A Derivative Array Approach for Linear Second Order Differential-Algebraic Systems*

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Abstract

We discuss the solution of linear second order differential-algebraic equations with variable coefficients. Since index reduction and order reduction for higher order higher index differential-algebraic systems do not commute, appropriate index reduction methods for higher order DAEs are required. We present an index reduction method based on derivative arrays that allows to determine an equivalent second order system of lower index in a numerical computable way. For such an equivalent second order system an appropriate order reduction method allows to formulate a suitable first order DAE system of low index that has the same solution components as the original second order system.

Keywords: differential-algebraic equation, second order system, index reduction, order reduction, strangeness index, strangeness-free system.

AMS subject classification: 65L80, 65L05, 34A09, 34A12

1 Introduction

Linear second order differential-algebraic equations of the form

$$M(t)\ddot{x} + C(t)\dot{x} + K(t)x = f(t), \quad t \in \mathbb{I}, \tag{1}$$

where $M, C, K \in C(\mathbb{I}, \mathbb{C}^{m\times n})$ and $f \in C(\mathbb{I}, \mathbb{C}^m)$ are sufficiently smooth on a compact interval $\mathbb{I} \subseteq \mathbb{R}$ with initial conditions

$$x(t_0) = x_0 \in \mathbb{C}^n, \quad \dot{x}(t_0) = \dot{x}_0 \in \mathbb{C}^n \quad \text{for } t_0 \in \mathbb{I}, \tag{2}$$

naturally arise in many technical applications as, e.g., in the simulation of electrical circuits [8, 9] or mechanical multibody systems [6, 14]. Here, $C^k(\mathbb{I}, \mathbb{C}^{m,n})$ denotes the set of k-times continuously differentiable functions from the interval $\mathbb{I}$ to the vector space of complex $m \times n$ matrices $\mathbb{C}^{m,n}$.
For the numerical (as well as analytical) solution, second order systems of the form (1) are usually transformed into first order systems by introducing new variables for the derivatives, as is the common practice in the classical theory of ordinary differential equations. For

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DAEs, however, the classical order reduction approach has to be performed with great care, since it may lead to a number of mathematical difficulties as has been discussed in several publications, see [1, 5, 13, 15, 16]. Moreover, the numerical solution of DAEs usually requires the reformulation of higher index DAEs as an equivalent system of lower index to be able to use standard integration methods suited for DAEs, see e.g. [2, 10, 11]. Therefore, the numerical solution of linear second order differential-algebraic systems of the form (1) typically requires the reduction to a first order system on the one hand and an index reduction for higher index systems on the other hand. But, for high order high index differential-algebraic systems the order reduction and index reduction do not commute as can be seen in the following example.

Example 1. We consider the linear second order system

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 1 \\ 0 & t & t \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1+t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \tag{3}$$

for $t \in [t_0, t_1]$ with $t_0 > 0$. System (3) has the unique solution components

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_2 - \ddot{f}_3 + t\ddot{f}_2 + 2\dot{f}_2 \\ f_3 - (1+t)f_2 + \ddot{f}_3 - t\ddot{f}_2 - 2\dot{f}_2 \end{bmatrix},$$

and x_1 is the solution of the second order ordinary differential equation $t\ddot{x}_1 + \dot{x}_1 + x_1 = f_1$ for given initial values $x_1(t_0) = x_{1,0}$ and $\dot{x}_1(t_0) = \dot{x}_{1,0}$. Hence, the minimum requirement for the existence of a continuous solution is that f_1 is continuous and f_2 and f_3 are twice continuously differentiable (corresponding to a strangeness index of $\mu = 2$). The classical order reduction for the second order system (3) yields a first order system of the form

$$\left[\begin{array}{ccc|ccc} t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & t & t & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1+t & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

In comparison to the solution of (3), this system has the additional solution components

$$\begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \dot{f}_2 - f_3^{(3)} + tf_2^{(3)} + 3\ddot{f}_2 \\ \dot{f}_3 - (t+1)\dot{f}_2 - f_2 + f_3^{(3)} - tf_2^{(3)} - 3\ddot{f}_2 \end{bmatrix},$$

i.e., the third derivative of the inhomogeneity is required (system (4) is of strangeness index $\tilde{\mu} = 3$). On the other hand, system (3) is equivalent to the system

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 - \ddot{f}_3 + (tf_2)'' \\ f_3 - tf_2 - f_2 + \ddot{f}_3 - (tf_2)'' \end{bmatrix},$$

a decoupled system of two algebraic equations and one differential equation. Now, introducing only $v_1 = \dot{x}_1$ as new variable, we get the first order system

$$\begin{bmatrix} t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 - \ddot{f}_3 + (tf_2)'' \\ f_3 - tf_2 - f_2 + \ddot{f}_3 - (tf_2)'' \\ 0 \end{bmatrix},$$

which is also a decoupled system of algebraic and differential equations and no further smoothness requirements are imposed.

Further examples are presented in [12] that show that the classical approach of introducing the derivatives of the unknown vector-valued function $x(t)$ as new variables may lead to higher smoothness requirements for the inhomogeneity $f(t)$ to ensure the existence of a solution that can even cause the loss of solvability of the system. By introducing only some new variables, however, this difficulty can be circumvented. An index reduction method and condensed forms for linear high order differential-algebraic systems are introduced in [12], which allow an identification of those higher order derivatives of variables that can be replaced to obtain a first order system without changing the smoothness requirements. But, the computation of this condensed form is not feasible for numerical solution methods as it involves the derivatives of computed transformation matrices. However, since the standard way to obtain a strangeness-free first order formulation by first introducing new variables for the derivatives to transform the system into a first order system and then applying the usual index reduction procedures to the first order system can fail due to a possible increase in the index, at first an index reduction of the higher order system should be used followed by an appropriate order reduction to obtain a suitable strangeness-free first order formulation. Recently, it has been shown in [15, 18] that also the direct discretization of the second order system may yield better numerical results and is able to prevent certain numerical difficulties as the failure of numerical methods, see also [1, 2, 17].
In this paper we will present a new index reduction method for linear second order differential-algebraic systems of the form (1) based on the derivatives of the coefficient matrices $M(t)$, $C(t)$ and $K(t)$ that allows the computation of an equivalent system of low index and in a second step also the formulation of a corresponding trimmed first order formulation. At first, we present the basic results of the analysis of linear second order differential-algebraic equations similar as in [12], including condensed forms and characteristic invariants as well as a strangeness-free formulation in Section 2. In Section 3, we introduce the derivative array approach which enables us to transform the linear second order system (1) into an equivalent strangeness-free second order system with the same solution set in a numerical computable way. Further, in Section 4, we present a trimmed first order formulation for linear strangeness-free second order systems. Throughout this paper, for ease of representation, we restrict to linear second order systems since they are most frequently used in practical applications. However, all presented ideas can also be extended to arbitrary linear k-th order systems and to nonlinear systems, see also [19].

2 Condensed Forms for Linear Second Order DAEs

In the following, we present the main results of the analysis of linear second order differential-algebraic systems of the form (1) similar as in [12]. The condensed forms given in [12] are used to derive the relationships between the global invariants of the triple of matrix-valued functions (M, C, K) and the local invariants of the derivative array as presented in Section 3. Since we use a slightly different stepwise index reduction procedure compared to [12] we state again the main ideas of the condensed forms and of the index reduction here. First of all, to derive condensed forms for triples $(M(t), C(t), K(t))$ of matrix-valued functions we need an appropriate equivalence relation.

Definition 2. *Two triples of matrix-valued functions* (M_1, C_1, K_1) *and* (M_2, C_2, K_2), *with* $M_i, C_i, K_i \in C(\mathbb{I}, \mathbb{C}^{m\times n})$, $i = 1, 2$ *are called* globally equivalent *if there exist pointwise nonsingular matrix-valued functions* $P \in C(\mathbb{I}, \mathbb{C}^{m\times m})$ *and* $Q \in C^2(\mathbb{I}, \mathbb{C}^{n\times n})$ *such that*

$$M_2 = PM_1Q, \quad C_2 = 2PM_1\dot{Q} + PC_1Q, \quad K_2 = PM_1\ddot{Q} + PC_1\dot{Q} + PK_1Q. \tag{5}$$

For equivalent matrix triples we write $(M_1, C_1, K_1) \sim (M_2, C_2, K_2)$.

Considering the action of the equivalence relation (5) locally at a fixed point $\hat{t} \in \mathbb{I}$, we take into account that for given matrices $\hat{P}$, $\hat{Q}$, $\hat{R}_1$ and $\hat{R}_2$ of appropriate size, using Hermite interpolation, we can always find matrix-valued functions P and Q, such that at a given value $t = \hat{t}$ we have $P(\hat{t}) = \hat{P}$, $Q(\hat{t}) = \hat{Q}$, $\dot{Q}(\hat{t}) = \hat{R}_1$ and $\ddot{Q}(\hat{t}) = \hat{R}_2$. Therefore, we can define local equivalence of matrix triples in the following way.

Definition 3. *Two matrix triples* (M_1, C_1, K_1) *and* (M_2, C_2, K_2) *with* $M_i, C_i, K_i \in \mathbb{C}^{m\times n}$, $i = 1, 2$, *are called* locally equivalent *if there exist nonsingular matrices* $P \in \mathbb{C}^{m\times m}$ *and* $Q \in \mathbb{C}^{n\times n}$ *and matrices* $R_1, R_2 \in \mathbb{C}^{n\times n}$ *such that*

$$M_2 = PM_1Q, \quad C_2 = 2PM_1R_1 + PC_1Q, \quad K_2 = PM_1R_2 + PC_1R_1 + PK_1Q. \tag{6}$$

Again, we write $(M_1, C_1, K_1) \sim (M_2, C_2, K_2)$ *if the context is clear.*

It has be shown in [16] that the relations (5) and (6) are equivalence relations on the set of tuples of matrix-valued functions, and on the set of tuples of matrices, respectively. For a linear second order differential-algebraic system of the form (1) a condensed form under local equivalence transformation (6) of the corresponding matrix triple $(M(\hat{t}), C(\hat{t}), K(\hat{t}))$ at a fixed point $\hat{t} \in \mathbb{I}$ has also been derived in [13, 16] (see Appendix A.1). In this way, second order differential-algebraic systems are locally characterized by their purely first and second order differential parts of size $d^{(1)}$ and $d^{(2)}$, by their algebraic part of size a, by undetermined and redundant parts of size u and v and by the strangeness parts of size $s^{(MCK)}$, $s^{(MC)}$, $s^{(MK)}$, and $s^{(CK)}$ due to the different possible couplings between the matrices M, C, and K. The quantities $s^{(MCK)}$, $s^{(MC)}$, $s^{(MK)}$, $s^{(CK)}$, $d^{(2)}$, $d^{(1)}$, a, v and u are called the *local characteristic values of the linear second order DAE* (1). These local characteristic values are invariant under the equivalence relation (6) and can be expressed in terms of ranks of matrices and dimensions of column spaces.

Lemma 4. *[13, 16] Let* $M, C, K \in \mathbb{C}^{m\times n}$ *and let*

V_1 *be a basis of* $\mathrm{kernel}(M^H)$,
V_2 *be a basis of* $\mathrm{kernel}(M)$,
V_3 *be a basis of* $\mathrm{kernel}(M^H) \cap \mathrm{kernel}(C^H)$,
V_4 *be a basis of* $\mathrm{kernel}(M) \cap \mathrm{kernel}(V_1^H C)$.

Then, the quantities

$r = \mathrm{rank}(M)$	*(rank of* M*)*
$a = \mathrm{rank}(V_3^H K V_4)$	*(algebraic part)*
$s^{(MCK)} = \dim(\mathrm{range}(M^H) \cap \mathrm{range}(C^H V_1) \cap \mathrm{range}(K^H V_3))$	*(strangeness of* M, C, K*)*
$s^{(CK)} = \mathrm{rank}(V_3^H K V_2) - a$	*(strangeness of* C, K*)*

$$
\begin{array}{ll}
d^{(1)} = \operatorname{rank}(V_1^H C V_2) - s^{(CK)} & \textit{(1st-order diff. part)} \\
s^{(MC)} = \operatorname{rank}(V_1^H C) - s^{(MCK)} - s^{(CK)} - d^{(1)} & \textit{(strangeness of } M, C\textit{)} \\
s^{(MK)} = \operatorname{rank}(V_3^H K) - a - s^{(MCK)} - s^{(CK)} & \textit{(strangeness of } M, K\textit{)} \\
d^{(2)} = r - s^{(MCK)} - s^{(MC)} - s^{(MK)} & \textit{(2nd-order diff. part)} \\
v = m - r - 2s^{(CK)} - d^{(1)} - 2s^{(MCK)} - s^{(MC)} - a - s^{(MK)} & \textit{(vanishing equations)} \\
u = n - r - s^{(CK)} - d^{(1)} - a & \textit{(undetermined part)}
\end{array}
$$

are invariant under the local equivalence relation (6)*.*

For triples $(M(t), C(t), K(t))$ of matrix-valued functions we can compute the local condensed form at any fixed value $\hat{t} \in \mathbb{I}$ and determine the local characteristic quantities so that we obtain functions

$$
r, a, d^{(2)}, d^{(1)}, s^{(MCK)}, s^{(CK)}, s^{(MC)}, s^{(MK)}, u, v : \mathbb{I} \to \mathbb{N}_0.
$$

In the following, we assume that these functions are constant over the interval $\mathbb{I}$, i.e.,

$$
\begin{array}{l}
r(t) \equiv r, \quad a(t) \equiv a, \quad d^{(1)}(t) \equiv d^{(1)}, \quad s^{(MCK)}(t) \equiv s^{(MCK)}, \\
s^{(CK)}(t) \equiv s^{(CK)}, \quad s^{(MC)}(t) \equiv s^{(MC)}, \quad s^{(MK)}(t) \equiv s^{(MK)}, \quad \text{for all } t \in \mathbb{I}.
\end{array}
\tag{7}
$$

From Lemma 4 it then follows that $d^{(2)}(t)$, $u(t)$ and $v(t)$ are also constant in $\mathbb{I}$. We will call (7) the *regularity conditions* for the triple $(M(t), C(t), K(t))$. Under this assumption a global condensed form for triples of matrix-valued functions via the global equivalence transformations (5) has been derived in [13, 16] (see Appendix A.1). The associated differential-algebraic system

$$
\tilde{M}(t)\ddot{\tilde{x}}(t) + \tilde{C}(t)\dot{\tilde{x}}(t) + \tilde{K}(t)\tilde{x}(t) = \tilde{f}(t), \tag{8}
$$

with coefficient matrices $(\tilde{M}, \tilde{C}, \tilde{K})$ in the global condensed form given in Lemma A.2 (see Appendix A.1) is given by

$$
\begin{array}{rl}
\ddot{x}_1 + C_{13}\dot{x}_3 + C_{14}\dot{x}_4 + C_{17}\dot{x}_7 + C_{18}\dot{x}_8 + K_{12}x_2 + K_{14}x_4 + K_{16}x_6 + K_{18}x_8 = f_1, & (a) \\
\ddot{x}_2 + C_{23}\dot{x}_3 + C_{24}\dot{x}_4 + C_{27}\dot{x}_7 + C_{28}\dot{x}_8 + K_{22}x_2 + K_{24}x_4 + K_{26}x_6 + K_{28}x_8 = f_2, & (b) \\
\ddot{x}_3 + C_{33}\dot{x}_3 + C_{34}\dot{x}_4 + C_{37}\dot{x}_7 + C_{38}\dot{x}_8 + K_{32}x_2 + K_{34}x_4 + K_{36}x_6 + K_{38}x_8 = f_3, & (c) \\
\ddot{x}_4 + C_{43}\dot{x}_3 + C_{44}\dot{x}_4 + C_{47}\dot{x}_7 + C_{48}\dot{x}_8 + K_{42}x_2 + K_{44}x_4 + K_{46}x_6 + K_{48}x_8 = f_4, & (d) \\
\dot{x}_5 + K_{52}x_2 + K_{54}x_4 + K_{56}x_6 + K_{58}x_8 = f_5, & (e) \\
\dot{x}_6 + K_{62}x_2 + K_{64}x_4 + K_{66}x_6 + K_{68}x_8 = f_6, & (f) \\
\dot{x}_1 + K_{72}x_2 + K_{74}x_4 + K_{76}x_6 + K_{78}x_8 = f_7, & (g) \\
\dot{x}_2 + K_{82}x_2 + K_{84}x_4 + K_{86}x_6 + K_{88}x_8 = f_8, & (h) \\
x_7 = f_9, & (i) \\
x_5 = f_{10}, & (j) \\
x_3 = f_{11}, & (k) \\
x_1 = f_{12}, & (l) \\
0 = f_{13}. & (m)
\end{array}
$$

If we differentiate the equations $(g) - (l)$ once, we can eliminate the corresponding derivatives $\dot{x}_1, \dot{x}_3, \dot{x}_5, \dot{x}_7, \ddot{x}_1, \ddot{x}_2$ in the equations $(a) - (e)$ and (g) and get the system

$$
-K_{72}\dot{x}_2 + (C_{14} - K_{74})\dot{x}_4 - K_{76}\dot{x}_6 + (C_{18} - K_{78})\dot{x}_8 + (K_{12} - \dot{K}_{72})x_2
$$

$$
\begin{aligned}
&+(K_{14}-\dot{K}_{74})x_4+(K_{16}-\dot{K}_{76})x_6+(K_{18}-\dot{K}_{78})x_8 = f_1-\dot{f}_7-C_{17}\dot{f}_9-C_{13}\dot{f}_{11},\\
-K_{82}\dot{x}_2+(C_{24}-K_{84})\dot{x}_4-K_{86}\dot{x}_6+(C_{28}-K_{88})\dot{x}_8+(K_{22}-\dot{K}_{82})x_2&\\
&+(K_{24}-\dot{K}_{84})x_4+(K_{26}-\dot{K}_{86})x_6+(K_{28}-\dot{K}_{88})x_8 = f_2-C_{27}\dot{f}_9-\dot{f}_8-C_{23}\dot{f}_{11},\\
\ddot{x}_3+C_{34}\dot{x}_4+C_{38}\dot{x}_8+K_{32}x_2+K_{34}x_4+K_{36}x_6+K_{38}x_8 &= f_3-C_{37}\dot{f}_9-C_{33}\dot{f}_{11},\\
\ddot{x}_4+C_{44}\dot{x}_4+C_{48}\dot{x}_8+K_{42}x_2+K_{44}x_4+K_{46}x_6+K_{48}x_8 &= f_4-C_{47}\dot{f}_9-C_{43}\dot{f}_{11},\\
K_{52}x_2+K_{54}x_4+K_{56}x_6+K_{58}x_8 &= f_5-\dot{f}_{10},\\
\dot{x}_6+K_{62}x_2+K_{64}x_4+K_{66}x_6+K_{68}x_8 &= f_6,\\
K_{72}x_2+K_{74}x_4+K_{76}x_6+K_{78}x_8 &= f_7-\dot{f}_{12},\\
\dot{x}_2+K_{82}x_2+K_{84}x_4+K_{86}x_6+K_{88}x_8 &= f_8,\\
x_7 &= f_9,\\
x_5 &= f_{10},\\
x_3 &= f_{11},\\
x_1 &= f_{12},\\
0 &= f_{13}.
\end{aligned}
$$

Thus, after one differentiation-and-elimination step we get an equivalent second order system

$$
M^{<1>}(t)\ddot{\tilde{x}}+C^{<1>}(t)\dot{\tilde{x}}+K^{<1>}(t)\tilde{x}=f^{<1>}(t),
$$

with $(M^{<1>},C^{<1>},K^{<1>};f^{<1>})$ being of the form

$$
\left(
\begin{bmatrix}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&I_{s^{(MK)}}&0&0&0&0&0\\
0&0&0&I_{d^{(2)}}&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{bmatrix},
\begin{bmatrix}
0&C&0&C&0&C&0&C\\
0&C&0&C&0&C&0&C\\
0&0&0&C&0&0&0&C\\
0&0&0&C&0&0&0&C\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&I_{d^{(1)}}&0&0\\
0&0&0&0&0&0&0&0\\
0&I_{s^{(MC)}}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{bmatrix},
\right.
$$

$$
\left.
\begin{bmatrix}
0&K&0&K&0&K&0&K\\
0&K&0&K&0&K&0&K\\
0&K&0&K&0&K&0&K\\
0&K&0&K&0&K&0&K\\
0&K&0&K&0&K&0&K\\
0&K&0&K&0&K&0&K\\
0&K&0&K&0&K&0&K\\
0&K&0&K&0&K&0&K\\
0&0&0&0&0&0&I_a&0\\
0&0&0&0&I_{s^{(CK)}}&0&0&0\\
0&0&I_{s^{(MK)}}&0&0&0&0&0\\
I_{s^{(MCK)}}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{bmatrix};
\begin{bmatrix}
f_1-\dot{f}_7-C_{17}\dot{f}_9-C_{13}\dot{f}_{11}\\
f_2-C_{27}\dot{f}_9-\dot{f}_8-C_{23}\dot{f}_{11}\\
f_3-C_{37}\dot{f}_9-C_{33}\dot{f}_{11}\\
f_4-C_{47}\dot{f}_9-C_{43}\dot{f}_{11}\\
f_5-\dot{f}_{10}\\
f_6\\
f_7-\dot{f}_{12}\\
f_8\\
f_9\\
f_{10}\\
f_{11}\\
f_{12}\\
f_{13}
\end{bmatrix}
\right). \tag{9}
$$

Note that, for convenience of expression, in (9) and in the following, we drop the subscripts of the elements of block matrices unless they are needed for clarification. The triple $(M^{<1>}, C^{<1>}, K^{<1>})$ in (9) can then again be transformed to the global condensed form and the reduction step is repeated, i.e., again, the derivatives of certain equations are added to other equations to eliminate the coupling between equations. Note, that the identity block of size $s^{(MK)}$ in the matrix $\tilde{M}$ is not eliminated in the first reduction step, as this would require the second derivative of equation (k). This block is eliminated in the second reduction step by differentiating equation (k) twice and eliminating $\ddot{x}_3$ in equation (c). Continuing this process, we obtain a sequence of triples of matrix-valued functions $(M^{<i>}(t), C^{<i>}(t), K^{<i>}(t))$, $i \in \mathbb{N}_0$, with corresponding *global characteristic values* $(r_i, d_i^{(1)}, a_i, s_i^{(MC)}, s_i^{(CK)}, s_i^{(MK)}, s_i^{(MCK)}, u_i, v_i)$, where $(M^{<0>}(t), C^{<0>}(t), K^{<0>}(t)) = (M(t), C(t), K(t))$. During this reduction procedure the identity blocks of size $s_i^{(MK)}$ are decomposed into two blocks of size s_i and s_{i-1}, with $s_i^{(MK)} = s_i + s_{i-1}$, and $s_0 = s_0^{(MK)}$, in such a way that s_{i-1} denotes the part of $s_i^{(MK)}$ that can be eliminated in the $(i+1)$-th reduction step, while s_i denotes the part that cannot be eliminated until reduction step $i+2$. Here, we set $s_{-1} = 0$ and for the following we use the convention that characteristic quantities with negative subscript are zero. The relations

$$
\begin{aligned}
\operatorname{rank}(M^{<i+1>}) &= r_{i+1} = r_i - s_i^{(MCK)} - s_i^{(MC)} - s_{i-1}, \\
\operatorname{rank}(K^{<i+1>}) &\geq a_{i+1} \geq a_i + s_i^{(CK)} + s_i^{(MCK)}
\end{aligned}
$$

guarantee that after a finite number of steps μ, the strangeness $s_\mu^{(MCK)}$, $s_\mu^{(MK)}$, $s_\mu^{(CK)}$ and $s_\mu^{(MC)}$ corresponding to $(M^{<\mu>}(t), C^{<\mu>}(t), K^{<\mu>}(t))$ vanish and the process becomes stationary. We call μ the *strangeness index* or *s-index* of the second order system of DAEs (1) and we call the final equivalent second order system of DAEs *strangeness-free.*

Remark 5. *Here, we differ slightly from the index reduction procedure described in [13, 16], where the identity block of size $s^{(MK)}$ is also completely eliminated in every reduction step. Thus, for one reduction step one or two differentiations of equations are required, depending on the occurrence of strangeness blocks. In this way, the index definition does not correspond to the differentiability requirements for the right hand side. In our approach the right-hand side is only differentiated once in each elimination step before the system is again transformed to global condensed form such that the strangeness index corresponds to the differentiability requirements for the right hand side, which is the case for all general index concepts.*

Theorem 6. *Consider the linear second order system* (1)*, suppose that the regularity conditions* (7) *hold, and let μ be the strangeness index of* (1)*. If $f \in C^\mu(\mathbb{I}, \mathbb{C}^m)$, then system* (1) *is equivalent (in the sense that there is a one-to-one correspondence between the solution sets) to a strangeness-free system of second order differential-algebraic equations of the form*

$$
\begin{aligned}
\ddot{\tilde{x}}_1 + \tilde{C}_{11}(t)\dot{\tilde{x}}_1 + \tilde{C}_{14}(t)\dot{\tilde{x}}_4 + \tilde{K}_{11}(t)\tilde{x}_1 + \tilde{K}_{12}(t)\tilde{x}_2 + \tilde{K}_{14}(t)\tilde{x}_4 &= \tilde{f}_1(t), \quad (d_\mu^{(2)}) \\
\dot{\tilde{x}}_2 + \tilde{K}_{21}(t)\tilde{x}_1 + \tilde{K}_{22}(t)\tilde{x}_2 + \tilde{K}_{24}(t)\tilde{x}_4 &= \tilde{f}_2(t), \quad (d_\mu^{(1)}) \\
\tilde{x}_3 &= \tilde{f}_3(t), \quad (a_\mu) \\
0 &= \tilde{f}_4(t), \quad (v_\mu)
\end{aligned}
\tag{10}
$$

where the inhomogeneity $\tilde{f} := [\tilde{f}_1^H, \ldots, \tilde{f}_4^H]^H$ is determined by $f^{(0)}, \ldots, f^{(\mu)}$. In particular, $d_\mu^{(2)}$, $d_\mu^{(1)}$ and a_μ are the number of second order differential, first order differential, and *algebraic components of the unknown* $\tilde{x} := [\tilde{x}_1^H, \ldots, \tilde{x}_4^H]^H$*, while* u_μ *is the dimension of the undetermined vector* $\tilde{x}_4$*, and* v_μ *is the number of conditions in the last equation.*

Proof. The proof is similar to the proof of [16, Theorem 2.12] with slight modifications concerning the definition of the strangeness index and the counting of the differentiations. $\square$

Using the strangeness-free form (10) we can analyze existence and uniqueness of solutions and consistency of initial conditions for linear second order differential-algebraic systems (1), see [13, 16]. Further, the strangeness-free form (10) allows the identification of those second order derivatives of variables that can be replaced to obtain a first order system that is strangeness-free without increasing the index.
Instead of actually performing the differentiation-and-elimination steps leading to (9) we can apply all equivalence transformations directly to the triple $(M(t), C(t), K(t))$ and construct a triple of matrix functions from which we can read off the whole sequence of characteristic values $(r_i,\, d_i^{(1)},\, a_i,\, s_i^{(MCK)},\, s_i^{(MC)},\, s_i^{(MK)},\, s_i^{(CK)}, u_i, v_i)$. This sequence of characteristic values can be further characterized in terms of ranks of block matrices of the matrix triple.

Lemma 7. *Let the functions* $M, C, K \in C(\mathbb{I}, \mathbb{C}^{m,\times n})$ *be sufficiently smooth and let the strangeness index* μ *be well-defined. Further, let the process leading to Theorem 6 yield a sequence* $(M^{<i>}, C^{<i>}, K^{<i>})$*,* $i \in \mathbb{N}_0$*, with* $(M^{<0>}, C^{<0>}, K^{<0>}) = (M, C, K)$ *and characteristic values* $(r_i, d_i^{(1)}, a_i, s_i^{(MCK)}, s_i^{(MC)}, s_i^{(MK)}, s_i^{(CK)}, u_i, v_i)$ *according to Lemma 4. The triple* $(M^{<i>}, C^{<i>}, K^{<i>})$ *of matrix-valued functions is globally equivalent to the triple*

$$\left(\begin{bmatrix} I_{s_i^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_i^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{s_{i-1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{s_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{d_i^{(2)}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$
\begin{bmatrix}
0 & 0 & C_{13}^{<i>} & C_{14}^{<i>} & C_{15}^{<i>} & 0 & 0 & C_{18}^{<i>} & C_{19}^{<i>} \\
0 & 0 & C_{23}^{<i>} & C_{24}^{<i>} & C_{25}^{<i>} & 0 & 0 & C_{28}^{<i>} & C_{29}^{<i>} \\
0 & 0 & C_{33}^{<i>} & C_{34}^{<i>} & C_{35}^{<i>} & 0 & 0 & C_{38}^{<i>} & C_{39}^{<i>} \\
0 & 0 & C_{43}^{<i>} & C_{44}^{<i>} & C_{45}^{<i>} & 0 & 0 & C_{48}^{<i>} & C_{49}^{<i>} \\
0 & 0 & C_{53}^{<i>} & C_{54}^{<i>} & C_{55}^{<i>} & 0 & 0 & C_{58}^{<i>} & C_{59}^{<i>} \\
0 & 0 & 0 & 0 & 0 & I_{s_i^{(CK)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{d_i^{(1)}} & 0 & 0 \\
I_{s_i^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s_i^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \qquad (11)
$$

$$
\begin{array}{cl}
\left[\begin{matrix}
0 & K_{12}^{<i>} & 0 & 0 & K_{15}^{<i>} & 0 & K_{17}^{<i>} & 0 & K_{19}^{<i>} \\
0 & K_{22}^{<i>} & 0 & 0 & K_{25}^{<i>} & 0 & K_{27}^{<i>} & 0 & K_{29}^{<i>} \\
0 & K_{32}^{<i>} & 0 & 0 & K_{35}^{<i>} & 0 & K_{37}^{<i>} & 0 & K_{39}^{<i>} \\
0 & K_{42}^{<i>} & 0 & 0 & K_{45}^{<i>} & 0 & K_{47}^{<i>} & 0 & K_{49}^{<i>} \\
0 & K_{52}^{<i>} & 0 & 0 & K_{55}^{<i>} & 0 & K_{57}^{<i>} & 0 & K_{59}^{<i>} \\
0 & K_{62}^{<i>} & 0 & 0 & K_{65}^{<i>} & 0 & K_{67}^{<i>} & 0 & K_{69}^{<i>} \\
0 & K_{72}^{<i>} & 0 & 0 & K_{75}^{<i>} & 0 & K_{77}^{<i>} & 0 & K_{79}^{<i>} \\
0 & K_{82}^{<i>} & 0 & 0 & K_{85}^{<i>} & 0 & K_{87}^{<i>} & 0 & K_{89}^{<i>} \\
0 & K_{92}^{<i>} & 0 & 0 & K_{95}^{<i>} & 0 & K_{97}^{<i>} & 0 & K_{99}^{<i>} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{a_i} & 0 \\
0 & 0 & 0 & 0 & 0 & I_{s_i^{(CK)}} & 0 & 0 & 0 \\
0 & 0 & I_{s_{i-1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{s_i} & 0 & 0 & 0 & 0 & 0 \\
I_{s_i^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right]\Bigg) &
\begin{matrix}
s_i^{(MCK)} \\ s_i^{(MC)} \\ s_{i-1} \\ s_i \\ d_i^{(2)} \\ s_i^{(CK)} \\ d_i^{(1)} \\ s_i^{(MCK)} \\ s_i^{(MC)} \\ a_i \\ s_i^{(CK)} \\ s_{i-1} \\ s_i \\ s_i^{(MCK)} \\ v_i
\end{matrix}
\end{array},
$$

where $s_i^{(MK)}$ *is separated into* $s_i^{(MK)} = s_i + s_{i-1}$ *and the last block columns have size* u_i*. We define*

$$
\begin{aligned}
\tilde{C}_{1j}^{<i>} &:= C_{1j}^{<i>} - K_{8j}^{<i>}, \quad j = 5, 9, \\
\tilde{C}_{2j}^{<i>} &:= C_{2j}^{<i>} - K_{9j}^{<i>}, \quad j = 5, 9, \\
\tilde{K}_{1j}^{<i>} &:= K_{1j}^{<i>} - \dot{K}_{8j}^{<i>} + K_{82}^{<i>} K_{9j}^{<i>} + K_{87}^{<i>} K_{7j}^{<i>}, \quad j = 2, 5, 7, 9, \\
\tilde{K}_{2j}^{<i>} &:= K_{2j}^{<i>} - \dot{K}_{9j}^{<i>} + K_{92}^{<i>} K_{9j}^{<i>} + K_{97}^{<i>} K_{7j}^{<i>}, \quad j = 2, 5, 7, 9, \\
\tilde{C}_1 &:= \begin{bmatrix} \tilde{C}_{15}^{<i>H} & \tilde{C}_{25}^{<i>H} & C_{35}^{<i>H} \end{bmatrix}^H, \\
\tilde{C}_2 &:= \begin{bmatrix} \tilde{C}_{19}^{<i>H} & \tilde{C}_{29}^{<i>H} & C_{39}^{<i>H} \end{bmatrix}^H,
\end{aligned}
$$

as well as

$$
\begin{aligned}
k_0 &= d_0^{(1)} + s_0^{(CK)}, & k_{i+1} &= \operatorname{rank} \tilde{C}_2, \\
e_0 &= d_0^{(1)} + s_0^{(MC)} + s_0^{(CK)} + s_0^{(MCK)}, & e_{i+1} &= \operatorname{rank}\left(\begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}\right).
\end{aligned}
$$

Then, let U and V be nonsingular matrix-valued functions of size $(s_i^{(MCK)} + s_i^{(MC)} + s_{i-1}, s_i^{(MCK)} + s_i^{(MC)} + s_{i-1})$ and $(d_i^{(2)} + u_i, d_i^{(2)} + u_i)$, respectively, such that

$$U^H \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} V = \begin{bmatrix} I_{e_{i+1}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Further, let U and V be partitioned into $U = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$ such that

$$\begin{bmatrix} U_1^H \\ U_2^H \\ U_3^H \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} I_{e_{i+1}-k_{i+1}} & 0 & 0 \\ 0 & I_{k_{i+1}} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and with a splitting of V_3 into $V_3 = \begin{bmatrix} V_{31} & V_{32} \end{bmatrix}$ with V_{31} of size $(d_i^{(2)} + u_i, d_i^{(2)} - e_{i+1} + k_{i+1})$ and V_{32} of size $(d_i^{(2)} + u_i, u_i - k_{i+1})$ we can define

$$\begin{aligned} &\begin{bmatrix} K_1 & K_2 & K_3 & K_4 & K_5 & K_6 \end{bmatrix} := \\ &\begin{bmatrix} U_3^H & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|cc|c} \tilde{K}_{15}^{<i>} & \tilde{K}_{17}^{<i>} & \tilde{K}_{12}^{<i>} & \tilde{K}_{19}^{<i>} \\ \tilde{K}_{25}^{<i>} & \tilde{K}_{27}^{<i>} & \tilde{K}_{22}^{<i>} & \tilde{K}_{29}^{<i>} \\ K_{35}^{<i>} & K_{37}^{<i>} & K_{32}^{<i>} & K_{39}^{<i>} \\ \hline K_{65}^{<i>} & K_{67}^{<i>} & K_{62}^{<i>} & K_{69}^{<i>} \\ K_{85}^{<i>} & K_{87}^{<i>} & K_{82}^{<i>} & K_{89}^{<i>} \end{array}\right] \begin{bmatrix} [V_1\ V_{31}] & 0 & 0 & 0 \\ 0 & I_{d_i^{(1)}} & 0 & 0 \\ 0 & 0 & I_{s_i^{(MC)}} & 0 \\ 0 & 0 & 0 & [V_2\ V_{32}] \end{bmatrix}, \end{aligned} \tag{12}$$

where the identity matrix on the left-hand side is of size $s_i^{(CK)} + s_i^{(MCK)}$. Further, we define

$$\begin{aligned} b_0 &= a_0, & b_{i+1} &= \operatorname{rank}\left(\begin{bmatrix} K_6 \end{bmatrix}\right), \\ p_0 &= a_0 + s_0^{(CK)}, & p_{i+1} &= \operatorname{rank}\left(\begin{bmatrix} K_5 & K_6 \end{bmatrix}\right), \\ t_0 &= a_0 + s_0^{(CK)} - s_0^{(MK)}, & t_{i+1} &= \operatorname{rank}\left(\begin{bmatrix} K_4 & K_5 & K_6 \end{bmatrix}\right), \\ d_0 &= a_0 + s_0^{(CK)}, & d_{i+1} &= \operatorname{rank}\left(\begin{bmatrix} K_3 & K_4 & K_5 & K_6 \end{bmatrix}\right), \\ h_0 &= a_0 + s_0^{(CK)} + s_0^{(MK)}, & h_{i+1} &= \operatorname{rank}\left(\begin{bmatrix} K_2 & K_3 & K_4 & K_5 & K_6 \end{bmatrix}\right), \\ c_0 &= a_0 + s_0^{(MCK)} + s_0^{(CK)} + s_0^{(MK)}, & c_{i+1} &= \operatorname{rank}\left(\begin{bmatrix} K_1 & K_2 & K_3 & K_4 & K_5 & K_6 \end{bmatrix}\right), \\ w_0 &= v_0, & w_{i+1} &= v_{i+1} - v_i, \\ q_0 &= e_0, & q_{i+1} &= e_{i+1} + c_i - s_i^{(CK)} - s_i^{(MCK)}. \end{aligned}$$

Then, we have

$$\begin{aligned} &r_{i+1} = r_i - s_i^{(MCK)} - s_i^{(MC)} - s_{i-1}, \\ &c_{i+1} = b_{i+1} + s_{i+1}^{(MCK)} + s_{i+1}^{(CK)} + s_{i+1}^{(MK)} - s_i, \\ &e_{i+1} = k_{i+1} + s_{i+1}^{(MC)} + s_{i+1}^{(MCK)}, \\ &a_{i+1} = a_i + s_i^{(CK)} + s_i^{(MCK)} + s_{i-1} + b_{i+1} = c_0 + \cdots + c_{i+1} - s_{i+1}^{(CK)} - s_{i+1}^{(MK)} - s_{i+1}^{(MCK)}, \\ &s_{i+1}^{(MCK)} = c_{i+1} - h_{i+1}, \\ &s_{i+1}^{(MC)} = e_{i+1} - k_{i+1} - c_{i+1} + h_{i+1}, \\ &s_{i+1} = h_{i+1} - d_{i+1}, \\ &s_{i+1}^{(MK)} = s_{i+1} + s_i, \\ &s_{i+1}^{(CK)} = d_{i+1} - b_{i+1}, \end{aligned}$$

$$
\begin{aligned}
d_{i+1}^{(2)} &= r_{i+1} - s_{i+1}^{(MCK)} - s_{i+1}^{(MC)} - s_{i+1}^{(MK)} = d_i^{(2)} - e_{i+1} + k_{i+1} - s_{i+1}, \\
d_{i+1}^{(1)} &= d_i^{(1)} + s_i^{(MC)} + k_{i+1} - s_{i+1}^{(CK)} \\
&= q_0 + \cdots + q_{i+1} - c_0 - \cdots - c_i - s_{i+1}^{(MCK)} - s_{i+1}^{(MC)} - s_{i+1}^{(CK)}, \\
w_{i+1} &= 2s_i^{(MCK)} + s_i^{(CK)} + s_i^{(MC)} + s_{i-1} - e_{i+1} - c_{i+1}, \\
u_{i+1} &= u_0 - b_1 - \cdots - b_{i+1}, \\
v_{i+1} &= v_0 + w_1 + \cdots + w_{i+1} = 2s_i^{(MCK)} + s_i^{(CK)} + s_i^{(MC)} + s_{i-1} - e_{i+1} - c_{i+1} + v_i.
\end{aligned}
$$

Proof. The proof of Lemma 7 is given in the Appendix A.2. □

3 Derivative Array Approach

The algebraic approach described in the previous section allows for the theoretical analysis of linear second order DAEs (1), but it cannot be used for the development of numerical methods as neither the inductive process of the reduction to the strangeness-free formulation (10) nor the global condensed form are obtained in a way that is feasible for numerical methods. Therefore, we look for other ways to compute the characteristic invariants of a given DAE as well as a canonical form similar to (10) in a numerically stable procedure. The basic idea due to Campbell [4] is to differentiate the differential-algebraic equation (1) a number of times and put the original DAE and its derivatives into a large system. Then, purely local invariants can be constructed via local equivalence transformations, which allow to determine the global invariants including the strangeness index, wherever they are defined. Further, it is also possible to derive a strangeness-free formulation using only local informations.
In the following, we consider matrix-valued functions $M, C, K \in C(\mathbb{I}, \mathbb{C}^{m,n})$ that are sufficiently smooth and we assume that the strangeness index μ is well-defined, i.e., the ranks are constant in the considered interval and none of the invariant values changes its value during the process. This can always be achieved by going to smaller intervals since there always exits open intervals $\mathbb{I}_j \subseteq \mathbb{I}$, $j \in \mathbb{N}$ with $\overline{\bigcup_{j\in\mathbb{N}} \mathbb{I}_j} = \mathbb{I}$, $\mathbb{I}_i \cap \mathbb{I}_j = \emptyset$ for $i \neq j$ such that the constant rank assumption holds for all $t \in \mathbb{I}_j$, $j \in \mathbb{N}$ and the following construction can be applied separately for each interval $\mathbb{I}_j$, see e.g. [11]. Differentiating the differential-algebraic equation (1) and putting the original DAE and its derivatives up to a sufficiently high order into a large system, we obtain the *derivative array* associated with the linear second order DAE (1) of the form

$$
\mathcal{M}_l(t)\ddot{z}_l + \mathcal{L}_l(t)\dot{z}_l + \mathcal{N}_l(t)z_l = g_l(t), \quad l \in \mathbb{N}_0, \tag{13}
$$

where $\mathcal{M}_l$, $\mathcal{L}_l$, $\mathcal{N}_l$, z_l and g_l are defined by

$$
\begin{aligned}
[\mathcal{M}_l]_{i,j} &:= \binom{i}{j} M^{(i-j)} + \binom{i}{j+1} C^{(i-j-1)} + \binom{i}{j+2} K^{(i-j-2)}, \quad i,j = 0,\dots,l, \\
[\mathcal{L}_l]_{i,j} &:= \begin{cases} C^{(i)} + iK^{(i-1)} & \text{for } i = 0,\dots,l,\ j = 0, \\ 0 & \text{otherwise,} \end{cases} \\
[\mathcal{N}_l]_{i,j} &:= \begin{cases} K^{(i)} & \text{for } i = 0,\dots,l,\ j = 0, \\ 0 & \text{otherwise,} \end{cases} \\
[z_l]_i &:= x^{(i)}, \quad i = 0,\dots,l, \\
[g_l]_i &:= f^{(i)}, \quad i = 0,\dots,l.
\end{aligned} \tag{14}
$$

Here, we use the convention that $\binom{i}{j} = 0$ for $i < 0$, $j < 0$ or $j > i$. For $l = 2$, for example, the extended system (13) is of the form

$$\begin{bmatrix} M & 0 & 0 \\ \dot{M} + C & M & 0 \\ \ddot{M} + 2\dot{C} + K & 2\dot{M} + C & M \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \dddot{x} \\ x^{(4)} \end{bmatrix} + \begin{bmatrix} C & 0 & 0 \\ \dot{C} + K & 0 & 0 \\ \ddot{C} + 2\dot{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \end{bmatrix} + \begin{bmatrix} K & 0 & 0 \\ \dot{K} & 0 & 0 \\ \ddot{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} f \\ \dot{f} \\ \ddot{f} \end{bmatrix}.$$

For every $l \in \mathbb{N}_0$ and every $t \in \mathbb{I}$, we can now determine the local characteristic values of the triple $(\mathcal{M}_l(t), \mathcal{L}_l(t), \mathcal{N}_l(t))$ by transforming it to the local condensed form given in Lemma A.1. These local quantities at a fixed point $\hat{t} \in \mathbb{I}$ are invariant under global equivalence transformations of the original triple $(M(t), C(t), K(t))$ of matrix-valued functions. To prove this, we use the following Lemmas.

Lemma 8. *[11, Lemma 3.28] Let $D = ABC$ be the product of three sufficiently smooth matrix valued functions of appropriate dimensions. Then*

$$D^{(i)} = \sum_{j=0}^{i} \sum_{k=0}^{i-j} \binom{i}{j}\binom{i-j}{k} A^{(j)} B^{(k)} C^{(i-j-k)}.$$

Lemma 9. *For all integers i, j, k, l with $i \geq 0$, $i \geq j \geq 0$, $i - j \geq k \geq 0$, we have*

$$\binom{i}{k}\binom{i-k}{l}\binom{i-k-l}{j} = \binom{i}{j}\binom{i-j}{k}\binom{i-j-k}{l},$$

$$\begin{aligned} \binom{i}{k}\binom{i-k}{l}\binom{i-k-l+2}{j+2} = {} & \binom{i}{j}\binom{i-j}{k}\binom{i-j-k}{l} + 2\binom{i}{j+1}\binom{i-j-1}{k}\binom{i-j-k-1}{l} \\ & + \binom{i}{j+2}\binom{i-j-2}{k}\binom{i-j-k-2}{l}, \end{aligned}$$

$$\binom{i}{k}\binom{i-k}{l}\binom{i-k-l+1}{j+2} = \binom{i}{j+1}\binom{i-j-1}{k}\binom{i-j-1-k}{l} + \binom{i}{j+2}\binom{i-j-2}{k}\binom{i-j-k-2}{l},$$

$$\binom{i}{k}\binom{i-k}{l}\binom{i-k-l}{j+2} = \binom{i}{j+2}\binom{i-j-2}{k}\binom{i-j-k-2}{l}.$$

Proof. The proof follows by straightforward calculations. □

Now, we can show that the local quantities of the triple $(\mathcal{M}_l(\hat{t}), \mathcal{L}_l(\hat{t}), \mathcal{N}_l(\hat{t}))$ are invariant under global equivalence transformations of the original triple $(M(t), C(t), K(t))$.

Theorem 10. *Consider two triples (M, C, K) and $(\tilde{M}, \tilde{C}, \tilde{K})$ of sufficiently smooth matrix-valued functions that are globally equivalent via the transformation*

$$\tilde{M} = PMQ, \quad \tilde{C} = PCQ + 2PM\dot{Q}, \quad \tilde{K} = PKQ + PC\dot{Q} + PM\ddot{Q}$$

according to Definition 2, with sufficiently smooth matrix-valued functions P and Q. Let $(\mathcal{M}_l, \mathcal{L}_l, \mathcal{N}_l)$ and $(\tilde{\mathcal{M}}_l, \tilde{\mathcal{L}}_l, \tilde{\mathcal{N}}_l)$, $l \in \mathbb{N}_0$, be the corresponding inflated triples constructed as in (14) *and introduce the block matrix functions*

$$[\Pi_l]_{i,j} = \binom{i}{j} P^{(i-j)}, \qquad [\Psi_l]_{i,j} = \begin{cases} \frac{i+2}{2} Q^{(i+1)} & \text{for } i = 0, \ldots, l,\ j = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

$$[\Theta_l]_{i,j} = \binom{i+2}{j+2} Q^{(i-j)}, \quad [\Sigma_l]_{i,j} = \begin{cases} Q^{(i+2)} & \text{for } i = 0, \ldots, l,\ j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$[\tilde{\mathcal{M}}_l(t), \tilde{\mathcal{L}}_l(t), \tilde{\mathcal{N}}_l(t)] = \Pi_l(t)[\mathcal{M}_l(t), \mathcal{L}_l(t), \mathcal{N}_l(t)] \begin{bmatrix} \Theta_l(t) & 2\Psi_l(t) & \Sigma_l(t) \\ 0 & \Theta_l(t) & \Psi_l(t) \\ 0 & 0 & \Theta_l(t) \end{bmatrix} \tag{16}$$

for every $t \in \mathbb{I}$, and the corresponding matrix triples are locally equivalent.

Proof. First, we note that all matrix-valued functions $\mathcal{M}_l, \mathcal{L}_l, \mathcal{N}_l, \tilde{\mathcal{M}}_l, \tilde{\mathcal{L}}_l, \tilde{\mathcal{N}}_l, \Pi_l, \Psi_l$, Θ_l and Σ_l are block lower triangular with the same block structure. Furthermore, $\mathcal{N}_l$, $\tilde{\mathcal{N}}_l$,$\mathcal{L}_l$, $\tilde{\mathcal{L}}_l$, Ψ_l and Σ_l have nonzero blocks only in the first block column. Using Lemma 8 we obtain

$$\begin{aligned}
\tilde{M}^{(i)} &= \sum_{k_1=0}^{i} \sum_{k_2=0}^{i-k_1} \binom{i}{k_1}\binom{i-k_1}{k_2} P^{(k_1)} M^{(k_2)} Q^{(i-k_1-k_2)}, \\
\tilde{C}^{(i)} &= \sum_{k_1=0}^{i} \sum_{k_2=0}^{i-k_1} \binom{i}{k_1}\binom{i-k_1}{k_2} \left[P^{(k_1)} C^{(k_2)} Q^{(i-k_1-k_2)} + 2P^{(k_1)} M^{(k_2)} Q^{(i+1-k_1-k_2)}\right], \\
\tilde{K}^{(i)} &= \sum_{k_1=0}^{i} \sum_{k_2=0}^{i-k_1} \binom{i}{k_1}\binom{i-k_1}{k_2} \left[P^{(k_1)} K^{(k_2)} Q^{(i-k_1-k_2)} + P^{(k_1)} C^{(k_2)} Q^{(i+1-k_1-k_2)}\right. \\
&\qquad\qquad \left. + P^{(k_1)} M^{(k_2)} Q^{(i+2-k_1-k_2)}\right].
\end{aligned}$$

Inserting the definitions, shifting and inverting the summations and applying Lemma 9 leads to

$$\begin{aligned}
&[\Pi_l \mathcal{M}_l \Theta_l]_{i,j} = \sum_{l_1=j}^{i} \sum_{l_2=j}^{l_1} [\Pi_l]_{i,l_1} [\mathcal{M}_l]_{l_1,l_2} [\Theta_l]_{l_2,j} \\
&= \sum_{l_1=j}^{i} \sum_{l_2=j}^{l_1} \binom{i}{l_1} P^{(i-l_1)} \left[\binom{l_1}{l_2} M^{(l_1-l_2)} + \binom{l_1}{l_2+1} C^{(l_1-l_2-1)} + \binom{l_1}{l_2+2} K^{(l_1-l_2-2)}\right] \binom{l_2+2}{j+2} Q^{(l_2-j)} \\
&= \sum_{k_1=0}^{i-j} \sum_{l_2=j}^{k_1+j} \binom{i}{k_1+j} P^{(i-k_1-j)} \left[\binom{k_1+j}{l_2} M^{(k_1+j-l_2)} + \binom{k_1+j}{l_2+1} C^{(k_1+j-l_2-1)}\right. \\
&\qquad\qquad \left. + \binom{k_1+j}{l_2+2} K^{(k_1+j-l_2-2)}\right] \binom{l_2+2}{j+2} Q^{(l_2-j)} \\
&= \binom{i}{j} \sum_{k_1=0}^{i-j} \sum_{k_2=0}^{i-j-k_1} \binom{i-j}{k_1}\binom{i-j-k_1}{k_2} P^{(k_1)} M^{(k_2)} Q^{(i-j-k_1-k_2)} \\
&+ \binom{i}{j+1} \sum_{k_1=0}^{i-j-1} \sum_{k_2=0}^{i-j-1-k_1} \binom{i-j-1}{k_1}\binom{i-j-1-k_1}{k_2} \left[P^{(k_1)} C^{(k_2)} Q^{(i-j-1-k_1-k_2)} + 2P^{(k_1)} M^{(k_2)} Q^{(i-j-k_1-k_2)}\right] \\
&+ \binom{i}{j+2} \sum_{k_1=0}^{i-j-2} \sum_{k_2=0}^{i-j-2-k_1} \binom{i-j-2}{k_1}\binom{i-j-2-k_1}{k_2} \left[P^{(k_1)} K^{(k_2)} Q^{(i-j-2-k_1-k_2)}\right. \\
&\qquad\qquad \left. + P^{(k_1)} C^{(k_2)} Q^{(i-j-1-k_1-k_2)} + P^{(k_1)} M^{(k_2)} Q^{(i-j-k_1-k_2)}\right] \\
&= \binom{i}{j} \tilde{M}^{(i-j)} + \binom{i}{j+1} \tilde{C}^{(i-j-1)} + \binom{i}{j+2} \tilde{K}^{(i-j-2)} = [\tilde{\mathcal{M}}_l]_{i,j}.
\end{aligned}$$

In the same way we get

$$
\begin{aligned}
&[\Pi_l\mathcal{L}_l\Theta_l]_{i,0} + [2\Pi_l\mathcal{M}_l\Psi_l]_{i,0} = \sum_{l_1=0}^{i}[\Pi_l]_{i,l_1}[\mathcal{L}_l]_{l_1,0}[\Theta_l]_{0,0} + 2\sum_{l_1=0}^{i}\sum_{l_2=0}^{l_1}[\Pi_l]_{i,l_1}[\mathcal{M}_l]_{l_1,l_2}[\Psi_l]_{l_2,0} \\
&= \sum_{l_1=0}^{i}\tbinom{i}{l_1}P^{(i-l_1)}\left[C^{(l_1)} + l_1K^{(l_1-1)}\right]Q \\
&\quad + 2\sum_{l_1=0}^{i}\sum_{l_2=0}^{l_1}\tbinom{i}{l_1}P^{(i-l_1)}\left[\tbinom{l_1}{l_2}M^{(l_1-l_2)} + \tbinom{l_1}{l_2+1}C^{(l_1-l_2-1)} + \tbinom{l_1}{l_2+2}K^{(l_1-l_2-2)}\right]\frac{l_2+2}{2}Q^{(l_2+1)} \\
&= \sum_{k_1=0}^{i}\sum_{k_2=0}^{i-k_1}\tbinom{i}{k_1}\tbinom{i-k_1}{k_2}\left[P^{(k_1)}C^{(k_2)}Q^{(i-k_1-k_2)} + 2P^{(k_1)}M^{(k_2)}Q^{(i+1-k_1-k_2)}\right] \\
&\quad + i\sum_{k_1=0}^{i-1}\sum_{k_2=0}^{i-1-k_1}\tbinom{i-1}{k_1}\tbinom{i-1-k_1}{k_2}\Big[P^{(k_1)}K^{(k_2)}Q^{(i-1-k_1-k_2)} \\
&\qquad\qquad + P^{(k_1)}C^{(k_2)}Q^{(i-k_1-k_2)} + P^{(k_1)}M^{(k_2)}Q^{(i+1-k_1-k_2)}\Big] \\
&= \tilde{C}^{(i)} + i\tilde{K}^{(i-1)} = [\tilde{\mathcal{L}}_l]_{i,0},
\end{aligned}
$$

and

$$
\begin{aligned}
&[\Pi_l\mathcal{N}_l\Theta_l]_{i,0} + [\Pi_l\mathcal{L}_l\Psi_l]_{i,0} + [\Pi_l\mathcal{M}_l\Sigma_l]_{i,0} = \\
&= \sum_{l_1=0}^{i}[\Pi_l]_{i,l_1}[\mathcal{N}_l]_{l_1,0}[\Theta_l]_{0,0} + \sum_{l_1=0}^{i}[\Pi_l]_{i,l_1}[\mathcal{L}_l]_{l_1,0}[\Psi_l]_{0,0} + \sum_{l_1=0}^{i}\sum_{l_2=0}^{l_1}[\Pi_l]_{i,l_1}[\mathcal{M}_l]_{l_1,l_2}[\Sigma_l]_{l_2,0} \\
&= \sum_{l_1=0}^{i}\tbinom{i}{l_1}P^{(i-l_1)}K^{(l_1)}Q + \sum_{l_1=0}^{i}\tbinom{i}{l_1}P^{(i-l_1)}\left[C^{(l_1)} + l_1K^{(l_1-1)}\right]Q^{(1)} \\
&\quad + \sum_{l_1=0}^{i}\sum_{l_2=0}^{l_1}\tbinom{i}{l_1}P^{(i-l_1)}\left[\tbinom{l_1}{l_2}M^{(l_1-l_2)} + \tbinom{l_1}{l_2+1}C^{(l_1-l_2-1)} + \tbinom{l_1}{l_2+2}K^{(l_1-l_2-2)}\right]Q^{(l_2+2)} \\
&= \sum_{k_1=0}^{i}\sum_{k_2=0}^{i-k_1}\tbinom{i}{k_1}\tbinom{i-k_1}{k_2}\left[P^{(k_1)}(K^{(k_2)}Q^{(i-k_1-k_2)} + C^{(k_2)}Q^{(i+1-k_1-k_2)} + M^{(k_2)}Q^{(i+2-k_1-k_2)})\right] \\
&= \tilde{K}^{(i)} = [\tilde{\mathcal{N}}_l]_{i,0}.
\end{aligned}
$$

□

As a consequence of Theorem 10 the local characteristic values ($\tilde{r}_l$, $\tilde{d}_l^{(1)}$, $\tilde{a}_l$, $\tilde{s}_l^{(MCK)}$, $\tilde{s}_l^{(MK)}$, $\tilde{s}_l^{(CK)}$, $\tilde{s}_l^{(MC)}$, $\tilde{u}_l$, $\tilde{v}_l$) of the inflated triple $(\mathcal{M}_l(\hat{t}), \mathcal{L}_l(\hat{t}), \mathcal{N}_l(\hat{t}))$ at a fixed point $\hat{t}$ are well-defined for equivalent triples of matrix-valued functions and for each $l \in \mathbb{N}_0$. These quantities are numerically computable via a number of numerical rank decisions. Next, we show how these local quantities of the inflated triple $(\mathcal{M}_l(\hat{t}), \mathcal{L}_l(\hat{t}), \mathcal{N}_l(\hat{t}))$ are related to the global characteristic values of the original triple (M, C, K) at the point $\hat{t}$. For convenience of representation we restrict ourselves in the following to the case that $\mu \leq 2$.

Theorem 11. *Let the functions $M, C, K \in C(\mathbb{I}, \mathbb{C}^{m,n})$ be sufficiently smooth with well-defined strangeness index $\mu \leq 2$ and global characteristic values $(r_i, d_i^{(1)}, a_i, s_i^{(MCK)}, s_i^{(CK)}, s_i^{(MC)}, s_i^{(MK)}, u_i, v_i)$, $i \in \mathbb{N}_0$. Furthermore, let $(\mathcal{M}_l(\hat{t}), \mathcal{L}_l(\hat{t}), \mathcal{N}_l(\hat{t}))$ be the corresponding inflated* *matrix triple at a fixed $\hat{t} \in \mathbb{I}$ with local characteristic values* ($\tilde{r}_l$, $\tilde{d}_l^{(1)}$, $\tilde{a}_l$, $\tilde{s}_l^{(MCK)}$,$\tilde{s}_l^{(CK)}$,$\tilde{s}_l^{(MC)}$, $\tilde{s}_l^{(MK)}$,$\tilde{u}_l$,$\tilde{v}_l$)*. Then, for $l = 0, 1, 2$, we have*

$$\begin{aligned}
\operatorname{rank}[\mathcal{M}_l] &= \tilde{r}_l = (l+1)m - \sum_{i=0}^{l} q_i - \sum_{i=0}^{l} c_i - \sum_{i=0}^{l} v_i, \\
\operatorname{rank}[\mathcal{M}_l, \mathcal{L}_l] &= (l+1)m - \sum_{i=0}^{l} c_i - \sum_{i=0}^{l} v_i, \\
\operatorname{rank}[\mathcal{M}_l, \mathcal{L}_l, \mathcal{N}_l] &= (l+1)m - \sum_{i=0}^{l} v_i,
\end{aligned} \tag{17}$$

using the definitions as in Lemma 7, and

$$\begin{aligned}
\tilde{d}_l^{(1)} &= k_l - p_l + b_l, \\
\tilde{a}_l &= b_l = c_l - s_l^{(MCK)} - s_l^{(CK)} - s_l^{(MK)} + s_{l-1}, \\
\tilde{s}_l^{(MCK)} &= \sum_{i=0}^{l} c_i - b_{l-1} - p_l - \sum_{i=1}^{l} (d_i - t_i), \\
\tilde{s}_l^{(CK)} &= b_{l-1} - b_l + p_l, \\
\tilde{s}_l^{(MC)} &= \sum_{i=0}^{l} q_i - \sum_{i=0}^{l} c_i + \sum_{i=1}^{l} (d_i - t_i) + p_l - k_l, \\
\tilde{s}_l^{(MK)} &= \sum_{i=1}^{l} (d_i - t_i), \\
\tilde{v}_l &= \sum_{i=0}^{l} v_i, \\
\tilde{u}_l &= (l+1)u_0 + (l+1)a_0 + lk_0 - \sum_{i=0}^{l} k_i - \sum_{i=0}^{l} b_i.
\end{aligned} \tag{18}$$

Proof. The proof is given in the Appendix A.2. □

From the relations (18) we can determine recursive formulas for the global characteristic values of the original matrix triple (M, C, K).

Corollary 12. *Let the strangeness index μ of the matrix triple (M, C, K) be well-defined with $\mu \leq 2$ and let $(\tilde{r}_l, \tilde{d}_l^{(1)}, \tilde{a}_l, \tilde{s}_l^{(MCK)}, \tilde{s}_l^{(CK)}, \tilde{s}_l^{(MC)}, \tilde{s}_l^{(MK)}, \tilde{u}_l, \tilde{v}_l)$, $l = 0, \ldots, \mu$ be the sequence of the local characteristic values of $(\mathcal{M}_l, \mathcal{L}_l, \mathcal{N}_l)$ for some $t \in \mathbb{I}$. Then for the sequence $(r_i, d_i^{(1)}, a_i, s_i^{(MCK)}, s_i^{(CK)}, s_i^{(MC)}, s_i^{(MK)}, u_i, v_i)$ of the global characteristic values of (M, C, K) it holds that*

$$\begin{aligned}
c_0 &= \tilde{a}_0 + \tilde{s}_0^{(MCK)} + \tilde{s}_0^{(CK)} + \tilde{s}_0^{(MK)}, \\
c_{i+1} &= (\tilde{a}_{i+1} - \tilde{a}_i) + (\tilde{s}_{i+1}^{(MCK)} - \tilde{s}_i^{(MCK)}) + (\tilde{s}_{i+1}^{(CK)} - \tilde{s}_i^{(CK)}) + (\tilde{s}_{i+1}^{(MK)} - \tilde{s}_i^{(MK)}), \\
q_0 &= \tilde{d}_0^{(1)} + \tilde{s}_0^{(MCK)} + \tilde{s}_0^{(CK)} + \tilde{s}_0^{(MC)},
\end{aligned}$$

$$
\begin{aligned}
&q_{i+1} = (\tilde{d}^{(1)}_{i+1} - \tilde{d}^{(1)}_{i}) + (\tilde{s}^{(MCK)}_{i+1} - \tilde{s}^{(MCK)}_{i}) + (\tilde{s}^{(CK)}_{i+1} - \tilde{s}^{(CK)}_{i}) + (\tilde{s}^{(MC)}_{i+1} - \tilde{s}^{(MC)}_{i}),\\
&v_0 = m - c_0 - q_0 - \tilde{r}_0,\\
&v_{i+1} = m - c_{i+1} - q_{i+1} - (\tilde{r}_{i+1} - \tilde{r}_i),\\
&s^{(MCK)}_i + s^{(CK)}_i + s_i = c_i - \tilde{a}_i,\\
&s^{(MCK)}_i + s^{(MC)}_i + s_{i-1} = q_i - \tilde{d}^{(1)}_i - \tilde{s}^{(CK)}_i.
\end{aligned} \tag{19}
$$

Proof. The relations follow directly from Theorem 11 and from the definitions in Theorem 7 since

$$
\begin{aligned}
&\tilde{a}_{i+1} - \tilde{a}_i = b_{i+1} - b_i,\\
&\tilde{s}^{(MCK)}_{i+1} - \tilde{s}^{(MCK)}_{i} + \tilde{s}^{(CK)}_{i+1} - \tilde{s}^{(CK)}_{i} = c_{i+1} - d_{i+1} + t_{i+1} - b_{i+1} + b_i,\\
&\tilde{s}^{(MK)}_{i+1} - \tilde{s}^{(MK)}_{i} = d_{i+1} - t_{i+1},\\
&\tilde{d}^{(1)}_{i+1} - \tilde{d}^{(1)}_{i} = k_{i+1} - k_i + p_i - p_{i+1} + b_{i+1} - b_i,\\
&\tilde{s}^{(MC)}_{i+1} - \tilde{s}^{(MC)}_{i} = q_{i+1} - c_{i+1} + d_{i+1} - t_{i+1} + p_{i+1} - p_i + k_i - k_{i+1},\\
&m - c_{i+1} - q_{i+1} - (\tilde{r}_{i+1} - \tilde{r}_i) = m - c_{i+1} - q_{i+1} - m + q_{i+1} + c_{i+1} + v_{i+1} = v_{i+1},\\
&c_i - \tilde{a}_i = c_i - c_i + s^{(MCK)}_i + s^{(CK)}_i + s^{(MK)}_i - s_{i-1} = s^{(MCK)}_i + s^{(CK)}_i + s_i,\\
&q_i - \tilde{d}^{(1)}_i - \tilde{s}^{(CK)}_i = q_i - k_i - b_{i-1} = s^{(MCK)}_i + s^{(MC)}_i + s_{i-1}.
\end{aligned}
$$

□

The recursive formulas (19) enable the determination of the strangeness index μ in a numerically computable way by determining the local characteristic values of the inflated triple $(\mathcal{M}_l, \mathcal{L}_l, \mathcal{N}_l)$ for each time $t \in \mathbb{I}$. The obtained system in the i-th reduction step is strangeness free if the sums $s^{(MCK)}_i + s^{(MC)}_i + s_{i-1}$ and $s^{(MCK)}_i + s^{(CK)}_i + s_i$ vanish, since then $s^{(MCK)}_i = s^{(CK)}_i = s^{(MK)}_i = s^{(MC)}_i = 0$, as all summands are nonnegative integer values. Finally, for the characteristic values of the strangeness-free system we get

$$
\begin{aligned}
a_\mu &= \sum_{i=0}^{\mu} c_i = \operatorname{rank}[\mathcal{M}_\mu, \mathcal{L}_\mu, \mathcal{N}_\mu] - \operatorname{rank}[\mathcal{M}_\mu, \mathcal{L}_\mu],\\
d^{(1)}_\mu &= \sum_{i=0}^{\mu} q_i - \sum_{i=0}^{\mu-1} c_i\\
&= \operatorname{rank}[\mathcal{M}_\mu, \mathcal{L}_\mu] - \tilde{r}_\mu + \operatorname{rank}[\mathcal{M}_{\mu-1}, \mathcal{L}_{\mu-1}] - \operatorname{rank}[\mathcal{M}_{\mu-1}, \mathcal{L}_{\mu-1}, \mathcal{N}_{\mu-1}],\\
v_\mu &= \tilde{v}_\mu - \tilde{v}_{\mu-1},\\
d^{(2)}_\mu &= m - a_\mu - d^{(1)}_\mu - v_\mu.
\end{aligned} \tag{20}
$$

Next, we want to extract a strangeness-free triple $(\hat{M}, \hat{C}, \hat{K})$ from the inflated system with characteristic values $\hat{r} = d^{(2)}_\mu$, $\hat{d}^{(1)} = d^{(1)}_\mu$, $\hat{a} = a_\mu$, $\hat{u} = u_\mu$, $\hat{v} = v_\mu$ and $\hat{s}^{(MCK)} = \hat{s}^{(CK)} = \hat{s}^{(MK)} = \hat{s}^{(MC)} = 0$ using only local information from $(\mathcal{M}_\mu(t), \mathcal{L}_\mu(t), \mathcal{N}_\mu(t))$.

Theorem 13. *Consider a linear second order differential-algebraic system* (1) *with well-defined strangeness-index* $\mu \leq 2$. *Then the inflated triple* $(\mathcal{M}_\mu, \mathcal{L}_\mu, \mathcal{N}_\mu)$ *associated with* (M, C, K) *has the following properties:*

1. *For all $t \in \mathbb{I}$ it holds that*

$$\operatorname{rank} \mathcal{M}_\mu(t) = (\mu+1)m - a_\mu - \tilde{v}_\mu - d_\mu^{(1)} - \sum_{i=0}^{\mu-1} c_i,$$

such that there exists a smooth matrix function Z with orthonormal columns and size $((\mu+1)m, a_\mu + \tilde{v}_\mu + d_\mu^{(1)} + \sum_{i=0}^{\mu-1} c_i)$ satisfying

$$Z^H \mathcal{M}_\mu = 0.$$

2. *For all $t \in \mathbb{I}$ it holds that*

$$\begin{aligned}\operatorname{rank} [\mathcal{M}_\mu(t), \mathcal{L}_\mu(t)] &= (\mu+1)m - a_\mu - \tilde{v}_\mu,\\ \operatorname{rank} [\mathcal{M}_\mu(t), \mathcal{L}_\mu(t), \mathcal{N}_\mu(t)] &= (\mu+1)m - \tilde{v}_\mu,\end{aligned}$$

such that without loss of generality Z can be partitioned into $Z = [Z_2, Z_3, Z_4]$, with Z_2 of size $((\mu+1)m, d_\mu^{(1)} + \sum_{i=0}^{\mu-1} c_i)$, Z_3 of size $((\mu+1)m, a_\mu)$ and Z_4 of size $((\mu+1)m, \tilde{v}_\mu)$ such that

$$Z_3^H \mathcal{L}_\mu = 0, \quad Z_4^H \mathcal{L}_\mu = 0, \quad Z_4^H \mathcal{N}_\mu = 0.$$

3. *For all $t \in \mathbb{I}$ we have*

$$\begin{aligned}\operatorname{rank} (Z_3^H \mathcal{N}_\mu \begin{bmatrix} I_n & 0 & \dots & 0 \end{bmatrix}^H) &= a_\mu,\\ \operatorname{rank} (Z_2^H \mathcal{L}_\mu \begin{bmatrix} I_n & 0 & \dots & 0 \end{bmatrix}^H) &= d_\mu^{(1)} + \sum_{i=0}^{\mu-1} c_i,\end{aligned}$$

such that there exists a smooth matrix function T_3 with orthonormal columns and size $(n, n-a_\mu)$, with $n - a_\mu = d_\mu^{(2)} + d_\mu^{(1)} + u_\mu$ satisfying

$$Z_3^H \mathcal{N}_\mu \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix}^H T_3 = 0.$$

4. *For all $t \in \mathbb{I}$ we have*

$$\operatorname{rank} (Z_2^H \mathcal{L}_\mu \begin{bmatrix} I_n & 0 & \dots & 0 \end{bmatrix}^H T_3) = d_\mu^{(1)},$$

such that there exists a smooth matrix function Z_1 of size $(d_\mu^{(1)} + \sum_{i=0}^{\mu-1} c_i, d_\mu^{(1)})$ with orthonormal columns such that

$$\operatorname{rank} (Z_1^H Z_2^H \mathcal{L}_\mu \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix}^H) = d_\mu^{(1)}.$$

Furthermore, there exists a smooth matrix function T_2 of size $(n - a_\mu, n - a_\mu - d_\mu^{(1)})$ with orthonormal columns, such that

$$Z_1^H Z_2^H \mathcal{L}_\mu \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix}^H T_3 T_2 = 0.$$

5. *For all $t \in \mathbb{I}$ it holds that* $\operatorname{rank}(MT_3T_2) = d_\mu^{(2)}$. *This implies the existence of a smooth matrix function Z_0 with orthonormal columns and size $(m, d_\mu^{(2)})$ such that $Z_0^H M$ has constant rank $d_\mu^{(2)}$.*

Proof. By assumption, the strangeness index is well-defined and the ranks of $\mathcal{M}_\mu$, $\mathcal{L}_\mu$ and $\mathcal{N}_\mu$ are constant in $\mathbb{I}$ with

$$\begin{aligned}
\operatorname{rank}[\mathcal{M}_\mu, \mathcal{L}_\mu, \mathcal{N}_\mu] &= (\mu+1)m - \tilde{v}_\mu, \\
\operatorname{rank}[\mathcal{M}_\mu, \mathcal{L}_\mu] &= (\mu+1)m - a_\mu - \tilde{v}_\mu, \\
\operatorname{rank}\mathcal{M}_\mu &= (\mu+1)m - a_\mu - \tilde{v}_\mu - d_\mu^{(1)} - \sum_{i=0}^{\mu-1} c_i,
\end{aligned}$$

due to Theorem 11 and Corollary 12 (see also the relations (20)). Thus, there exists a continuous matrix-valued function Z of size $((\mu+1)m, a_\mu + \tilde{v}_\mu + d_\mu^{(1)} + \sum_{i=0}^{\mu-1} c_i)$, whose columns form a basis of $\operatorname{corange}\mathcal{M}_\mu$, i.e., $Z^H\mathcal{M}_\mu = 0$. Without loss of generality the matrix Z can be partitioned into $Z = [Z_2, Z_3, Z_4]$, with Z_2 of size $((\mu+1)m, d_\mu^{(1)} + \sum_{i=0}^{\mu-1} c_i)$, Z_3 of size $((\mu+1)m, a_\mu)$ and Z_4 of size $((\mu+1)m, \tilde{v}_\mu)$, such that

$$Z_3^H\mathcal{L}_\mu = 0, \quad Z_4^H\mathcal{L}_\mu = 0, \quad Z_4^H\mathcal{N}_\mu = 0,$$

i.e., the columns of the matrices Z_4 and Z_3 form bases of $\operatorname{corange}([\mathcal{M}_\mu, \mathcal{L}_\mu, \mathcal{N}_\mu])$, and of $\operatorname{corange}([\mathcal{M}_\mu, \mathcal{L}_\mu])$, respectively. First, we note that multiplication of (13) for $l = \mu$ by Z_3^H gives

$$Z_3^H\mathcal{N}_\mu z_\mu = Z_3^H g_\mu.$$

The only nontrivial entries in $\mathcal{N}_\mu$ are in the first block column belonging to the original unknown x. Hence, we get purely algebraic equations for x. Lemma 4 and (18) give that

$$\operatorname{rank}(Z_3^H\mathcal{N}_\mu \begin{bmatrix} I_n & 0 & \dots & 0 \end{bmatrix}^H) = \tilde{a}_\mu + \tilde{s}_\mu^{(MCK)} + \tilde{s}_\mu^{(CK)} + \tilde{s}_\mu^{(MK)} = a_\mu, \tag{21}$$

thus, with Z_3 we obtain the complete set of algebraic equations. Next, we must get $d_\mu^{(1)}$ first order differential equations and $d_\mu^{(2)}$ second order differential equations to complete these algebraic equations to a strangeness-free differential-algebraic system. In a similar way multiplication of (13) with the matrix Z_2^H yields

$$Z_2^H\mathcal{L}_\mu\dot{z}_\mu + Z_2^H\mathcal{N}_\mu z_\mu = Z_2^H g_\mu.$$

Again, the only non-zero entries of $\mathcal{L}_\mu$ are in the first block column belonging to the first order derivative $\dot{x}$. Lemma 4 and (18) give

$$\operatorname{rank}(Z_2^H\mathcal{L}_\mu) = \tilde{d}_\mu^{(1)} + \tilde{s}_\mu^{(CK)} + \tilde{s}_\mu^{(MC)} + \tilde{s}_\mu^{(MCK)} = \sum_{i=0}^{\mu} q_i = d_\mu^{(1)} + \sum_{i=0}^{\mu-1} c_i.$$

So far we have shown the first three parts of Theorem 13. To show part 4 and 5 of the Theorem we distinguish between systems of strangeness index $\mu = 0, 1$ and 2. Let $(\tilde{M}, \tilde{C}, \tilde{K})$ be a normal form of the triple (M, C, K) according to the global condensed form given in

Lemma A.2. with corresponding inflated triples $(\tilde{\mathcal{M}_\mu}, \tilde{\mathcal{L}_\mu}, \tilde{\mathcal{N}_\mu})$. Due to Theorem 10, there exist matrices Π, Θ, Ψ, and Σ such that

$$\tilde{\mathcal{M}_\mu} = \Pi \mathcal{M}_\mu \Theta, \quad \tilde{\mathcal{L}_\mu} = \Pi \mathcal{L}_\mu \Theta + 2\Pi \mathcal{M}_\mu \Psi, \quad \tilde{\mathcal{N}_\mu} = \Pi \mathcal{N}_\mu \Theta + \Pi \mathcal{L}_\mu \Psi + \Pi \mathcal{M}_\mu \Sigma,$$

according to (16). For $\mu = 0$ the triple $(\tilde{\mathcal{M}_0}, \tilde{\mathcal{L}_0}, \tilde{\mathcal{N}_0})$ is of the form

$$\left(\begin{bmatrix} I_{d_\mu^{(2)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} C & 0 & C & C \\ 0 & I_{d_\mu^{(1)}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} K & K & 0 & K \\ K & K & 0 & K \\ 0 & 0 & I_{a_\mu} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right). \tag{22}$$

Let Π and Θ be partitioned as $\Pi := [\Pi_1^H, \Pi_2^H, \Pi_3^H, \Pi_4^H]^H$ and $\Theta := [\Theta_1, \Theta_2, \Theta_3, \Theta_4]$ according to (22). Then, setting

$$Z_2 = \Pi_2^H, \quad Z_3 = \Pi_3^H, \quad Z_4 = \Pi_4^H,$$

yields

$$\begin{aligned} Z_4^H \mathcal{M}_0 = 0, \quad Z_4^H \mathcal{L}_0 = 0, \quad Z_4^H \mathcal{N}_0 = 0, \\ Z_3^H \mathcal{M}_0 = 0, \quad Z_3^H \mathcal{L}_0 = 0, \quad Z_2^H \mathcal{M}_0 = 0, \end{aligned}$$

as well as

$$\begin{aligned} \operatorname{rank}(Z_3^H \mathcal{N}_0) &= \operatorname{rank} \begin{bmatrix} 0 & 0 & I_{a_\mu} & 0 \end{bmatrix} = a_\mu, \\ \operatorname{rank}(Z_2^H \mathcal{L}_0) &= \operatorname{rank} \begin{bmatrix} 0 & I_{d_\mu^{(1)}} & 0 & 0 \end{bmatrix} = d_\mu^{(1)}. \end{aligned}$$

Further, setting $T_3 = [\Theta_1, \Theta_2, \Theta_4]$ we get

$$\operatorname{rank}(Z_2^H \mathcal{L}_0 T_3) = \operatorname{rank} \begin{bmatrix} 0 & I_{d_\mu^{(1)}} & 0 \end{bmatrix} = d_\mu^{(1)},$$

and with $Z_1 = I_{d_\mu^{(1)}}$ and $T_2 = \begin{bmatrix} I_{d_\mu^{(2)}} & 0 \\ 0 & 0 \\ 0 & I_{u_\mu} \end{bmatrix}$ we get $Z_1^H Z_2^H \mathcal{L}_0 T_3 T_2 = 0$, and

$$\operatorname{rank}(M T_3 T_2) = \operatorname{rank} \begin{bmatrix} I_{d_\mu^{(2)}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = d_\mu^{(2)}.$$

Finally, setting $Z_0^H = \begin{bmatrix} I_{d_\mu^{(2)}} & 0 & 0 & 0 \end{bmatrix}$ yields

$$\operatorname{rank}(Z_0^H M) = \operatorname{rank} \begin{bmatrix} I_{d_\mu^{(2)}} & 0 & 0 & 0 \end{bmatrix} = d_\mu^{(2)}.$$

In the case $\mu = 1$ the triple $(\tilde{\mathcal{M}_1}, \tilde{\mathcal{L}_1}, \tilde{\mathcal{N}_1})$ is in the form (33) given in the proof of Theorem 11. Let Π and Θ be partitioned as

$$\Pi := \left[\Pi_1^H, \ldots, \Pi_{28}^H\right]^H, \quad \Theta := [\Theta_1, \ldots, \Theta_{18}],$$

corresponding to the structure of block matrices in (33). Then, setting

$$
\begin{aligned}
Z &= [\Pi_1^H, \Pi_2^H, \Pi_3^H, \Pi_7^H, \dots, \Pi_{15}^H, \Pi_{24}^H, \dots, \Pi_{28}^H], \\
Z_4 &= [\Pi_{3,2}^H, \Pi_{7,2}^H, \Pi_{9,2}^H, \Pi_{15}^H, \Pi_{28}^H], \\
Z_3 &= [\Pi_{3,1}^H, \Pi_{7,1}^H, \Pi_{9,1}^H, \Pi_{11}^H, \dots, \Pi_{14}^H], \\
Z_2 &= [\Pi_1^H, \Pi_2^H, \Pi_8^H, \Pi_{10}^H, \Pi_{24}^H, \dots, \Pi_{27}^H],
\end{aligned}
$$

where again $\Pi_{.,1}$ and $\Pi_{.,2}$ denote the parts of $\Pi_.$ that after one more block decomposition of the matrices in (33) corresponds to the range and nullspace of block rows, respectively (see also the proof of Theorem 11), we have

$$
\begin{aligned}
\operatorname{rank}(Z_3^H \mathcal{N}_1) &= c_0 + c_1 = a_\mu, \\
\operatorname{rank}(Z_2^H \mathcal{L}_1) &= e_1 + d_0^{(1)} + s_0^{(MC)} + c_0 = d_\mu^{(1)} + c_0.
\end{aligned}
$$

Setting $T_3 = [\Theta_{2,2}, \Theta_{4,2}, \Theta_{5,2}, \Theta_{7,2}, \Theta_{9,2}, \Theta_{10,2}]$ we get

$$
\operatorname{rank}(Z_2^H \mathcal{L}_1 \begin{bmatrix} I_n & 0 \end{bmatrix}^H T_3) = d_0^{(1)} + s_0^{(MC)} + e_1 = d_\mu^{(1)},
$$

and further choosing Z_1 such that $Z_2 Z_1 = [\Pi_1^H, \Pi_2^H, \Pi_8^H, \Pi_{10}^H]$ we have

$$
\operatorname{rank}(Z_1^H Z_2^H \mathcal{L}_1) = d_\mu^{(1)}.
$$

If we choose T_2 such that $T_3 T_2 = [\Theta_{5,2}, \Theta_{10,2}]$, then we have

$$
Z_1^H Z_2^H \mathcal{L}_1 \begin{bmatrix} I_n & 0 \end{bmatrix}^H T_3 T_2 = 0,
$$

and

$$
\operatorname{rank}(M T_3 T_2) = d_0^{(2)} = d_\mu^{(2)}.
$$

Finally, there exists a smooth matrix function Z_0 of size $(m, d_\mu^{(2)})$ with orthonormal columns such that

$$
\operatorname{rank}(Z_0^H M) = d_\mu^{(2)}.
$$

For s-index $\mu = 2$ the triple $(\tilde{\mathcal{M}}_2, \tilde{\mathcal{L}}_2, \tilde{\mathcal{N}}_2)$ is in the form (34) and Π and Θ are partitioned into

$$
\Pi := \begin{bmatrix} \Pi_1^H, \dots, \Pi_{44}^H \end{bmatrix}^H, \quad \Theta := [\Theta_1, \dots, \Theta_{28}]
$$

according to the block structure of (34). Then, by setting

$$
\begin{aligned}
Z &= [\Pi_1^H, \dots, \Pi_6^H, \Pi_{10}^H, \dots, \Pi_{18}^H, \Pi_{20}^H, \Pi_{23}^H, \Pi_{25}^H, \Pi_{27}^H, \dots, \Pi_{31}^H, \Pi_{44}^H], \\
Z_4 &= [\Pi_{3,2}^H, \Pi_{6,2}^H, \Pi_{10,2}^H, \Pi_{12,2}^H, \Pi_{18}^H, \Pi_{20,3}^H, \Pi_{23,3}^H, \Pi_{25,3}^H, \Pi_{31}^H, \Pi_{44}^H], \\
Z_3 &= [\Pi_{3,1}^H, \Pi_{6,1}^H, \Pi_{10,1}^H, \Pi_{12,1}^H, \Pi_{14}^H, \dots, \Pi_{17}^H, \Pi_{20,2}^H, \Pi_{23,2}^H, \Pi_{25,2}^H], \\
Z_2 &= [\Pi_1^H, \Pi_2^H, \Pi_4^H, \Pi_5^H, \Pi_{11}^H, \Pi_{13}^H, \Pi_{20,1}^H, \Pi_{23,1}^H, \Pi_{25,1}^H, \Pi_{27}^H, \dots, \Pi_{30}^H],
\end{aligned}
$$

we have

$$\begin{aligned}
&\operatorname{rank}(Z_3^H \mathcal{N}_2) = c_0 + c_1 + c_2 = a_\mu,\\
&\operatorname{rank}(Z_2^H \mathcal{L}_2) = e_1 + e_2 + d_0^{(1)} + s_0^{(MC)} + c_0 + b_1 = d_2^{(1)} + c_0 + c_1 - s_1 = d_\mu^{(1)} + c_0 + c_1.
\end{aligned}$$

Further, setting $T_3 = [\Theta_{2,2}, \Theta_{4,2}, \Theta_{5,2}, \Theta_{6,2}, \Theta_{8,2}, \Theta_{10,2}, \Theta_{11,2}, \Theta_{12,2}]$ yields

$$Z_3^H \mathcal{N}_2 \begin{bmatrix} I_n & 0 & 0 \end{bmatrix}^H T_3 = 0$$

and we get

$$\operatorname{rank}(Z_2^H \mathcal{L}_2 \begin{bmatrix} I_n & 0 & 0 \end{bmatrix}^H T_3) = d_2^{(1)}.$$

In a similar way as before, choosing Z_1 such that $Z_2 Z_1 = [\Pi_1^H, \Pi_2^H, \Pi_4^H, \Pi_5^H, \Pi_{11}^H, \Pi_{13}^H]$ we have

$$\operatorname{rank}(Z_1^H Z_2^H \mathcal{L}_2) = d_\mu^{(1)},$$

and if we choose T_2 such that $T_3 T_2 = [\Theta_{5,2}, \Theta_{10,2}]$ we have

$$Z_1^H Z_2^H \mathcal{L}_2 \begin{bmatrix} I_n & 0 & 0 \end{bmatrix}^H T_3 T_2 = 0,$$

as well as

$$\operatorname{rank}(M T_3 T_2) = d_\mu^{(2)}.$$

Again, there exists a smooth matrix function Z_0 of size $(m, d_\mu^{(2)})$ with orthonormal columns such that

$$\operatorname{rank}(Z_0^H M) = d_\mu^{(2)}.$$

□

From the results of Theorem 13 we can construct a triple of matrix-valued functions

$$(\hat{M}, \hat{C}, \hat{K}) = \left(\begin{bmatrix} \hat{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \\ 0 \end{bmatrix} \right), \tag{23}$$

with entries

$$\begin{aligned}
&\hat{M}_1 = Z_0^H M, \quad \hat{C}_1 = Z_0^H C, \quad \hat{K}_1 = Z_0^H K,\\
&\hat{C}_2 = Z_1^H Z_2^H \mathcal{L}_\mu \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix}^H, \quad \hat{K}_2 = Z_1^H Z_2^H \mathcal{N}_\mu \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix}^H,\\
&\hat{K}_3 = Z_3^H \mathcal{N}_\mu \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix}^H,
\end{aligned}$$

which has the same size as the original triple (M, C, K). We can show that this triple is strangeness-free with the same characteristic values as the strangeness-free system (10).

Theorem 14. *Let the strangeness index μ of (M, C, K) be well-defined with $\mu \leq 2$ and global characteristic values $(r_i, d_i^{(1)}, a_i, s_i^{(MCK)}, s_i^{(MC)}, s_i^{(MK)}, s_i^{(CK)}, u_i, v_i)$, for $i = 0, \ldots, \mu$. Then, the triple $(\hat{M}, \hat{C}, \hat{K})$, constructed as in (23), has a well-defined strangeness index $\hat{\mu} = 0$ and the global characteristic values of $(\hat{M}(t), \hat{C}(t), \hat{K}(t))$ are given by*

$$(\hat{r}, \hat{d}^{(1)}, \hat{a}, \hat{s}^{(MCK)}, \hat{s}^{(MC)}, \hat{s}^{(MK)}, \hat{s}^{(CK)}, \hat{v}) = (d_\mu^{(2)}, d_\mu^{(1)}, a_\mu, 0, 0, 0, 0, v_\mu)$$

uniformly for all $t \in \mathbb{I}$.

Proof. In the following, we omit the argument t. By construction the columns of T_3 defined in Theorem 13 form a basis of kernel $\hat{K}_3$ and the columns of T_2 form a basis of kernel $(\hat{C}_2 T_3)$. We consider the matrix $T = T_3 T_2$. Because $\hat{M}_1$ has full row rank we can split T without loss of generality into $T = \begin{bmatrix} T_1' & T_4' \end{bmatrix}$ in such a way that $\hat{M}_1 T_1'$ is nonsingular. Choosing T_3' such that $\hat{K}_3 T_3'$ is also nonsingular and T_2' such that $\hat{C}_2 T_2'$ is nonsingular and $\hat{K}_3 T_2' = 0$ we get a nonsingular matrix $\hat{T} = \begin{bmatrix} T_1' & T_2' & T_3' & T_4' \end{bmatrix}$. By multiplication with this matrix from the right we get the following local equivalence

$$\begin{aligned}
(\hat{M}, \hat{C}, \hat{K}) &= \left(\begin{bmatrix} \hat{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \\ 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} \hat{M}_1 T_1' & \hat{M}_1 T_2' & \hat{M}_1 T_3' & \hat{M}_1 T_4' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{C}_1 T_1' & \hat{C}_1 T_2' & \hat{C}_1 T_3' & \hat{C}_1 T_4' \\ \hat{C}_2 T_1' & \hat{C}_2 T_2' & \hat{C}_2 T_3' & \hat{C}_2 T_4' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right. \\
&\qquad \left. \begin{bmatrix} \hat{K}_1 T_1' & \hat{K}_1 T_2' & \hat{K}_1 T_3' & \hat{K}_1 T_4' \\ \hat{K}_2 T_1' & \hat{K}_2 T_2' & \hat{K}_2 T_3' & \hat{K}_2 T_4' \\ \hat{K}_3 T_1' & \hat{K}_3 T_2' & \hat{K}_3 T_3' & \hat{K}_3 T_4' \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} \hat{M}_1 T_1' & \hat{M}_1 T_2' & \hat{M}_1 T_3' & \hat{M}_1 T_4' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{C}_1 T_1' & \hat{C}_1 T_2' & \hat{C}_1 T_3' & \hat{C}_1 T_4' \\ 0 & \hat{C}_2 T_2' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right. \\
&\qquad \left. \begin{bmatrix} \hat{K}_1 T_1' & \hat{K}_1 T_2' & \hat{K}_1 T_3' & \hat{K}_1 T_4' \\ \hat{K}_2 T_1' & \hat{K}_2 T_2' & \hat{K}_2 T_3' & \hat{K}_2 T_4' \\ 0 & 0 & \hat{K}_3 T_3' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} \hat{M}_1 T_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & \star & \star & \star \\ 0 & \hat{C}_2 T_2' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \hat{K}_3 T_3' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_{d_\mu^{(2)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & \star & \star & \star \\ 0 & I_{d_\mu^{(1)}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & I_{a_\mu} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).
\end{aligned}$$

From the last triple we obtain $\hat{r} = d_\mu^{(2)}$, $\hat{d}^{(1)} = d_\mu^{(1)}$, $\hat{a} = a_\mu$, $\hat{s}^{(MCK)} = \hat{s}^{(MC)} = \hat{s}^{(MK)} = \hat{s}^{(CK)} = 0$ and $\hat{v} = v_\mu$ from Lemma 4. $\square$

Thus, we have derived an index reduction method that allows us to extract a strangeness-free triple from the original triple of matrix-valued functions and its derivatives. The matrix-valued functions Z_0, Z_1, Z_2 and Z_3 as given in Theorem 13 can be determined via numerical rank decisions, e.g., using a singular value decomposition or a rank revealing QR decomposition, see e.g. [7]. Setting the inhomogeneities $\hat{f}_1 = Z_0^H f$, $\hat{f}_2 = Z_1^H Z_2^H g_\mu$, $\hat{f}_3 = Z_3^H g_\mu$ and $\hat{f}_4 = 0$ accordingly (assuming that the system is solvable) we obtain a differential-algebraic system

$$\hat{M}(t)\ddot{x} + \hat{C}(t)\dot{x} + \hat{K}(t)x = \hat{f}(t), \tag{24}$$

from the inflated differential-algebraic equation (13). Setting $\hat{f}_4 = 0$ in (24) can be seen as a regularization, since an unsolvable problem is replaced by a solvable one. System (24) is strangeness-free and has the same size and also the same solution set as the original system (1) since only transformations from the left are involved. Concluding, we give an example to illustrate the index reduction procedure.

Example 15. We consider again the linear second order system (3) of strangeness index $\mu = 2$ with characteristic values $d_\mu^{(2)} = 1$, $d_\mu^{(1)} = 0$, $a_\mu = 2$, $v_\mu = 0$ and $u_\mu = 0$. The matrix triple corresponding to the extended system (13) is given by

$$(\mathcal{M}_2(t), \mathcal{L}_2(t), \mathcal{N}_2(t)) =$$
$$\left(\left[\begin{array}{ccc|ccc|ccc} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & t & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & t & t & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 3 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1+t & 1 & 0 & 2 & 2 & 0 & t & t \end{array}\right], \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ \hline 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 1+t & 1 & & \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 2 & 0 & & \end{array}\right], \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 1+t & 1 & & \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{array}\right] \right).$$

We have

$$\begin{aligned} \operatorname{rank}\left[\mathcal{M}_2(t), \mathcal{L}_2(t), \mathcal{N}_2(t)\right] &= 9 = (\mu+1)m - \tilde{v}_\mu, \\ \operatorname{rank}\left[\mathcal{M}_2(t), \mathcal{L}_2(t)\right] &= 7 = (\mu+1)m - a_\mu - \tilde{v}_\mu, \\ \operatorname{rank}\left[\mathcal{M}_2(t)\right] &= 6 = (\mu+1)m - d_\mu^{(1)} - c_0 - c_1 - a_\mu - \tilde{v}_\mu, \end{aligned}$$

independent of $t \in \mathbb{I}$ and we can choose

$$\begin{aligned} Z_3^H &= \begin{bmatrix} 0 & -1 & 0 & 0 & -2 & 0 & 0 & -t & 1 \\ 0 & -t & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ Z_2^H &= \begin{bmatrix} 0 & -1 & 0 & 0 & -t & 1 & 0 & 0 & 0 \end{bmatrix}, \\ T_3 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^H, \quad T_2 = 1. \end{aligned}$$

Then, we have

$$
\begin{aligned}
&\operatorname{rank}\left(Z_3^H \mathcal{N}_2 \begin{bmatrix} I_n & 0 & 0 \end{bmatrix}^H\right) = \operatorname{rank}\left(\begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2 = a_\mu, \\
&\operatorname{rank}\left(Z_2^H \mathcal{L}_2 \begin{bmatrix} I_n & 0 & 0 \end{bmatrix}^H\right) = \operatorname{rank}\left(\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\right) = 1 = d_\mu^{(1)} + c_0 + c_1, \\
&\operatorname{rank}\left(Z_2^H \mathcal{L}_2 \begin{bmatrix} I_n & 0 & 0 \end{bmatrix}^H T_3\right) = \operatorname{rank}\left(\begin{bmatrix} 0 \end{bmatrix}\right) = 0 = d_\mu^{(1)}, \\
&\operatorname{rank}(M T_3 T_2) = \operatorname{rank}\left(\begin{bmatrix} t & 0 & 0 \end{bmatrix}^H\right) = 1 = d_\mu^{(2)}.
\end{aligned}
$$

Finally, choosing $Z_0^H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ we get a strangeness-free system of the form

$$
\begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} f_1 \\ -f_2 - 2\dot{f}_2 - t\ddot{f}_2 + \ddot{f}_3 \\ -tf_2 + f_3 \end{bmatrix}, \tag{25}
$$

with the same solution as the original system (3).

Remark 16. *The proof of Theorem 11, and consequently also of Corollary 12, Theorem 13 and Theorem 14, is given only for DAEs of strangeness index $\mu \leq 2$ for ease of representation. However, the results are also valid for systems of arbitrary high index. To prove the results for linear second order systems* (1) *of arbitrary strangeness index $\mu > 2$ the construction of a global canonical form analogous to the form given in [11, Theorem 3.21] is more convenient, but until now it is not clear how a suitable condensed form can be constructed.*

Remark 17. *The derivative array approach presented in this section can also be extended to arbitrary linear high order differential-algebraic systems. The inflated system corresponding to* (13) *can be obtained in the same way by differentiating the original k-th order system and ordering the derivatives of the coefficient matrices in such a way that only the leading coefficient matrix has a lower triangular block structure and all other coefficient matrices of the inflated system have entries only in the first block columns. Then, the results of Theorem 10 also hold for k-th order systems and a Hypothesis similar to Theorem 13 can be formulated that allows an index reduction for linear k-th order systems by choosing suitable projections in the same way as for linear second order systems. For the theoretical analysis of linear k-th order differential-algebraic systems see also [13, 16]. Further, a corresponding Hypothesis for nonlinear second order differential-algebraic systems has been formulated in [19].*

4 Order Reduction for Linear Second Order DAEs

The numerical solution of higher order differential-algebraic systems either requires the direct numerical solution of the higher order system by appropriate numerical methods, see e.g. [15, 18], or a suitable transformation into a first order system that does not increase the index. Since most of the numerical methods suited for the solution of DAEs are constructed for first order systems and these methods are well-studied, in general a transformation into a first order system is desired. Furthermore, for a robust solution the numerical methods require differential-algebraic systems of low index, such that besides the order reduction also an index reduction is required. In Example 1 we have seen that for higher order DAEs the classical order reduction by introducing new variables $v = \dot{x}$ for the derivatives can lead to an increase in the index of the DAE corresponding to higher smoothness requirements for the inhomogeneity $f(t)$ that even can cause the loss of solvability of the system, see also [13]. For k-th order linear DAEs it has been shown in [13] that if μ is the strangeness index of the tuple of matrix-valued functions associated with the k-th order DAE system, then the maximal possible increase in the strangeness index $\tilde{\mu}$ of the first order system obtained by the classical order reduction procedure is $\tilde{\mu} \leq \mu + k - 1$. In [13] it has been proposed to use the strangeness-free condensed form (10) given in Theorem 6 for the identification of those second order derivatives of variables that can be replaced to obtain a first order system that is strangeness-free without increasing the index. By introducing the new variable $\tilde{v} = \dot{\tilde{x}}_1$ for the strangeness-free system (10) we obtain a first order system in the variables $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{v})$ that is also strangeness-free The drawback of this approach is that there is no computational feasible method to compute the condensed form (10), except if the structure can be used, since the derivatives of computed transformation matrices are used during the transformations. Further, the strangeness-free system (10) does not have the same solution x as the original second order system (1), but a transformed solution $\tilde{x} = Q^{-1}x$. Nevertheless, the results suggest that first index reduction and then order reduction should be used for a proper treatment of second order systems.

In the following, we will use the index reduction based on derivative arrays derived in Section 3 to obtain a strangeness-free second order system (24), which can then be used to construct a trimmed first order formulation in a numerical feasible way. We assume that we have locally computed a strangeness-free second order system (24) with matrix triple of the form

$$(\hat{M}, \hat{C}, \hat{K}) = \left(\begin{bmatrix} \hat{M}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \\ 0 \end{bmatrix} \right).$$

To find a suitable first order formulation, we first have to identify the second order differential variables. As the matrices $\hat{M}_1$, $\hat{C}_2$ and $\hat{K}_3$ have full row rank due to construction (see Theorem 13) there exists a pointwise unitary matrix-valued function $Q \in C(\mathbb{I}, \mathbb{C}^{n \times n})$ that is sufficiently smooth, such that

$$\begin{bmatrix} \hat{M}_1 \\ \hat{C}_2 \\ \hat{K}_3 \end{bmatrix} Q = \begin{bmatrix} M_{11} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ K_{31} & K_{32} & K_{33} & 0 \end{bmatrix}, \tag{26}$$

where the matrix-valued functions M_{11} of size $d_\mu^{(2)} \times d_\mu^{(2)}$, C_{22} of size $d_\mu^{(1)} \times d_\mu^{(1)}$ and K_{33} of size $a_\mu \times a_\mu$ are pointwise nonsingular. With the corresponding basis transformation

$$x = Q\hat{x}, \quad \dot{x} = Q\dot{\hat{x}} + \dot{Q}\hat{x}, \quad \ddot{x} = Q\ddot{\hat{x}} + 2\dot{Q}\dot{\hat{x}} + \ddot{Q}\hat{x},$$

we get the equivalent system

$$
\begin{aligned}
&\underbrace{\begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{M}Q} \underbrace{\begin{bmatrix} \ddot{\hat{x}}_1 \\ \ddot{\hat{x}}_2 \\ \ddot{\hat{x}}_3 \\ \ddot{\hat{x}}_4 \end{bmatrix}}_{\ddot{\hat{x}}} + \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{C}Q+2\hat{M}\dot{Q}} \underbrace{\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{bmatrix}}_{\dot{\hat{x}}} \\
&\qquad + \underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{K}Q+\hat{C}\dot{Q}+\hat{M}\ddot{Q}} \underbrace{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \end{bmatrix}}_{\hat{f}},
\end{aligned} \tag{27}
$$

where the second order differential variables $\hat{x}_1$ are explicitly specified. By introducing the new variable $\hat{v} = \dot{\hat{x}}_1$ we can transform the system (27) into first order form

$$
\begin{bmatrix} M_{11} & C_{11} & C_{12} & C_{13} & C_{14} \\ 0 & C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{v}} \\ \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{bmatrix} + \begin{bmatrix} 0 & K_{11} & K_{12} & K_{13} & K_{14} \\ 0 & K_{21} & K_{22} & K_{23} & K_{24} \\ 0 & K_{31} & K_{32} & K_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \\ 0 \end{bmatrix}. \tag{28}
$$

Here, we have

$$
\begin{aligned}
&\hat{v} = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} (\dot{Q}^H x + Q^H \dot{x}) = \dot{Q}_1^H x + Q_1^H \dot{x} = \tfrac{d}{dt}(Q_1^H x), \\
&\dot{\hat{v}} = \tfrac{d^2}{dt^2}(Q_1^H x),
\end{aligned}
$$

with $Q_1 = Q \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^H$. Therefore, system (28) is equivalent to

$$
\begin{bmatrix} \hat{M}Q_1 & \hat{C}Q + 2\hat{M}\dot{Q} \\ 0 & J \end{bmatrix} \begin{bmatrix} \frac{d^2}{dt^2}(Q_1^H x) \\ \frac{d}{dt}(Q^H x) \end{bmatrix} + \begin{bmatrix} 0 & \hat{K}Q + \hat{C}\dot{Q} + \hat{M}\ddot{Q} \\ -I & 0 \end{bmatrix} \begin{bmatrix} \frac{d}{dt}(Q_1^H x) \\ Q^H x \end{bmatrix} = \begin{bmatrix} \hat{f} \\ 0 \end{bmatrix}, \tag{29}
$$

with $J = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}$. Now, introducing another variable $v = Q_1^H \dot{x} = \frac{d}{dt}(Q_1^H x) - \dot{Q}_1^H x$, the first equation of (29) becomes

$$
\hat{M}Q_1 \dot{v} + (\hat{C} + \hat{M}(Q_1 \dot{Q}_1^H + 2\dot{Q}Q^H))\dot{x} + (\hat{K} + \hat{M}(Q_1 \ddot{Q}_1^H + 2\dot{Q}\dot{Q}^H + \ddot{Q}Q^H))x = \hat{f},
$$

where we have used that $\dot{Q}Q^H + Q\dot{Q}^H = 0$, as Q is unitary. Thus, we get a first order system in the original variable x and in v of the form

$$
\begin{bmatrix} \hat{M}Q_1 & \tilde{C} \\ 0 & Q_1^H \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & \tilde{K} \\ -I & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} \hat{f} \\ 0 \end{bmatrix},
$$

with

$$
\begin{aligned}
\tilde{C} &= \hat{C} + \hat{M}(Q_1 \dot{Q}_1^H + 2\dot{Q}Q^H) = \hat{C} + \hat{M}(Q_1 \dot{Q}_1^H - 2Q\dot{Q}^H), \\
\tilde{K} &= \hat{K} + \hat{M}(Q_1 \ddot{Q}_1^H + 2\dot{Q}\dot{Q}^H + \ddot{Q}Q^H) = \hat{K} + \hat{M}(Q_1 \ddot{Q}_1^H - Q\ddot{Q}^H),
\end{aligned}
$$

using that $\ddot{Q}Q^H + 2\dot{Q}\dot{Q}^H + Q\ddot{Q}^H = 0$. In addition, it holds that

$$Q_1\dot{Q}_1^H - 2Q\dot{Q}^H = QJ^HJ\dot{Q}^H - 2Q\dot{Q}^H = Q\begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & -2I & 0 & 0 \\ 0 & 0 & -2I & 0 \\ 0 & 0 & 0 & -2I \end{bmatrix}\dot{Q}^H,$$

$$Q_1\ddot{Q}_1^H - Q\ddot{Q}^H = QJ^HJ\ddot{Q}^H - Q\ddot{Q}^H = Q\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}\ddot{Q}^H,$$

such that we get

$$\tilde{C} = \hat{C} - \hat{M}Q\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 2I & 0 & 0 \\ 0 & 0 & 2I & 0 \\ 0 & 0 & 0 & 2I \end{bmatrix}\dot{Q}^H = \hat{C} - \begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\dot{Q}^H = \hat{C} - \hat{M}Q\dot{Q}^H,$$

$$\tilde{K} = \hat{K} - \hat{M}Q\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\ddot{Q}^H = \hat{K}.$$

Altogether, we have constructed a first order formulation in the original variable x and v, only using the coefficient matrices of the strangeness-free formulation (24) and the unitary transformation matrix Q. Due to construction this first order system is strangeness-free.

Theorem 18. *Consider a strangeness-free linear second order differential-algebraic system* (24) *with matrix-valued functions $\hat{M}, \hat{C}, \hat{K} \in C(\mathbb{I}, \mathbb{C}^{m,n})$ and right-hand side $\hat{f} \in C(\mathbb{I}, \mathbb{C}^m)$. Further, let $Q \in C^1(\mathbb{I}, \mathbb{C}^{n,n})$ be a unitary matrix-valued function that decomposes $\hat{M}, \hat{C}, \hat{K}$ as in* (26)*. Then, the* trimmed first order formulation

$$\begin{bmatrix} \hat{M}Q_1 & \hat{C} + \hat{M}\dot{Q}Q^H \\ 0 & Q_1^H \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & \hat{K} \\ -I & 0 \end{bmatrix}\begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} \hat{f} \\ 0 \end{bmatrix}, \tag{30}$$

is also strangeness-free, with $Q_1 = Q[I\ 0\ 0\ 0]^H$, and the characteristic values are given by $d_\mu = 2d_\mu^{(2)} + d_\mu^{(1)}, a_\mu, v_\mu$ and u_μ.

Proof. The proof follows directly from the construction of the trimmed first order formulation (30). Setting $\hat{x} = Q^H x$ and $\hat{v} = \frac{d}{dt}(Q_1^H x) = v + \dot{Q}_1^H x$, we obtain

$$\begin{bmatrix} M_{11} & C_{11} & C_{12} & C_{13} & C_{14} \\ 0 & C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_{d_\mu^{(2)}} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\hat{v}} \\ \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{bmatrix} + \begin{bmatrix} 0 & K_{11} & K_{12} & K_{13} & K_{14} \\ 0 & K_{21} & K_{22} & K_{23} & K_{24} \\ 0 & K_{31} & K_{32} & K_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -I_{d_\mu^{(2)}} & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{v} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \\ 0 \end{bmatrix},$$

which is clearly strangeness-free, since M_{11}, C_{22} and K_{33} are nonsingular. $\square$

Theorem 18 allows to construct a strangeness-free first order formulation directly from the coefficients of the strangeness-free second order system (24). The trimmed first order formulation (30) is of minimal possible size, has the same solution component x as the original system (1), and no further smoothness requirements for the inhomogeneity are required.

Example 19. For the second order system (3) given in Example 1 we have computed an equivalent strangeness-free formulation (25) using the derivative array approach. Following Theorem 18 a trimmed first order formulation (with $Q = I$) for system (3) is given by

$$\left[\begin{array}{c|ccc} t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{v} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline -1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} v \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ -f_2 - 2\dot{f}_2 - t\ddot{f}_2 + \ddot{f}_3 \\ -tf_2 + f_3 \\ 0 \end{bmatrix},$$

which is strangeness-free and has the same solution components x_1, x_2, and x_3 as the original system (3).

Remark 20. *For strangeness-free linear k-th order system, the trimmed order reduction formalism can also be applied successively to the k-th order system to reduce the order by one in each reduction step. In this process the derivative of order $(k-1)$ of the transformation matrix Q, chosen similar as in* (26)*, will occur. In the constant coefficient case structure preserving staircase forms for matrix tuples are given in [3], that allow trimmed linearizations for arbitrary high order systems in the context of matrix polynomials. For the variable coefficient case, however, it is not clear if such structure preserving staircase forms exist and how trimmed first order formulations can be derived in this case.*

5 Conclusions

In this paper we have discussed the solution of linear second order differential-algebraic equations with variable coefficients. Since index reduction and order reduction for higher order higher index DAEs do not commute, appropriate index reduction methods for higher order DAEs are required. We have presented a numerically computable way to determine a strangeness-free normal form using the derivative array approach for linear second order DAEs. For differential-algebraic systems with well-defined strangeness index μ the complete structural information on the global characteristic values of the triple $(M(t), C(t), K(t))$ can be obtained from the local information of an inflated triples and it is possible to derive a strangeness-free differential-algebraic system using only local information. For this strangeness-free second order system a trimmed first order formulation is derived that is strangeness-free and has the same solution components as the original second order system. Concluding, for an appropriate treatment of higher order differential-algebraic system the index reduction should be carried out at first, either by transforming to the condensed form (10) or by using the derivative arrays (13), followed possibly by a suitable order reduction to obtain a strangeness-free first order system.

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A Appendix

A.1 Condensed Forms

Lemma A.1. *[13, 16] Consider matrices $M, C, K \in \mathbb{C}^{m \times n}$. Then the matrix triple (M, C, K) is locally equivalent via equivalence transformation* (6) *to a matrix triple $(\hat{M}, \hat{C}, \hat{K})$ of the following local condensed form*

$$
\Bigg(\begin{bmatrix}
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{s^{(MK)}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{bmatrix}
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & 0 & C & I_{s^{(CK)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0 \\
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{31}
$$

$$
\left.\begin{bmatrix}
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & 0 & I_a & 0 \\
0 & 0 & 0 & 0 & I_{s^{(CK)}} & 0 & 0 & 0 \\
0 & 0 & I_{s^{(MK)}} & 0 & 0 & 0 & 0 & 0 \\
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right)
\quad
\begin{array}{l}
s^{(MCK)} \\ s^{(MC)} \\ s^{(MK)} \\ d^{(2)} \\ s^{(CK)} \\ d^{(1)} \\ s^{(MCK)} \\ s^{(MC)} \\ a \\ s^{(CK)} \\ s^{(MK)} \\ s^{(MCK)} \\ v
\end{array},
$$

where the quantities $s^{(MCK)}$, $s^{(MC)}$, $s^{(MK)}$, $s^{(CK)}$, $d^{(2)}$, $d^{(1)}$, a *and* v *are nonnegative integers and the last column in each matrix is of width* u.

Lemma A.2. *[13, 16] Let the matrix-valued functions* $M, C, K \in C(\mathbb{I}, \mathbb{C}^{m,\times n})$ *be sufficiently smooth, and suppose that the regularity conditions* (7) *hold for the local characteristic values of* (M, C, K). *Then,* (M, C, K) *is globally equivalent to a triple of matrix-valued functions* $(\tilde{M}, \tilde{C}, \tilde{K})$ *of the condensed form*

$$
\left(\begin{bmatrix}
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{s^{(MK)}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\right.
$$

$$
\begin{bmatrix}
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & 0 & 0 & I_{s^{(CK)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0 \\
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{32}
$$

$$
\left.\begin{array}{c}
\begin{bmatrix}
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & 0 & 0 & 0 & 0 & 0 & I_a & 0\\
0 & 0 & 0 & 0 & I_{s^{(CK)}} & 0 & 0 & 0\\
0 & 0 & I_{s^{(MK)}} & 0 & 0 & 0 & 0 & 0\\
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}\right)
\qquad
\begin{array}{l}
s^{(MCK)}\\ s^{(MC)}\\ s^{(MK)}\\ d^{(2)}\\ s^{(CK)}\\ d^{(1)}\\ s^{(MCK)}\\ s^{(MC)}\\ a\\ s^{(CK)}\\ s^{(MK)}\\ s^{(MCK)}\\ v
\end{array}.
$$

Here, all blocks are again functions on $\mathbb{I}$ and the last block columns have size u.

A.2 Proofs of Lemma 7 and Theorem 11

Proof of Lemma 7: From Lemma A.2 it follows that each triple $(M^{<i>}, C^{<i>}, K^{<i>})$, $i \in \mathbb{N}_0$ is globally equivalent to the form (11). The identity blocks of size $s_i^{(MK)}$ are decomposed into two identity blocks of size s_i and s_{i-1}, such that in one differentiation and elimination step the block of size s_{i-1} can be eliminated. To prove the relations for the characteristic quantities we first perform one differentiation and elimination step for system (11) and thus get the matrix triple

$$
\left(\begin{bmatrix}
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&I_{s_i}&0&0&0&0&0\\
0&0&0&0&I_{d_i^{(2)}}&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{bmatrix},
\begin{bmatrix}
0&0&0&0&\tilde{C}_{15}^{<i>}&0&0&0&\tilde{C}_{19}^{<i>}\\
0&0&0&0&\tilde{C}_{25}^{<i>}&0&0&0&\tilde{C}_{29}^{<i>}\\
0&0&0&0&C_{35}^{<i>}&0&0&0&C_{39}^{<i>}\\
0&0&0&0&C_{45}^{<i>}&0&0&0&C_{49}^{<i>}\\
0&0&0&0&C_{55}^{<i>}&0&0&0&C_{59}^{<i>}\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&I_{d_i^{(1)}}&0&0\\
0&0&0&0&0&0&0&0&0\\
0&I_{s_i^{(MC)}}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{bmatrix},\right.
$$

$$
\left.\begin{bmatrix}
0 & \tilde{K}_{12}^{<i>} & 0 & 0 & \tilde{K}_{15}^{<i>} & 0 & \tilde{K}_{17}^{<i>} & 0 & \tilde{K}_{19}^{<i>} \\
0 & \tilde{K}_{22}^{<i>} & 0 & 0 & \tilde{K}_{25}^{<i>} & 0 & \tilde{K}_{27}^{<i>} & 0 & \tilde{K}_{29}^{<i>} \\
0 & K_{32}^{<i>} & 0 & 0 & K_{35}^{<i>} & 0 & K_{37}^{<i>} & 0 & K_{39}^{<i>} \\
0 & K_{42}^{<i>} & 0 & 0 & K_{45}^{<i>} & 0 & K_{47}^{<i>} & 0 & K_{49}^{<i>} \\
0 & K_{52}^{<i>} & 0 & 0 & K_{55}^{<i>} & 0 & K_{57}^{<i>} & 0 & K_{59}^{<i>} \\
0 & K_{62}^{<i>} & 0 & 0 & K_{65}^{<i>} & 0 & K_{67}^{<i>} & 0 & K_{69}^{<i>} \\
0 & K_{72}^{<i>} & 0 & 0 & K_{75}^{<i>} & 0 & K_{77}^{<i>} & 0 & K_{79}^{<i>} \\
0 & K_{82}^{<i>} & 0 & 0 & K_{85}^{<i>} & 0 & K_{87}^{<i>} & 0 & K_{89}^{<i>} \\
0 & K_{92}^{<i>} & 0 & 0 & K_{95}^{<i>} & 0 & K_{97}^{<i>} & 0 & K_{99}^{<i>} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{a_i} & 0 \\
0 & 0 & 0 & 0 & 0 & I_{s_i^{(CK)}} & 0 & 0 & 0 \\
0 & 0 & I_{s_{i-1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{s_i} & 0 & 0 & 0 & 0 & 0 \\
I_{s_i^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right) .
$$

In the following, we omit the subscript i. Permutation of the resulting triple of matrix-valued functions yields

$$
\left(\left[\begin{array}{cccc|ccccc}
I_{d_i^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{s_i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
\left[\begin{array}{cccc|ccccc}
C_{55} & 0 & 0 & C_{59} & 0 & 0 & 0 & 0 & 0 \\
C_{45} & 0 & 0 & C_{49} & 0 & 0 & 0 & 0 & 0 \\
\tilde{C}_{15} & 0 & 0 & \tilde{C}_{19} & 0 & 0 & 0 & 0 & 0 \\
\tilde{C}_{25} & 0 & 0 & \tilde{C}_{29} & 0 & 0 & 0 & 0 & 0 \\
C_{35} & 0 & 0 & C_{39} & 0 & 0 & 0 & 0 & 0 \\
0 & I_{d_i^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{s_i^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],\right.
$$

$$
\left.\left[\begin{array}{cccc|ccccc}
K_{55} & K_{57} & K_{52} & K_{59} & 0 & 0 & 0 & 0 & 0 \\
K_{45} & K_{47} & K_{42} & K_{49} & 0 & 0 & 0 & 0 & 0 \\
\tilde{K}_{15} & \tilde{K}_{17} & \tilde{K}_{12} & \tilde{K}_{19} & 0 & 0 & 0 & 0 & 0 \\
\tilde{K}_{25} & \tilde{K}_{27} & \tilde{K}_{22} & \tilde{K}_{29} & 0 & 0 & 0 & 0 & 0 \\
K_{35} & K_{37} & K_{32} & K_{39} & 0 & 0 & 0 & 0 & 0 \\
K_{75} & K_{77} & K_{72} & K_{79} & 0 & 0 & 0 & 0 & 0 \\
K_{95} & K_{97} & K_{92} & K_{99} & 0 & 0 & 0 & 0 & 0 \\
K_{65} & K_{67} & K_{62} & K_{69} & 0 & 0 & 0 & 0 & 0 \\
K_{85} & K_{87} & K_{82} & K_{89} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{s_i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{s_{i-1}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{a_i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_i^{(CK)}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_i^{(MCK)}}
\end{array}\right]\right) .
$$

The so obtained triple has to be transformed to global condensed form (32). We can restrict ourselves to the upper left blocks and use global equivalence transformations to separate the corresponding nullspaces. In the following, we only specify the blocks we are using in the transformations for convenience. Thus, we consider the triple of matrix-valued functions

$$
\left(\begin{bmatrix} I_{d^{(2)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} C_{55} & 0 & 0 & C_{59} \\ C_{45} & 0 & 0 & C_{49} \\ \tilde{C}_{15} & 0 & 0 & \tilde{C}_{19} \\ \tilde{C}_{25} & 0 & 0 & \tilde{C}_{29} \\ C_{35} & 0 & 0 & C_{39} \\ 0 & I_{d^{(1)}} & 0 & 0 \\ 0 & 0 & I_{s^{(MC)}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} K_{55} & K_{57} & K_{52} & K_{59} \\ K_{45} & K_{47} & K_{42} & K_{49} \\ \tilde{K}_{15} & \tilde{K}_{17} & \tilde{K}_{12} & \tilde{K}_{19} \\ \tilde{K}_{25} & \tilde{K}_{27} & \tilde{K}_{22} & \tilde{K}_{29} \\ K_{35} & K_{37} & K_{32} & K_{39} \\ K_{75} & K_{77} & K_{72} & K_{79} \\ K_{95} & K_{97} & K_{92} & K_{99} \\ K_{65} & K_{67} & K_{62} & K_{69} \\ K_{85} & K_{87} & K_{82} & K_{89} \end{bmatrix}\right)
$$

$$
= \left(\begin{bmatrix} I_{d^{(2)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} C & 0 & 0 & C \\ C & 0 & 0 & C \\ \tilde{C}_1 & 0 & 0 & \tilde{C}_2 \\ 0 & I_{d^{(1)}} & 0 & 0 \\ 0 & 0 & I_{s^{(MC)}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \end{bmatrix}\right),
$$

with $\tilde{C}_1 = \begin{bmatrix}\tilde{C}_{15}^H & \tilde{C}_{25}^H & C_{35}^H\end{bmatrix}^H$ and $\tilde{C}_2 = \begin{bmatrix}\tilde{C}_{19}^H & \tilde{C}_{29}^H & C_{39}^H\end{bmatrix}^H$. Using equivalence transformations this triple is equivalent to the following triple

$$
\left(\left[\begin{array}{cc|cccc} I_{e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{r_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cccc} 0 & C & 0 & 0 & 0 & C \\ 0 & C & 0 & 0 & 0 & C \\ 0 & C & 0 & 0 & 0 & C \\ \hline 0 & 0 & 0 & 0 & I_{e_1} & 0 \\ I_{e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{d^{(1)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{s^{(MC)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cccc} K & K & K & K & K & K \\ K & K & K & K & K & K \\ K & K & K & K & K & K \\ \hline K & K & K & K & K & K \\ K & K & K & K & K & K \\ K & K & K & K & K & K \\ \hline K & K & K & K & K & K \\ K & K & K & K & K & K \\ K & K & K & K & K & K \end{array}\right]\right),
$$

with $\operatorname{rank}[\tilde{C}_1\ \tilde{C}_2] = e_{i+1} = e_1 + e_2$, $e_1 = k_{i+1}$, and $r_1 = d_i^{(2)} - e_2$. Rearanging the block rows and summarizing the last two block rows using the definition in (12) yields

$$
\sim \left(\left[\begin{array}{cc|cccc} I_{e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{r_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cccc} 0 & C & 0 & 0 & 0 & C \\ 0 & C & 0 & 0 & 0 & C \\ 0 & C & 0 & 0 & 0 & C \\ \hline 0 & 0 & 0 & 0 & I_{e_1} & 0 \\ I_{e_2} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{d^{(1)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{s^{(MC)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cccc} K & K & K & K & K & K \\ K & K & K & K & K & K \\ K & K & K & K & K & K \\ \hline K & K & K & K & K & K \\ K & K & K & K & K & K \\ \hline K & K & K & K & K & K \\ K & K & K & K & K & K \\ K_1 & K_2 & K_3 & K_4 & K_5 & K_6 \end{array}\right]\right),
$$

$$
\sim \left(\left[\begin{array}{cc|ccccc} I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|ccccc} 0 & C & 0 & 0 & 0 & C & C \\ 0 & C & 0 & 0 & 0 & C & C \\ 0 & C & 0 & 0 & 0 & C & C \\ \hline 0 & 0 & 0 & 0 & I_{e_1} & 0 & 0 \\ I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{d^{(1)}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{s^{(MC)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],\right.
$$

$$
\left.\left[\begin{array}{cc|ccccc}
K & K & K & K & K & 0 & K \\
K & K & K & K & K & 0 & K \\
K & K & K & K & K & 0 & K \\ \hline
K & K & K & K & K & 0 & K \\
K & K & K & K & K & 0 & K \\ \hline
K & K & K & K & K & 0 & K \\
K & K & K & K & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & I_{r_2} & 0 \\
K & K & K & K & \tilde{K}_5 & 0 & 0
\end{array}\right]\right), \quad \text{with } \operatorname{rank} K_6 = r_2,
$$

$$
\sim \left(\left[\begin{array}{cc|cccccc}
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{r_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \left[\begin{array}{cc|cccccc}
0 & C & 0 & 0 & 0 & 0 & C & C \\
0 & C & 0 & 0 & 0 & 0 & C & C \\
0 & C & 0 & 0 & 0 & 0 & C & C \\ \hline
0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{s_2} & 0 & 0 \\
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & I_{d^{(1)}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{s^{(MC)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \right.
$$

$$
\left.\left[\begin{array}{cc|cccccc}
K & K & K & K & 0 & K & 0 & K \\
K & K & K & K & 0 & K & 0 & K \\
K & K & K & K & 0 & K & 0 & K \\ \hline
K & K & K & K & 0 & K & 0 & K \\
K & K & K & K & 0 & K & 0 & K \\
K & K & K & K & 0 & K & 0 & K \\ \hline
K & K & K & K & 0 & K & 0 & K \\
K & K & K & K & 0 & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & 0 & I_{r_2} & 0 \\
0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0 \\
K & K & K & \tilde{K}_4 & 0 & 0 & 0 & 0
\end{array}\right]\right), \quad \text{with } \operatorname{rank} \tilde{K}_5 = r_3, \text{ and } s_2 = e_1 - r_3,
$$

$$
\sim \left(\left[\begin{array}{cc|ccccccc}
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \left[\begin{array}{cc|ccccccc}
0 & C & 0 & 0 & 0 & 0 & 0 & C & C \\
0 & C & 0 & 0 & 0 & 0 & 0 & C & C \\
0 & C & 0 & 0 & 0 & 0 & 0 & C & C \\ \hline
0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{s_2} & 0 & 0 \\
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & I_{d^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{s_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \right.
$$

$$
\left.\left[\begin{array}{cc|ccccccc}
K & K & K & 0 & K & 0 & K & 0 & K \\
K & K & K & 0 & K & 0 & K & 0 & K \\
K & K & K & 0 & K & 0 & K & 0 & K \\ \hline
K & K & K & 0 & K & 0 & K & 0 & K \\
K & K & K & 0 & K & 0 & K & 0 & K \\
K & K & K & 0 & K & 0 & K & 0 & K \\ \hline
K & K & K & 0 & K & 0 & K & 0 & K \\
K & K & K & 0 & K & 0 & K & 0 & K \\
K & K & K & 0 & K & 0 & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_2} & 0 \\
0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0 \\
0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0 \\
K & K & \tilde{K}_3 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\right), \quad \text{with } \operatorname{rank} \tilde{K}_4 = r_4, \text{ and } s_3 = s_i^{(MC)} - r_4,
$$

$$
\sim\left(\left[\begin{array}{cc|cccccccc}
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & I_{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
\left[\begin{array}{cc|cccccccc}
0 & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C\\
0 & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C\\
0 & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_2} & 0 & 0\\
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & I_{r_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I_{s_4} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I_{s_3} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],\right.
$$

$$
\left.\left[\begin{array}{cc|cccccccc}
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
\hline
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
\hline
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & K & 0 & K & 0 & K & 0 & K & 0 & K\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I_{r_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
K & \tilde{K}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\right), \quad \text{with } \operatorname{rank}\tilde{K}_3 = r_5, \text{ and } s_4 = d_i^{(1)} - r_5,
$$

$$
\sim\left(\left[\begin{array}{ccc|cccccccc}
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & I_{r_6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I_{r_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
\left[\begin{array}{ccc|cccccccc}
0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C\\
0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C\\
0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C\\
0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_2} & 0 & 0\\
I_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & I_{r_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & I_{s_4} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I_{s_3} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],\right.
$$

$$
\left.\left[\begin{array}{ccc|cccccccc}
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
\hline
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
\hline
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I_{r_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & I_{r_6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\tilde{K}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\right), \quad \text{with } \operatorname{rank}\tilde{K}_2 = r_6,\ r_8 = r_1 - r_6,
$$

$$
\sim \left(\left[\begin{array}{cccc|cccccccc}
I_{r_7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s_6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{r_6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{r_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \right.
$$

$$
\left[\begin{array}{cccc|cccccccc}
0 & 0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & 0 & 0 & 0 & 0 & C & C \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_2} & 0 & 0 \\
I_{r_7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s_6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & I_{r_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{s_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
$$

$$
\left. \left[\begin{array}{cccc|cccccccc}
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
\hline
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
\hline
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K & 0 & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{r_4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{r_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{r_6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{r_7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \right),
$$

with $\operatorname{rank} \tilde{K}_1 = r_7$, and $s_6 = e_2 - r_7$. With the given definitions we then have that

$$
\begin{aligned}
r_2 &= b_{i+1}, & r_6 &= h_{i+1} - d_{i+1},\\
r_3 &= p_{i+1} - b_{i+1}, & r_7 &= c_{i+1} - h_{i+1},\\
r_4 &= t_{i+1} - p_{i+1}, & e_1 &= k_{i+1},\\
r_5 &= d_{i+1} - t_{i+1}, & e_2 &= e_{i+1} - k_{i+1},
\end{aligned}
$$

and thus we get

$$
\begin{aligned}
&a_{i+1} = a_i + s_i^{(CK)} + s_i^{(MCK)} + s_{i-1} + r_2,\\
&s_{i+1}^{(MCK)} = r_7,\\
&s_{i+1}^{(MC)} = s_6 = e_2 - r_7,\\
&s_{i+1}^{(MK)} = r_6 + s_i,\\
&s_{i+1}^{(CK)} = r_5 + r_4 + r_3,\\
&d_{i+1}^{(2)} = r_8 = d_i^{(2)} - e_2 - r_6,\\
&d_{i+1}^{(1)} = s_2 + s_3 + s_4 = d_i^{(1)} + s_i^{(MC)} + e_1 - r_3 - r_4 - r_5,\\
&w_{i+1} = v_{i+1} - v_i\\
&\qquad = 2s_i^{(MCK)} + s_i^{(CK)} + s_i^{(MC)} + s_{i-1} - e_1 - e_2 - r_2 - r_3 - r_4 - r_5 - r_6 - r_7.
\end{aligned}
$$

$\square$

Proof of Theorem 11:
Due to Theorem 10 we may assume without loss of generality that the triple (M, C, K) is already given in the global condensed form (32). Then, for fixed $\hat{t} \in \mathbb{I}$, we have to determine the local characteristic quantities of $(\mathcal{M}_l(\hat{t}), \mathcal{L}_l(\hat{t}), \mathcal{N}_l(\hat{t}))$. In the following, we will omit the argument $\hat{t}$ and we use local equivalence transformations of the form (6) on the inflated triple. For $l = 0$ it is immediately clear that

$$
\begin{aligned}
&\tilde{r}_0 = \operatorname{rank} \mathcal{M}_0 = m - q_0 - c_0 - v_0,\\
&\operatorname{rank}\,[\mathcal{M}_0, \mathcal{L}_0] = m - a_0 - s_0^{(CK)} - s_0^{(MCK)} - s_0^{(MK)} - v_0 = m - c_0 - v_0,\\
&\operatorname{rank}\,[\mathcal{M}_0, \mathcal{L}_0, \mathcal{N}_0] = m - v_0,
\end{aligned}
$$

and the local characteristic values of $(\mathcal{M}_0, \mathcal{L}_0, \mathcal{N}_0)$ correspond to the global characteristic values of (M, C, K). For $l = 1$ we have to consider the inflated triple

$$
(\mathcal{M}_1, \mathcal{L}_1, \mathcal{N}_1) = \left(\begin{bmatrix} M & 0 \\ \dot{M} + C & M \end{bmatrix}, \begin{bmatrix} C & 0 \\ \dot{C} + K & 0 \end{bmatrix}, \begin{bmatrix} K & 0 \\ \dot{K} & 0 \end{bmatrix} \right),
$$

with (M, C, K) in global condensed form (32). The identity blocks in the matrix M allow to eliminate all other entries in the corresponding block rows of $\mathcal{M}_1$ by local equivalence transformations. Further eliminations using the identity blocks of the global condensed form and block decompositions yield the following matrix triple, where we only state the first block columns of $\mathcal{L}_1$ and $\mathcal{N}_1$ since all other entries are zero

$$
\left(\left[\begin{array}{cccccccccc|cccc}
0 & . & . & . & . & . & . & . & . & . & & & & \\
. & 0 & . & . & . & . & . & . & . & . & & & & \\
. & 0 & . & . & . & . & . & . & . & . & & & & \\
. & . & I & . & . & . & . & . & . & . & & & & \\
. & . & . & I & . & . & . & . & . & . & & & & \\
. & . & . & . & I & . & . & . & . & . & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
. & . & . & . & . & & & & & & & & & \\
\hline
. & . & 0 & 0 & 0 & . & . & 0 & 0 & 0 & I & & & \\
. & . & 0 & 0 & 0 & . & . & 0 & 0 & 0 & & I & & \\
. & . & 0 & 0 & 0 & . & . & 0 & 0 & 0 & & & I & \\
. & . & 0 & 0 & 0 & . & . & 0 & 0 & 0 & & & & I \\
. & . & . & . & . & I & . & . & . & . & & & & \\
. & . & . & . & . & . & I & . & . & . & & & & \\
I & . & & & & & & & & & & & & \\
. & I & & & & & & & & & & & & \\
. & . & & & & & & & & & & & & \\
. & . & & & & & & & & & & & & \\
. & . & & & & & & & & & & & & \\
. & . & & & & & & & & & & & & \\
. & . & & & & & & & & & & & &
\end{array} \right] \right. , \tag{33}
$$

$$
\begin{array}{cc}
\left(
\begin{bmatrix}
\cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & I_{k_1} & 0 \\
\cdot & 0 & 0 & I_{\tilde{e}_1} & 0 & \cdot & 0 & 0 & 0 & 0 \\
\cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\
\cdot & \cdot & 0 & 0 & C & \cdot & \cdot & 0 & 0 & C \\
\cdot & \cdot & 0 & 0 & C & \cdot & \cdot & 0 & 0 & C \\
\cdot & \cdot & 0 & 0 & C & \cdot & \cdot & 0 & 0 & C \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot & \cdot \\
0 & \cdot & & & & & & & & \\
\cdot & I & & & & & & & & \\
\cdot & \cdot & & & & & & & & \\
\cdot & \cdot & & & & & & & & \\
\cdot & \cdot & & & & & & & & \\
\cdot & \cdot & & & & & & & & \\
\cdot & \cdot & & & & & & & & \\
\hline
\cdot & K & 0 & \dot{C} & \dot{C} & \cdot & K & 0 & \dot{C} & \dot{C} \\
\cdot & K & 0 & \dot{C} & \dot{C} & \cdot & K & 0 & \dot{C} & \dot{C} \\
\cdot & K & 0 & \dot{C} & \dot{C} & \cdot & K & 0 & \dot{C} & \dot{C} \\
\cdot & K & 0 & \dot{C} & \dot{C} & \cdot & K & 0 & \dot{C} & \dot{C} \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & I & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
I & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix},
\begin{bmatrix}
\cdot & \tilde{K} & \cdot & \tilde{K} & \tilde{K} & \cdot & \tilde{K} & \cdot & \tilde{K} & \tilde{K} \\
\cdot & \tilde{K} & \cdot & \tilde{K} & \tilde{K} & \cdot & \tilde{K} & \cdot & \tilde{K} & \tilde{K} \\
\cdot & \tilde{K} & \cdot & \tilde{K} & \tilde{K} & \cdot & \tilde{K} & \cdot & \tilde{K} & \tilde{K} \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & K & \cdot & K & K & \cdot & K & \cdot & K & K \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & I & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
I & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \dot{K} & \cdot & \dot{K} & \dot{K} & \cdot & \dot{K} & \cdot & \dot{K} & \dot{K} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\right), &
\begin{array}{c}
k_1 \\ \tilde{e}_1 \\ \tilde{s}_5 \\ \tilde{s}_6 \\ \tilde{e}_1 \\ \tilde{d}_2 \\ \tilde{s}_4 \\ \tilde{d}_1 \\ \tilde{s}_1 \\ \tilde{s}_2 \\ a_0 \\ \tilde{s}_4 \\ \tilde{s}_3 \\ \tilde{s}_1 \\ v_0 \\
\tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \\ \tilde{d}_2 \\ \tilde{s}_4 \\ \tilde{d}_1 \\ \tilde{s}_1 \\ \tilde{s}_2 \\ a_0 \\ \tilde{s}_4 \\ \tilde{s}_3 \\ \tilde{s}_1 \\ v_0
\end{array}
\end{array}
$$

with dimensions $\tilde{s}_1 = s_0^{(MCK)}$, $\tilde{s}_2 = s_0^{(MC)}$, $\tilde{s}_3 = s_0^{(MK)}$, $\tilde{s}_4 = s_0^{(CK)}$, $\tilde{s}_5 = s_0^{(MCK)} + s_0^{(MC)} - e_1$, $\tilde{s}_6 = s_0^{(MK)} - e_1 + k_1$, $\tilde{e}_1 = e_1 - k_1$, $\tilde{d}_1 = d_0^{(1)}$, and $\tilde{d}_2 = d_0^{(2)}$. Now, we can read off the corresponding ranks as

$$
\begin{aligned}
\tilde{r}_1 = \operatorname{rank} \mathcal{M}_1 &= 2m - c_0 - 2v_0 - e_0 - 2s_0^{(MCK)} - s_0^{(MC)} - s_0^{(CK)} - s_0^{(MK)} - a_0 \\
&= 2m - c_0 - c_1 - v_0 - v_1 - e_0 - e_1 - a_0 - s_0^{(MK)} \\
&= 2m - c_0 - c_1 - v_0 - v_1 - e_0 - (e_1 + c_0 - s_0^{(CK)} - s_0^{(MCK)}), \\
\operatorname{rank} [\mathcal{M}_1, \mathcal{L}_1] &= 2m - c_0 - s_0^{(CK)} - 2s_0^{(MCK)} - 2v_0 - s_0^{(MC)} + e_1 \\
&= 2m - c_0 - c_1 - v_0 - v_1, \\
\operatorname{rank} [\mathcal{M}_1, \mathcal{L}_1, \mathcal{N}_1] &= 2m - 2v_0 - 2s_0^{(MCK)} - s_0^{(CK)} - s_0^{(MC)} + e_1 + c_1 \\
&= 2m - v_0 - v_1.
\end{aligned}
$$

Let $\Pi \in \mathbb{C}^{2m \times 2m}$ and $\Theta, \Psi, \Sigma \in \mathbb{C}^{2n \times 2n}$ be block matrices, such that

$$
(\tilde{\mathcal{M}}_1, \tilde{\mathcal{L}}_1, \tilde{\mathcal{N}}_1) = (\Pi \mathcal{M}_1 \Theta, \Pi \mathcal{L}_1 \Theta + 2\Pi \mathcal{M}_1 \Psi, \Pi \mathcal{N}_1 \Theta + \Pi \mathcal{L}_1 \Psi + \Pi \mathcal{M}_1 \Sigma)
$$

is in the locally transformed form (33). Further, let Π and Θ be partitioned as

$$
\Pi := \left[\Pi_1^H, \ldots, \Pi_{28}^H\right]^H, \quad \Theta := [\Theta_1, \ldots, \Theta_{18}],
$$

corresponding to the row and column structure of the block matrices in (33). Then we have

$$\begin{aligned}
[\Pi_1^H, \Pi_2^H, \Pi_3^H, \Pi_7^H, \ldots, \Pi_{15}^H, \Pi_{24}^H, \ldots, \Pi_{28}^H]^H \mathcal{M}_1 &= 0, \\
\mathcal{M}_1[\Theta_8, \Theta_9, \Theta_{10}, \Theta_{15}, \ldots, \Theta_{18}] &= 0, \\
[\Pi_3^H, \Pi_7^H, \Pi_9^H, \Pi_{11}^H, \ldots, \Pi_{15}^H, \Pi_{28}^H]^H \mathcal{L}_1 &= 0, \\
[\Pi_1^H, \Pi_2^H, \Pi_3^H, \Pi_7^H, \ldots, \Pi_{15}^H, \Pi_{24}^H, \ldots, \Pi_{28}^H]^H \mathcal{L}_1[\Theta_{10}, \Theta_{15}, \ldots, \Theta_{18}] &= 0.
\end{aligned}$$

This means that the columns of the matrix $V_1 := [\Pi_1^H, \Pi_2^H, \Pi_3^H, \Pi_7^H, \ldots, \Pi_{15}^H, \Pi_{24}^H, \ldots, \Pi_{28}^H]$ form a basis of $\mathrm{kernel}(\mathcal{M}_1^H)$, the columns of $V_2 := [\Theta_8, \Theta_9, \Theta_{10}, \Theta_{15}, \ldots, \Theta_{18}]$ form a basis of $\mathrm{kernel}(\mathcal{M}_1)$, the columns of $V_3 := [\Pi_3^H, \Pi_7^H, \Pi_9^H, \Pi_{11}^H, \ldots, \Pi_{15}^H, \Pi_{28}^H]$ form a basis of $\mathrm{kernel}(\mathcal{M}_1^H) \cap \mathrm{kernel}(\mathcal{L}_1^H)$, and the columns of $V_4 := [\Theta_{10}, \Theta_{15}, \ldots, \Theta_{18}]$ form a basis of $\mathrm{kernel}(\mathcal{M}_1) \cap \mathrm{kernel}(V_1^H \mathcal{L}_1)$. Therefore, using Lemma 4 and with the definitions of Lemma 7 we have

$$\begin{aligned}
&\tilde{a}_1 = \mathrm{rank}(V_3^H \mathcal{N}_1 V_4) = b_1, \\
&\tilde{s}_1^{(MCK)} = \dim(\mathrm{range}(\mathcal{M}_1^H) \cap \mathrm{range}(\mathcal{L}_1^H V_1) \cap \mathrm{range}(\mathcal{N}_1^H V_3)) = c_0 + c_1 - b_0 - d_1 + t_1 - p_1, \\
&\tilde{s}_1^{(CK)} = \mathrm{rank}(V_3^H \mathcal{N}_1 V_2) - \tilde{a}_1 = a_0 + p_1 - b_1, \\
&\tilde{d}_1^{(1)} = \mathrm{rank}(V_1^H \mathcal{L}_1 V_2) - \tilde{s}_1^{(CK)} = k_1 + a_0 - p_1 - a_0 + b_1 = k_1 - p_1 + b_1, \\
&\tilde{s}_1^{(MC)} = \mathrm{rank}(V_1^H \mathcal{L}_1) - \tilde{s}_1^{(MCK)} - \tilde{s}_1^{(CK)} - \tilde{d}_1^{(1)} = q_0 + q_1 - c_0 - c_1 + d_1 - t_1 + p_1 - k_1, \\
&\tilde{s}_1^{(MK)} = \mathrm{rank}(V_3^H \mathcal{N}_1) - \tilde{a}_1 - \tilde{s}_1^{(MCK)} - \tilde{s}_1^{(CK)} = d_1 - t_1, \\
&\tilde{v}_1 = 2m - \tilde{r}_1 - 2\tilde{s}_1^{(CK)} - \tilde{d}_1^{(1)} - 2\tilde{s}_1^{(MCK)} - \tilde{s}_1^{(MC)} - \tilde{a}_1 - \tilde{s}_1^{(MK)} = v_0 + v_1, \\
&\tilde{u}_1 = 2n - \tilde{r}_1 - \tilde{s}_1^{(CK)} - \tilde{d}_1^{(1)} - \tilde{a}_1 = 2u_0 + a_0 + k_0 - k_1 - b_1.
\end{aligned}$$

Finally, for $l = 2$ we have to consider the inflated triple

$$(\mathcal{M}_2, \mathcal{L}_2, \mathcal{N}_2) = \left(\begin{bmatrix} M & 0 & 0 \\ \dot{M} + C & M & 0 \\ \ddot{M} + 2\dot{C} + K & 2\dot{M} + C & M \end{bmatrix}, \begin{bmatrix} C & 0 & 0 \\ \dot{C} + K & 0 & 0 \\ \ddot{C} + 2\dot{K} & 0 & 0 \end{bmatrix}, \begin{bmatrix} K & 0 & 0 \\ \dot{K} & 0 & 0 \\ \ddot{K} & 0 & 0 \end{bmatrix} \right).$$

Again, the identities in the diagonal blocks of $\mathcal{M}_2$ allow to eliminate all another entries in the corresponding block rows of $\mathcal{M}_2$ without altering $\mathcal{L}_2$ or $\mathcal{N}_2$. Further eliminations using identity blocks in the global condensed form and block decompositions using local equivalence transformations yield the matrix triple $(\tilde{\mathcal{M}}_2, \tilde{\mathcal{L}}_2, \tilde{\mathcal{N}}_2)$ of the form

$$
\left(\left[\begin{array}{cccccccccccc|cccccccc|cccc}
0 & . & . & . & . & . & . & . & . & . & . & . \\
. & 0 & . & . & . & . & . & . & . & . & . & . \\
. & . & 0 & . & . & . & . & . & . & . & . & . \\
. & . & 0 & . & . & . & . & . & . & . & . & . \\
. & . & 0 & . & . & . & . & . & . & . & . & . \\
. & . & 0 & . & . & . & . & . & . & . & . & . \\
. & . & . & I & . & . & . & . & . & . & . & . \\
. & . & . & . & I & . & . & . & . & . & . & . \\
. & . & . & . & . & I & . & . & . & . & . & . \\
. & . & . & . & . & . \\
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\end{array}\right]\right) ,
&
\begin{array}{c}
k_1 \\ \tilde{e}_1 \\ \tilde{s}_7 \\ k_2 \\ \tilde{e}_2 \\ \tilde{s}_8 \\ \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{d}_2 \\ \tilde{s}_4 \\ \tilde{d}_1 \\ \tilde{s}_1 \\ \tilde{s}_2 \\ a_0 \\ \tilde{s}_4 \\ \tilde{s}_3 \\ \tilde{s}_1 \\ v_0 \\
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\tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \\ \tilde{d}_2 \\ \tilde{s}_4 \\ \tilde{d}_1 \\ \tilde{s}_1 \\ \tilde{s}_2 \\ a_0 \\ \tilde{s}_4 \\ \tilde{s}_3 \\ \tilde{s}_1 \\ v_0
\end{array}
\end{array}
$$

with dimensions $\tilde{e}_2 = e_2 - k_2$, $\tilde{s}_7 = s_0^{(MCK)} + s_0^{(MC)} - e_1 - e_2$, $\tilde{s}_8 = s_0^{(MK)} - e_1 + k_1 - e_2 + k_2$. We can read off the corresponding ranks to be

$$
\begin{aligned}
\tilde{r}_2 = \operatorname{rank} \tilde{\mathcal{M}}_2 &= 3m - q_0 - 2c_0 - 3v_0 - 3s_0^{(MCK)} - 2s_0^{(MC)} - s_0^{(CK)} - s_0^{(MK)} + k_1 \\
&= 3m - c_0 - c_1 - c_2 - v_0 - v_1 - v_2 - q_0 - q_1 - q_2, \\
\operatorname{rank}\,[\tilde{\mathcal{M}}_2, \tilde{\mathcal{L}}_2] &= 3m - c_0 - 3v_0 - 4s_0^{(MCK)} - 2s_0^{(MC)} - 2s_0^{(CK)} - s_0^{(MK)} + e_1 + e_2 + k_1 + b_1 \\
&= 3m - c_0 - c_1 - c_2 - v_0 - v_1 - v_2, \\
\operatorname{rank}\,[\tilde{\mathcal{M}}_2, \tilde{\mathcal{L}}_2, \tilde{\mathcal{N}}_2]& \\
&= 3m - 3v_0 - 4s_0^{(MCK)} - 2s_0^{(MC)} - 2s_0^{(CK)} - s_0^{(MK)} + c_1 + b_1 + k_1 + c_2 + e_1 + e_2
\end{aligned}
$$

$= 3m - v_0 - v_1 - v_2.$

Again, let $\tilde{\Pi} \in \mathbb{C}^{3m\times 3m}$ and $\tilde{\Theta}, \tilde{\Psi}, \tilde{\Sigma} \in \mathbb{C}^{3n\times 3n}$ be the corresponding block matrices that locally transform the inflated triple $(\mathcal{M}_2, \mathcal{L}_2, \mathcal{N}_2)$ to the form (34), i.e.,

$$(\tilde{\mathcal{M}}_2, \tilde{\mathcal{L}}_2, \tilde{\mathcal{N}}_2) = (\tilde{\Pi}\mathcal{M}_2\tilde{\Theta}, \tilde{\Pi}\mathcal{L}_2\tilde{\Theta} + 2\tilde{\Pi}\mathcal{M}_2\tilde{\Psi}, \tilde{\Pi}\mathcal{N}_2\tilde{\Theta} + \tilde{\Pi}\mathcal{L}_2\tilde{\Psi} + \tilde{\Pi}\mathcal{M}_2\tilde{\Sigma}),$$

and $\tilde{\Pi}$ and $\tilde{\Theta}$ be partitioned as

$$\tilde{\Pi} := \left[\Pi_1^H, \ldots, \Pi_{44}^H\right]^H, \quad \tilde{\Theta} := [\Theta_1, \ldots, \Theta_{28}],$$

corresponding to the row and column structure of the block matrices in (34). Then we have

$$[\Pi_1^H, ..., \Pi_6^H, \Pi_{10}^H, ..., \Pi_{18}^H, \Pi_{20}^H, \Pi_{23}^H, \Pi_{25}^H, \Pi_{27}^H, ..., \Pi_{31}^H, \Pi_{44}^H]^H \mathcal{M}_2 = 0,$$
$$\mathcal{M}_2[\Theta_{11}, \Theta_{12}, \Theta_{19}, \Theta_{20}, \Theta_{25}, ..., \Theta_{28}] = 0,$$

such that the columns of $V_1 := [\Pi_1^H, ..., \Pi_6^H, \Pi_{10}^H, ..., \Pi_{18}^H, \Pi_{20}^H, \Pi_{23}^H, \Pi_{25}^H, \Pi_{27}^H, ..., \Pi_{31}^H, \Pi_{44}^H]$ form a basis of kernel$(\mathcal{M}_2^H)$ and the columns of $V_2 := [\Theta_{11}, \Theta_{12}, \Theta_{19}, \Theta_{20}, \Theta_{25}, ..., \Theta_{28}]$ form a basis of kernel$(\mathcal{M}_2)$. Further decompositions of rows and columns of $\tilde{\mathcal{L}}_2$, where $\Pi_{.,2}$, $\Theta_{.,2}$ denote the parts of the rows and columns of $\tilde{\Pi}$ and $\tilde{\Theta}$, respectively, that corresponds to null-rows or null-columns after the decomposition, yield

$$[\Pi_3^H, \Pi_6^H, \Pi_{10}^H, \Pi_{12}^H, \Pi_{14}^H, ..., \Pi_{18}^H, \Pi_{20,2}^H, \Pi_{23,2}^H, \Pi_{25,2}^H, \Pi_{31}^H, \Pi_{44}^H]^H \mathcal{L}_2 = 0,$$

as well as

$$V_1^H \mathcal{L}_2[\Theta_{12,2}, \Theta_{19}, \Theta_{20}, \Theta_{25}, ..., \Theta_{28}] = 0,$$

such that the columns of

$$V_3 := [\Pi_3^H, \Pi_6^H, \Pi_{10}^H, \Pi_{12}^H, \Pi_{14}^H, ..., \Pi_{18}^H, \Pi_{20,2}^H, \Pi_{23,2}^H, \Pi_{25,2}^H, \Pi_{31}^H, \Pi_{44}^H]$$

form a basis of kernel$(\mathcal{M}_2^H) \cap$ kernel$(\mathcal{L}_2^H)$, and the columns of the matrix V_4 given by $V_4 := [\Theta_{12,2}, \Theta_{19}, \Theta_{20}, \Theta_{25}, ..., \Theta_{28}]$ form a basis of kernel$(\mathcal{M}_2) \cap$ kernel$(V_1^H \mathcal{L}_2)$. Therefore, we have

$$\begin{aligned}
\operatorname{rank}(V_1^H \mathcal{L}_2) &= q_0 + q_1 + q_2, \\
\operatorname{rank}(V_3^H \mathcal{N}_2) &= c_0 + c_1 + c_2,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{a}_2 &= \operatorname{rank}(V_3^H \mathcal{N}_2 V_4) = b_2, \\
\tilde{s}_2^{(MCK)} &= \dim(\operatorname{range}(\mathcal{M}_2^H) \cap \operatorname{range}(\mathcal{L}_2^H V_1) \cap \operatorname{range}(\mathcal{N}_2^H V_3)) \\
&= c_0 + c_1 + c_2 - b_1 - d_1 + t_1 - d_2 + t_2 - p_2, \\
\tilde{s}_2^{(CK)} &= \operatorname{rank}(V_3^H \mathcal{N}_2 V_2) - \tilde{a}_2 = b_1 + p_2 - b_2, \\
\tilde{d}_2^{(1)} &= \operatorname{rank}(V_1^H \mathcal{L}_2 V_2) - \tilde{s}_2^{(CK)} = k_2 + b_1 - b_1 - p_2 + b_2, \\
\tilde{s}_2^{(MC)} &= \operatorname{rank}(V_1^H \mathcal{L}_2) - \tilde{s}_2^{(MCK)} - \tilde{s}_2^{(CK)} - \tilde{d}_2^{(1)} \\
&= q_0 + q_1 + q_2 - c_0 - c_1 - c_2 + d_1 - t_1 + d_2 - t_2 - k_2 + p_2, \\
\tilde{s}_2^{(MK)} &= \operatorname{rank}(V_3^H \mathcal{N}_2) - \tilde{a}_2 - \tilde{s}_2^{(MCK)} - \tilde{s}_2^{(CK)} = d_1 - t_1 + d_2 - t_2, \\
\tilde{v}_2 &= 3m - \tilde{r}_2 - 2\tilde{s}_2^{(CK)} - \tilde{d}_2^{(1)} - 2\tilde{s}_2^{(MCK)} - \tilde{s}_2^{(MC)} - \tilde{a}_2 - \tilde{s}_2^{(MK)} = v_0 + v_1 + v_2, \\
\tilde{u}_2 &= 3n - \tilde{r}_2 - \tilde{s}_2^{(CK)} - \tilde{d}_2^{(1)} - \tilde{a}_2 = 3u_0 + 2a_0 + k_0 - k_1 - k_2 - b_1 - b_2.
\end{aligned}$$

□