POSITIVITY PRESERVING BALANCED TRUNCATION FOR DESCRIPTOR SYSTEMS

TIMO REIS AND ELENA VIRNIK

Abstract. We propose a model reduction method for positive systems that ensures the positivity of the reduced-order model. In the standard as well as in the descriptor case, for continuous-time and discrete-time systems, our approach is based on constructing diagonal solutions of Lyapunov inequalities. These are linear matrix inequalities (LMIs), which are shown to be feasible. Positivity and stability are preserved and an error bound in the $\mathcal{H}_\infty$-norm is provided.

Key words. Positive systems; descriptor systems; structured model reduction; linear matrix inequalities (LMI); balanced truncation; singular perturbation balanced truncation; continuous-time systems; discrete-time systems; error bound;

1. Introduction. We consider linear time-invariant positive descriptor systems in continuous-time

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

and in discrete-time

$$\begin{aligned} Ex(t+1) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are real constant coefficient matrices. In the continuous-time case, the state x, input u and output y are real-valued vector functions. In the discrete-time case x, u and y are real-valued vector sequences. We focus on (internally) positive systems. These are systems whose state and output variables take only nonnegative values at all times t for any nonnegative initial state and any nonnegative input, [17, 21, 25].

Positive systems arise naturally in many applications such as pollutant transport, chemotaxis, pharmacokinetics, Leontief input-output models, population models and compartmental systems, [2, 5, 6, 8, 12, 17, 21]. In these models, the variables represent concentrations, population numbers of bacteria or cells or, in general, measures that are per se nonnegative. Positive systems are subject to ongoing research by many authors, [1,13,14,17,21,28–30,32,33]. Recent advances on control theoretical issues have been made especially in the positive discrete-time case. Yet, there are still many open problems, in particular for standard positive systems in continuous-time. One such problem in the continuous-time as well as in the discrete-time case is model reduction which preserves the positivity of a system. In the present paper, we generalize the model reduction methods of standard balanced truncation and singular perturbation balanced truncation such that positivity is preserved. Our technique uses a linear matrix inequality (LMI) approach and we show that stability is preserved and an error bound in the H_∞-norm is provided. Furthermore, we provide a generalization of this positivity preserving model reduction to the case of positive descriptor systems.

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Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG. E-mail: {reis,virnik}@math.tu-berlin.de.

This paper is organized as follows. In Section 2, we introduce the notation, define the positivity concept and give a well-known characterization. In Section 3, we introduce positivity-preserving balanced truncation and singular perturbation balanced truncation for standard systems and thereafter, we generalize the methods to positive descriptor systems in Section 4. The applicability of the proposed methods is demonstrated by means of several examples in Section 5.

2. Preliminaries. Throughout the paper $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the spaces of $n\times m$ real and complex matrices. The open right half-plane is denoted by $\mathbb{C}_+$ and the complex open unit disc by $\mathbb{D}$. The matrix A^T denotes the transpose of $A\in\mathbb{R}^{n\times m}$, and $A^{-T}=(A^{-1})^T$. An identity matrix of order n is denoted by I_n or simply by I. The zero $n\times m$ ($n\times n$) matrix is denoted by $0_{n,m}$ (0_n) or simply by 0. By $\operatorname{rank}(A)$ we denote the rank and by $\operatorname{im}(A)$ the image of a matrix A. For $A=[a_{ij}]_{i=1,\dots,n,j=1,\dots,m}\in\mathbb{R}^{n\times m}$ and $I=[i_1,\dots,i_k]\in\{1,\dots,n\}^k$, $J=[j_1,\dots,j_l]\in\{1,\dots,m\}^l$, we define the matrix $A(I,J)\in\mathbb{R}^{k\times l}$ by

$$A(I,J)=[a_{i_rj_s}]_{r=1,\dots,k,s=1,\dots,l}.$$

A matrix $A\in\mathbb{R}^{m\times n}$ is called *nonnegative (positive)* and we write $A\geq 0$ ($A>0$) if all entries are nonnegative (positive). A matrix $A\in\mathbb{R}^{n\times n}$ is called *Z-matrix* if its off-diagonal entries are nonpositive. In the literature, a matrix for which $-A$ is a Z-matrix sometimes is called L-matrix, Metzler matrix or essentially positive matrix, see, e.g., [7, 17, 21, 34]. Throughout this paper we will use the term $-Z$-matrix. Let $B\in\mathbb{R}^{n\times n}$ $B\geq 0$ with spectral radius $\rho(B)$. A matrix A of the form $A=\alpha I-B$, with $\alpha>0$, and $\alpha\geq\rho(B)$ is called *M-matrix*. If $\alpha>\rho(B)$ then A is a *nonsingular* M-matrix, if $\alpha=\rho(B)$ then A is a *singular* M-matrix. The class of M-matrices is a subclass of the Z-matrices. Accordingly, a matrix $A\in\mathbb{R}^{n\times n}$ for which $-A$ is an M-matrix is called a $-M$-matrix. Note that for a nonsingular M-matrix A, we have $A^{-1}\geq 0$ [34]. Further, for Hermitian matrices $P,Q\in\mathbb{C}^{n\times n}$, we write $P\succ Q$ ($P\succeq Q$) if $P-Q$ is positive (semi)definite.
Let $H_{\infty,d}$ be the space of all transfer functions that are analytic and bounded on $\mathbb{C}\backslash\overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is the closed unit ball around the origin. The continuous-time and discrete-time *H_∞-norms* are defined by

$$\|\boldsymbol{G}\|_{\infty,c}=\sup_{s\in\mathbb{C}^+}\|\boldsymbol{G}(s)\|_2,\quad\text{and}\quad\|\boldsymbol{G}\|_{\infty,d}=\sup_{z\in\mathbb{C}\backslash\overline{\mathbb{D}}}\|\boldsymbol{G}(z)\|_2,\tag{2.1}$$

respectively, where $\|\cdot\|_2$ denotes the spectral matrix norm.

2.1. Descriptor systems. Let the matrix quintuple $[\,E\,,A\,,B\,,C\,,D]$ denote the system (1.1) or (1.2), respectively. The function $\boldsymbol{G}(\lambda)=C(\lambda E-A)^{-1}B+D$ is called *transfer function* and λ is called *frequency variable*. Conversely, the quintuple $[\,E\,,A\,,B\,,C\,,D]$ is called *realization* of $\boldsymbol{G}$. Note that the frequency variable is denoted by s in continuous-time and by z in discrete-time. A transfer function $\boldsymbol{G}$ is called *proper* if it is bounded on some complex half-plane and *strictly proper* if additionally $\lim_{\lambda\to\infty}\boldsymbol{G}(\lambda)=0$ holds.

A matrix pair $(E,A)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$ is called *regular* if $\det(\lambda E-A)\neq 0$ for some $\lambda\in\mathbb{C}$. It is called *singular* otherwise. In this paper we restrict our considerations to square and regular pencils. In this case, the pair (E,A) can be transformed into Weierstraß canonical form, see e.g. [10, 15, 22], i.e., there exist regular matrices

$W, T \in \mathbb{R}^{n\times n}$ such that

$$(E, A) = \left(W \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T, W \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T \right), \tag{2.2}$$

where $N \in \mathbb{R}^{n_\infty \times n_\infty}$ is nilpotent with index of nilpotency ν. The number ν is called *index* of the matrix pair (E, A) and is denoted by $\mathrm{ind}(E, A)$. The eigenvalues of J are called the *finite eigenvalues* of (E, A). By n_f and n_∞ we denote the dimensions of the deflating subspaces of (E, A) corresponding to the finite and infinite eigenvalues, respectively. The matrix

$$P_r = T^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T, \tag{2.3}$$

is the spectral projector onto the right deflating subspace of (E, A) corresponding to the finite eigenvalues.

We call a matrix pair (E, A) *c-stable* if it is regular and all its finite eigenvalues have negative real part. We call a matrix pair (E, A) *d-stable* if its finite spectral radius, i.e., the largest modulus of a finite eigenvalue, is less than one, i.e., $\rho_f(E, A) < 1$. Note that a matrix A is called c-stable (d-stable) if (I, A) is c-stable (d-stable). We call a realization of a continuous-time (discrete-time) system $[\, E,\, A\,,\, B\,,\, C\,,\, D]$ stable, if (E, A) is c-stable (d-stable).

For regular (E, A) and $\hat{\lambda} \in \mathbb{R}$ such that $\hat{\lambda}E - A$ is non-singular, we define

$$\hat{E} = (\hat{\lambda}E - A)^{-1}E, \quad \hat{A} = (\hat{\lambda}E - A)^{-1}A, \quad \hat{B} = (\hat{\lambda}E - A)^{-1}B. \tag{2.4}$$

Note, that the matrices $\hat{E}$ and $\hat{A}$ commute, see, e.g., [11, 22].

Let $M \in \mathbb{R}^{n\times n}$ have index ν, i.e., $\mathrm{ind}(M, I) = \nu$. The Drazin inverse $M^D \in \mathbb{R}^{n\times n}$ of M, see, e.g., [11, 16], is uniquely defined by the properties:

$$M^D M = M M^D, \quad M^D M M^D = M^D, \quad M^D M^{\nu+1} = M^\nu. \tag{2.5}$$

Note that, via the Jordan canonical form, the matrix $M \in \mathbb{R}^{n\times n}$ has a representation $M = T^{-1}\,\mathrm{diag}(J, N)T$, where J is square and invertible and N is nilpotent. The Drazin inverse is then given by [11]

$$M^D = T^{-1} \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} T, \tag{2.6}$$

For $\hat{E}$, $\hat{A}$ as defined in (2.4) and their corresponding Drazin inverses, the following properties hold, see, e.g., [22]:

$$\hat{E}\hat{A}^D = \hat{A}^D\hat{E}, \quad \hat{E}^D\hat{A} = \hat{A}\hat{E}^D, \quad \hat{E}^D\hat{A}^D = \hat{A}^D\hat{E}^D. \tag{2.7}$$

Note that if we form matrix products such as $\hat{E}^D\hat{E}, \hat{E}^D\hat{A}, \hat{E}^D\hat{B}$, then the terms in $\hat{\lambda}$ cancel out, so that these products do not depend on the specific choice of $\hat{\lambda}$, [22, 36]. Furthermore, note that $\hat{E}^D\hat{E} = P_r$ is the spectral projector defined in (2.3), [26, 36].

From [27], we have that the transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B + D$ can be additively decomposed as $\boldsymbol{G}(s) = \boldsymbol{G}_{sp}(s) + \boldsymbol{P}(s)$, where $\boldsymbol{G}_{sp}(s)$ is the strictly proper

and $\boldsymbol{P}(s)$ is the polynomial part of $\boldsymbol{G}(s)$. By using (2.6), it can be derived [36] that the strictly proper part $\boldsymbol{G}_{sp}(s)$ of the transfer function $\boldsymbol{G}(s)$ can be written as

$$\boldsymbol{G}_{sp}(s) = \sum_{k=1}^{\infty} (CP_r)(\hat{E}^D\hat{A})^{k-1}(\hat{E}^D\hat{B})s^{-k} = (CP_r)(sI - \hat{E}^D\hat{A})^{-1}(\hat{E}^D\hat{B}), \tag{2.8}$$

and the polynomial part can be written as

$$\begin{aligned} \boldsymbol{P}(s) =& -C(I-P_r)\sum_{k=0}^{\nu-1}(\hat{E}\hat{A}^D)^k(I-P_r)\hat{A}^D\hat{B}s^k + D \\ =& C(I-P_r)(s(I-P_r)\hat{E}\hat{A}^D - I)^{-1}(I-P_r)\hat{A}^D\hat{B} + D. \end{aligned} \tag{2.9}$$

2.2. Positive systems. Here we introduce the basic facts about positive standard and descriptor systems.

DEFINITION 2.1 (Positivity). *We call the continuous-time system (1.1) with* $\operatorname{ind}(E,A) = \nu$ positive *if for all* $t \in \mathbb{R}_+$ *we have* $x(t) \geq 0$ *and* $y(t) \geq 0$ *for any input function* $u \in C^\nu$ *such that* $u^{(i)}(\tau) \geq 0$ *for* $i = 0, \ldots, \nu-1$ *and* $0 \leq \tau \leq t$ *and any consistent initial value* $x_0 \geq 0$.
The discrete-time system (1.2) with $\operatorname{ind}(E,A) = \nu$ *is called* positive *if for all* $t \in \mathbb{N}_0$ *we have* $x(t) \geq 0$ *and* $y(t) \geq 0$ *for any input sequence* $u(\tau) \geq 0$ *for* $0 \leq \tau \leq t+\nu-1$ *and any consistent initial value* $x_0 \geq 0$.

The following theorem states a well-known characterization of positive systems in the standard case, that is $E = I$.

THEOREM 2.2 ([17,21]). *The continuous-time system* $[\,I\,,\,A\,,\,B\,,\,C\,,\,D\,]$ *is positive if and only if* A *is a* $-Z$*-matrix and* $B, C, D \geq 0$. *The discrete-time system* $[\,I\,,\,A\,,\,B\,,\,C\,,\,D\,]$ *is positive if and only if* A, B, C, $D \geq 0$.

A corresponding characterization of positivity in the descriptor case was given in [36].

THEOREM 2.3. *Consider the system* $[\,E\,,\,A\,,\,B\,,\,C\,,\,D\,]$ *in continuous-time or in discrete-time with* (E,A) *regular of* $\operatorname{ind}(E,A) = \nu$ *and let* $\hat{E}$, $\hat{A}$, $\hat{B}$ *be defined as in (2.4). In the continuous-time case, assume that*

(i) $(I - \hat{E}^D\hat{E})(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{B} \leq 0$ *for* $i = 0, \ldots, \nu-1$,
(ii) $\hat{E}^D\hat{E} \geq 0$,
(iii) $D \geq 0$.

Then, the continuous-time system $[\,E\,,\,A\,,\,B\,,\,C\,,\,D\,]$ *is positive if and only if*

1. *there exists a scalar* $\alpha \geq 0$ *such that* $\bar{M} := -\alpha I + (\hat{E}^D\hat{A} + \alpha\hat{E}^D\hat{E})$ *is a* $-Z$*-matrix,*
2. $\hat{E}^D\hat{B} \geq 0$,
3. C *is nonnegative on the subspace* $\mathcal{X}$ *defined by*

$$\mathcal{X} := \operatorname{im}_+[\hat{E}^D\hat{E}, -(I-\hat{E}^D\hat{E})\hat{A}^D\hat{B}, \ldots, -(I-\hat{E}^D\hat{E})(\hat{E}\hat{A}^D)^{\nu-1}\hat{A}^D\hat{B}], \tag{2.10}$$

where for $W \in \mathbb{R}^{n\times q}$, *the space* $\operatorname{im}_+ W$ *is defined by*

$$\operatorname{im}_+ W := \{w_1 \in \mathbb{R}^n \mid \exists w_2 \in \mathbb{R}^q, w_2 \geq 0 : Ww_2 = w_1\}.$$

In the discrete-time case, assume that $\hat{E}^D\hat{E} \geq 0$. *Then, the discrete-time system* $[\,E\,,\,A\,,\,B\,,\,C\,,\,D\,]$ *is positive if and only if* $\hat{E}^D\hat{A}$, $\hat{E}^D\hat{B}$, $D \geq 0$ *and* C *is nonnegative on* $\mathcal{X}$ *as defined in (2.10).*

To motivate the assumption $\hat{E}^D\hat{E} \geq 0$ in Theorem 2.3, consider the homogeneous system $E\dot{x} = Ax$. If we require that this system has a nonnegative solution for any initial value $x_0 \geq 0$ (instead of any consistent initial value $x_0 = \hat{E}^D\hat{E}v \geq 0$), then $\hat{E}^D\hat{E} \geq 0$ turns out to be a necessary condition [35]. Moreover, from the point of view of applications, it is preferable to prescribe an initial condition that is just nonnegative instead of one that is nonnegative on some special subspace.

3. Balanced truncation for positive standard systems. In this section, we restrict our considerations to so-called *standard systems*, that is, a system (1.1) or (1.2) with $E = I$. In Section 4, we treat the general descriptor case.

Continuous-time case. Consider the standard system $[\,I\,,\,A\,,\,B\,,\,C\,,\,D\,]$ in continuous-time and assume that A is c-stable. The method of balanced truncation is based on Lyapunov equations. It generates c-stable reduced order systems. However, positivity is not preserved in general. Instead, we consider model reduction based on Lyapunov inequalities

$$A\mathcal{P} + \mathcal{P}A^T + BB^T \preceq 0, \quad A^T\mathcal{Q} + \mathcal{Q}A + C^TC \preceq 0, \tag{3.1}$$

with diagonal matrices $\mathcal{P}, \mathcal{Q} \succeq 0$. In the following we show that for positive systems the equations (3.1) are solvable. Moreover, we show the existence of a positive diagonal transformation $T \in \mathbb{R}^{n\times n}$ such that for the transformed system $[\,I\,,\,A_b\,,\,B_b\,,\,C_b\,,\,D_b\,]$ given by

$$A_b = T^{-1}AT, \quad B_b = T^{-1}B, \quad C_b = CT, \quad \text{and} \quad D_b = D, \tag{3.2}$$

the corresponding Lyapunov inequalities

$$A_b\mathcal{P}_b + \mathcal{P}_bA_b^T + B_bB_b^T \preceq 0, \quad A_b^T\mathcal{Q}_b + \mathcal{Q}_bA_b + C_b^TC_b \preceq 0, \tag{3.3}$$

are fulfilled for

$$\mathcal{P}_b = \operatorname{diag}(\Sigma, \Sigma_c, 0_{n_o}, 0_{n_{co}}), \quad \mathcal{Q}_b = \operatorname{diag}(\Sigma, 0_{n_o}, \Sigma_o, 0_{n_{co}}) \tag{3.4}$$

with $0 \prec \Sigma_c \in \mathbb{R}^{n_c\times n_c}$, $0 \prec \Sigma_o \in \mathbb{R}^{n_o\times n_o}$ and

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k) \text{ for some } \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0. \tag{3.5}$$

We will call $[\,I\,,\,A_b\,,\,B_b\,,\,C_b\,,\,D_b\,]$ a *positive balanced realization.*

THEOREM 3.1. *Consider the c-stable continuous-time positive standard system (1.1). Then, there exists a diagonal matrix $T \succ 0$ such that the positive system $[\,I\,,\,A_b\,,\,B_b\,,\,C_b\,,\,D_b\,]$ given by (3.2) is positive balanced, i.e. there exist diagonal matrices $\mathcal{P}_b \succeq 0$, $\mathcal{Q}_b \succeq 0$ as in (3.4), such that the Lyapunov inequalities in (3.3) hold.*

Proof. A $-M$-matrix is diagonally stable, i.e., there exist diagonal positive definite matrices X, Y such that

$$AX + XA^T \prec 0 \quad \text{and} \quad A^TY + YA \prec 0,$$

see, e.g. [3, 7]. In particular, there exist diagonal matrices $\mathcal{P} \succeq 0$, $\mathcal{Q} \succeq 0$ such that (3.1) holds. We define a permutation matrix Π such that

$$\Pi^T\mathcal{P}\Pi = \operatorname{diag}(X_{11}, X_{22}, 0_{n_o}, 0_{n_{co}}), \quad \Pi^T\mathcal{Q}\Pi = \operatorname{diag}(Y_{11}, 0_{n_c}, Y_{33}, 0_{n_{co}})$$

with the additional property that $X_{11} = \operatorname{diag}(x_1, \ldots, x_k)$, $Y_{11} = \operatorname{diag}(y_1, \ldots, y_k)$ satisfy $x_1 y_1 \geq x_2 y_2 \geq \ldots \geq x_k y_k > 0$. Setting $\bar{T} = \operatorname{diag}((X_{11} Y_{11}^{-1})^{\frac{1}{4}}, I_{n_c}, I_{n_o}, I_{n_{co}})$ and $T = \Pi \bar{T}$, we have that $\mathcal{P}_b = T^{-1} \mathcal{P} T^{-T}$, $\mathcal{Q}_b = T^T \mathcal{Q} T$ have the desired form. The transformed system is given by $[\, I\,,\, A_b\,,\, B_b\,,\, C_b\,,\, D_b\,]$ as defined in (3.2). Since A_b is a $-Z$-matrix and $B_b, C_b, D_b \geq 0$, the transformed system is again positive by Theorem 2.2. ∎

The numbers $\sigma_1, \ldots, \sigma_k$ play the role of *Hankel singular values* for standard balanced truncation [20]. Consider a partition

$$A_b = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_b = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C_b = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{3.6}$$

where $A_{11} \in \mathbb{R}^{\ell \times \ell}$ and either $\ell = k$ or $\ell < k$ such that $\sigma_{\ell+1} < \sigma_\ell$. The matrices B and C are partitioned accordingly. By means of balanced realizations, reduced-order models

$$\begin{aligned} \dot{\widetilde{x}}(t) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(t), \\ \widetilde{y}(t) &= \widetilde{C}\widetilde{x}(t) + \widetilde{D}u(t) \end{aligned} \tag{3.7}$$

can now be constructed, where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ are defined by

$$\widetilde{A} = A_{11}, \quad \widetilde{B} = B_1, \quad \widetilde{C} = C_1, \quad \widetilde{D} = D. \tag{3.8}$$

An alternative method for the construction of reduced-order models is

$$\begin{aligned} \widetilde{A} &= A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad & \widetilde{B} &= B_1 - A_{12} A_{22}^{-1} B_2, \\ \widetilde{C} &= C_1 - C_2 A_{22}^{-1} A_{21}, & \widetilde{D} &= D - C_2 A_{22}^{-1} B_2. \end{aligned} \tag{3.9}$$

For the reduced-order models, we have the following result.

THEOREM 3.2. *Let $[\, I\,,\, A\,,\, B\,,\, C\,,\, D\,]$ be a realization of $\boldsymbol{G}(s)$ that is c-stable. Moreover, let A_b, B_b, C_b, D_b be constructed as in (3.2), such that (3.3) holds for $\mathcal{P}_b \succeq 0$, $\mathcal{Q}_b \succeq 0$ as in (3.4). Let $[\, I\,,\, \widetilde{A}\,,\, \widetilde{B}\,,\, \widetilde{C}\,,\, \widetilde{D}\,]$ be the realization that is either constructed via (3.8) or (3.9). Then, the system $[\, I\,,\, \widetilde{A}\,,\, \widetilde{B}\,,\, \widetilde{C}\,,\, \widetilde{D}\,]$ is positive and the transfer function $\widetilde{\boldsymbol{G}}(s) = \widetilde{C}(sI_\ell - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}$ satisfies*

$$\|\boldsymbol{G} - \widetilde{\boldsymbol{G}}\|_{\infty,c} \leq 2 \sum_{i=\ell+1}^{k} \sigma_i. \tag{3.10}$$

Proof. It suffices to show the positivity of the reduced-order systems. For a proof of the error bound in (3.10), we refer to [24].
The reduced-order system defined in (3.8) is again positive, since $\widetilde{B} \geq 0$, $\widetilde{C} \geq 0$, $\widetilde{D} \geq 0$ and $\widetilde{A}$ is a $-M$-matrix as a submatrix of a $-M$-matrix.
The positivity of the reduced-order system defined in (3.9) can be seen as follows: The $-M$-matrix property of $\widetilde{A}$ is preserved, since it is a Schur complement of A [34]. Furthermore, since A_{22} is also a $-M$-matrix, we have $A_{22}^{-1} \leq 0$, and hence, $\widetilde{B}, \widetilde{C}, \widetilde{D} \geq 0$. ∎

The main difference between the reduced-order models (3.8) and (3.9) is that the model (3.8) is exact for $s = \infty$ meaning that $\boldsymbol{G}(\infty) = \widetilde{\boldsymbol{G}}(\infty)$, whereas (3.9) is exact

at $s = 0$. In balanced truncation based on Lyapunov equations, the first method is called *standard balanced truncation*, whereas the second method is called *singular perturbation balanced truncation*.
Note that for the computation of positive reduced-order models, there is no need to compute a balanced realization explicitely. Instead, for solutions

$$\mathcal{P} = \operatorname{diag}(p_1, \ldots, p_n) \succeq 0 \quad \text{and} \quad \mathcal{Q} = \operatorname{diag}(q_1, \ldots, q_n) \succeq 0$$

of (3.1), indices $\{i_1, \ldots, i_n\}$ have to be found such that we have $p_{i_1} q_{i_1} \geq \ldots \geq p_{i_\ell} q_{i_\ell} > p_{i_{\ell+1}} q_{i_{\ell+1}} \geq \ldots \geq p_{i_n} q_{i_n}$. Then, a reduced-order model (3.7) can be obtained in the following way: Let $I_1 = [i_1, \ldots, i_\ell]$, $I_2 = [i_{\ell+1}, \ldots, i_n]$ and

$$\begin{aligned} \bar{A}_{11} &= A(I_1, I_1), & \bar{A}_{12} &= A(I_1, I_2), & \bar{B}_1 &= B(I_1, [1, \ldots, m]), \\ \bar{A}_{21} &= A(I_2, I_1), & \bar{A}_{22} &= A(I_2, I_2), & \bar{B}_2 &= B(I_2, [1, \ldots, m]), \\ \bar{C}_1 &= C([1, \ldots, p], I_1), & \bar{C}_2 &= C([1, \ldots, p], I_2). \end{aligned} \tag{3.11}$$

and either define the reduced-order model by

$$[\, I \,,\, \widetilde{A} \,,\, \widetilde{B} \,,\, \widetilde{C} \,,\, \widetilde{D} \,] = [\, I \,,\, \bar{A}_{11} \,,\, \bar{B}_1 \,,\, \bar{C}_1 \,,\, D \,] \tag{3.12}$$

or by

$$\begin{aligned} &[\, I \,,\, \widetilde{A} \,,\, \widetilde{B} \,,\, \widetilde{C} \,,\, \widetilde{D} \,] \\ =&[\, \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21} \,,\, \bar{B}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2 \,,\, \bar{C}_1 - \bar{C}_2\bar{A}_{22}^{-1}\bar{A}_{21} \,,\, D - \bar{C}_2\bar{A}_{22}^{-1}\bar{B}_2 \,]. \end{aligned} \tag{3.13}$$

These systems are linked to the reduced-order models (3.8) and (3.9), respectively, via a positive diagonal state-space transformation. Therefore, the positivity as well as the error bound (3.10) are still valid.

Discrete-time case. Consider a positive system $[\, I \,,\, A \,,\, B \,,\, C \,,\, D \,]$ in discrete-time and assume that A is d-stable. We consider model reduction based on discrete-time Lyapunov inequalities

$$A\mathcal{P}A^T - \mathcal{P} + BB^T \preceq 0, \quad A^T\mathcal{Q}A - \mathcal{Q} + C^TC \preceq 0, \tag{3.14}$$

which are solved for diagonal $\mathcal{P}, \mathcal{Q} \succeq 0$. As in the continuous-time case, we consider a *positive balanced realization*, i.e. $[\, I \,,\, A_b \,,\, B_b \,,\, C_b \,,\, D_b \,] = [\, I \,,\, T^{-1}AT \,,\, T^{-1}B \,,\, CT \,,\, D \,]$ with diagonal $T \succeq 0$, such that the Lyapunov inequalities

$$A_b\mathcal{P}_bA_b^T - \mathcal{P}_b + B_bB_b^T \preceq 0, \quad A_b^T\mathcal{Q}_bA_b - \mathcal{Q}_b + C_b^TC_b \preceq 0 \tag{3.15}$$

are fulfilled for $\mathcal{P}_b$, $\mathcal{Q}_b$ as in (3.4).
Consider a partition of the balanced system as in (3.6) and assume that either $\ell = k$ or $\ell < k$ with $\sigma_{\ell+1} < \sigma_\ell$. A reduced-order model

$$\begin{aligned} \widetilde{x}(t+1) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(t), \\ \widetilde{y}(t) &= \widetilde{C}\widetilde{x}(t) + \widetilde{D}u(t), \end{aligned} \tag{3.16}$$

can either be constructed via (3.8) or, alternatively, via

$$\begin{aligned} \widetilde{A} &= A_{11} + A_{12}(I_{n-r} - A_{22})^{-1}A_{21}, & \widetilde{B} &= B_1 + A_{12}(I_{n-r} - A_{22})^{-1}B_2, \\ \widetilde{C} &= C_1 + C_2(I_{n-r} - A_{22})^{-1}A_{21}, & \widetilde{D} &= D + C_2(I_{n-r} - A_{22})^{-1}B_2. \end{aligned} \tag{3.17}$$

In the case where discrete-time Lyapunov equations are considered instead of inequalities, the reduced-order model constructed via (3.8) is referred to as standard balanced truncation and (3.17) as singular perturbation [20,24]. For the reduced-order models, we have an analogous result as in Theorem 3.2.

THEOREM 3.3. *Consider the d-stable discrete-time positive standard system (1.1). Then, there exists a diagonal $T \succ 0$ such that the positive system $[\, I\,,\, A_b\,,\, B_b\,,\, C_b\,,\, D_b\,]$ given by (3.2) is positive balanced, i.e. there exist diagonal $\mathcal{P}_b \succeq 0$, $\mathcal{Q}_b \succeq 0$ as in (3.4), such that the Lyapunov inequalities in (3.15) hold.*
Let $[\, I\,,\, \widetilde{A}\,,\, \widetilde{B}\,,\, \widetilde{C}\,,\, \widetilde{D}\,]$ be the realization that is either constructed by (3.8) or (3.17). Then, the system $[\, I\,,\, \widetilde{A}\,,\, \widetilde{B}\,,\, \widetilde{C}\,,\, \widetilde{D}\,]$ is positive and the transfer function $\widetilde{\boldsymbol{G}}(z) = \widetilde{C}(zI_\ell - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}$ satisfies

$$\|\boldsymbol{G} - \widetilde{\boldsymbol{G}}\|_{\infty,d} \leq 2 \sum_{i=\ell+1}^{k} \sigma_i. \tag{3.18}$$

Proof. The existence of a diagonal matrix $T \succ 0$ such that $[\, I\,,\, A_b\,,\, B_b\,,\, C_b\,,\, D_b\,]$ given by (3.2) is positive balanced can be obtained analogously to Theorem 3.1.
Moreover, a proof of the error bound (3.10) can be found in [4]. It remains to show that the reduced-order system is positive.
The reduced-order system defined via (3.8) is positive, since $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ are submatrices of positive matrices. The positivity of the reduced-order model constructed via (3.17) can be observed as follows. By the stability assumption, we have that $\rho(A_{22}) \leq \rho(A) < 1$. Hence, $I_{n-\ell} - A_{22}$ is an M-matrix and $(I_{n-\ell} - A_{22})^{-1} \geq 0$. Therefore, we obtain $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \geq 0$. ∎

As in the continuous-time case, the actual transformation to a balanced system $[\, I\,,\, A_b\,,\, B_b\,,\, C_b\,,\, D_b\,]$ does not have to be performed, but only the submatrices in (3.11) have to be considered. Reduced-order systems then can be constructed according to (3.12) or, alternatively, via

$$\begin{aligned}[\, \bar{A}_{11} + \bar{A}_{12}(I_{n-r} - \bar{A}_{22})^{-1}\bar{A}_{21}\,,\; \bar{B}_1 + \bar{A}_{12}(I_{n-r} - \bar{A}_{22})^{-1}\bar{B}_2\,,\\ \bar{C}_1 + \bar{C}_2(I_{n-r} - \bar{A}_{22})^{-1}\bar{A}_{21}\,,\; D + \bar{C}_2(I_{n-r} - \bar{A}_{22})^{-1}\bar{B}_2\,]\end{aligned}$$

Let us finally give a remark on the Lyapunov inequalities (3.1) and (3.14). It is clear that their solutions are not unique and one should look for solutions $\mathcal{P} = \mathrm{diag}(p_1, \ldots, p_n)$, $\mathcal{Q} = \mathrm{diag}(q_1, \ldots, q_n)$ such that $\sqrt{\mathcal{P}\mathcal{Q}}$ has a large number of small diagonal elements. This yields components of the state which are candidates to truncate. A good heuristic for this is the minimization of the trace of $\mathcal{P}$ and $\mathcal{Q}$ [9]. For getting even sharper bounds, the Lyapunov inequalities can be solved once more while now minimizing the sum of those diagonal elements of $\mathcal{P}$ and $\mathcal{Q}$ corresponding to the candidates for truncation.

4. Model reduction for positive descriptor systems.

Continuous-time case. In this section, for positive continuous-time descriptor systems, we first prove the existence of a positive balanced realization. Based on this, we define a reduced-order system and show that it is positive and that it yields the usual H_∞ error bound in Lemma 3.2. The construction of reduced-order models is

based on the continuous-time Lyapunov inequalities

$$(\hat{E}^D\hat{A})X + X(\hat{E}^D\hat{A})^T + \hat{E}^D\hat{B}\hat{B}^T(\hat{E}^D)^T \preceq 0, \tag{4.1a}$$

$$(\hat{E}^D\hat{A})^TY + Y(\hat{E}^D\hat{A}) + P_r^TC^TCP_r \preceq 0, \tag{4.1b}$$

with diagonal matrices $X \succeq 0$, $Y \succeq 0$.
Note that if the above relations become equalities and, additionally, the solutions fulfill $X = P_rXP_r^T$ and $Y = P_l^TYP_l$, where P_l is a spectral projector onto the left deflating subspace of (E, A) corresponding to the finite eigenvalues, then we obtain the generalized Lyapunov equations in [27]. However, for diagonal solutions of the corresponding Lyapunov inequalities this will not necessarily be the case.

THEOREM 4.1. *Consider the c-stable continuous-time system* $[\,E\,,\ A\,,\,B\,,\,C\,,\,D\,]$ *that is positive in the sense of Theorem 2.3. Then, there exists a diagonal matrix* $T \succ 0$ *such that the system* $[\,E_b\,,\,A_b\,,\,B_b\,,\,C_b\,,\,D_b\,]$ *with*

$$E_b = ET, \quad A_b = AT, \quad B_b = B, \quad C_b = CT \quad \textit{and} \quad D_b = D \tag{4.2}$$

is positive and there exist diagonal matrices $\mathcal{P}_b \succeq 0$, $\mathcal{Q}_b \succeq 0$ *partitioned as in (3.4) such that the following Lyapunov inequalities hold*

$$(\hat{E}_b^D\hat{A}_b)\mathcal{P}_b + \mathcal{P}_b(\hat{E}_b^D\hat{A}_b)^T + \hat{E}_b^D\hat{B}_b\hat{B}_b^T(\hat{E}_b^D)^T \preceq 0, \tag{4.3a}$$

$$(\hat{E}^D\hat{A}_b)^T\mathcal{Q}_b + \mathcal{Q}_b(\hat{E}_b^D\hat{A}_b) + P_{r,b}^TC_b^TC_bP_{r,b} \preceq 0, \tag{4.3b}$$

where $\hat{E}_b, \hat{A}_b, \hat{B}_b$ *are obtained from the matrices* E_b, A_b, B_b *as in (2.4) and* $P_{r,b} = T^{-1}P_rT$.

Proof. From [36, Theorem 4.8], we have that for a positive c-stable system, if $P_r \geq 0$, then there exist diagonal matrices $\tilde{X}, \tilde{Y} \succ 0$, such that

$$(\hat{E}^D\hat{A})\tilde{X} + \tilde{X}(\hat{E}^D\hat{A})^T \preceq 0, \quad (\hat{E}^D\hat{A})^T\tilde{Y} + \tilde{Y}(\hat{E}^D\hat{A}) \preceq 0,$$

and the above inequalities are strict on $\operatorname{im} P_r$, i.e., for any $0 \neq v \in \operatorname{im} P_r$, we have

$$v^T(\hat{E}^D\hat{A}\tilde{X} + \tilde{X}\hat{A}^T(\hat{E}^D)^T)v < 0, \quad v^T(\hat{A}^T(\hat{E}^D)^T\tilde{Y} + \tilde{Y}\hat{E}^D\hat{A})v < 0.$$

Since, by assumption, we have $\hat{E}^D\hat{B}\hat{B}^T(\hat{E}^D)^T \geq 0$ and $P_r^TC^TCP_r \geq 0$ and since both terms are projected onto $\operatorname{im} P_r$, we obtain the existence of diagonal matrices $X \succeq 0$, $Y \succeq 0$ such that the Lyapunov inequalities (4.1) hold. By using the same argument as in the proof of Theorem 3.1, there exist diagonal matrices $\mathcal{P} \succeq 0$, $\mathcal{Q} \succeq 0$ and $T \succ 0$ such that by transforming the system matrices as in (4.2), we obtain (4.3), where $\hat{E}_b, \hat{A}_b, \hat{B}_b$ are constructed as in (2.4) from the matrices E_b, A_b, B_b and $P_{r,b} = T^{-1}P_rT$ is the corresponding spectral projector as in (2.3). Since T is diagonal with positive diagonal entries, the transformed system is again positive. ∎

From now on, we consider the system $[\,E_b\,,\,A_b\,,\,B_b\,,\,C_b\,,\,D_b\,]$ from Theorem 4.1. Since $R = E_bP_{r,b} + A_b(I - P_{r,b}$ is invertible, the state equation of the system can be scaled by the matrix R^{-1}. Then we obtain an equivalent system $[\,E\,,\,A\,,\,B\,,\,C\,,\,D\,]$ with system matrices $E := R^{-1}E_b$, $A := R^{-1}A_b$, $B := R^{-1}B_b$. The multiplication with the corresponding spectral projector P_r and its complementary projector $(I-P_r)$, respectively, leads to an equivalent system of two equations

$$\begin{aligned} P_r\dot{x}(t) &= \hat{E}^D\hat{A}x(t) + \hat{E}^D\hat{B}u(t), \\ (I - P_r)\hat{E}\hat{A}^D\dot{x}(t) &= (I - P_r)x(t) + (I - P_r)\hat{A}^D\hat{B}u(t). \end{aligned} \tag{4.4}$$

Consider a partitioning as in (3.6) but for the matrices $\hat{E}^D\hat{A}, \hat{E}^D\hat{B}, CP_r$:

$$\hat{E}^D\hat{A} = \begin{bmatrix} [\hat{E}^D\hat{A}]_{11} & [\hat{E}^D\hat{A}]_{12} \\ [\hat{E}^D\hat{A}]_{21} & [\hat{E}^D\hat{A}]_{22} \end{bmatrix}, \ \hat{E}^D\hat{B} = \begin{bmatrix} [\hat{E}^D\hat{B}]_1 \\ [\hat{E}^D\hat{B}]_2 \end{bmatrix}, \ CP_r = \begin{bmatrix} [CP_r]_1 & [CP_r]_2 \end{bmatrix}, \tag{4.5}$$

where $[\hat{E}^D\hat{A}]_{11} \in \mathbb{R}^{\ell\times\ell}$ and ℓ is chosen as in the standard case in (3.6). The matrices $\hat{E}^D\hat{B}$ and CP_r are partitioned accordingly.

Our aim is to construct a reduction method that allows to obtain an H_∞ error bound as in the standard case. This is possible, for instance, if the polynomial part $\boldsymbol{P}(s)$ of the transfer function $\boldsymbol{G}(s)$ remains unchanged, whereas the strictly proper part $\boldsymbol{G}_{sp}(s)$ is reduced as in the standard case [31]. In this case, the polynomial parts of the original and the reduced transfer functions cancel out in the H_∞ norm and we obtain the usual H_∞ error bound. Note that, since (E, A) was assumed to be c-stable, we have that $\hat{E}^D\hat{A}$ has only stable eigenvalues except for possibly several eigenvalues zero that correspond to the eigenvalue ∞ of (E, A) [36]. To obtain an H_∞ error bound, these must not be reduced and, hence, we have to make sure that the block $[\hat{E}^D\hat{A}]_{22}$ is regular.

We partition the spectral projector P_r and the matrices $\hat{E}\hat{A}^D, \hat{A}^D\hat{B}$ conformably with the partitioning of the matrix $\hat{E}^D\hat{A}$,

$$P_r = \begin{bmatrix} [P_r]_{11} & [P_r]_{12} \\ [P_r]_{21} & [P_r]_{22} \end{bmatrix}, \ \hat{E}\hat{A}^D = \begin{bmatrix} [\hat{E}\hat{A}^D]_{11} & [\hat{E}\hat{A}^D]_{12} \\ [\hat{E}\hat{A}^D]_{21} & [\hat{E}\hat{A}^D]_{22} \end{bmatrix}, \ \hat{A}^D\hat{B} = \begin{bmatrix} [\hat{A}^D\hat{B}]_1 \\ [\hat{A}^D\hat{B}]_2 \end{bmatrix}. \tag{4.6}$$

The following Lemma 4.2, in particular, states that $[P_r]_{22}$ is regular whenever $[\hat{E}^D\hat{A}]_{22}$ is regular.

LEMMA 4.2. *Let the matrix $\hat{E}^D\hat{A}$ and the nonnegative projector P_r be partitioned as in (4.5) and (4.6), respectively, such that $[\hat{E}^D\hat{A}]_{22}$ is regular. Then, $[P_r]_{22}$ is a (regular) diagonal matrix with positive diagonal entries.*

Proof. We have that $P_r \geq 0$ is a projector. Hence, there exists a permutation matrix $Q \in \Pi_n$ such that QP_rQ^T is in the following canonical form [18, 19]

$$QP_rQ^T = \begin{bmatrix} \pi_{11} & 0 & \dots & 0 & \pi_{1k} & 0 \\ 0 & \pi_{22} & \ddots & \vdots & \pi_{2k} & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & \pi_{k,k} & \pi_{k,k+1} & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 \\ \pi_{k+2,1} & \dots & \dots & \dots & \pi_{k+2,k+1} & 0 \end{bmatrix}. \tag{4.7}$$

We use the permutation matrix Q to obtain a corresponding permutation of $\hat{E}^D\hat{A}$ and partition it accordingly

$$Q\hat{E}^D\hat{A}Q^T = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1,k+2} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{k+2,1} & \dots & & A_{k+2,k+2} \end{bmatrix}. \tag{4.8}$$

Since $P_r\hat{E}^D\hat{A} = \hat{E}^D\hat{A}$ and $\pi_{k+1,k+1} = 0$, we have that $\pi_{k+1,k+1}A_{k+1,i} = A_{k+1,i} = 0$ for $i = 1, \dots, k+2$. Furthermore, since $\hat{E}^D\hat{A}P_r = \hat{E}^D\hat{A}$ and $\pi_{k+2,k+2} = 0$, we

have that $A_{i,k+2}\pi_{k+2,k+2} = A_{i,k+2} = 0$ for $i = 1, \ldots, k+2$. This implies that whenever we choose a regular part of $\hat{E}^D\hat{A}$, then the corresponding part of P_r will have a positive diagonal by construction. Furthermore, since $\hat{E}^D\hat{A}P_r = \hat{E}^D\hat{A}$ we have that $A_{ii}\pi_{ii} = A_{ii}$ for $i = 1, \ldots, k-1$ and hence, $\operatorname{rank}(A_{ii}) \leq \operatorname{rank}(\pi_{ii}) = 1$. Since $\operatorname{rank}(\hat{E}^D\hat{A}) = \operatorname{rank}(P_r)$ we conclude that $\operatorname{rank}(A_{ii}) = \operatorname{rank}(\pi_{ii}) = 1$. Hence, for a regular part of $\hat{E}^D\hat{A}$, we can pick at most one row/column from each block row/column in (4.8). By construction, the corresponding part of P_r will be also regular. Moreover, this part will be diagonal with positive diagonal entries. □

The following Lemma 4.3 states the special structure of the matrices in (4.4) induced by the characterization of positivity in Theorem 2.3 [35].

LEMMA 4.3. *Consider the system matrices in (4.4) and let ν be the index of nilpotency of $(I - P_r)(\hat{E}\hat{A}^D)$. Then, assuming that P_r is in canonical form (4.7), for $i = 0, \ldots, \nu - 1$, we have*

$$(I - P_r)(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ *_1 \\ *_2 \end{bmatrix} \leq 0, \tag{4.9}$$

where $*_1, *_2$ denote some unspecified entries.

The special structures of $(I - P_r)\hat{A}^D\hat{B}$ and $(I - P_r)(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{B}$ given in (4.9) lead to the following facts. Let

$$(I - P_r)\hat{A}^D\hat{B} = \begin{bmatrix} [(I - P_r)\hat{A}^D\hat{B}]_1 \\ [(I - P_r)\hat{A}^D\hat{B}]_2 \end{bmatrix}$$

be partitioned according to $\hat{E}^D\hat{A}$ in (4.5). Since the part $[(I - P_r)\hat{A}^D\hat{B}]_2$ corresponds to the regular part P_{22}, by considering the canonical form (4.7) of P_r and the corresponding form of $(I - P_r)\hat{A}^D\hat{B}$ in (4.9), the term $[(I - P_r)\hat{A}^D\hat{B}]_2$ must be zero. By (4.9), we have that

$$(\hat{E}\hat{A}^D)^i(I - P_r)\hat{A}^D\hat{B} = \begin{bmatrix} [\hat{E}\hat{A}^D]_{11} & [\hat{E}\hat{A}^D]_{12} \\ [\hat{E}\hat{A}^D]_{21} & [\hat{E}\hat{A}^D]_{22} \end{bmatrix} \begin{bmatrix} [(I - P_r)\hat{A}^D\hat{B}]_1 \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}, \tag{4.10}$$

and we conclude that $[\hat{E}\hat{A}^D]_{21}[(I - P_r)\hat{A}^D\hat{B}]_1 = 0$.

Since $(I - P_r)\hat{E}\hat{A}^D$ is nilpotent, the second equation of (4.4) has the solution

$$(I - P_r)x(t) = -\sum_{i=1}^{\nu-1}(\hat{E}\hat{A}^D)^i(I - P_r)\hat{A}^D\hat{B}u^{(i)}(t),$$

and by (4.10) is equivalent to

$$\begin{bmatrix} (I - P_{11})[\hat{E}\hat{A}^D]_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = (I - P_r)x(t) + \begin{bmatrix} [(I - P_r)\hat{A}^D\hat{B}]_1 \\ 0 \end{bmatrix} u(t). \tag{4.11}$$

System (4.4) is therefore equivalent to the following decoupled system

$$\dot{x}_f(t) = A_f x_f(t) + B_f u(t), \quad E_\infty \dot{x}_\infty(t) = x_\infty(t) + B_\infty u(t), \tag{4.12}$$

where

$$\begin{aligned} x_f(t) &:= P_r x(t), & A_f &:= \hat{E}^D \hat{A}, & B_f &:= \hat{E}^D \hat{B}, \\ x_\infty(t) &:= (I - P_r)x(t), & E_\infty &:= \begin{bmatrix} (I - P_{11})[\hat{E}\hat{A}^D]_{11} & 0 \\ 0 & 0 \end{bmatrix}, & B_\infty &:= \begin{bmatrix} [(I - P_r)\hat{A}^D\hat{B}]_1 \\ 0 \end{bmatrix}. \end{aligned} \tag{4.13}$$

Furthermore, we set

$$C_f := CP_r, \quad C_\infty := C(I - P_r). \tag{4.14}$$

Using (2.8) and (2.9), we have $\boldsymbol{G}(s) = \boldsymbol{G}_{sp}(s) + \boldsymbol{P}(s)$ with

$$\boldsymbol{G}_{sp}(s) = C_f(sI - A_f)^{-1}B_f, \qquad \boldsymbol{P}(s) = C_\infty(sE_\infty - I)^{-1}B_\infty + D.$$

The first equation of the system in (4.12) is a standard system on the subspace $\operatorname{im} P_r$. Requiring that $[\hat{E}^D\hat{A}]_{22} = [A_f]_{22}$ is regular, we can apply the reduction scheme in (3.9) to the system $[\, I,\, A_f,\, B_f,\, C_f,\, 0\,]$. We obtain a reduced-order system $[\, I\,,\, \widetilde{A}_f\,,\, \widetilde{B}_f\,,\, \widetilde{C}_f\,,\, \widetilde{D}_f\,]$. Since A_f is c-stable on $\operatorname{im} P_r$ and $[\, I\,,\, A_f\,,\, B_f\,,\, C_f\,,\, 0\,]$ is balanced on $\operatorname{im} P_r$, we have that $\tilde{\boldsymbol{G}}_{sp}(s) = \tilde{C}_f(sI - \tilde{A}_f)^{-1}\tilde{B}_f + \tilde{D}_f$ satisfies the error bound in Lemma 3.2.

The second equation of (4.12) corresponds to the system $[\, E_\infty\,,\, I\,,\, B_\infty\,,\, C_\infty\,,\, D\,]$. Assuming the same partitioning as for $[\, I\,,\, A_f\,,\, B_f\,,\, C_f\,,\, 0\,]$, we can apply the standard balanced truncation reduction scheme in (3.8). We obtain the reduced-order system $[\, \widetilde{E}_\infty\,,\, I\,,\, \widetilde{B}_\infty\,,\, \widetilde{C}_\infty\,,\, D\,]$.
That $\widetilde{\boldsymbol{P}}(s) = \widetilde{C}_\infty(s\widetilde{E}_\infty - I)^{-1}\widetilde{B}_\infty + D$ satisfies $\widetilde{\boldsymbol{P}}(s) = \boldsymbol{P}(s)$ can be verified as follows. By (4.9) we know that if P_r is in canonical form and the matrices $\hat{E}\hat{A}^D, \hat{A}^D\hat{B}$ are permuted and partitioned accordingly, we have that

$$(\hat{E}\hat{A}^D)^k(I - P_r)\hat{A}^D\hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ * \end{bmatrix}.$$

Therefore, if we consider a partitioning as in (4.11) and, furthermore, use (2.9), we obtain that

$$\begin{aligned} \boldsymbol{P}(s) &= -C(I - P_r)\sum_{k=0}^{\nu-1}(\hat{E}\hat{A}^D)^k(I - P_r)\hat{A}^D\hat{B}s^k + D \\ &= -\begin{bmatrix} C_{\infty_1} & C_{\infty_2} \end{bmatrix}\begin{bmatrix} [E_\infty B_\infty]_1 \\ 0 \end{bmatrix} = -C_{\infty_1}[E_\infty B_\infty]_1 = \widetilde{\boldsymbol{P}}(s). \end{aligned}$$

Note that, in particular, this proves that Assertion (i) of Theorem 2.3 still holds for the reduced-order system.

We obtain a corresponding descriptor system $[\, \widetilde{E}\,,\, \widetilde{A}\,,\, \widetilde{B}\,,\, \widetilde{C}\,,\, \widetilde{D}\,]$ of $\operatorname{ind}(\widetilde{E}, \widetilde{A}) = \operatorname{ind}(E, A) = \nu$, which is equal to the degree of the polynomial $\boldsymbol{P}(s)$, by setting the reduced-order spectral projector $\widetilde{P}_r$ to

$$\widetilde{P}_r := [P_r]_{11} - [P_r]_{12}[P_r]_{22}^{-1}[P_r]_{21}, \tag{4.15}$$

and

$$\begin{aligned} \widetilde{E} &:= \widetilde{P}_r + \widetilde{E}_\infty, \quad \widetilde{A} := \widetilde{A}_f + (I - \widetilde{P}_r), \quad \widetilde{B} := \widetilde{B}_f + \widetilde{B}_\infty, \\ \widetilde{C} &:= \widetilde{C}_f + \widetilde{C}_\infty, \quad \widetilde{D} := \widetilde{D}_f + D. \end{aligned} \tag{4.16}$$

For the transfer function $\widetilde{\boldsymbol{G}}(s)$ of the reduced-order system we obtain

$$\widetilde{\boldsymbol{G}}(s) = \widetilde{\boldsymbol{G}}_{sp}(s) + \widetilde{\boldsymbol{P}}(s) = \widetilde{C}(s\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D},$$

and since $\boldsymbol{P}(s) = \widetilde{\boldsymbol{P}}(s)$ we have $\|\boldsymbol{G} - \widetilde{\boldsymbol{G}}\|_{\infty,c} = \|\boldsymbol{G}_{sp} - \widetilde{\boldsymbol{G}}_{sp}\|_{\infty,c}$. Therefore, the error bound (3.10) holds true.

We still have to show that the thus obtained reduced-order system in (4.16) is again positive in the sense of Theorem 2.3.

The matrix $\widetilde{P}_r$ is again a nonnegative projector [19]. Furthermore, the projector $\widetilde{P}_r$ has the following properties that are essential for the positivity of the reduced-order system.

LEMMA 4.4. *Let $\widetilde{P}_r$ be defined as in (4.15) and the reduced-order system matrices as in (4.16). Then, the following relations hold:*

1. $\widetilde{P}_r\widetilde{A}_f = \widetilde{A}_f\widetilde{P}_r = \widetilde{A}_f$;
2. $\widetilde{P}_r\widetilde{B}_f = \widetilde{B}_f$;
3. $\widetilde{C}_f\widetilde{P}_r = \widetilde{C}_f$.

Proof. For Relation 1. we have to take into account that $A_f = \hat{E}^D\hat{A}$ and use the relations for the partitioned block matrices that arise from the property $P_r\hat{E}^D\hat{A} = \hat{E}^D\hat{A}P_r = \hat{E}^D\hat{A}$, i.e.,

$$\begin{aligned} \begin{bmatrix} [A_f]_{11} & [A_f]_{12} \\ [A_f]_{21} & [A_f]_{22} \end{bmatrix} &= \begin{bmatrix} [P_r]_{11}[A_f]_{11} + [P_r]_{12}[A_f]_{21} & [P_r]_{11}[A_f]_{12} + [P_r]_{12}[A_f]_{22} \\ [P_r]_{21}[A_f]_{11} + [P_r]_{22}[A_f]_{21} & [P_r]_{21}[A_f]_{12} + [P_r]_{22}[A_f]_{22} \end{bmatrix} \\ &= \begin{bmatrix} [A_f]_{11}[P_r]_{11} + [A_f]_{12}[P_r]_{21} & [A_f]_{11}[P_r]_{12} + [A_f]_{12}[P_r]_{22} \\ [A_f]_{21}[P_r]_{11} + [A_f]_{22}[P_r]_{21} & [A_f]_{21}[P_r]_{12} + [A_f]_{22}[P_r]_{22} \end{bmatrix}. \end{aligned} \tag{4.17}$$

Exemplarily, we prove the relation $\widetilde{P}_r\widetilde{A}_f = \widetilde{A}_f$, since the other relations follow analogously, see [35, 36] for details. We have

$$\begin{aligned} \widetilde{P}_r\widetilde{A}_f =& ([P_r]_{11} - [P_r]_{12}[P_r]_{22}^{-1}[P_r]_{21})([A_f]_{11} - [A_f]_{12}[A_f]_{22}^{-1}[A_f]_{21}) \\ =& [P_r]_{11}[A_f]_{11} - [P_r]_{12}[P_r]_{22}^{-1}[P_r]_{21}[A_f]_{11} - [P_r]_{11}[A_f]_{12}[A_f]_{22}^{-1}[A_f]_{21} \\ &+ [P_r]_{12}[P_r]_{22}^{-1}[P_r]_{21}[A_f]_{12}[A_f]_{22}^{-1}[A_f]_{21}, \end{aligned}$$

where plugging in the relations from (4.17), we obtain

$$\begin{aligned} \widetilde{P}_r\widetilde{A}_f =& [P_r]_{11}[A_f]_{11} - [P_r]_{12}[P_r]_{22}^{-1}[A_f]_{21} + [P_r]_{12}[A_f]_{21} - [A_f]_{12}[A_f]_{22}^{-1}[A_f]_{21} \\ &+ [P_r]_{12}[A_f]_{21} + [P_r]_{12}[P_r]_{22}^{-1}[A_f]_{21} - [P_r]_{12}[A_f]_{21} \\ =& \widetilde{A}_f, \end{aligned}$$

where for the last equality we have used that $[P_r]_{11}[A_f]_{11} + [P_r]_{12}[A_f]_{21} = [A_f]_{11}$. ∎

Due to $E_\infty = (I - P_r)\hat{E}\hat{A}^D$ and $P_rE_\infty = 0$ we have $\widetilde{P}_r\widetilde{E}_\infty = \widetilde{E}_\infty\widetilde{P}_r = 0$. Therefore, Lemma 4.4 implies that the matrices $\widetilde{E}$ and $\widetilde{A}$ commute. Using the properties

of $\widetilde{P}_r$ we therefore have that $\widetilde{E}\dot{\widetilde{x}}(t) = \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(t)$ is equivalent to the decoupled system

$$\widetilde{P}_r\dot{\widetilde{x}}(t) = \widetilde{A}_f\widetilde{x}(t) + \widetilde{B}_f u(t), \quad \widetilde{E}_\infty\dot{\widetilde{x}}(t) = (I - \widetilde{P}_r)\widetilde{x}(t) + \widetilde{B}_\infty u(t).$$

We have already shown that Assumptions (i)-(ii) of Theorem 2.3 hold for the reduced-order system. It remains to show Relations 1.-3. in Theorem 2.3.

To this end, note that by Lemma 4.2, $[P_r]_{22}$ is a diagonal matrix with a strictly positive diagonal. Hence, from the relation

$$A_f + \alpha P_r = \hat{E}^D\hat{A} + \alpha P_r \geq 0$$

we conclude that $[A_f]_{22}$ must be a $-Z$-matrix. Since $A_f = \hat{E}^D\hat{A}$ has only stable eigenvalues except for the eigenvalue 0 that corresponds to the eigenvalue ∞ of (E, A) and since $[A_f]_{22}$ is regular, it must be c-stable and therefore, a $-M$-matrix and we have $[A_f]_{22}^{-1} \leq 0$. By using Lemma 4.4 and the relations $\widetilde{P}_r[P_r]_{11} = [P_r]_{11}\widetilde{P}_r = \widetilde{P}_r$ and $\widetilde{P}_r[P_r]_{12} = [P_r]_{21}\widetilde{P}_r = 0$ that can be verified by direct calculation, we obtain

$$\begin{aligned}\widetilde{A}_f + \alpha\widetilde{P}_r =& \widetilde{P}_r(\widetilde{A}_f + \alpha\widetilde{P}_r)\widetilde{P}_r \\ =& \widetilde{P}_r([A_f]_{11} + \alpha[P_r]_{11} - [A_f]_{12}[A_f]_{22}^{-1}[A_f]_{21} - \alpha[P_r]_{12}[P_r]_{22}^{-1}[P_r]_{21})\widetilde{P}_r \\ =& \widetilde{P}_r([A_f]_{11} + \alpha[P_r]_{11} - ([A_f]_{12} + \alpha[P_r]_{12})[A_f]_{22}^{-1}([A_f]_{21} + \alpha[P_r]_{21}))\widetilde{P}_r \geq 0,\end{aligned}$$

since $A_f + \alpha P_r \geq 0$.

Next, we show that $\widetilde{B}_f \geq 0$. By Lemma 4.4 we know that $\widetilde{P}_r\widetilde{B}_f = \widetilde{B}_f$. Then, by using $\widetilde{P}_r[P_r]_{12} = 0$, we obtain

$$\widetilde{B}_f = \widetilde{P}_r([B_f]_1 - [A_f]_{12}[A_f]_{22}^{-1}[B_f]_2) = \widetilde{P}_r([B_f]_1 - ([A_f]_{12} + \alpha[P_r]_{12})[A_f]_{22}^{-1}[B_f]_2) \geq 0.$$

Similarly, by Lemma 4.4 and since $[P_r]_{21}\widetilde{P}_r = 0$ we have that

$$\begin{aligned}\widetilde{C}_f =& ([C_f]_1 - X[C_f]_2[A_f]_{22}^{-1}[A_f]_{21})\widetilde{P}_r = ([C_f]_1 - X[C_f]_2[A_f]_{22}^{-1}([A_f]_{21} + \alpha[P_r]_{21}))\widetilde{P}_r \\ \geq& 0.\end{aligned}$$

Finally, $\widetilde{D} = \widetilde{D}_f + D \geq 0$ holds, since $D \geq 0$ and $\widetilde{D}_f = -[C_f]_2[A_f]_{22}^{-1}[B_f]_2 \geq 0$.

We have shown that the reduced-order system as defined in 4.16 is again positive. The strictly proper part $\boldsymbol{G}_{sp}(s)$ of the transfer function $\boldsymbol{G}(s)$ is reduced as in the standard case, whereas the polynomial part $\boldsymbol{P}(s)$ remains unchanged, which leads to the usual H_∞ error bound as for standard balanced truncation in Lemma 3.2.

Discrete-time case. For d-stable discrete-time descriptor systems which satisfy the assumptions of Theorem 2.3, we consider positivity-preserving model reduction based on Lyapunov inequalities of the form

$$\begin{aligned}(\hat{E}^D\hat{A})\mathcal{P}(\hat{E}^D\hat{A})^T - \mathcal{P} + \hat{E}^D\hat{B}\hat{B}^T(\hat{E}^D)^T \preceq 0, \\ (\hat{E}^D\hat{A})^T\mathcal{Q}(\hat{E}^D\hat{A}) - \mathcal{Q} + P_r^TC^TCP_r \preceq 0.\end{aligned} \tag{4.18}$$

with diagonal matrices $\mathcal{P} \succeq 0$, $\mathcal{Q} \succeq 0$. If $P_r \geq 0$ then, by Theorem 2.3, we have $\hat{E}^D\hat{A} \geq 0$. Since (E, A) is d-stable, we also have that $\rho(\hat{E}^D\hat{A}) < 1$. Hence, as in the

standard discrete-time case in Theorem 3.3, there exists a positive realization with $\mathcal{P}_b$ and $\mathcal{Q}_b$ as in (3.4)

From now on, we assume that $\mathcal{P}$ and $\mathcal{Q}$ of the form (3.4) solve the Lyapunov inequalities (4.18). Scaling the system by $R^{-1} = (EP_r + A(I - P_r))^{-1}$ as in the continuous-time case, and further multiplying the state equation with P_r and $(I-P_r)$, respectively leads to the equivalent system

$$\begin{aligned} P_r x(t+1) &= \hat{E}^D \hat{A} x(t) + \hat{E}^D \hat{B} u(t), \\ (I - P_r)\hat{E}\hat{A}^D x(t+1) &= (I - P_r)x(t) + (I - P_r)\hat{A}^D \hat{B} u(t). \end{aligned} \tag{4.19}$$

By using the conditions for $(I-P_r)\hat{A}^D\hat{B}$ and $(I-P_r)(\hat{E}\hat{A}^D)\hat{A}^D\hat{B}$ derived in Section 3, for the continuous-time case, we deduce that the system in (4.19) is equivalent to

$$\begin{aligned} x_f(t+1) &= A_f x_f(t) + B_f u(t), \\ E_\infty x_\infty(t+1) &= x_\infty(t) + B_\infty u(t), \end{aligned} \tag{4.20}$$

where the systems $[\, I\,,\, A_f\,,\, B_f\,,\, C_f\,,\, 0\,]$ for $\boldsymbol{G}_{sp}(s)$ and $[\, E_\infty\,,\, I\,,\, B_\infty,\, C_\infty,\, D\,]$ for $\boldsymbol{P}(s)$ are given by the matrices in (4.13) and (4.14).

As in the continuous-time case, we reduce the strictly proper part $\boldsymbol{G}_{sp}(s)$ of $\boldsymbol{G}(s)$ using standard singular perturbation balanced truncation, whereas the polynomial part $\boldsymbol{P}(s)$ remains unchanged. We show that we obtain a reduced-order positive descriptor system that approximates the original system with the usual H_∞ error bound in Lemma 3.3.

Consider a partitioning as in (4.5) and (4.6). As in the continuous-time case we choose the block $[A_f]_{22}$ regular and therefore, $\rho([A_f]_{22}) < 1$ and we have that $(I - [A_f]_{22})$ is an M-matrix with $(I - [A_f]_{22}) \geq 0$.

The first equation of the system in (4.20) is a standard system on the subspace $\operatorname{im} P_r$. If the block $[P_r]_{22}$ contains ones on the diagonal, we first apply the balanced truncation scheme in (3.8) to the corresponding part of the system. The truncated projector is again a nonnegative projector and also the system is again positive. Therefore, without loss of generality, we may assume that the diagonal entries of $[P_r]_{22}$ are strictly less than 1.

We apply the reduction scheme in (3.17) and obtain a reduced-order system

$$[\, I\,,\, \widetilde{A}_f\,,\, \widetilde{B}_f\,,\, \widetilde{C}_f\,,\, \widetilde{D}_f\,] \quad \text{with} \quad \widetilde{\boldsymbol{G}}_{sp}(s) = \widetilde{C}_f(sI - \widetilde{A}_f)^{-1}\widetilde{B}_f + \widetilde{D}_f.$$

Since A_f is d-stable on $\operatorname{im} P_r$ and $[\, I\,,\, A_f\,,\, B_f\,,\, C_f\,,\, 0\,]$ is balanced on $\operatorname{im} P_r$ we have that $\widetilde{\boldsymbol{G}}_{sp}(s)$ yields the error bound in Lemma 3.3.

As in the continuous-time case, we partition the system $[\, E_\infty\,,\, B_\infty\,,\, C_\infty\,,\, D\,]$ according to $[\, I\,,\, A_f\,,\, B_f\,,\, C_f\,,\, 0\,]$ and reduce it by standard balanced truncation in (3.8). We obtain the reduced-order system

$$[\, \widetilde{E}_\infty\,,\, \widetilde{B}_\infty\,,\, \widetilde{C}_\infty\,,\, D\,] \quad \text{with} \quad \widetilde{\boldsymbol{P}}(s) = \widetilde{C}_\infty(s\widetilde{E}_\infty - I)^{-1}\widetilde{B}_\infty + D = \boldsymbol{P}(s).$$

To obtain a corresponding descriptor system we set the reduced-order spectral projector to

$$\widetilde{P}_r := [P_r]_{11} + [P_r]_{12}(I - [P_r]_{22})^{-1}[P_r]_{21}. \tag{4.21}$$

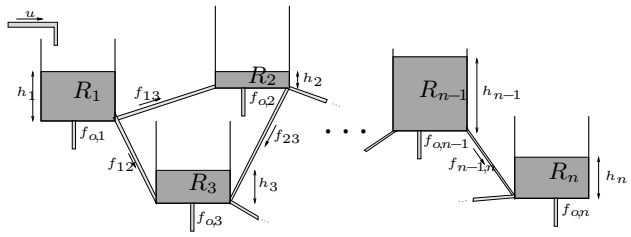

FIG. 5.1. *System of n water reservoirs*

In [19] it was shown that $\widetilde{P}_r$ is again a projector and that $\widetilde{P}_r \geq 0$ if $P_r \geq 0$. The reduced-order descriptor system $[\,\widetilde{E}\,,\,\widetilde{A}\,,\,\widetilde{B}\,,\,\widetilde{C}\,,\,\widetilde{D}\,]$ is then given by the matrices in (4.16).

We still have to show that the thus obtained reduced-order system is again positive in the sense of Theorem 2.3. As in the standard case we have that $I - [A_f]_{22}$ is an M-matrix and hence, $\widetilde{A}_f$, $\widetilde{B}_f$, $\widetilde{C}_f$, $\widetilde{D}_f \geq 0$.

For the reduced-order discrete-time descriptor system defined by the matrices in (4.16), we can obtain an equivalent system of two decoupled equations by using the properties of the projector $\widetilde{P}_r$ given in the following result whose proof is analogous to the proof of Lemma 4.4.

LEMMA 4.5. *Let $\widetilde{P}_r$ be defined as in (4.21) and $\widetilde{A}_f$, $\widetilde{B}_f$ be the reduced-order matrices from singular perturbation balanced truncation. Then, we have*

1. $\widetilde{P}_r\widetilde{A}_f = \widetilde{A}_f\widetilde{P}_r = \widetilde{A}_f$;
2. $\widetilde{P}_r\widetilde{B}_f = \widetilde{B}_f$.

By using Lemma 4.5, we obtain that $\widetilde{E}$ and $\widetilde{A}$ commute and the reduced-order state equation $\widetilde{E}\widetilde{x}(t{+}1) = \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(t)$, is equivalent to $\widetilde{P}_r\widetilde{x}(t{+}1) = \widetilde{A}_f\widetilde{x}(t) + \widetilde{B}u(t)$, $\widetilde{E}_\infty\widetilde{x}(t{+}1) = (I - \widetilde{P}_r)\widetilde{x}(t) + \widetilde{B}_\infty u(t)$.

We have shown that the discrete-time reduced-order system as defined in (4.16) is again positive. The strictly proper part $\boldsymbol{G}_{sp}(s)$ of the transfer function $\boldsymbol{G}(s)$ is reduced as in the standard case, whereas the polynomial part $\boldsymbol{P}(s)$ remains unchanged, which leads to the usual H_∞ error bound as for standard balanced truncation in Lemma 3.3.

5. Examples. In this section we present some numerical examples to demonstrate the properties of the discussed model reduction approaches for positive systems.

EXAMPLE 5.1 (Continuous-time). *Consider a system of n water reservoirs such as schematically shown in Figure 5.1. All reservoirs $R_1, \ldots, R_n$ are assumed to be located on the same level. The base area of R_i and its fill level are denoted by a_i and h_i, respectively. The first reservoir R_1 has an inflow u which is the input of the system, and for each $i \in \{1, \ldots, n\}$, R_i has an outflow $f_{o,i}$ through a pipe with diameter $d_{o,i}$. The output of the system is assumed to be the sum of all outflows. Furthermore, each R_i and R_j are connected by a pipe with diameter $d_{ij} = d_{ji} \geq 0$. The direct flow from R_i to R_j is denoted by f_{ij}. We assume that the flow depends linearly on the difference between the pressures on both ends. This leads to the equations*

$$f_{ij}(t) = d_{ij}^2 \cdot c \cdot (h_i(t) - h_j(t)), \quad f_{o,i}(t) = d_{o,i}^2 \cdot c \cdot (h_i(t) - h_j(t)),$$

where c is a constant that depends on the viscosity and density of the medium and gravity. The fill level of R_i thus satisfies the following differential equation

$$\dot{h}_i(t) = \frac{c}{a_i}\left(-d_{o,i}^2 h_i(t) + \sum_{j=1}^{n} d_{ij}^2 (h_j(t) - h_i(t))\right) + f_{Ii}(t)\frac{1}{a_i}\delta_{1i} f_I(t),$$

where δ_{1i} denotes the Kronecker symbol, and $f_I(t)$ denotes the inflow, i.e. $f_I(t) = u(t)$. We assume that the state is composed of the fill levels and the inflow. For our illustrative computation, we have constructed the presented compartment model with ten states. We assume that we have two well connected substructures each consisting of five reservoirs, where each reservoir is connected with every other reservoir by a pipe of diameter 1. The substructures are connected with each other by a pipe of diameter 0.01 between reservoirs one and ten. For simplicity reasons, we set all base areas of the reservoirs to 1 and also $c = 1$. This leads to a descriptor system $[\,E\,,\,A\,,\,B\,,\,C\,,\,D\,]$ with $E, A \in \mathbb{R}^{11,11}$, $B, C^T \in \mathbb{R}^{11,1}$ and $D = 0$.

The reduced-order system obtained by the procedure as described in Section 4 is given by $\widetilde{E} = \mathrm{diag}(I_4, 0_1)$ and

$$\widetilde{A} = \begin{bmatrix} -5.01 & 1.32 & 1.32 & 1.32 & 1.32 & 0 \\ 0.76 & -5 & 1 & 1 & 1 & 0 \\ 0.76 & 1 & -5 & 1 & 1 & 0 \\ 0.76 & 1 & 1 & -5 & 1 & 0 \\ 0.76 & 1 & 1 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0.45 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 2.22 \\ 2.90 \\ 2.90 \\ 2.90 \\ 2.90 \\ 1 \end{bmatrix}^T, \quad \widetilde{D} = 0.$$

and $D = 0_1$. The frequency responses, i.e., the transfer function $\boldsymbol{G}(s)$ at values

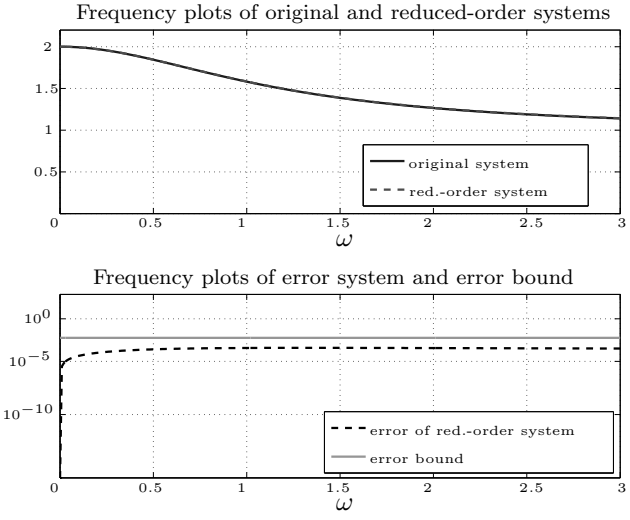

FIG. 5.2. *Frequency plot showing original and reduced-order models.*

$s = j\omega$, for $\omega \in [0, 3]$, of the original and of the reduced-order models are depicted in the upper diagram of Figure 5.2. The lower diagram shows the frequency response of the error system along with the error bound 0.0162.

As an example in discrete-time, we consider the well-known Leslie model [23], which describes the time evolution of age-structured populations.

EXAMPLE 5.2 (Discrete-time). *Let the time $t \in \mathbb{N}_0$ describe the reproduction season (year) and let $x_i(t)$, $i = 1, \ldots, n$, represent the number of individuals of age i at time t. We assume constant survival rates s_i, $i = 0, \ldots, n-1$, i.e., the fraction of individuals of age i that survive for at least one year, and fertility rates f_i, $i = 1, \ldots, n$, i.e., the mean number of offspring born from an individual at age i. For purely illustrative purposes of this example, we use the estimated data given in [17, p. 118] for squirrel reproduction. Furthermore, we assume that immigration into the considered tribe can only happen at birth, i.e., the input is a positive multiple of the first unit vector, and as the output we take the total population, i.e., the sum of the population numbers over all ages. Thus, the aging process is described by the following difference equations*

$$x_{i+1}(t+1) = s_i x_i(t), \quad i = 1, \ldots, n-1, \tag{5.1}$$

and the first state equation takes into account reproduction and immigration

$$x_1(t+1) = s_0(f_1 x_1(t) + f_2 x_2(t) + \ldots + f_n x_n(t) + f_I(t)). \tag{5.2}$$

where $f_I(t) = u(t)$. The system matrices for squirrel reproduction in [17, p. 118] are given by $E = \operatorname{diag}(I_{10}, 0_1)$ and

$$A = \begin{bmatrix} 0.24 & 0.48 & 0.76 & 0.76 & 0.76 & 0.76 & 0.76 & 0.76 & 0.72 & 0.64 & 0.4 \\ 0.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.30 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad D = 0.$$

The reduced-order system obtained by the procedure as described in Section 4 is given by $\widetilde{E} = \operatorname{diag}(I_5, 0_1)$

$$\widetilde{A} = \begin{bmatrix} 0.24 & 1.25 & 3.74 & 5.32 & 10.41 & 0 \\ 0.092 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.23 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0.19 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 2.16 \\ 5.61 \\ 10.61 \\ 15.1 \\ 29.54 \\ 1 \end{bmatrix}^T, \quad \widetilde{D} = 0.$$

The frequency responses, i.e., the transfer function $\boldsymbol{G}(z)$ at values $z = e^{j\omega}$, for $\omega \in [0, 2\pi]$, of the original and of the reduced-order models are depicted in Figure 5.3. The lower diagram shows the frequency response of the error system along with the error bound.

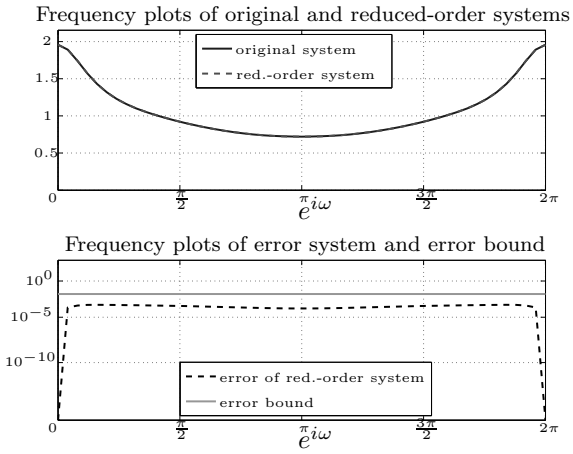

FIG. 5.3. *Frequency plot showing original and reduced-order models.*

6. Conclusion. In this paper, we have presented a model reduction approach that preserves the positivity of continuous-time as well as of discrete-time systems in the standard and in the descriptor case. In particular, we first have extended standard balanced truncation and singular perturbation balanced truncation methods to preserve positivity of standard systems. The proposed approach is based on the existence of a diagonal solution of Lyapunov inequalities, which may be used instead of Lyapunov equations in the classical approach. In this method, along with positivity, also stability is preserved and an error bound in the H_∞ norm is provided. Furthermore, we have generalized this positivity preserving model reduction technique to positive descriptor systems. The additive decomposition of the transfer function into a strictly proper and a polynomial part allows to use the results established for the standard case. The strictly proper part may be reduced as in the standard case, whereas the polynomial part remains unchanged. This guarantees the same H_∞ error bound as in the standard case. A corresponding descriptor system, obtained via recomposition the two reduced parts, is shown to be positive. The functionality of the proposed method has been illustrated by some numerical examples.

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