(X)$, is immediate from the statements (ii) and (iii) below. Let $\mathcal{S} \in \mathcal{K}(X)$. Then

(ii) to each $\epsilon > 0$ there exists a $\delta > 0$ such that for any subset $\widetilde{S}$ of X, $\widetilde{\mathcal{S}} \subseteq \mathcal{S}_\delta$ implies $\mathcal{Z}(\widetilde{\mathcal{S}}) \subseteq \mathcal{Z}_\epsilon(\mathcal{S})$;

(iii) to each $\epsilon > 0$ there exists a $\delta > 0$ such that for any subset $\widetilde{S}$ of X, $\mathcal{S} \subseteq \widetilde{\mathcal{S}}_\delta$ implies $\mathcal{Z}(\mathcal{S}) \subseteq \mathcal{Z}_\epsilon(\widetilde{\mathcal{S}})$.

Proof of (ii): Suppose the statement fails. Then there are an $\epsilon > 0$ and sequences $\widetilde{x}_j \in X$, $x_j \in \mathcal{S}$ and $\widetilde{z}_j \in \Omega$ such that

$$d(\widetilde{x}_j, x_j) < 1/j, \text{ and } f(\widetilde{x}_j, \widetilde{z}_j) = 0, \text{ and } |\widetilde{z}_j - z| \geq \epsilon \text{ for all } z \in \mathcal{Z}(\mathcal{S}). \tag{A.1}$$

Since $\mathcal{S}$ is compact and compact sets are bounded, the sequence $\widetilde{x}_j$ is bounded too. Hence, by the assumption made in (c), all $\widetilde{z}_j$ are contained in a compact set $K \subset \Omega$. By compactness there are converging subsequences $x_{j_k} \to x \in \mathcal{S}$ and $\widetilde{z}_{j_k} \to \widetilde{z} \in K$. From (A.1) it follows that $\widetilde{x}_{j_k} \to x$, and $f(x, \widetilde{z}) = 0$ (hence $\widetilde{z} \in \mathcal{Z}(\mathcal{S})$), and $|\widetilde{z} - z| \geq \epsilon$ for all $z \in \mathcal{Z}(\mathcal{S})$, a contradiction.

Proof of (iii): By compactness we have $\mathcal{Z}(\mathcal{S}) \subset \cup_{j=1}^{q} \mathcal{D}_{z_j}(\epsilon/2)$ for some $z_1, \ldots, z_q \in \mathcal{Z}(\mathcal{S})$. Let $x_j \in \mathcal{S}$ be such that $f(x_j, z_j) = 0$ and let $\delta = \min_j \delta(x_j, z_j, \epsilon/2)$, where $\delta(x_j, z_j, \epsilon/2)$ satisfies (i). To each j there is some $\widetilde{x}_j \in \widetilde{\mathcal{S}}$ with $d(\widetilde{x}_j, x_j) \leq \delta$ since we assume that $\mathcal{S} \subseteq \widetilde{\mathcal{S}}_\delta$. By definition of δ there exists to each j a $\widetilde{z}_j \in \mathcal{D}_{z_j}(\epsilon/2)$ with $f(\widetilde{x}_j, \widetilde{z}_j) = 0$. Hence $z_j \in \mathcal{Z}_{\epsilon/2}(\widetilde{\mathcal{S}})$, which implies $\mathcal{D}_{z_j}(\epsilon/2) \subset \mathcal{Z}_\epsilon(\widetilde{\mathcal{S}})$. This yields (iii).

(d). Let n denote the constant number of zeros of the functions $f(x, \cdot)$. A connected component $\mathcal{C}$ of $\mathcal{Z}(\mathcal{S})$ is closed and open with respect to the topology on $\mathcal{Z}(\mathcal{S})$ induced by the topology on $\mathbb{C}$. Hence there are open subsets $\mathcal{U}, \mathcal{V}$ of $\mathbb{C}$ such that $\mathcal{C} = \mathcal{Z}(\mathcal{S}) \cap \mathcal{U}$ and $\mathcal{Z}(\mathcal{S}) \setminus \mathcal{C} = \mathcal{Z}(\mathcal{S}) \cap \mathcal{V}$. Let $\mathcal{X}_m$ be the set of $\widetilde{x} \in X$ such that at least m zeros of $f(\widetilde{x}, \cdot)$ are contained in $\mathcal{U}$ and at least $n - m$ zeros are contained in $\mathcal{V}$, counting multiplicities. Claim (a) yields that the sets $\mathcal{X}_m$ are open subsets of X. Hence the sets $\mathcal{W}_m := \mathcal{X}_m \cap \mathcal{S}$ are open subsets of $\mathcal{S}$. Furthermore, $\mathcal{W}_m$ is the set of $\widetilde{x} \in \mathcal{S}$ such that at least m zeros of $f(\widetilde{x}, \cdot)$ are contained in $\mathcal{C}$ and at least $n - m$ zeros are contained in $\mathcal{Z}(\mathcal{S}) \setminus \mathcal{C}$. However, since each $f(\widetilde{x},\cdot)$ has n zeros, it follows that $\mathcal{W}_m$ is the set of $\widetilde{x} \in \mathcal{S}$ such that precisely m zeros of $f(\widetilde{x},\cdot)$ are contained in $\mathcal{C}$. Hence the sets $\mathcal{W}_m$ are pairwise disjoint and form an open covering of $\mathcal{S}$. Since $\mathcal{S}$ is connected it follows that $\mathcal{W}_m = \mathcal{S}$ for some m. □

Remark A.1. The following example shows that the assumption made in Claim (c) of Lemma A.1 is necessary for the well definedness and the continuity of the map $\mathcal{K}(X) \ni \mathcal{S} \mapsto \mathcal{Z}(\mathcal{S}) \in \mathcal{K}(\mathbb{C})$.

Let $X_1 = [-1,1]$, $X_2 = [1,2]$, $\Omega_1 = \mathbb{C}$, $\Omega_2 = \{z \in \mathbb{C};\ |z| < 1\}$ and

$$\mathcal{Z}_j(\mathcal{S}) = \{\, z \in \Omega_j;\ \ f(x,z) = 0 \ \text{ for some } x \in \mathcal{S}\,\}, \quad \mathcal{S} \subseteq X_j, \ \ j = 1,2,$$

where $f(x,z) = z(1-xz)$. Then for $x \in X_j$,

$$\mathcal{Z}_1(\{x\}) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{0, 1/x\} & \text{otherwise,} \end{cases} \qquad \mathcal{Z}_2(\{x\}) = \begin{cases} \{0\} & \text{if } x = 1, \\ \{0, 1/x\} & \text{otherwise.} \end{cases}$$

In both cases X_j is bounded and $\mathcal{Z}_j(X_j)$ is not contained in a compact subset of Ω_j. Furthermore, $\mathcal{Z}_1(X_1)$ is not compact, $\lim_{x\to 0} \mathcal{Z}_1(\{x\})$ does not exist, and $\lim_{x\to 1} \mathcal{Z}_2(\{x\}) = \{0,1\} \neq \mathcal{Z}_2(\{1\})$.

Continuity of eigenvalues. Next, we consider eigenvalues of matrix sets. For $\mathcal{S} \subseteq \mathbb{C}^{n\times n}$ let

$$\sigma(\mathcal{S}) := \bigcup_{A\in\mathcal{S}} \sigma(A) = \{\, z \in \mathbb{C};\ \ z \in \sigma(A) \text{ for some } A \in \mathcal{S}\,\}.$$

Let $\|\cdot\|$ be a norm on $\mathbb{C}^{n\times n}$. Then $d(A,\widetilde{A}) = \|A - \widetilde{A}\|$ is a metric on $\mathbb{C}^{n\times n}$. If $\mathcal{S}$ is bounded, i.e. $\|A\| \le r$ for all $A \in \mathcal{S}$ and a fixed $r > 0$, then $\sigma(\mathcal{S})$ is contained in the compact disk $\mathcal{D}_0(R)$, where $R = \max\{\varrho(A);\ A \in \mathbb{C}^{n\times n},\ \|A\| \le r\}$. Hence, all statements of the proposition below follow by specializing Lemma A.1 to the function $f(A,z) = \det(zI - A)$.

Proposition A.2.

(a) *Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n\times n}$ of algebraic multiplicity m. Let $\epsilon > 0$ be such that the disks $\mathcal{D}_z(\epsilon)$ contains no eigenvalue of A different from λ. Then there is a $\delta > 0$ such that for all $\widetilde{A} \in \mathbb{C}^{n\times n}$ satisfying $\|A - \widetilde{A}\| \le \delta$ the disk $\mathcal{D}_\lambda(\epsilon)$ contains precisely m eigenvalues of $\widetilde{A}$, counting multiplicities.*

(b) *Let $\mathcal{S}$ be a a bounded subset of $\mathbb{C}^{n\times n}$. Then $cl(\sigma(\mathcal{S})) = \sigma(cl(\mathcal{S}))$.*

(c) *If $\mathcal{S}$ is compact then $\sigma(\mathcal{S})$ is also compact. Furthermore, the map*

$$\sigma : \mathcal{K}(\mathbb{C}^{n\times n}) \to \mathcal{K}(\mathbb{C}), \qquad \mathcal{S} \overset{\sigma}{\longmapsto} \sigma(\mathcal{S})$$

is continuous.

(d) *Let $\mathcal{S}$ be a connected subset of $\mathbb{C}^{n\times n}$ and let $\mathcal{C} \subseteq \mathbb{C}$ be a connected component of $\sigma(\mathcal{S})$. Suppose there is an $A \in \mathcal{S}$ such that $\mathcal{C}$ contains precisely m eigenvalues of A counting algebraic multiplicities. Then for any $\widetilde{A} \in \mathcal{S}$ the set $\mathcal{C}$ contains precisely m eigenvalues of $\widetilde{A}$.*

Proof of Theorem 4.1. By (4.4) and (4.5) the rational matrix function $G(s) = C(sI - A)^{-1}B$ can be written as

$$G(s) = \sum_{\nu\in\sigma(A)} \left(\frac{CP_\nu B}{s-\nu} + \sum_{\ell=2}^{\ell_\nu} \frac{CN_\nu^{\ell-1}B}{(s-\nu)^\ell} \right) = (s-\lambda)^{-\ell_\lambda}(\Gamma_\lambda + H(s)),$$

where

$$\Gamma_\lambda = \begin{cases} CP_\lambda B & \text{if } \ell_\lambda = 1, \\ CN_\lambda^{\ell_\lambda - 1} B & \text{otherwise,} \end{cases}$$

and

$$H(s) := H_0(s) + \sum_{\nu \in \sigma(A)\setminus\{\lambda\}} (s-\lambda)^{\ell_\lambda} \left(\frac{CP_\nu B}{s-\nu} + \sum_{\ell=2}^{\ell_\nu} \frac{CN_\nu^{\ell-1}B}{(s-\nu)^\ell} \right),$$

$$H_0(s) := \begin{cases} 0 & \text{if } \ell_\lambda = 1, \\ (s-\lambda)^{\ell_\lambda - 1} CP_\lambda B + \sum_{\ell=2}^{\ell_\lambda - 1} (s-\lambda)^{\ell_\lambda - \ell} CN_\lambda^{\ell-1} B & \text{otherwise.} \end{cases}$$

Note that $H(\lambda) = 0$. According to (2.5) the following equivalence holds for all $s \in \mathbb{C} \setminus \sigma(A)$ and any $\Delta \in \mathbb{C}^{l \times q}$.

$$s \in \sigma(A + B\Delta C) \;\Leftrightarrow\; 1 \in \sigma(\Delta G(s)) \;\Leftrightarrow\; (s-\lambda)^{\ell_\lambda} \in \sigma(\Delta(\Gamma_\lambda + H(s))) \tag{A.2}$$

Let $R > 0$ be such that the disk $\mathcal{D}_\lambda(R)$ contains no eigenvalue of A different from λ. Then the function $s \mapsto H(s)$ is holomorphic in an open neighborhood of $\mathcal{D}_\lambda(R)$. Let $\mathcal{B} := \{\, \Delta_0 \in \mathbf{\Delta};\ \|\Delta_0\| \le 1 \,\}$ and let $r_1, r_2 > 0$ be such that

$$\max\{\, \varrho(\Delta_0\, (\Gamma_\lambda + H(s)));\ \ \Delta_0 \in \mathcal{B},\ s \in \mathcal{D}_\lambda(R) \,\} < r_1 < r_2. \tag{A.3}$$

Then, since $\mathbf{\Delta}$ is a cone, we have for any $\delta > 0$,

$$\max\{\, \varrho(\Delta\, (\Gamma_\lambda + H(s)));\ \ \Delta \in \mathbf{\Delta},\ \|\Delta\| \le \delta,\ s \in \mathcal{D}_\lambda(R) \,\} < r_1\, \delta < r_2\, \delta. \tag{A.4}$$

Suppose $s \in \sigma(A + B\Delta C) \cap \mathcal{D}_\lambda(R)$ for some $\Delta \in \mathbf{\Delta}$ with $\|\Delta\| \le \delta$. Then (A.2) and (A.4) yield $|s-\lambda|^{\ell_\lambda} < r_1\, \delta$. Hence,

$$\sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta) \cap \mathcal{D}_\lambda(R) \subset K_1(\delta) \subset K_2(\delta), \tag{A.5}$$

where $K_j(\delta) := \{\, s \in \mathbb{C};\ \ |s-\lambda|^{\ell_\lambda} < r_j \delta \,\}$, $j = 1, 2$. Suppose $0 < \delta \le R^{\ell_\lambda}/r_2$. Then $K_1(\delta) \subset K_2(\delta) \subset \mathcal{D}_\lambda(R)$. Hence, by (A.5),

$$\sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta) \cap \mathcal{D}_\lambda(R) = \sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta) \cap K_j(\delta), \quad j = 1, 2.$$

It follows that $\sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta) \cap \partial K_1(\delta) = \emptyset$. The latter implies that each connected component of $\sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta) \cap K_1(\delta)$ is also a connected component of $\sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta)$. However, since each connected component of $\sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta)$ contains at least one eigenvalue of A and λ is the only eigenvalue in $K_1(\delta) \subset \mathcal{D}_\lambda(R)$, we have

$$\mathcal{C}_\lambda(\delta) = \sigma^{c}_{\mathbf{\Delta}}(A,B,C;\delta) \cap K_j(\delta), \quad j = 1, 2. \tag{A.6}$$

Next, we want to apply Lemma A.1. To this end we define a metric space (X, d) by $X := [0, R^{\ell_\lambda}/r_2] \times \mathcal{B}$ and $d((\delta_1, \Delta_1), (\delta_2, \Delta_2)) := |\delta_1 - \delta_2| + \|\Delta_1 - \Delta_2\|$. Furthermore, we set $\Omega_j := \{z \in \mathbb{C};\ |z| < r_j\}$, $j = 1, 2$, and define a continuous family of holomorphic functions by

$$f : X \times \Omega_2 \to \mathbb{C}, \qquad f((\delta, \Delta_0), z) = \det(z^{\ell_\lambda} I - \Delta_0(\Gamma_\lambda + H(\lambda + \delta^{1/\ell_\lambda} z))).$$

Note that for $\delta \ne 0$ the map $z \mapsto \lambda + \delta^{1/\ell_\lambda} z$ is a bijection between Ω_j and $K_j(\delta)$, $j = 1, 2$. On replacing in (A.2) s by $\lambda + \delta^{1/\ell_\lambda} z$ and Δ by $\delta \Delta_0$ with $\Delta_0 \in \mathcal{B}$ we obtain that the following statements (a) and (b) are equivalent for $z \ne 0$, $\delta \ne 0$ and $j = 1, 2$.

(a) $z \in \Omega_j$ and $f((\delta, \Delta_0), z) = 0$.
(b) $\lambda + \delta^{1/\ell_\lambda} z \in K_j(\delta)$ and $\lambda + \delta^{1/\ell_\lambda} z \in \sigma(A + B(\delta\, \Delta_0)C)$.

The statements (a) and (b) both hold if $\Delta_0 = 0$ and $z = 0$. For any non-empty subset $\mathcal{S}$ of X let

$$\mathcal{Z}(\mathcal{S}) := \{\, z \in \mathbb{C};\ \ f((\delta, \Delta_0), z) = 0 \ \text{ for some } (\delta, \Delta_0) \in \mathcal{S} \,\}.$$

Then from (A.6) and the equivalence of (a) and (b) we obtain

$$\mathcal{C}_\lambda(\delta) = \lambda + \delta^{1/\ell_\lambda}\, \mathcal{Z}(\{\delta\} \times \mathcal{B}).$$

Note that the set $\mathcal{Z}(X) = \bigcup_{0 \le \delta \le R^{\ell_\lambda}/r_2} \mathcal{Z}(\{\delta\} \times \mathcal{B})$ is contained in the closure of Ω_1, which is a compact subset of Ω_2. As δ tends to 0 the compact set $\{\delta\} \times \mathcal{B}$ tends to $\{0\} \times \mathcal{B}$ with respect to the Hausdorff metric induced by d. Thus, by Claim (c) of Lemma A.1,

$$\lim_{\delta \to 0} \frac{\mathcal{C}_\lambda(\delta) - \lambda}{\delta^{1/\ell_\lambda}} = \lim_{\delta \to 0} \mathcal{Z}(\{\delta\} \times \mathcal{B}) = \mathcal{Z}(\{0\} \times \mathcal{B}) = \mathcal{L}_\lambda. \qquad \Box$$

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