BALANCED TRUNCATION OF LINEAR SECOND-ORDER SYSTEMS: A HAMILTONIAN APPROACH

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Abstract. We present a formal procedure for structure-preserving model reduction of linear second-order control problems that appear in a variety of physical contexts, e.g., vibromechanical systems or electrical circuit design. Typical balanced truncation methods that project onto the subspace of the largest Hankel singular values fail to preserve the problem's physical structure and may suffer from lack of stability. In this paper we adopt the framework of generalized Hamiltonian systems that covers the class of relevant problems and that allows for a generalization of balanced truncation to second-order problems. It turns out that the Hamiltonian structure, stability and passivity are preserved if the truncation is done by imposing a holonomic constraint on the system rather than standard Galerkin projection.

Key words. structure-preserving model reduction, generalized Hamiltonian systems, balanced truncation, strong confinement, invariant manifolds, Hankel norm approximation.

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1. Introduction.

Model reduction is a major issue for control, optimization and simulation of large-scale systems. In particular for linear time-invariant systems, balanced truncation is a well-established tool for deriving reduced (i.e., low-dimensional) models that have an input-output behaviour similar to the original model [1, 2]; see also [3] and the referenced therein. The general idea of balanced truncation is to restrict the system onto the subspace that is easily controllable and observable. Moreover the method is known to preserve certain properties of the original system such as stability or passivity and gives an error bound that is easily computable. A serious problem of balanced truncation is that there is no straightforward generalization to second-order systems. Second-order equations occur in modelling and control of many physical systems, e.g., electrical circuits, structural mechanics or vibro-acoustic models; see, e.g., [4, 5]. The main objective here is to derive reduced models that remain physically interpretable, i.e., that inherit the interaction structure of the original model. Ordinary balanced truncation proceeds by recasting the equations in first-order form, but structure-preservation is achieved only under very specific conditions that are too restrictive for actual applications [6, 7].

In this article we shall argue that an obvious (and in some sense natural) first-order formulation of a second-order control system that is amenable to balanced truncation takes into account the system's Hamiltonian structure. This leads to a formulation of the control problem as a generalized Hamiltonian system [8], and it turns out that balanced truncation can be applied to a linear generalized Hamiltonian system preserving its structure and properties such as stability or passivity.

The outline of the article is as follows: In Section 2 we introduce the conceptual framework of generalized Hamiltonian systems and balancing. Section 3 deals with the problem of restricting the equations of motion to the most controllable and observable subspace by using constraints; for this purpose we employ a singular perturbation

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argument and show in Section 3.1 that the dynamics are confined to the essential subspace as the negligible Hankel singular values go to zero; the approach is contrasted with directly imposing a holonomic constraint (Sec. 3.2). Stability of the reduced systems is a subtle issue and we have devoted a separate chapter to it (Sec. 4). Last but not least, we prove in Section 5 that the transfer function associated with the original system converges in H^∞ to the transfer function of the constrained system in the limit of vanishing small Hankel singular values. The article concludes with the discussion of two numerical examples in Section 6.

2. Set-up: generalized Hamiltonian systems. Given a smooth Hamiltonian

$$H\colon \mathbf{R}^{2n} \supseteq \mathbf{X} \to \mathbf{R}\,, \quad H(x) = \frac{1}{2} x^T E x$$

with $E = E^T \succ 0$ ("$\succ$" means positive definite) we consider systems of the form

$$\begin{aligned} \dot{x}(t) &= (J - D)\,\nabla H(x(t)) + Bu(t) \\ y(t) &= C \nabla H(x(t))\,, \end{aligned} \tag{2.1}$$

where $J \in \mathbf{R}^{2n\times 2n}$ is an invertible skew-symmetric matrix, $D \in \mathbf{R}^{2n\times 2n}$ is symmetric positive semidefinite, $B \in \mathbf{R}^{2n\times m}$ and $C \in \mathbf{R}^{l\times 2n}$ (all constant). We suppose that both the control function $u(\cdot) \in \mathbf{R}^m$ and the observable $y(\cdot) \in \mathbf{R}^l$ are in $L^2(\mathbf{R})$. For $C = B^T$ systems of type (2.1) are called port-Hamiltonian (see [8]).

Second-order systems. As can be readily checked, the second-order system

$$\begin{aligned} M\ddot{q}(t) + R\dot{q}(t) + Kq(t) &= B_2 u(t) \\ y(t) &= C_1 q(t) + C_2 \dot{q}(t) \end{aligned} \tag{2.2}$$

with $M, R, K \in \mathbf{R}^{n\times n}$ being symmetric positive definite, $B_2 \in \mathbf{R}^{n\times m}$ and $C_1, C_2 \in \mathbf{R}^{l\times n}$ is an instance of (2.1), where $x = (q, Mv)$ with (q, v) being coordinates on $T\mathbf{R}^n$. The total energy is given by the quadratic Hamiltonian

$$H(q,p) = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} q^T K q\,, \quad p = Mv\,. \tag{2.3}$$

Furthermore

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}, \quad D = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R \end{pmatrix} \tag{2.4}$$

in (2.1) and control and observable matrices in (2.1) and (2.2) are related by

$$B = \begin{pmatrix} \mathbf{0} \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 K^{-1} & C_2 \end{pmatrix}. \tag{2.5}$$

Given a solution of (2.2), the energy balance

$$\frac{d}{dt} H(q(t), M\dot{q}(t)) = -2\gamma(\dot{q}(t)) + \dot{q}(t)^T B_1 u(t)$$

holds, where

$$\gamma(v) = \frac{1}{2} v^T R v > 0\,.$$

denotes the Rayleigh dissipation function. As the total energy $H \geq 0$ is bounded from below, it follows that (2.2) with outputs $y = B_2^T \dot{q}$, i.e., $C = B^T$ is a passive state space system with storage function H; see [9] for details.

2.1. Balancing transformations. We shall make precise what it means that the second-order system (2.2) is easily controllable or observable, and we do so by studying controllability and observability of the equivalent Hamiltonian system (2.1). For M, R, K in (2.2) being symmetric and positive definite matrices, the first-order system (2.1) is stable, i.e., all eigenvalues of the constant matrix

$$A = (J - D)\nabla^2 H(x)\,, \quad \nabla^2 H(x) = \begin{pmatrix} K & \mathbf{0} \\ \mathbf{0} & M^{-1} \end{pmatrix}$$

are lying in the open left complex half-plane (see Sec. 4). The controllability function

$$L_c(x) = \min_{u \in L^2} \int_{-\infty}^{0} |u(t)|^2\, dt\,, \quad x(-\infty) = 0,\, x(0) = x$$

then measures the minimum energy that is needed to steer the system from $x(-\infty) = 0$ to $x(0) = x$. In turn, the observability function

$$L_o(x) = \int_0^{\infty} |y(t)|^2\, dt\,, \quad x(0) = x,\, u \equiv 0$$

measures the control-free energy of the output as the system evolves from $x(0) = x$ to $x(\infty) = 0$ (asymptotic stability). It is easy to see that

$$L_c(x) = x^T W_{\mathrm{c}}^{-1} x\,, \quad L_o(x) = x^T W_{\mathrm{o}} x\,,$$

where the controllability Gramian W_{c} and the observability W_{o} are the unique symmetric solutions of the Lyapunov equations

$$AW_{\mathrm{c}} + W_{\mathrm{c}} A^T = -BB^T\,, \quad A^T W_{\mathrm{o}} + W_{\mathrm{o}} A = -Q^T Q$$

with the shorthand $Q = C\nabla^2 H$. Moore [10] has shown that if $W_{\mathrm{c}}, W_{\mathrm{o}} \succ 0$ (positive definiteness = complete controllability/observability) there exists a coordinate transformation $x \mapsto Tx$, such that the two Gramians become equal and diagonal,

$$T^{-1} W_{\mathrm{c}} T^{-T} = T^T W_{\mathrm{o}} T = \mathrm{diag}(\sigma_1, \ldots, \sigma_{2n})\,,$$

where the Hankel singular values (HSV) $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{2n} > 0$ are independent of the choice of coordinates. A convenient way to express the balancing transformation is due to Moore [10]: Noting that Σ^2 contains the positive eigenvalues of the product $W_{\mathrm{c}} W_{\mathrm{o}}$ we decompose the two Gramians according to

$$W_{\mathrm{c}} = XX^T\,, \quad W_{\mathrm{o}} = YY^T$$

and do a singular value decomposition (SVD) of the matrix $Y^T X$, i.e.,

$$Y^T X = U \Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}. \tag{2.6}$$

The partitioning $\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$ and $\Sigma_2 = \mathrm{diag}(\sigma_{d+1}, \ldots, \sigma_{2n})$ indicates which singular values are important and which are negligible. The remaining matrices satisfy $U_1^T U_1 = V_1^T V_1 = \mathbf{1}_{d \times d}$ and $U_2^T U_2 = V_2^T V_2 = \mathbf{1}_{r \times r}$ with $r = 2n - d$. In terms of the SVD the balancing transformation T and its inverse $S = T^{-1}$ take the form

$$T = XV\Sigma^{-1/2}\,, \quad S = \Sigma^{-1/2} U^T Y^T\,. \tag{2.7}$$

It can be readily seen that the balancing transformation leaves the structure of the equations of motion (2.1) unchanged and preserves both stability and passivity. In the balanced variables $z = Sx$ our Hamiltonian system reads

$$\begin{aligned} \dot{z}(t) &= (\tilde{J} - \tilde{D})\nabla\tilde{H}(z(t)) + \tilde{B}u(t) \\ y(t) &= \tilde{C}\nabla\tilde{H}(z(t)) \end{aligned} \tag{2.8}$$

with the balanced Hamiltonian $\tilde{H}(z) = H(Tz)$, i.e.,

$$\tilde{H}(\xi) = \frac{1}{2} z^T \tilde{E} z\,, \quad \tilde{E} = T^T E T\,, \tag{2.9}$$

where $E = \nabla^2 H(x)$. The transformed coefficients are given by

$$\tilde{J} = SJS^T\,, \quad \tilde{R} = SRS^T\,, \quad \tilde{B} = SB\,, \quad \tilde{C} = CS^T\,. \tag{2.10}$$

3. Balanced truncation of Hamiltonian systems. Balancing amounts to changing coordinates such that those states that are least influenced by the input also have least influence on the output. Accordingly *balanced truncation* consists in first balancing the system, and then truncating the least observable and controllable states, which have little effect on the input-output behaviour.

3.1. Strong confinement limit. There are several ways that lead to a truncated (i.e., dimension-reduced) system; standard approaches such as Galerkin projection or naive singular perturbation methods, however, fail to preserve the systems inherent Hamiltonian structure. In mechanics, a natural way to restrict a system to a subspace is by means of constraints [11].

From a physical viewpoint it makes sense to study the limit of vanishing small Hankel singular values, i.e., to gradually eliminate the least observable and controllable states. To this end, we scale the HSV uniformly according to

$$(\sigma_1, \ldots, \sigma_d, \sigma_{d+1}, \ldots, \sigma_{2n}) \mapsto (\sigma_1, \ldots, \sigma_d, \epsilon\sigma_{d+1}, \ldots, \epsilon\sigma_{2n})\,,$$

i.e., in (2.6)–(2.7) we replace Σ_2 by $\epsilon\Sigma_2$ and partition the thus obtained balancing matrices accordingly,

$$S(\epsilon) = \begin{pmatrix} S_{11} & S_{12} \\ \epsilon^{-1/2} S_{21} & \epsilon^{-1/2} S_{22} \end{pmatrix}, \quad T(\epsilon) = \begin{pmatrix} T_{11} & \epsilon^{-1/2} T_{12} \\ T_{21} & \epsilon^{-1/2} T_{22} \end{pmatrix}.$$

Splitting the state variables $z = (z_1, z_2)$ in the same fashion and omitting the free variable in what follows the balanced equations of motion (2.8) take the form

$$\begin{aligned} \dot{z}_1^\epsilon &= (\tilde{J}_{11} - \tilde{D}_{11})\frac{\partial\tilde{H}^\epsilon}{\partial z_1} + \frac{1}{\sqrt{\epsilon}}(\tilde{J}_{12} - \tilde{D}_{12})\frac{\partial\tilde{H}^\epsilon}{\partial z_2} + \tilde{B}_1 u \\ \dot{z}_2^\epsilon &= \frac{1}{\sqrt{\epsilon}}(\tilde{J}_{21} - \tilde{D}_{21})\frac{\partial\tilde{H}^\epsilon}{\partial z_1} + \frac{1}{\epsilon}(\tilde{J}_{22} - \tilde{D}_{22})\frac{\partial\tilde{H}^\epsilon}{\partial z_2} + \frac{1}{\sqrt{\epsilon}}\tilde{B}_2 u \\ y^\epsilon &= \tilde{C}_1\frac{\partial\tilde{H}^\epsilon}{\partial z_1} + \frac{1}{\sqrt{\epsilon}}\tilde{C}_2\frac{\partial\tilde{H}^\epsilon}{\partial z_2} \end{aligned} \tag{3.1}$$

with the scaled Hamiltonian

$$\tilde{H}^\epsilon(z) = \frac{1}{2} z^T \tilde{E}^\epsilon z\,, \quad \tilde{E}^\epsilon = \begin{pmatrix} \tilde{E}_{11} & \epsilon^{-1/2}\tilde{E}_{12} \\ \epsilon^{-1/2}\tilde{E}_{21} & \epsilon^{-1}\tilde{E}_{22} \end{pmatrix}.$$

Boundedness of the total energy implies that $z_2 = \mathcal{O}(\sqrt{\epsilon})$ as $\epsilon \to 0$. This suggests to introduce new variables $(\zeta_1, \zeta_2) = (z_1, \epsilon^{-1/2} z_2)$ in terms of which (3.1) becomes

$$\begin{aligned}
\dot{\zeta}_1^\epsilon &= (\tilde{J}_{11} - \tilde{D}_{11})\frac{\partial \tilde{H}}{\partial \zeta_1} + \frac{1}{\epsilon}(\tilde{J}_{12} - \tilde{D}_{12})\frac{\partial \tilde{H}}{\partial \zeta_2} + \tilde{B}_1 u \\
\dot{\zeta}_2^\epsilon &= \frac{1}{\epsilon}(\tilde{J}_{21} - \tilde{D}_{21})\frac{\partial \tilde{H}}{\partial \zeta_1} + \frac{1}{\epsilon^2}(\tilde{J}_{22} - \tilde{D}_{22})\frac{\partial \tilde{H}}{\partial \zeta_2} + \frac{1}{\epsilon}\tilde{B}_2 u \\
y^\epsilon &= \tilde{C}_1 \frac{\partial \tilde{H}}{\partial \zeta_1} + \frac{1}{\epsilon}\tilde{C}_2 \frac{\partial \tilde{H}}{\partial \zeta_2}\,.
\end{aligned} \tag{3.2}$$

where $\tilde{H}(\zeta) = \tilde{H}^\epsilon(\zeta_1, \sqrt{\epsilon}\zeta_2)$ is independent of ϵ.

Limiting equation. Equation (3.2) is an instance of a slow/fast system, and we seek an effective equation for the slow variable $z_1 = \zeta_1$. Let us start with some preliminary considerations: given a solution of (3.2), the following energy balance

$$\begin{aligned}
\frac{d\tilde{H}}{dt} =& -\left(\frac{\partial \tilde{H}}{\partial \zeta_1}\right)^T \tilde{D}_{11}\frac{\partial \tilde{H}}{\partial \zeta_1} - \frac{2}{\epsilon}\left(\frac{\partial \tilde{H}}{\partial \zeta_1}\right)^T \tilde{D}_{12}\frac{\partial \tilde{H}}{\partial \zeta_2} - \frac{1}{\epsilon^2}\left(\frac{\partial \tilde{H}}{\partial \zeta_2}\right)^T \tilde{D}_{22}\frac{\partial \tilde{H}}{\partial \zeta_2} \\
&+ \left(\frac{\partial \tilde{H}}{\partial \zeta_1}\right)^T \tilde{B}_1 u + \frac{1}{\epsilon}\left(\frac{\partial \tilde{H}}{\partial \zeta_2}\right)^T \tilde{B}_2 u
\end{aligned}$$

holds. Assuming that $\tilde{H}$ remains bounded as ϵ goes to zero and that the fast dynamics is uniformly hyperbolic for all ζ_1, we conclude that (see [12, 13])

$$\frac{\partial \tilde{H}}{\partial \zeta_2} \to 0 \quad \text{as} \quad \epsilon \to 0\,.$$

Latter implies that the dynamics admit a stable invariant manifold given by

$$\zeta_2 = -\tilde{E}_{22}^{-1}\tilde{E}_{21}\zeta_1\,.$$

Inserting the last identity into (3.2), we obtain a closed equation for $z_1 = \zeta_1$, viz.,

$$\begin{aligned}
\dot{z}_1 &= (\tilde{J}_{11} - \tilde{D}_{11})(\tilde{E}_{11} - \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21})z_1 + \tilde{B}_1 u \\
y &= \tilde{C}_1(\tilde{E}_{11} - \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21})z_1\,.
\end{aligned} \tag{3.3}$$

Equation (3.3) is Hamiltonian with

$$\bar{H}(z_1) = \frac{1}{2} z_1^T \bar{E} z_1\,, \quad \bar{E} = \tilde{E}_{11} - \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21} \tag{3.4}$$

Notice that $\tilde{J}_{11} = -\tilde{J}_{11}^T$ and $\tilde{D}_{11} = \tilde{D}_{11}^T \succcurlyeq 0$ are simply the original structure and friction matrices restricted to the subspace of the most controllable and observable states. That is, in the limit of vanishing small HSV the dynamics are confined to the controllable and observable subspace. Note that the confined system is passive if the original system was, i.e., if $C = B^T$ in (2.1). Moreover $E = E^T \succ 0$ implies $\bar{E} = \bar{E}^T \succ 0$ for the Schur complement. We now give a systematic derivation of (3.3); as for the stability issue, we refer to Section 4.

Derivation. We suppose that $u \in L^2([0,\infty[,\mathbf{R}^m)$. We aim at a perturbative expansion of the solutions to (3.2). To this end we observe that the infinitesimal generator $\mathcal{L}^\epsilon$ that generates the semigroup of solutions can be split acccording to

$$\mathcal{L}^\epsilon = \mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1 + \frac{1}{\epsilon^2}\mathcal{L}_2$$

with

$$\begin{aligned}
\mathcal{L}_0 &= \left(Z_{11} + \tilde{B}_1 u\right)^T \frac{\partial}{\partial \zeta_1} \\
\mathcal{L}_1 &= Z_{12}^T \frac{\partial}{\partial \zeta_1} + \left(Z_{21} + \tilde{B}_2 u\right)^T \frac{\partial}{\partial \zeta_2} \\
\mathcal{L}_2 &= Z_{22}^T \frac{\partial}{\partial \zeta_2}
\end{aligned}$$

and the shorthand

$$Z_{ij} = (\tilde{J}_{ij} - \tilde{D}_{ij})\frac{\partial \tilde{H}}{\partial \zeta_j}\,.$$

Consider the following Cauchy problem[1]

$$\partial_t v^\epsilon(\zeta,t) = \mathcal{L}^\epsilon v^\epsilon(\zeta,t)\,, \quad v^\epsilon(\zeta,0) = f(\zeta) \tag{3.5}$$

that is fully equivalent to the Hamiltonian system (3.2) by the method of characteristics and for which we seek a perturbative expansion of the form

$$v^\epsilon = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots\,.$$

Plugging the ansatz in the backward equation (3.5) and equating equal powers of ϵ yields a hierarchy of equations the first three of which are

$$\mathcal{L}_2 v_0 = 0 \tag{3.6}$$
$$\mathcal{L}_2 v_1 = -\mathcal{L}_1 v_0 \tag{3.7}$$
$$\mathcal{L}_2 v_2 = -\mathcal{L}_0 v_0 - \mathcal{L}_1 v_1 + \partial_t v_0\,. \tag{3.8}$$

We proceed step by step: since $\mathcal{L}_2$ is a differential operator in ζ_2 only and the nullspace of $\tilde{J}_{22} - \tilde{D}_{22}$ is empty, the only functions that solve (3.6) are constant in ζ_2, i.e., $v_0 = v_0(\zeta_1, t)$. By the Fredholm alternative [14], equation (3.7) has a solution if and only if the right hand side is orthogonal to the kernel of the L^2-adjoint $\mathcal{L}_2^*$, where orthogonality is meant in the L^2 sense. Since the fast subsystem

$$\dot{\zeta}_2(t) = (\tilde{J}_{22} - \tilde{D}_{22})(\tilde{E}_{21}\eta + \tilde{E}_{22}\zeta_2(t))\,, \quad \zeta_2(0) = \zeta_2\,, \tag{3.9}$$

corresponding to $\mathcal{L}_2$, is asymptotically stable (see Sec. 4, Cor. 4.5) for each $\zeta_1 = \eta$ fixed, the dynamics converge exponentially fast to the invariant subspace given by

$$S = \left\{\zeta_2 \in \mathbf{R}^{2n-d} \;\middle|\; \tilde{E}_{22}\zeta_2 = -\tilde{E}_{21}\eta\right\}\,.$$

[1] We assume that $\mathcal{L}^\epsilon$ is equipped with appropriate boundary conditions.

Solvability of (3.7) therefore requires that the right hand is zero when we integrate it against any function that is in the nullspace of $\mathcal{L}_2^*$, i.e., an invariant measure of the fast dynamics. By stability the invariant measure is unique and is given by

$$d\rho_\eta(\zeta_2) = \det \tilde{E}_{22}\, \delta(\tilde{E}_{21}\eta + \tilde{E}_{22}\zeta_2)\, d\zeta_2$$

with $\delta(\cdot)$ being the Dirac mass. As v_0 is independent of ζ_2 it follows that

$$\int_{\mathbf{R}^{2n-d}} \mathcal{L}_1 v_0\, d\rho_\eta = 0\,,$$

i.e., the solvability condition $\mathcal{L}_1 v_0 \perp \ker \mathcal{L}_2^*$ is met. To solve equation (3.7) we first of all observe that v_1 must be of the form (see [15])

$$v_1(\zeta_1, \zeta_2, t) = \phi(\zeta_1, \zeta_2)^T \nabla v_0(\zeta_1, t) + \psi(\zeta_1, t)\,,$$

where $\psi \in \ker \mathcal{L}_2$ plays no role in what follows so we set it to zero. Equation (3.7) can now be recast as an equation for ϕ: $\mathbf{X} \to \mathbf{R}^d$, the so-called cell problem

$$\mathcal{L}_2 \phi = -Z_{12}^T\,. \tag{3.10}$$

In (3.5), the initial condition is independent of ϵ, therefore $v_1(\zeta, 0) = 0$ which leaves the only possible choices $v_0 = c$ or $\phi = 0$. If we exclude the trivial stationary solution $v_0(x,t)$ being constant, consistency of (3.10) requires that $Z_{12} = 0$, i.e., the initial conditions are restricted to the invariant subspace S. To conclude, the Fredholm alternative for equation (3.8) entails the solvability condition

$$\int_{\mathbf{R}^{2n-d}} (\partial_t v_0 - \mathcal{L}_0 v_0 - \mathcal{L}_1 \phi \nabla v_0)\, d\rho_\eta = 0$$

which, for $\phi = 0$, can be recast as

$$\partial_t v_0(\zeta_1, t) = \left(\bar{Z} + \tilde{B}_1 u\right)^T \nabla v_0(\zeta_1, t) \tag{3.11}$$

with the abbreviation

$$\bar{Z} = (\tilde{J}_{11} - \tilde{D}_{11})(\tilde{E}_{11} - \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21})\zeta_1\,.$$

Upon reinterpetation as a control system, (3.11) equals (3.3).

Secular Terms. The derivation of the effective equation (3.11) relies on specific assumptions regarding the initial conditions in (3.5), namely, being independent of ϵ. If we drop this assumption, the solution of the cell problem (3.10) reads

$$\phi(\zeta_1, \zeta_2) = -(\tilde{J}_{12} - \tilde{D}_{12})(\tilde{J}_{22} - \tilde{D}_{22})^{-1}\zeta_2$$

where invertibility of $(\tilde{J}_{22} - \tilde{D}_{22})$ is due to Corollary 4.5 below, and we have omitted integration constants $C(\zeta_1)$. The Fredholm alternative for (3.8) then yields

$$\partial_t v_0 = \left(\bar{W} + \bar{B}u\right)^T \nabla v_0\,,$$

with

$$\begin{aligned}
\bar{W} &= \left(\tilde{J}_{11} - \tilde{D}_{11} - (\tilde{J}_{12} - \tilde{D}_{12})(\tilde{J}_{22} - \tilde{D}_{22})^{-1}(\tilde{J}_{21} - \tilde{D}_{21})\right) \nabla \bar{H} \\
\bar{B} &= \tilde{B}_1 - (\tilde{J}_{12} - \tilde{D}_{12})(\tilde{J}_{22} - \tilde{D}_{22})^{-1}\tilde{B}_2\,.
\end{aligned}$$

That is, for $\phi \neq 0$ the corresponding control system turns out to be

$$\begin{aligned} \dot{z}_1 &= (\bar{J} - \bar{D})\nabla\bar{H}(z_1) + \bar{B}u \\ y &= \bar{C}\nabla\bar{H}(z_1) + \bar{F}u\,. \end{aligned} \tag{3.12}$$

where

$$\bar{J} = \frac{1}{2}(\bar{L} - \bar{L}^T) \quad \text{and} \quad \bar{D} = -\frac{1}{2}(\bar{L} + \bar{L}^T) \tag{3.13}$$

are antisymmetric and symmetric parts of the Schur complement of $L = \tilde{J} - \tilde{D}$ and

$$\begin{aligned} \bar{C} &= \tilde{C}_1 - \tilde{C}_2(\tilde{J}_{22} - \tilde{D}_{22})^{-1}(\tilde{J}_{21} - \tilde{D}_{21}) \\ \bar{F} &= -\tilde{C}_2(\tilde{J}_{22} - \tilde{D}_{22})^{-1}\tilde{B}_2 \end{aligned} \tag{3.14}$$

are the effective observability coefficients with an additional feed-through term (cf. [16]). The extra terms in the effective equation are secular terms that occur if

$$\frac{\partial\tilde{H}}{\partial\zeta_2} = \mathcal{O}(\epsilon)$$

in (3.2) such that the singular term in the slow equation becomes essentially $\mathcal{O}(1)$.

3.2. Hard constraints. We shall briefly mention yet another method so as to impose the controllability/observability constraint on the balanced system (2.8). In (2.6), set $\Sigma_2 = \mathbf{0}$ which renders the equation rank-deficient,

$$Y^T X = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1\Sigma_1 V_1^T$$

with $U_1^T U_1 = V_1^T V_1 = \mathbf{1}_{d\times d}$. We define $T_1 \in \mathbf{R}^{2n\times d}$ and $S_1 \in \mathbf{R}^{d\times 2n}$ by

$$T_1 = XV_1\Sigma_1^{-1/2}\,, \quad S_1 = \Sigma_1^{-1/2}U_1^T Y^T\,,$$

where the reduced balancing transformation satisfies $S_1 W_{\mathrm{c}} S_1^T = \Sigma_1 = T_1^T W_{\mathrm{c}} T_1$ with $S_1 T_1 = \mathbf{1}_{d\times d}$, and we introduce local coordinates $z_1 = S_1 x$ on the essential subspace that is defined as the orthogonal complement of the nullspace of $Y^T X$. The canonical way to constrain a mechanical system is by restricting the Hamiltonian and the corresponding structure matrix (i.e., the symplectic form). Restricting the Hamiltonian (2.3) according to $x = T_1 z_1$ we thus obtain the restricted system

$$\begin{aligned} \dot{z}_1(t) &= (\tilde{J}_{11} - \tilde{D}_{11})\nabla\tilde{H}_1(z_1(t)) + \tilde{B}_1 u(t) \\ y(t) &= \tilde{C}_1\nabla\tilde{H}_1(z_1(t)) \end{aligned} \tag{3.15}$$

where $\tilde{J}_{11} = S_1 J S_1^T$, $\tilde{D}_{11} = S_1 D S_1^T$, $\tilde{B}_1 = S_1 B$, $\tilde{C}_1 = C S_1^T$ as in (3.3), and

$$\tilde{H}_1(z_1) = \tilde{H}(z_1, 0)\,, \quad \tilde{H}_1(z_1) = \frac{1}{2} z_1^T \tilde{E}_{11} z_1 \tag{3.16}$$

denotes the restricted Hamiltonian. Note that the thus constrained system shares all properties of (3.3) in terms of structure preservation and passivity.

4. Stability issues. It is well-known that balanced truncation for linear systems preserves asymptotic stability of the original system, and we may ask whether the reduced systems (3.3) and (3.15) preserve stability, too.

We first of all show that the original system is stable: Suppose J and E are real and non-singular $2n \times 2n$ matrices, where $J = -J^T$ is skew-symmetric and $E = E^T$ is symmetric. Given a real, positive semi-definite matrix $D \in \mathbf{R}^{2n\times 2n}$, we consider a perturbed eigenvalue problem of the form

$$(J - \mu D)Ev = \lambda v\,. \tag{4.1}$$

THEOREM 4.1 (Maddocks & Overton 1995). *Suppose that*

$$u^*EDEu > 0 \quad \forall \text{ eigenvectors } u \text{ of } JE \text{ with pure imaginary eigenvalues.}$$

Then, for all $\mu > 0$ and counting algebraic multiplicity, the number of eigenvalues λ of $(J-\mu D)E$ in the open right half-plane $\mathbf{C}^+$ equals the number of negative eigenvalues of E. Furthermore no eigenvalue of $(J-\mu D)E$ is pure imaginary for $\mu > 0$.

For the proof using an homotopy argument, we refer to [17]. We have

LEMMA 4.2. *The system (2.1) with the Hamiltonian (2.3) and the matrices* (2.4) *is asymptotically stable, i.e., all eigenvalues of $(J-D)E$ lie in the open left half-plane.*

Proof. The matrix E is positive definite and symmetric, that is, all its eigenvalues are positive. Then, according to Theorem 4.1, it suffices to prove that $u^*EDEu > 0$ on the invariant subspace of JE that corresponds to pure imaginary eigenvalues. We prove this by contradiction: Assume the contrary, i.e., there is an eigenvector $u = (u_1, u_2)$ of JE with associated imaginary eigenvalue and $u^*EDEu = 0$. Since

$$u^*EDEu = u_2^* M^{-1} D M^{-1} u_2\,,$$

we conclude that $u_2 = 0$. But then

$$JEu = \begin{pmatrix} 0 \\ -Ku_1 \end{pmatrix}.$$

Hence $u_1 = 0$, which contradicts that u is an eigenvector. □

4.1. Stability preservation. The reasoning in the proof of Lemma 4.2 heavily relies on the specific form of the unbalanced friction matrix D, and we cannot use the same trick to prove stability of the dimension-reduced systems (3.3) and (3.15).

In fact it immediately recognize that (asymptotic) stability cannot be preserved, as, e.g., a block diagonal balancing transformation with $k = n$ and $S_{12} = \mathbf{0} \in \mathbf{R}^{n\times n}$ would lead to the reduced system $\dot{z}_1 = \tilde{B}_1 u$. Thus the dimension-reduced system is only neutrally stable in general. If, however, J_{11} is non-singular, asymptotic stability is indeed preserved. We shall discuss the two scenarios separately; the case of a singular structure matrix is treated later on in the text.

Nonsingular structure matrix. We suppose that $\tilde{J}_{11} = S_1 J S_1^T$ is invertible which implies that the dimension k of the reduced system is even. Let $Q = Q^T \succ 0$ denote the Hessian of the reduced Hamiltonian, i.e., either $Q = \tilde{E}_{11}$ the restricted Hessian or $Q = \bar{E}$ the Schur complement. The following can be proved:

LEMMA 4.3. *Let $\tilde{J}_{11} \in \mathbf{R}^{d\times d}$ invertible and $\tilde{D}_{11}$ be given as in (3.15). For any $d \times d$ matrix $Q = Q^T \succ 0$ the matrix $P = (\tilde{J}_{11} - \tilde{Q}_{11})Q$ is stable, i.e., all eigenvalues of P lie in the open left half-plane.*

Proof. Since Q is symmetric, positive definite, we may write

$$Q^{1/2}(\tilde{J}_{11}Q)Q^{-1/2} = Q^{1/2}\tilde{J}_{11}Q^{1/2}\,,$$

i.e., $\tilde{J}_{11}Q$ is similar to a skew-symmetric matrix and so its spectrum lies entirely on the imaginary axis. Moreover $\tilde{J}_{11}$ is non-singular so we can exclude 0 to be an eigenvalue. Let v be an eigenvector of $\tilde{J}_{11}Q$ with pure imaginary eigenvalue. The vector

$$w = Qv$$

solves the eigenvalue problem

$$Q^{1/2}\tilde{J}_{11}Q^{1/2}w = \lambda w\,, \quad \lambda \neq 0\,.$$

Upon multiplying the last equation with w^* from the left, we obtain

$$w^*Q^{1/2}\tilde{J}_{11}Q^{1/2}w = \lambda\|w\|^2\,.$$

Since the right hand side of this equation is non-zero by construction, we have

$$w^*Q^{1/2}\tilde{J}_{11}Q^{1/2}w \neq 0. \tag{4.2}$$

Theorem 4.1 guarantees that $P = (\tilde{J}_{11} - \tilde{D}_{11})Q$ is stable if $v^*Q\tilde{D}_{11}Qv > 0$, where v is an arbitrary eigenvector of $\tilde{J}_{11}Q$. We assume that $v^*Q\tilde{D}_{11}Qv = 0$ and aim for a contradiction. The corresponding condition for w reads

$$w^*Q^{1/2}\tilde{D}_{11}Q^{1/2}w = 0\,.$$

Since $\tilde{D}_{11} = S_{12}RS_{12}^T$ with $R = R^T \succ 0$ we conclude that $Q^{1/2}w$ lies in the nullspace of S_{12}^T. But $\tilde{J}_{11} = S_{11}S_{12}^T - S_{12}S_{11}^T$, and therefore the left hand side of (4.2) is zero which yields the contradiction. □

COROLLAR 4.4. *The reduced Hamiltonian systems (3.3) and (3.15) are stable whenever $\tilde{J}_{11}$ is non-singular.*

Singular structure matrix. The requirement that the restricted structure matrix is invertible is rather restrictive. For instance, k may be odd in case of which $\tilde{J}_{11} \in \mathbf{R}^{k\times k}$ has at least one eigenvalue zero. As we have argued it may even happen that $\tilde{J}_{11} - \tilde{D}_{11}$ vanishes identically, although S_1 has maximum rank. To treat the general case general we assume that the $d \times d$ matrix

$$\tilde{J}_{11} - \tilde{D}_{11} = S_1(J - D)S_1^T$$

has reduced rank $d - s$. That is, the autonomous system (i.e., for $u = 0$) admits certain conserved quantities $C_i : \mathbf{R}^d \to \mathbf{R}$, $i = 1, \ldots, s$ (Casimirs) that satisfy

$$\nabla C_i(z_1)^T(\tilde{J}_{11} - \tilde{D}_{11}) = 0\,, \quad i = 1, \ldots, s\,.$$

Clearly, the C_i are of the form $C_i(z_1) = b_i^T z_1$. Accordingly the dynamics are unstable (i.e., not asymptotically stable) in the directions orthogonal to the level sets of the $C_i(z_1)$. The system is still completely controllable though, and it is easy to see that we can the transform the equations of motion to assume the following form [18]

$$\begin{aligned}\dot{\xi} &= (\hat{J}(C) - \hat{D}(C))\frac{\partial \hat{H}}{\partial \xi} + \hat{B}_1(C)u\\ \dot{C} &= \hat{B}_2(C)u\,,\end{aligned}$$

where $\xi = (\xi_1, \ldots, \xi_{d-s})$ are local coordinates on the hyperplane $C(z_1) = C$, and $\hat{J} - \hat{D}$ is the restriction of $\tilde{J}_{11} - \tilde{D}_{11}$ to the hyperplane. Since $\hat{J}$ is invertible and $\hat{D} \succcurlyeq 0$, the restricted system remains stable along the ξ-direction.

4.2. Invariant manifold. Recalling the singular perturbation argument from Section 3.1, it remains to prove that the subspace of uncontrollable/unobservable states to which the dynamics collapses as the small HSV go to zero is indeed an stable invariant manifold. In other words, we have to show that the eigenvalues of the matrix $(\tilde{J}_{22}-\tilde{D}_{22})\tilde{E}_{22})$ have strictly negative real part. We suppose the original system (2.1) is minimal such that all the HSV in (2.6) are strictly positive. In accordance with (2.7) we define $T_2 \in \mathbf{R}^{2n\times(2n-d)}$ and $S_2 \in \mathbf{R}^{(2n-d)\times 2n}$ by

$$T_2 = XV_2\Sigma_2^{-1/2}\,, \quad S_2 = \Sigma_2^{-1/2}U_2^TY^T\,,$$

which are the balancing transformation on the fast (i.e., least controllable/observable) subspace. By positivity of $E = \nabla^2 H(x)$ the matrix $\tilde{E}_{22} = T_2^T E T_2$ is symmetric positive definite. Employing the notation $S_2 = (S_{21}, S_{22})$ while assuming that $S_{22} \neq \mathbf{0}$ structure and friction matrix on the fast subspace take the form

$$\tilde{D}_{22} = S_{22}RS_{22}^T\,, \quad \tilde{J}_{22} = S_{21}S_{22}^T - S_{22}S_{21}^T\,.$$

The following is a straight consequence of Lemma 4.3.

COROLLAR 4.5. *Let $\tilde{J}_{22}$ be non-singular and $\tilde{D}_{22} \succcurlyeq 0$. Then the fast subsystem (3.9) admits a unique stable invariant manifold S.*

Proof. Obviously S is an invariant manifold of (3.9). To show that it is unique and asymptotically stable it suffices to show that both $\tilde{J}_{22}-\tilde{D}_{22}$ and $(\tilde{J}_{22}-\tilde{D}_{22})\tilde{E}_{22}$ are stable and have no have no eigenvalues on the imaginary axis. Adapting the proof of Lemma 4.3 with $Q = \mathbf{1}$ or $Q = \tilde{E}_{22}$, respectively, yields the result. □

The last statement guarantees that (3.2) is hyperbolic in the sense that, as ϵ goes to zero, the fast dynamics contracts exponentially to its stationary point conditional on the fixed value of the slow variable. By invertibility of $\tilde{J}_{22}-\tilde{R}_{22}$, the family of the stationary points is uniquely determined by S.

5. Minimal realization. Before we conclude with some numerical examples we demonstrate that the dimension-reduced system (3.3) provides a minimal realization of the original system when the weakly controllable/obervable modes become completely uncontrollable/unobservable. That is, we prove that, in the limit of vanishing small Hankel singular values, the transfer function of the original Hamiltonian system (2.1) converges to the reduced transfer function associated with (3.3). The transfer function associated with (2.1) reads

$$G(s) = CE\,(s-(J-D)E)^{-1}\,B\,, \quad E = \nabla^2 H\,. \tag{5.1}$$

It can be regarded as the solution to (2.1) in the Laplace domain with zero initial conditions considered as a input/output mapping $G : L^2([0,\infty[,\mathbf{R}^m) \to L^2([0,\infty[,\mathbf{R}^l)$; see [1] for details. Provided that $(J-D)E$ is stable, the transfer function is analytic in the open right half-plane, and the H^∞ norm of G is defined as the supremum of the largest singular value of the transfer function on the imaginary axis [2],

$$||G||_\infty = \sup_{\omega\in\mathbf{R}} \{\sigma_{\max}\,(G(i\omega))\}\,.$$

Convergence. We now prove convergence of (2.1) to the reduced system (3.3) in terms of their input/output behaviour in the H^∞ norm. The transfer function of the singularly perturbed system (3.2) given by

$$G^\epsilon(s) = \tilde{C}^\epsilon\tilde{E}(s-(\tilde{J}^\epsilon-\tilde{D}^\epsilon)\tilde{E})^{-1}\tilde{B}^\epsilon\,, \tag{5.2}$$

where the scaled coefficient matrices read

$$\tilde{J}^\epsilon - \tilde{D}^\epsilon = \begin{pmatrix} \tilde{J}_{11} - \tilde{D}_{11} & \epsilon^{-1}(\tilde{J}_{12} - \tilde{D}_{12}) \\ \epsilon^{-1}(\tilde{J}_{21} - \tilde{D}_{21}) & \epsilon^{-2}(\tilde{J}_{22} - \tilde{D}_{22}) \end{pmatrix},$$

and

$$\tilde{B}^\epsilon = \begin{pmatrix} \tilde{B}_1 \\ \epsilon^{-1}\tilde{B}_2 \end{pmatrix}, \quad \tilde{C}^\epsilon = \begin{pmatrix} \tilde{C}_1 & \epsilon^{-1}\tilde{C}_2 \end{pmatrix}.$$

Note that $\tilde{E} = \nabla^2 \tilde{H}$ does not depend on ϵ. Moreover, $G^{\epsilon=1}$ coincides with the transfer function G of the original system (2.1). We have

THEOREM 5.1. *Let G^ϵ be the scaled transfer function (5.2), and denote by $\bar{G}$ the reduced transfer function of the limit system (3.3), i.e.,*

$$\bar{G}(s) = \tilde{C}_1 Q (s - (\tilde{J}_{11} - \tilde{D}_{11})Q)^{-1} \tilde{B}_1 \tag{5.3}$$

with $Q = \tilde{E}_{11} - \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21}$ being the Schur complement of $\tilde{E}$. Then

$$\|G^\epsilon - \bar{G}\|_\infty \to 0 \quad \textit{as} \quad \epsilon \to 0.$$

Proof. We show that $G^\epsilon = \bar{G} + \mathcal{O}(\epsilon)$ by Taylor expanding G^ϵ around $\epsilon = 0$; the calculation is tedious but straightforward. First of all recall that the inverse of a partitioned matrix $X \in \mathbf{R}^{2n\times 2n}$ can be written as

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}^{-1} = \begin{pmatrix} Y^{-1} & -X_{11}^{-1}X_{12}Z^{-1} \\ -X_{22}^{-1}X_{21}Z^{-1} & Z^{-1} \end{pmatrix}$$

with the shorthands $Y = X_{11} - X_{12}X_{22}^{-1}X_{21}$ and $Z = X_{22} - X_{21}X_{11}^{-1}X_{12}$. Hence

$$\begin{pmatrix} s - \tilde{A}_{11}^\epsilon & \tilde{A}_{12}^\epsilon \\ \tilde{A}_{21}^\epsilon & s - \tilde{A}_{22}^\epsilon \end{pmatrix}^{-1} =: \begin{pmatrix} W_{11}^\epsilon & W_{12}^\epsilon \\ W_{21}^\epsilon & W_{22}^\epsilon \end{pmatrix}$$

with $\tilde{A}^\epsilon = (\tilde{J}^\epsilon - \tilde{D}^\epsilon)\tilde{E}$. This yields

$$\begin{aligned} G^\epsilon(s) = &\underbrace{(\tilde{C}_1\tilde{E}_{11} + \frac{1}{\epsilon}\tilde{C}_2\tilde{E}_{21})W_{11}^\epsilon\tilde{B}_1}_{(i)} + \underbrace{(\tilde{C}_1\tilde{E}_{12} + \frac{1}{\epsilon}\tilde{C}_2\tilde{E}_{22})W_{21}^\epsilon\tilde{B}_1}_{(ii)} \\ &+ \underbrace{\frac{1}{\epsilon}(\tilde{C}_1\tilde{E}_{11} + \frac{1}{\epsilon}\tilde{C}_2\tilde{E}_{21})W_{12}^\epsilon\tilde{B}_2}_{(iii)} + \underbrace{\frac{1}{\epsilon}(\tilde{C}_1\tilde{E}_{12} + \frac{1}{\epsilon}\tilde{C}_2\tilde{E}_{22})W_{22}^\epsilon\tilde{B}_2}_{(iv)}, \end{aligned} \tag{5.4}$$

where, e.g., W_{11}^ϵ is given by

$$\begin{aligned} W_{11}^\epsilon = \Big[& s - (\tilde{J}_{11} - \tilde{D}_{11})\tilde{E}_{11} - \frac{1}{\epsilon}(\tilde{J}_{12} - \tilde{D}_{12})\tilde{E}_{21} \\ & - \left((\tilde{J}_{11} - \tilde{D}_{11})\tilde{E}_{12} + \frac{1}{\epsilon}(\tilde{J}_{12} - \tilde{D}_{12})\tilde{E}_{22}\right) \\ & \times \left(s - \frac{1}{\epsilon}(\tilde{J}_{21} - \tilde{D}_{21})\tilde{E}_{12} - \frac{1}{\epsilon^2}(\tilde{J}_{22} - \tilde{D}_{22})\tilde{E}_{22}\right)^{-1} \\ & \times \left(\frac{1}{\epsilon}(\tilde{J}_{21} - \tilde{D}_{21})\tilde{E}_{11} + \frac{1}{\epsilon^2}(\tilde{J}_{22} - \tilde{D}_{22})\tilde{E}_{21}\right)\Big]^{-1} \end{aligned}$$

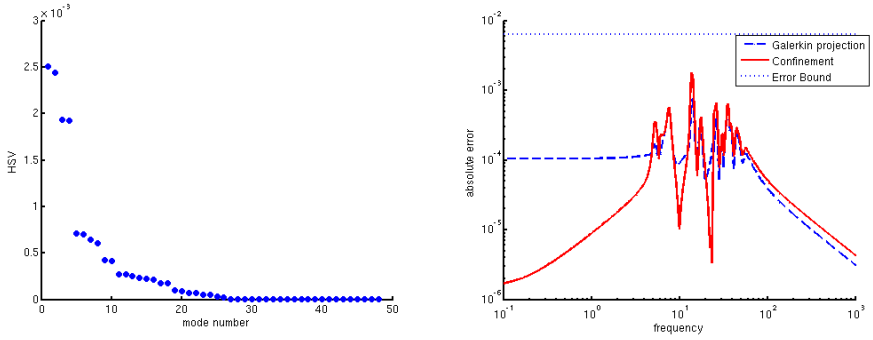

FIG. 5.1. *Hankel singular values (left panel) and spectral norms of 8-th order approximations (right panel) of the 48-dimensional building model. The dashed line shows the error bound.*

The result follows upon Taylor expansion around $\epsilon = 0$; we give only a short sketch for the derivation: the first term in (5.4) yields up to terms of order ϵ

$$(i) \approx \tilde{C}_1 \tilde{E}_{11} \left[s - (\tilde{J}_{11} - \tilde{D}_{11})(\tilde{E}_{11} - \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21}) \right]^{-1} \tilde{B}_1 \,.$$

Expanding the second term, we find up to order ϵ:

$$(ii) \approx -\tilde{C}_1 \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21} \left[s - (\tilde{J}_{11} - \tilde{D}_{11})(\tilde{E}_{11} - \tilde{E}_{12}\tilde{E}_{22}^{-1}\tilde{E}_{21}) \right]^{-1} \tilde{B}_1 \,.$$

All remaining terms in $(i)-(ii)$ and the expansion of $(iii)-(iv)$ are formally of order ϵ or of higher orders. Thus the transfer function of the full system can be recast as

$$G^\epsilon(s) = \tilde{C}_1 Q (s - (\tilde{J}_{11} - \tilde{E}_{11})Q)^{-1} \tilde{B}_1 + \epsilon \rho^\epsilon,$$

where ρ^ϵ is uniformly bounded in ϵ, since all matrices involved remain non-singular. Hence we can bound ρ^ϵ by its largest singular value (that is bounded in ϵ). Using the triangle inequality we therefore conclude that $\|G^\epsilon - \bar{G}_{\rm r}\|_\infty \to 0$ as $\epsilon \to 0$. □

6. Numerical illustration. For second-order systems of the form

$$\begin{aligned} M\ddot{q}(t) + R\dot{q}(t) + Kq(t) &= B_2 u(t) \\ y(t) &= C_1 q(t) + C_2 \dot{q}(t) \end{aligned}$$

with $M, R, K \in \mathbf{R}^{n\times n}$ being all symmetric positive definite, $B_2 \in \mathbf{R}^{n\times m}$ and $C_i \in \mathbf{R}^{l\times n}$, we compare four different model reduction methods: standard balanced truncation, reduction by strong confinement (Sec. 3.1), the method of [16] that is recovered in the discussion of secular terms (end of Sec. 3.1) and constraints (Sec. 3.2). For this purpose we recast our second-order system as an equivalent Hamiltonian system

$$\begin{aligned} \dot{x}(t) &= (J - D)\nabla H(x(t)) + Bu(t) \\ y(t) &= C\nabla H(x(t))\,, \end{aligned}$$

with $x = (q, M\dot{q})$ and the Hamiltonian function

$$H = \frac{1}{2} x_2^T M^{-1} x_2 + \frac{1}{2} x_1^T K x_1 \,.$$

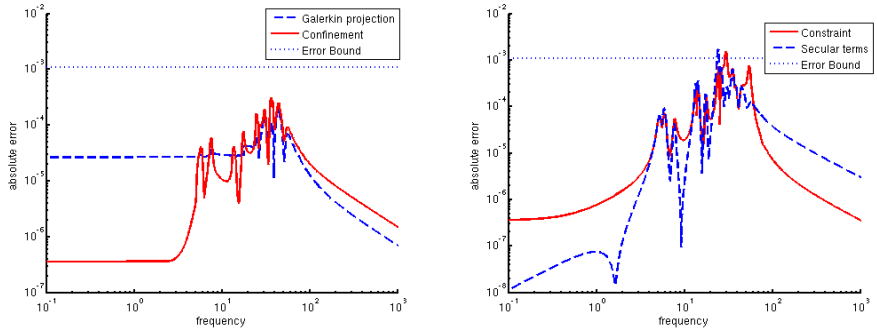

FIG. 6.1. *Building model: comparison of balanced truncation / confinement (left panel) and secular system / constraint (right panel) for an approximant of order* $k = 18$.

Choosing $J = -J^T$ to be the canonical structure matrix

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} \in \mathbf{R}^{2n \times 2n}$$

the definition of the remaining coefficients follows accordingly. For benchmarking we choose two model systems from structural mechanics: the international space station (ISS) model and the building model that are both described in [19]. The comparison is made by computing the spectral norm of $\delta\bar{G}_k = G - \bar{G}_k$ in the frequency domain, where $\bar{G}_k : L^2([0,\infty[, \mathbf{R}^m) \to L^2([0,\infty[, \mathbf{R}^l)$ stands for the k-th order transfer function associated with one of the reduction schemes; see [1] for various notions of approximation errors. For instance, for standard balanced truncation we have

$$\bar{G}_k(s) = \tilde{F}_1(s - \tilde{A}_{11})^{-1}\tilde{B}_1\,,$$

where $\tilde{F}_1 \in \mathbf{R}^{l\times k}$ consists of the first k columns of $F = C\nabla^2 H$ after balancing, $\tilde{A}_{11} \in \mathbf{R}^{k\times k}$ denotes the upper left $k \times k$ block of $A = (J - D)\nabla^2 H$ after balancing and $\tilde{B}_1 \in \mathbf{R}^{k\times m}$ is simply the balanced and truncated input matrix B; the transfer function of the full system is given by (5.1).

We start by our comparison by computing the Hankel singular values and spectral norms for the building model ($m = l = 1$ and $n = 24$). Figure 5.1 shows the error of a 8-th order approximations for standard balanced truncation (Galerkin projection) and the confined system (3.3). Both methods meet the usual upper H^∞ bound

$$\|G - \bar{G}_k\|_\infty \leq 2(\sigma_{k+1} + \ldots + \sigma_{2n})\,,$$

that is due to Glover [20] and that holds for standard balanced truncation and, under suitable assumptions, singular perturbation approaches such as the strong confinement method of Section 3.1. The reduced model involving the secular terms and the constrained system yield similar results for low-order approximations (slightly worse though) but it turns out that the error exceeds the upper bound as the order of the reduced model is increased (see Fig. 6.1); compare also the numerical studies in the recent article [6]. As for the secular system a possible explanation for this behaviour are inconsistent initial conditions, for the definition of the transfer function relies on zero initial conditions; cf. the discussion of the cell problem at the end of Section 3.1. The same effect is observed for the 270-dimensional ISS model ($m = l = 3$ and

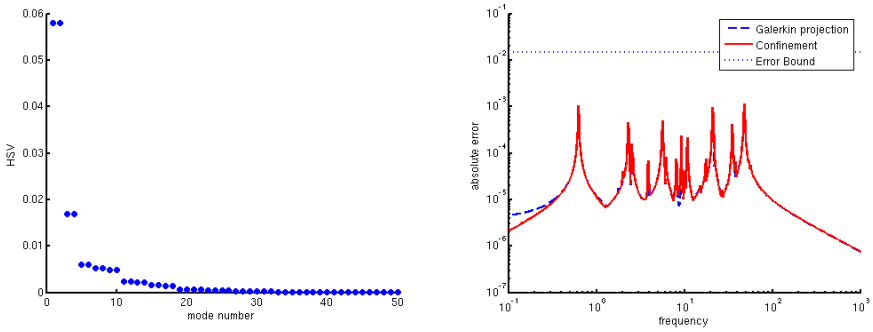

FIG. 6.2. *First 50 HSV (left panel) and spectral norms of 18-th order approximations (right panel) of the 270-dimensional ISS model. The dashed line shows the balanced truncation bound.*

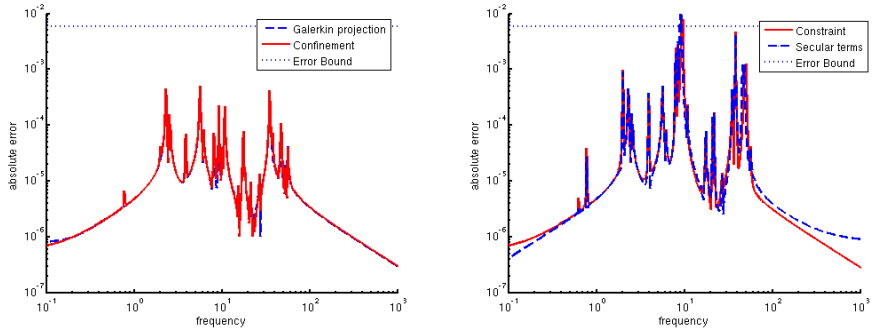

FIG. 6.3. *ISS model: comparison of balanced truncation / confinement (left panel) and secular system / constraint (right panel) for an approximant of order* $k = 26$.

$n = 135$): increasing the number of modes in the approximant renders the spectral norm of the error to exceed the balanced truncation bound as Figure 6.3 indicates. The corresponding Hankel singular values together with the Galerkin-projected and the confined approximants for $k = 18$ are depicted in Figure 6.2.

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