

# NONNEGATIVITY OF SCHUR COMPLEMENTS OF NONNEGATIVE IDEMPOTENT MATRICES\*

SHMUEL FRIEDLAND<sup>†</sup> AND ELENA VIRNIK<sup>‡</sup>

**Abstract.** Let  $A$  be a nonnegative idempotent matrix. We show that the Schur complement of a submatrix, using the Moore-Penrose inverse, is a nonnegative idempotent matrix if the submatrix has a positive diagonal. Similar results for the Schur complement of any submatrix of  $A$  are no longer true in general.

**Key words.** Nonnegative idempotent matrices; Schur complement; Moore-Penrose inverse; generalized inverse

**AMS subject classifications.** 15A09; 15A15; 15A48

**1. Introduction.** Let  $\langle n \rangle := \{1, \dots, n\}$  and assume that  $\alpha \subset \langle n \rangle$ ,  $\alpha^c := \langle n \rangle \setminus \alpha$ ,  $\beta \subset \langle n \rangle$  are three nonempty sets. For  $A \in \mathbb{R}^{n \times n}$ , denote by  $A[\alpha, \beta]$  the submatrix of  $A$  composed of the rows and columns indexed by the sets  $\alpha$  and  $\beta$ , respectively. Assume that  $A[\alpha, \alpha]$  is invertible. Then, the  $\alpha$  Schur complement of  $A$ , which is equal to the Schur complement of  $A[\alpha, \alpha]$ , is given by

$$A(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{-1}A[\alpha, \alpha^c]. \quad (1.1)$$

If  $A[\alpha, \alpha]$  is not invertible we define

$$A_{\text{ginv}}(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{\text{ginv}}A[\alpha, \alpha^c], \quad (1.2)$$

for some semi-inverse  $A[\alpha, \alpha]^{\text{ginv}}$  [1]. The  $\alpha$  Moore-Penrose Schur complement of  $A$  is defined as

$$A_{\dagger}(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{\dagger}A[\alpha, \alpha^c],$$

where  $A[\alpha, \alpha]^{\dagger}$  is the Moore-Penrose inverse of  $A[\alpha, \alpha]$  [3, 5, 6].

Assume that  $A$  is a nonnegative idempotent matrix, i.e.,  $A^2 = A \in \mathbb{R}_+^{n \times n}$ . In this note we show that if  $A[\alpha, \alpha]$  has a positive diagonal then  $A_{\dagger}(\alpha)$  is a nonnegative idempotent matrix. We give an example of  $A$ , where  $A[\alpha, \alpha]$  has a nonpositive diagonal, and  $A_{\dagger}(\alpha)$  has positive and negative entries. We show that for certain  $A[\alpha, \alpha]$  with a nonpositive diagonal, which includes the above example, one can define a semi-inverse such that  $A_{\text{ginv}}(\alpha)$  is nonnegative and idempotent. We do not know if this result holds in general. Our results follow from Flor's theorem [4], using manipulations with block matrices. Our study was motivated by the analysis of positive differential-algebraic equations (DAEs) [2, 7].

**2. Main result.** First, we recall the following facts [1]. For  $U \in \mathbb{R}^{m \times n}$ , a matrix  $U^{\text{ginv}} \in \mathbb{R}^{n \times m}$  is called a semi-inverse of  $U$  if the following conditions hold

$$UU^{\text{ginv}}U = U, \quad U^{\text{ginv}}UU^{\text{ginv}} = U^{\text{ginv}}. \quad (2.1)$$

If  $0 \neq U = \mathbf{x}\mathbf{y}^{\top}$  then

$$U^{\dagger} = \frac{1}{(\mathbf{x}^{\top}\mathbf{x})(\mathbf{y}^{\top}\mathbf{y})}\mathbf{y}\mathbf{x}^{\top}.$$

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<sup>2</sup>Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 322 Science and Engineering Offices (M/C 249), 851 S. Morgan Street Chicago, IL 60607-7045. E-mail: friedlan@uic.edu, and Visiting Professor, Berlin Mathematical School, Institut für Mathematik, Technische Universität Berlin, Strasse des 17. Juni 136, D-10623 Berlin, FRG.

<sup>3</sup>Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG. E-mail: virnik@math.tu-berlin.de.

If we assume that  $U$  is a direct sum of matrices  $U = \bigoplus_{i=1}^s U_i$ , then  $U^\dagger = \bigoplus_{i=1}^s U_i^\dagger$ .

For our main result we need the following simplification of Flor's theorem [4].

LEMMA 2.1. *Any nonzero nonnegative idempotent matrix  $B \in \mathbb{R}_+^{n \times n}$  is permutationally similar to the following  $3 \times 3$  block matrix*

$$P := \begin{bmatrix} J & JG & 0 \\ 0 & 0 & 0 \\ FJ & FJG & 0 \end{bmatrix}, \quad J \in \mathbb{R}_+^{n_1 \times n_1}, G \in \mathbb{R}_+^{n_1 \times n_2}, F \in \mathbb{R}_+^{n_3 \times n_1}, \quad (2.2)$$

where  $n = n_1 + n_2 + n_3$ ,  $1 \leq n_1, 0 \leq n_2, 0 \leq n_3$ .  $F, G$  are arbitrary nonnegative matrices, and  $J$  is a direct sum of  $k \geq 1$  rank one positive idempotent matrices  $J_i \in \mathbb{R}_+^{l_i \times l_i}$ , i.e.,

$$J = \bigoplus_{i=1}^k J_i, \quad J_i = \mathbf{u}_i \mathbf{v}_i^\top, \quad \mathbf{0} < \mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}_+^{l_i}, \quad \mathbf{v}_i^\top \mathbf{u}_i = 1, \quad i = 1, \dots, k. \quad (2.3)$$

*Proof.* Flor's theorem states that  $B$  is permutationally similar to the following block matrix [4]

$$C := \begin{bmatrix} J & JG_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F_1 J & F_1 JG_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here,  $J \in \mathbb{R}_+^{n_1 \times n_1}$  is of the form (2.3),  $G_1 \in \mathbb{R}_+^{n_1 \times m_2}$ ,  $F_1 \in \mathbb{R}_+^{n_3 \times n_1}$  are arbitrary nonnegative matrices, and the last  $m_4$  rows and columns of  $C$  are zero. Hence,  $n_1 + m_2 + n_3 + m_4 = n$  and  $0 \leq m_2, n_3, m_4$ . If  $m_4 = 0$  then  $C$  is of the form (2.2). It remains to show that  $C$  is permutationally similar to  $P$  if  $m_4 > 0$ .

Interchanging the last row and column of  $C$  with the  $(n_1 + m_2 + 1)$ -st row and column of  $C$  we obtain a matrix  $C_1$ . Then, we interchange the  $(n-1)$ -st row and column of  $C_1$  with the  $(n_1 + m_2 + 2)$ -nd row and column of  $C_1$ . We continue this process until we obtain the idempotent matrix  $P$  with  $n_2 = m_2 + m_4$  zero rows located at the rows  $n_1 + 1, \dots, n_1 + n_2$ . It follows that  $P$  is of the form

$$P := \begin{bmatrix} J & G & 0 \\ 0 & 0 & 0 \\ F & H & 0 \end{bmatrix}, \quad G \in \mathbb{R}_+^{n_1 \times n_2}, F \in \mathbb{R}_+^{n_3 \times n_1}, H \in \mathbb{R}_+^{n_3 \times n_3}.$$

Since  $P^2 = P$  we have that

$$G = JG, \quad F = FJ, \quad H = FG = (FJ)(JG) = FJG.$$

Hence,  $P$  is of the form (2.2).  $\square$

THEOREM 2.2. *Let  $A \in \mathbb{R}_+^{n \times n}$  be a nonnegative idempotent matrix. We assume that for  $\emptyset \neq \alpha \subsetneq \langle n \rangle$ , the submatrix  $A[\alpha, \alpha]$  has a positive diagonal. Then  $A_\dagger(\alpha)$  is a nonnegative idempotent matrix. Furthermore,*

$$\text{rank } A_\dagger(\alpha) = \text{rank } A - \text{rank } A[\alpha, \alpha]. \quad (2.4)$$

*Proof.* Without loss of generality we may assume that  $A$  is of the form (2.2). Since  $A[\alpha, \alpha]$  has a positive diagonal, we deduce that  $A[\alpha, \alpha]$  is a submatrix of  $J$ . First we consider the special case  $A[\alpha, \alpha] = J$ . Using the identity  $JJ^\dagger J = J$ , we obtain that  $A_\dagger(\alpha) = 0$ . Since  $\text{rank } A = \text{rank } J$ , also the equality in (2.4) holds.

Let  $J, F, G$  be defined as in (2.2) and assume now that  $A[\alpha, \alpha]$  is a strict submatrix of  $J$ . In the following, for an integer  $j$  we write  $j + \langle m \rangle$  for the index set  $\{j + 1, \dots, j + m\}$ .

Let  $\alpha' := \langle n_1 \rangle \setminus \alpha$ ,  $\beta := n_1 + \langle n_2 \rangle$  and  $\gamma := n_1 + n_2 + \langle n_3 \rangle$ . Then,

$$\begin{aligned} A[\alpha^c, \alpha]A[\alpha, \alpha]^\dagger A[\alpha, \alpha^c] &= \begin{bmatrix} J[\alpha', \alpha] \\ 0 \\ (FJ)[\gamma, \alpha] \end{bmatrix} J[\alpha, \alpha]^\dagger \begin{bmatrix} J[\alpha, \alpha'] & (JG)[\alpha, \beta] & 0 \end{bmatrix} \\ &= \begin{bmatrix} J[\alpha', \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] & J[\alpha', \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] & (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] & 0 \end{bmatrix}. \end{aligned}$$

On the other hand, we have

$$A[\alpha^c, \alpha^c] = \begin{bmatrix} J[\alpha', \alpha'] & (JG)[\alpha', \beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma, \alpha'] & FJG & 0 \end{bmatrix}.$$

Thus, the nonnegativity of  $A_\dagger(\alpha)$  is equivalent to the following, (entrywise), inequalities

$$J[\alpha', \alpha'] \geq J[\alpha', \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'], \quad (2.5)$$

$$(JG)[\alpha', \beta] \geq J[\alpha', \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta], \quad (2.6)$$

$$(FJ)[\gamma, \alpha'] \geq (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'], \quad (2.7)$$

$$FJG \geq (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta]. \quad (2.8)$$

Without loss of generality, we may assume that  $J$  is permuted such that the indices of the first  $q$  blocks  $J_i$  are contained in  $\alpha^c$ , the indices of the following blocks  $J_i$  for  $i = q+1, \dots, q+p$  are split between  $\alpha$  and  $\alpha^c$  and the indices of the blocks  $J_i$  for  $i = q+p+1, \dots, q+p+\ell = k$  are contained in  $\alpha$ . Partitioning the vectors  $\mathbf{u}_i, \mathbf{v}_i$  in (2.3) according to  $\alpha$  and  $\alpha^c$  as

$$\mathbf{u}_i^\top = (\mathbf{a}_i^\top, \mathbf{x}_i^\top), \quad \mathbf{v}_i^\top = (\mathbf{b}_i^\top, \mathbf{y}_i^\top) \quad \text{for } i = q+1, \dots, q+p,$$

we obtain that

$$J[\alpha', \alpha'] = (\oplus_{i=1}^q J_i) \oplus_{i=q+1}^{q+p} \mathbf{a}_i \mathbf{b}_i^\top, \quad J[\alpha, \alpha] = (\oplus_{j=q+1}^{q+p} \mathbf{x}_j \mathbf{y}_j^\top) \oplus_{i=q+p+1}^{q+p+\ell} J_i.$$

Note that

$$q = \text{rank } J - \text{rank } A[\alpha, \alpha] = \text{rank } A - \text{rank } A[\alpha, \alpha]. \quad (2.9)$$

We will only consider the case  $q, p, \ell > 0$ , as other cases follow similarly. We have

$$J[\alpha, \alpha]^\dagger = (\oplus_{i=q+1}^{q+p} \frac{1}{(\mathbf{x}_i^\top \mathbf{x}_i)(\mathbf{y}_i^\top \mathbf{y}_i)} \mathbf{y}_i \mathbf{x}_i^\top) \oplus_{i=q+p+1}^{q+p+\ell} \frac{1}{(\mathbf{u}_i^\top \mathbf{u}_i)(\mathbf{v}_i^\top \mathbf{v}_i)} \mathbf{v}_i \mathbf{u}_i^\top, \quad (2.10)$$

$$J[\alpha, \alpha'] = \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p} \mathbf{x}_i \mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix}, \quad J[\alpha', \alpha] = \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p} \mathbf{a}_i \mathbf{y}_i^\top & 0 \end{bmatrix}, \quad (2.11)$$

and hence,

$$J[\alpha', \alpha]J[\alpha, \alpha]^\dagger = \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p} \frac{1}{\mathbf{x}_i^\top \mathbf{x}_i} \mathbf{a}_i \mathbf{x}_i^\top & 0 \end{bmatrix}, \quad (2.12)$$

$$J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p} \frac{1}{\mathbf{y}_i^\top \mathbf{y}_i} \mathbf{y}_i \mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix}, \quad (2.13)$$

$$J[\alpha', \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = \begin{bmatrix} 0 & 0 \\ 0 & \oplus_{i=q+1}^{q+p} \mathbf{a}_i \mathbf{b}_i^\top \end{bmatrix}. \quad (2.14)$$

Therefore, we obtain

$$J[\alpha', \alpha'] - J[\alpha', \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = \begin{bmatrix} \oplus_{i=1}^q J_i & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \quad (2.15)$$

which proves (2.5).

We now show the inequalities (2.6) and (2.7). First, we observe that  $JG$  and  $FJ$  have the following block form

$$JG = \begin{bmatrix} \mathbf{u}_1 \mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_k \mathbf{g}_k^\top \end{bmatrix}, \quad FJ = [ \mathbf{f}_1 \mathbf{v}_1^\top \quad \cdots \quad \mathbf{f}_k \mathbf{v}_k^\top ], \quad \mathbf{g}_i \in \mathbb{R}_+^{n_2}, \mathbf{f}_i \in \mathbb{R}_+^{n_3} \text{ for } i = 1, \dots, k. \quad (2.16)$$

Hence, we obtain

$$(JG)[\alpha, \beta] = \begin{bmatrix} \mathbf{x}_{q+1} \mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{x}_{q+p} \mathbf{g}_{q+p}^\top \\ \mathbf{u}_{q+p+1} \mathbf{g}_{q+p+1}^\top \\ \vdots \\ \mathbf{u}_k \mathbf{g}_k^\top \end{bmatrix}, \quad (2.17)$$

$$(JG)[\alpha', \beta] = \begin{bmatrix} \mathbf{u}_1 \mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_q \mathbf{g}_q^\top \\ \mathbf{a}_{q+1} \mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{u}_{q+p} \mathbf{g}_{q+p}^\top \end{bmatrix}, \quad (2.18)$$

$$(FJ)[\gamma, \alpha] = [ \mathbf{f}_{q+1} \mathbf{y}_{q+1}^\top \quad \cdots \quad \mathbf{f}_{q+p} \mathbf{y}_{q+p}^\top \quad \mathbf{f}_{q+p+1} \mathbf{v}_{q+p+1}^\top \quad \cdots \quad \mathbf{f}_k \mathbf{v}_k^\top ], \quad (2.19)$$

$$(FJ)[\gamma, \alpha'] = [ \mathbf{f}_1 \mathbf{v}_1^\top \quad \cdots \quad \mathbf{f}_q \mathbf{v}_q^\top \quad \mathbf{f}_{q+1} \mathbf{b}_{q+1}^\top \quad \cdots \quad \mathbf{f}_{q+p} \mathbf{b}_{q+p}^\top ]. \quad (2.20)$$

We use (2.13) to deduce that

$$(FJ)[\gamma, \alpha] J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = [ 0 \quad \cdots \quad 0 \quad \mathbf{f}_{q+1} \mathbf{b}_{q+1}^\top \quad \cdots \quad \mathbf{f}_{q+p} \mathbf{b}_{q+p}^\top ].$$

Therefore, we have

$$(FJ)[\gamma, \alpha'] - (FJ)[\gamma, \alpha] J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = [ \mathbf{f}_1 \mathbf{v}_1^\top \quad \cdots \quad \mathbf{f}_q \mathbf{v}_q^\top \quad 0 \quad \cdots \quad 0 ]. \quad (2.21)$$

Similarly, using (2.12), we obtain

$$(JG)[\alpha', \beta] - J[\alpha', \alpha] J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] = \begin{bmatrix} \mathbf{u}_1 \mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_q \mathbf{g}_q^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (2.22)$$

Hence, the inequalities (2.6) and (2.7) hold.

We now show the last inequality (2.8). To this end, we observe that

$$FJG = (FJ)(JG) = \sum_{i=1}^k \mathbf{f}_i \mathbf{g}_i^\top. \quad (2.23)$$

Multiplying (2.10), (2.17) and (2.19) we obtain that

$$(FJ)[\gamma, \alpha] J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] = \sum_{i=q+1}^k \mathbf{f}_i \mathbf{g}_i^\top.$$

Hence,

$$FJG - (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger(JG)[\alpha, \beta] = \sum_{i=1}^q \mathbf{f}_i \mathbf{g}_i^\top \geq 0. \quad (2.24)$$

In particular, this proves that (2.8) holds.

It is left to show that  $A_\dagger(\alpha)$  is an idempotent matrix. Clearly, if  $q = 0$  then  $A_\dagger(\alpha) = 0$ . So  $A_\dagger(\alpha)$  is a trivial idempotent matrix, and (2.9) yields (2.4).

Assuming finally that  $q > 0$ , it follows that  $A_\dagger(\alpha)$  has the block form (2.2) with  $J = \bigoplus_{i=1}^q J_i \oplus 0$ . Hence  $A_\dagger(\alpha)$  is an idempotent matrix whose rank is  $q$ , and (2.9) yields (2.4).  $\square$

**COROLLARY 2.3.** *Let  $A \in \mathbb{R}_+^{n \times n}$ ,  $A \neq 0$  be idempotent. If  $\alpha \subsetneq \langle n \rangle$  is chosen such that  $A[\alpha, \alpha]$  is an invertible matrix, then  $A[\alpha, \alpha]$  is diagonal.*

*Proof.* Note that the number  $\ell$  in the proof of Theorem 2.2 is either zero or the corresponding blocks  $J_i$  are positive  $1 \times 1$  matrices for  $i = q + p + 1, \dots, q + p + \ell$ . Furthermore, for the split blocks, we also have that  $\mathbf{x}_i \mathbf{y}_i^\top \in \mathbb{R}^{1 \times 1}$ , for  $i = q + 1, \dots, q + p$ , since  $\mathbf{x}_i \mathbf{y}_i^\top$  is of rank 1. Therefore,  $A[\alpha, \alpha]$  is diagonal.  $\square$

**COROLLARY 2.4.** *Let  $A \in \mathbb{R}_+^{n \times n}$ ,  $A \neq 0$  be idempotent. If  $\alpha \subsetneq \langle n \rangle$  is chosen such that  $A[\alpha, \alpha]$  is a regular matrix, then the standard Schur complement (1.1) is nonnegative.*

**COROLLARY 2.5.** *Let  $A \in \mathbb{R}_+^{n \times n}$ ,  $A \neq 0$  be idempotent. Choose  $\alpha \subsetneq \langle n \rangle$ , such that  $I - A[\alpha, \alpha]$  is regular. Then,  $\tilde{A}(\alpha)$  defined by*

$$\tilde{A}(\alpha) := A[\alpha^c, \alpha^c] + A[\alpha^c, \alpha](I - A[\alpha, \alpha])^{-1}A[\alpha, \alpha^c] \quad (2.25)$$

*is a nonnegative idempotent matrix.*

To prove this Corollary 2.5 we need the following fact for idempotent matrices, which is probably known.

**LEMMA 2.6.** *Let  $A \in \mathbb{R}^{n \times n}$ ,  $A \neq 0$  be idempotent given as a  $2 \times 2$  block matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ . Assume that  $I - A_{22} \in \mathbb{R}^{n-m}$  is regular. Then  $B := A_{11} + A_{12}(I - A_{22})^{-1}A_{21}$  is idempotent.*

*Proof.* Let

$$E = (I - A_{22})^{-1}A_{21}, \quad D = A_{21} + A_{22}E, \quad z = \begin{bmatrix} x \\ Ex \end{bmatrix} \in \mathbb{R}^n, \quad x \text{ any vector in } \mathbb{R}^m.$$

Note that  $Az = \begin{bmatrix} Bx \\ Dx \end{bmatrix}$ . As  $A^2z = Az$  and  $x$  is an arbitrary vector, we obtain the equalities

$$A_{11}B + A_{12}D = B, \quad A_{21}B + A_{22}D = D. \quad (2.26)$$

From the second equality of (2.26) we obtain  $D = EB$ . Substituting this equality into the first equality of (2.26) we obtain that  $B^2 = B$ .  $\square$

*Proof of Corollary 2.5.* The assumption that  $I - A[\alpha, \alpha]$  is regular implies that  $A[\alpha, \alpha]$  does not have an eigenvalue 1, i.e.,  $\rho(A[\alpha, \alpha]) < 1$ . Hence,  $I - A[\alpha, \alpha]$  is an  $M$ -matrix [1] and  $(I - A[\alpha, \alpha])^{-1} \geq 0$ . The assertion of Corollary 2.5 now follows using Lemma 2.6.  $\square$

### 3. Additional results.

**3.1. An example.** In this subsection we assume that the nonnegative idempotent matrix  $A$  is of the special form

$$A := \begin{bmatrix} J & JG \\ 0 & 0 \end{bmatrix}. \quad (3.1)$$

Furthermore, we assume that  $A[\alpha, \alpha]$  has a zero on its main diagonal. We give an example where  $A_\dagger(\alpha)$  may fail to be nonnegative. To this end, we first start with the following known result.

LEMMA 3.1. Let  $A \in \mathbb{R}^{n \times n}$  be a singular matrix of the following form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0_{(n-p) \times p} & 0_{(n-p) \times (n-p)} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{p \times p}, A_{12} \in \mathbb{R}^{p \times (n-p)}, \text{ for some } 1 \leq p < n.$$

Then  $(A^\dagger)^\top$  has the same block form as  $A$ .

*Proof.* Let  $r = \text{rank } A$ . So  $r \leq p$ . Then the reduced singular value decomposition of  $A$  is of the form  $U_r \Sigma_r V_r^\top$ , where  $U_r, V_r \in \mathbb{R}^{n \times r}$ ,  $U_r^\top U_r = V_r V_r^\top = I_r$  and  $\Sigma_r$  is a diagonal matrix, whose diagonal entries are the positive singular values of  $A$ .

Clearly,  $AA^\top = \begin{bmatrix} A_{11}A_{11}^\top + A_{12}A_{12}^\top & 0 \\ 0 & 0 \end{bmatrix}$ . Hence all eigenvectors of  $AA^\top$ , corresponding to positive eigenvalues are of the form  $(\mathbf{x}^\top, \mathbf{0}^\top)^\top$ ,  $\mathbf{x} \in \mathbb{R}^p$ . Thus  $U_r^\top = [U_{r1}^\top \ 0_{r \times (n-p)}]$  where  $U_{r1} \in \mathbb{R}^{p \times r}$ . Recall that  $A^\dagger = V_r \Sigma_r^{-1} U_r^\top$ . The above form of  $U_r$  establishes the lemma.  $\square$

In the following example we permute some rows and columns of  $A$ , in order to find the Schur complement of the right lower block.

EXAMPLE 3.2. Consider a nonnegative idempotent matrix in the block form

$$B = \left[ \begin{array}{cc|cc} \mathbf{u}_1 \mathbf{v}_1^\top & 0 & \mathbf{u}_1 \mathbf{s}_1^\top & 0 \\ 0 & \mathbf{a}_2 \mathbf{b}_2^\top & \mathbf{a}_2 \mathbf{s}_2^\top & \mathbf{a}_1 \mathbf{t}_2^\top & \mathbf{a}_2 \mathbf{y}_2^\top \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x}_2 \mathbf{b}_2^\top & \mathbf{x}_2 \mathbf{s}_2^\top & \mathbf{x}_1 \mathbf{t}_2^\top & \mathbf{x}_2 \mathbf{y}_2^\top \end{array} \right].$$

Then,

$$B[\alpha, \alpha] = \begin{bmatrix} 0 & 0 \\ \mathbf{x}_1 \mathbf{t}_2^\top & \mathbf{x}_2 \mathbf{y}_2^\top \end{bmatrix}, \quad B[\alpha, \alpha]^\dagger = \begin{bmatrix} 0 & \frac{\mathbf{t}_2 \mathbf{x}_2^\top}{(\mathbf{x}_2^\top \mathbf{x}_2)(\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2)} \\ 0 & \frac{\mathbf{y}_2 \mathbf{x}_2^\top}{(\mathbf{x}_2^\top \mathbf{x}_2)(\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2)} \end{bmatrix},$$

and

$$B[\alpha^c, \alpha] B[\alpha, \alpha]^\dagger B[\alpha, \alpha^c] = \begin{bmatrix} 0 & \frac{\mathbf{t}_1^\top \mathbf{t}_2 \mathbf{u}_1 \mathbf{b}_2^\top}{\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2} & \frac{\mathbf{t}_1^\top \mathbf{t}_2 \mathbf{u}_1 \mathbf{s}_2^\top}{\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2} \\ 0 & \mathbf{a}_2 \mathbf{b}_2^\top & \mathbf{a}_2 \mathbf{s}_2^\top \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence  $B_\dagger(\alpha)_{11} > 0$ ,  $B_\dagger(\alpha)_{12} \leq 0$  and the Moore-Penrose inverse Schur complement is neither nonnegative nor nonpositive if  $\mathbf{t}_1^\top \mathbf{t}_2 > 0$ .

**3.2. Nonnegativity of semi-inverse Schur complement.** In this section we extend the results of Section 2 for idempotent matrices of the form (2.2) for some Schur complements with zero diagonal entries. We start with the following simple observation.

PROPOSITION 3.3. Let the assumptions of Lemma 3.1 hold. Suppose that

$$A_{11}(A_{11})^\dagger A_{12} = A_{12}.$$

Then  $A^{\text{ginv}} = \begin{bmatrix} (A_{11})^\dagger & 0_{p \times (n-p)} \\ 0_{(n-p) \times p} & 0_{(n-p) \times (n-p)} \end{bmatrix}$  is a semi-inverse of  $A$ . In particular any principle submatrix of an idempotent matrix as in (3.1) with at least one zero diagonal element has a semi-inverse of this form.

*Proof.* The proposition follows by checking the conditions in (2.1).  $\square$

Note that condition  $A_{11}(A_{11})^\dagger A_{12} = A_{12}$  holds in general for idempotent matrices  $A$  of the form as in (3.1).

The following Theorem states the general result of this subsection.

THEOREM 3.4. Let  $A \in \mathbb{R}_+^{n \times n}$  be of the form (2.2), where  $n_2 + n_3 \geq 1$  and the condition in (2.3) holds. Furthermore, let  $\alpha_1 \subset \langle n \rangle$  be of the following form

$$\begin{aligned} \text{either } & \alpha_1 = \alpha \cup \beta, \emptyset \neq \beta \subseteq n_1 + \langle n_2 \rangle, \\ \text{or } & \alpha_1 = \alpha \cup \gamma, \emptyset \neq \gamma \subseteq n_1 + n_2 + \langle n_3 \rangle, \end{aligned} \quad (3.2)$$

where  $\alpha \subseteq \langle n_1 \rangle$ . Then, there exists a semi-inverse  $A^{\text{ginv}}[\alpha_1, \alpha_1]$  of  $A[\alpha_1, \alpha_1]$  such that  $A_{\text{ginv}}(\alpha_1)$  as defined in (1.2) is a nonnegative idempotent matrix. The rank of  $A_{\text{ginv}}(\alpha_1)$  is equal to the multiplicity of the eigenvalue 1 in  $A[\alpha', \alpha']$ , where  $\alpha' = \langle n_1 \rangle \setminus \alpha$ . In particular, if 1 is not an eigenvalue of  $A[\alpha', \alpha']$ , then  $A_{\text{ginv}}(\alpha) = 0$ .

*Proof.* First we consider the case that  $\alpha_1 = \alpha \cup \beta$ . If  $\alpha = \emptyset$ , then  $A[\alpha_1, \alpha_1]$  and  $A[\alpha_1, \alpha_1]^{\text{ginv}}$  are zero matrices for any semi-inverse and  $A_{\text{ginv}}(\alpha_1) = A[\alpha_1^c, \alpha_1^c]$ . Using the proof of Theorem 2.2 we obtain that  $A_{\text{ginv}}(\alpha_1)$  is a nonnegative idempotent matrix of rank  $k$ .

Assuming now that  $\alpha \neq \emptyset$ , we observe that  $A[\alpha_1, \alpha_1]$  satisfies the assumption of Proposition 3.3. Defining  $A[\alpha_1, \alpha_1]^{\text{ginv}}$  as in Proposition 3.3 and following the arguments of the proof of Theorem 2.2 we deduce the theorem in this case.

We assume now that  $\alpha_1 = \alpha \cup \gamma$ . If  $\alpha = \emptyset$  we obtain that  $A_{\text{ginv}}(\alpha_1)$  is a nonnegative idempotent matrix of rank  $k$  as above. Assuming finally that  $\alpha \neq \emptyset$ , we have that  $A[\alpha_1, \alpha_1]^{\top}$  satisfies the assumption of Proposition 3.3. Define  $(A[\alpha_1, \alpha_1]^{\top})^{\text{ginv}}$  as in Proposition 3.3 and let  $A[\alpha_1, \alpha_1]^{\text{ginv}} := ((A[\alpha_1, \alpha_1]^{\top})^{\text{ginv}})^{\top}$ . Repeating the arguments of the proof of Theorem 2.2 we deduce the theorem in this case.  $\square$

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