NONNEGATIVITY OF SCHUR COMPLEMENTS OF NONNEGATIVE IDEMPOTENT MATRICES*

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Abstract. Let A be a nonnegative idempotent matrix. We show that the Schur complement of a submatrix, using the Moore-Penrose inverse, is a nonnegative idempotent matrix if the submatrix has a positive diagonal. Similar results for the Schur complement of any submatrix of A are no longer true in general.

Key words. Nonnegative idempotent matrices; Schur complement; Moore-Penrose inverse; generalized inverse

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1. Introduction.

Let $\langle n \rangle := \{1, \ldots, n\}$ and assume that $\alpha \subset \langle n \rangle, \alpha^c := \langle n \rangle \backslash \alpha, \beta \subset \langle n \rangle$ are three nonempty sets. For $A \in \mathbb{R}^{n \times n}$, denote by $A[\alpha, \beta]$ the submatrix of A composed of the rows and columns indexed by the sets α and β, respectively. Assume that $A[\alpha, \alpha]$ is invertible. Then, the α Schur complement of A, which is equal to the Schur complement of $A[\alpha, \alpha]$, is given by

$$A(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{-1}A[\alpha, \alpha^c]. \tag{1.1}$$

If $A[\alpha, \alpha]$ is not invertible we define

$$A_{\mathrm{ginv}}(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{\mathrm{ginv}}A[\alpha, \alpha^c], \tag{1.2}$$

for some semi-inverse $A[\alpha, \alpha]^{\mathrm{ginv}}$ [1]. The α Moore-Penrose Schur complement of A is defined as

$$A_{\dagger}(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{\dagger}A[\alpha, \alpha^c],$$

where $A[\alpha, \alpha]^{\dagger}$ is the Moore-Penrose inverse of $A[\alpha, \alpha]$ [3, 5, 6].

Assume that A is a nonnegative idempotent matrix, i.e., $A^2 = A \in \mathbb{R}_+^{n \times n}$. In this note we show that if $A[\alpha, \alpha]$ has a positive diagonal then $A_{\dagger}(\alpha)$ is a nonnegative idempotent matrix. We give an example of A, where $A[\alpha, \alpha]$ has a nonpositive diagonal, and $A_{\dagger}(\alpha)$ has positive and negative entries. We show that for certain $A[\alpha, \alpha]$ with a nonpositive diagonal, which includes the above example, one can define a semi-inverse such that $A_{\mathrm{ginv}}(\alpha)$ is nonnegative and idempotent. We do not know if this result holds in general. Our results follow from Flor's theorem [4], using manipulations with block matrices. Our study was motivated by the analysis of positive differential-algebraic equations (DAEs) [2, 7].

2. Main result.

First, we recall the following facts [1]. For $U \in \mathbb{R}^{m \times n}$, a matrix $U^{\mathrm{ginv}} \in \mathbb{R}^{n \times m}$ is called a semi-inverse of U if the following conditions hold

$$UU^{\mathrm{ginv}}U = U, \quad U^{\mathrm{ginv}}UU^{\mathrm{ginv}} = U^{\mathrm{ginv}}. \tag{2.1}$$

If $0 \neq U = \mathbf{x}\mathbf{y}^{\top}$ then

$$U^{\dagger} = \frac{1}{(\mathbf{x}^{\top}\mathbf{x})(\mathbf{y}^{\top}\mathbf{y})}\mathbf{y}\mathbf{x}^{\top}.$$

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If we assume that U is a direct sum of matrices $U = \oplus_{i=1}^s U_i$, then $U^\dagger = \oplus_{i=1}^s U_i^\dagger$.

For our main result we need the following simplification of Flor's theorem [4].

LEMMA 2.1. *Any nonzero nonnegative idempotent matrix $B \in \mathbb{R}_+^{n\times n}$ is permutationally similar to the following 3×3 block matrix*

$$P := \begin{bmatrix} J & JG & 0 \\ 0 & 0 & 0 \\ FJ & FJG & 0 \end{bmatrix}, \; J \in \mathbb{R}_+^{n_1\times n_1}, G \in \mathbb{R}_+^{n_1\times n_2}, F \in \mathbb{R}_+^{n_3\times n_1}, \tag{2.2}$$

where $n = n_1 + n_2 + n_3, 1 \le n_1, 0 \le n_2, 0 \le n_3$. F, G are arbitrary nonnegative matrices, and J is a direct sum of $k \ge 1$ rank one positive idempotent matrices $J_i \in \mathbb{R}_+^{l_i\times l_i}$, i.e.,

$$J = \oplus_{i=1}^k J_i, \; J_i = \mathbf{u}_i\mathbf{v}_i^\top, \mathbf{0} < \mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}_+^{l_i}, \; \mathbf{v}_i^\top \mathbf{u}_i = 1, \; i = 1, \dots, k. \tag{2.3}$$

Proof. Flor's theorem states that B is permutationally similar to the following block matrix [4]

$$C := \begin{bmatrix} J & JG_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F_1 J & F_1 J G_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, $J \in \mathbb{R}_+^{n_1\times n_1}$ is of the form (2.3), $G_1 \in \mathbb{R}_+^{n_1\times m_2}, F_1 \in \mathbb{R}_+^{n_3\times n_1}$ are arbitrary nonnegative matrices, and the last m_4 rows and columns of C are zero. Hence, $n_1 + m_2 + n_3 + m_4 = n$ and $0 \le m_2, n_3, m_4$. If $m_4 = 0$ then C is of the form (2.2). It remains to show that C is permutationally similar to P if $m_4 > 0$.

Interchanging the last row and column of C with the $(n_1 + m_2 + 1)$-st row and column of C we obtain a matrix C_1. Then, we interchange the $(n-1)$-st row and column of C_1 with the $(n_1 + m_2 + 2)$-nd row and column of C_1. We continue this process until we obtain the idempotent matrix P with $n_2 = m_2 + m_4$ zero rows located at the rows $n_1 + 1, \dots, n_1 + n_2$. It follows that P is of the form

$$P := \begin{bmatrix} J & G & 0 \\ 0 & 0 & 0 \\ F & H & 0 \end{bmatrix}, G \in \mathbb{R}_+^{n_1\times n_2}, F \in \mathbb{R}_+^{n_3\times n_1}, H \in \mathbb{R}_+^{n_3\times n_3}.$$

Since $P^2 = P$ we have that

$$G = JG, \; F = FJ, \; H = FG = (FJ)(JG) = FJG.$$

Hence, P is of the form (2.2). ☐

THEOREM 2.2. *Let $A \in \mathbb{R}_+^{n\times n}$ be a nonnegative idempotent matrix. We assume that for $\emptyset \ne \alpha \subsetneqq \langle n\rangle$, the submatrix $A[\alpha, \alpha]$ has a positive diagonal. Then $A_\dagger(\alpha)$ is a nonnegative idempotent matrix. Furthermore,*

$$\operatorname{rank} A_\dagger(\alpha) = \operatorname{rank} A - \operatorname{rank} A[\alpha, \alpha]. \tag{2.4}$$

Proof. Without loss of generality we may assume that A is of the form (2.2). Since $A[\alpha, \alpha]$ has a positive diagonal, we deduce that $A[\alpha, \alpha]$ is a submatrix of J. First we consider the special case $A[\alpha, \alpha] = J$. Using the identity $JJ^\dagger J = J$, we obtain that $A_\dagger(\alpha) = 0$. Since $\operatorname{rank} A = \operatorname{rank} J$, also the equality in (2.4) holds.

Let J, F, G be defined as in (2.2) and assume now that $A[\alpha, \alpha]$ is a strict submatrix of J. In the following, for an integer j we write $j + \langle m\rangle$ for the index set $\{j + 1, \dots, j + m\}$.

Let $\alpha' := \langle n_1\rangle\backslash\alpha$, $\beta := n_1 + \langle n_2\rangle$ and $\gamma := n_1 + n_2 + \langle n_3\rangle$. Then,

$$A[\alpha^c,\alpha]A[\alpha,\alpha]^\dagger A[\alpha,\alpha^c] = \begin{bmatrix} J[\alpha',\alpha] \\ 0 \\ (FJ)[\gamma,\alpha] \end{bmatrix} J[\alpha,\alpha]^\dagger \begin{bmatrix} J[\alpha,\alpha'] & (JG)[\alpha,\beta] & 0 \end{bmatrix}$$
$$= \begin{bmatrix} J[\alpha',\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] & J[\alpha',\alpha]J[\alpha,\alpha]^\dagger (JG)[\alpha,\beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] & (FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger (JG)[\alpha,\beta] & 0 \end{bmatrix}.$$

On the other hand, we have

$$A[\alpha^c,\alpha^c] = \begin{bmatrix} J[\alpha',\alpha'] & (JG)[\alpha',\beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma,\alpha'] & FJG & 0 \end{bmatrix}.$$

Thus, the nonnegativity of $A_\dagger(\alpha)$ is equivalent to the following, (entrywise), inequalities

$$J[\alpha',\alpha'] \geq J[\alpha',\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'], \tag{2.5}$$
$$(JG)[\alpha',\beta] \geq J[\alpha',\alpha]J[\alpha,\alpha]^\dagger (JG)[\alpha,\beta], \tag{2.6}$$
$$(FJ)[\gamma,\alpha'] \geq (FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'], \tag{2.7}$$
$$FJG \geq (FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger (JG)[\alpha,\beta]. \tag{2.8}$$

Without loss of generality, we may assume that J is permuted such that the indices of the first q blocks J_i are contained in α^c, the indices of the following blocks J_i for $i = q+1,\dots,q+p$ are split between α and α^c and the indices of the blocks J_i for $i = q+p+1,\dots,q+p+\ell = k$ are contained in α. Partitioning the vectors $\mathbf{u}_i, \mathbf{v}_i$ in (2.3) according to α and α^c as

$$\mathbf{u}_i^\top = (\mathbf{a}_i^\top, \mathbf{x}_i^\top), \quad \mathbf{v}_i^\top = (\mathbf{b}_i^\top, \mathbf{y}_i^\top) \ \text{ for } \ i = q+1,\dots,q+p,$$

we obtain that

$$J[\alpha',\alpha'] = (\oplus_{i=1}^{q} J_i) \oplus_{i=q+1}^{q+p} \mathbf{a}_i\mathbf{b}_i^\top,\ J[\alpha,\alpha] = (\oplus_{j=q+1}^{q+p} \mathbf{x}_i\mathbf{y}_i^\top) \oplus_{i=q+p+1}^{q+p+\ell} J_i.$$

Note that

$$q = \operatorname{rank} J - \operatorname{rank} A[\alpha,\alpha] = \operatorname{rank} A - \operatorname{rank} A[\alpha,\alpha]. \tag{2.9}$$

We will only consider the case $q, p, \ell > 0$, as other cases follow similarly. We have

$$J[\alpha,\alpha]^\dagger = (\oplus_{i=q+1}^{q+p} \frac{1}{(\mathbf{x}_i^\top\mathbf{x}_i)(\mathbf{y}_i^\top\mathbf{y}_i)}\mathbf{y}_i\mathbf{x}_i^\top) \oplus_{i=q+p+1}^{q+p+\ell} \frac{1}{(\mathbf{u}_i^\top\mathbf{u}_i)(\mathbf{v}_i^\top\mathbf{v}_i)}\mathbf{v}_i\mathbf{u}_i^\top, \tag{2.10}$$
$$J[\alpha,\alpha'] = \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p}\mathbf{x}_i\mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix},\ J[\alpha',\alpha] = \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p}\mathbf{a}_i\mathbf{y}_i^\top & 0 \end{bmatrix}, \tag{2.11}$$

and hence,

$$J[\alpha',\alpha]J[\alpha,\alpha]^\dagger = \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p}\frac{1}{\mathbf{x}_i^\top\mathbf{x}_i}\mathbf{a}_i\mathbf{x}_i^\top & 0 \end{bmatrix}, \tag{2.12}$$
$$J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] = \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p}\frac{1}{\mathbf{y}_i^\top\mathbf{y}_i}\mathbf{y}_i\mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix}, \tag{2.13}$$
$$J[\alpha',\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] = \begin{bmatrix} 0 & 0 \\ 0 & \oplus_{i=q+1}^{q+p}\mathbf{a}_i\mathbf{b}_i^\top \end{bmatrix}. \tag{2.14}$$

Therefore, we obtain

$$J[\alpha',\alpha'] - J[\alpha',\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] = \begin{bmatrix} \oplus_{i=1}^{q} J_i & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \tag{2.15}$$

which proves (2.5).

We now show the inequalities (2.6) and (2.7). First, we observe that JG and FJ have the following block form

$$JG = \begin{bmatrix} \mathbf{u}_1\mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_k\mathbf{g}_k^\top \end{bmatrix}, \ FJ = \begin{bmatrix} \mathbf{f}_1\mathbf{v}_1^\top & \cdots & \mathbf{f}_k\mathbf{v}_k^\top \end{bmatrix}, \ \mathbf{g}_i \in \mathbb{R}_+^{n_2}, \mathbf{f}_i \in \mathbb{R}_+^{n_3} \text{ for } i = 1, \dots, k. \tag{2.16}$$

Hence, we obtain

$$(JG)[\alpha, \beta] = \begin{bmatrix} \mathbf{x}_{q+1}\mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{x}_{q+p}\mathbf{g}_{q+p}^\top \\ \mathbf{u}_{q+p+1}\mathbf{g}_{q+p+1}^\top \\ \vdots \\ \mathbf{u}_k\mathbf{g}_k^\top \end{bmatrix}, \tag{2.17}$$

$$(JG)[\alpha', \beta] = \begin{bmatrix} \mathbf{u}_1\mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_q\mathbf{g}_q^\top \\ \mathbf{a}_{q+1}\mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{u}_{q+p}\mathbf{g}_{q+p}^\top \end{bmatrix}, \tag{2.18}$$

$$(FJ)[\gamma, \alpha] = \begin{bmatrix} \mathbf{f}_{q+1}\mathbf{y}_{q+1}^\top & \cdots & \mathbf{f}_{q+p}\mathbf{y}_{q+p}^\top & \mathbf{f}_{q+p+1}\mathbf{v}_{q+p+1}^\top & \cdots & \mathbf{f}_k\mathbf{v}_k^\top \end{bmatrix}, \tag{2.19}$$

$$(FJ)[\gamma, \alpha'] = \begin{bmatrix} \mathbf{f}_1\mathbf{v}_1^\top & \cdots & \mathbf{f}_q\mathbf{v}_q^\top & \mathbf{f}_{q+1}\mathbf{b}_{q+1}^\top & \cdots & \mathbf{f}_{q+p}\mathbf{b}_{q+p}^\top \end{bmatrix}. \tag{2.20}$$

We use (2.13) to deduce that

$$(FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = \begin{bmatrix} 0 & \cdots & 0 & \mathbf{f}_{q+1}\mathbf{b}_{q+1}^\top & \cdots & \mathbf{f}_{q+p}\mathbf{b}_{q+p}^\top \end{bmatrix}.$$

Therefore, we have

$$(FJ)[\gamma, \alpha'] - (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = \begin{bmatrix} \mathbf{f}_1\mathbf{v}_1^\top & \cdots & \mathbf{f}_q\mathbf{v}_q^\top & 0 & \cdots & 0 \end{bmatrix}. \tag{2.21}$$

Similarly, using (2.12), we obtain

$$(JG)[\alpha', \beta] - J[\alpha', \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] = \begin{bmatrix} \mathbf{u}_1\mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_q\mathbf{g}_q^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.22}$$

Hence, the inequalities (2.6) and (2.7) hold.

We now show the last inequality (2.8). To this end, we observe that

$$FJG = (FJ)(JG) = \sum_{i=1}^{k} \mathbf{f}_i\mathbf{g}_i^\top. \tag{2.23}$$

Multiplying (2.10), (2.17) and (2.19) we obtain that

$$(FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] = \sum_{i=q+1}^{k} \mathbf{f}_i\mathbf{g}_i^\top.$$

Hence,

$$FJG - (FJ)[\gamma,\alpha]J[\alpha,\alpha]^{\dagger}(JG)[\alpha,\beta] = \sum_{i=1}^{q} \mathbf{f}_i \mathbf{g}_i^{\top} \geq 0. \tag{2.24}$$

In particular, this proves that (2.8) holds.

It is left to show that $A_{\dagger}(\alpha)$ is an idempotent matrix. Clearly, if $q = 0$ then $A_{\dagger}(\alpha) = 0$. So $A_{\dagger}(\alpha)$ is a trivial idempotent matrix, and (2.9) yields (2.4).

Assuming finally that $q > 0$, it follows that $A_{\dagger}(\alpha)$ has the block form (2.2) with $J = \oplus_{i=1}^{q} J_i \oplus 0$. Hence $A_{\dagger}(\alpha)$ is an idempotent matrix whose rank is q, and (2.9) yields (2.4). ∎

Corollary 2.3. *Let $A \in \mathbb{R}_{+}^{n\times n}, A \neq 0$ be idempotent. If $\alpha \subsetneqq \langle n\rangle$ is chosen such that $A[\alpha,\alpha]$ is an invertible matrix, then $A[\alpha,\alpha]$ is diagonal.*

Proof. Note that the number ℓ in the proof of Theorem 2.2 is either zero or the corresponding blocks J_i are positive 1×1 matrices for $i = q+p+1,\dots,q+p+\ell$. Furthermore, for the split blocks, we also have that $\mathbf{x}_i \mathbf{y}_i^T \in \mathbb{R}^{1\times 1}$, for $i = q+1,\dots,q+p$, since $\mathbf{x}_i \mathbf{y}_i^T$ is of rank 1. Therefore, $A[\alpha,\alpha]$ is diagonal. ∎

Corollary 2.4. *Let $A \in \mathbb{R}_{+}^{n\times n}, A \neq 0$ be idempotent. If $\alpha \subsetneqq \langle n\rangle$ is chosen such that $A[\alpha,\alpha]$ is a regular matrix, then the standard Schur complement (1.1) is nonnegative.*

Corollary 2.5. *Let $A \in \mathbb{R}_{+}^{n\times n}, A \neq 0$ be idempotent. Choose $\alpha \subsetneqq \langle n\rangle$, such that $I - A[\alpha,\alpha]$ is regular. Then, $\tilde{A}(\alpha)$ defined by*

$$\tilde{A}(\alpha) := A[\alpha^c,\alpha^c] + A[\alpha^c,\alpha](I - A[\alpha,\alpha])^{-1}A[\alpha,\alpha^c] \tag{2.25}$$

is a nonnegative idempotent matrix.

To prove this Corollary 2.5 we need the following fact for idempotent matrices, which is probably known.

Lemma 2.6. *Let $A \in \mathbb{R}^{n\times n}, A \neq 0$ be idempotent given as a 2×2 block matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Assume that $I - A_{22} \in \mathbb{R}^{n-m}$ is regular. Then $B := A_{11} + A_{12}(I - A_{22})^{-1}A_{21}$ is idempotent.*

Proof. Let

$$E = (I - A_{22})^{-1}A_{21},\ D = A_{21} + A_{22}E,\ z = \begin{bmatrix} x \\ Ex \end{bmatrix} \in \mathbb{R}^n,\ x \text{ any vector in } \mathbb{R}^m.$$

Note that $Az = \begin{bmatrix} Bx \\ Dx \end{bmatrix}$. As $A^2 z = Az$ and x is an arbitrary vector, we obtain the equalities

$$A_{11}B + A_{12}D = B, \quad A_{21}B + A_{22}D = D. \tag{2.26}$$

From the second equality of (2.26) we obtain $D = EB$. Substituting this equality into the first equality of (2.26) we obtain that $B^2 = B$. ∎

Proof of Corollary 2.5. The assumption that $I - A[\alpha,\alpha]$ is regular implies that $A[\alpha,\alpha]$ does not have an eigenvalue 1, i.e., $\rho(A[\alpha,\alpha]) < 1$. Hence, $I - A[\alpha,\alpha]$ is an M-matrix [1] and $(I - A[\alpha,\alpha])^{-1} \geq 0$. The assertion of Corollary 2.5 now follows using Lemma 2.6. ∎

3. Additional results.

3.1. An example. In this subsection we assume that the nonnegative idempotent matrix A is of the special form

$$A := \begin{bmatrix} J & JG \\ 0 & 0 \end{bmatrix}. \tag{3.1}$$

Furthermore, we assume that $A[\alpha,\alpha]$ has a zero on its main diagonal. We give an example where $A_{\dagger}(\alpha)$ may fail to be nonnegative. To this end, we first start with the following known result.

Lemma 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a singular matrix of the following form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{bmatrix},\ A_{11} \in \mathbb{R}^{p\times p}, A_{12} \in \mathbb{R}^{p\times(n-p)},\ \textit{for some } 1 \le p < n.$$

Then $(A^\dagger)^\top$ has the same block form as A.

Proof. Let $r = \operatorname{rank} A$. So $r \le p$. Then the reduced singular value decomposition of A is of the form $U_r\Sigma_r V_r^\top$, where $U_r, V_r \in \mathbb{R}^{n\times r}, U_r^\top U_r = V_r V_r^\top = I_r$ and Σ_r is a diagonal matrix, whose diagonal entries are the positive singular values of A.

Clearly, $AA^\top = \begin{bmatrix} A_{11}A_{11}^\top + A_{12}A_{12}^\top & 0 \\ 0 & 0 \end{bmatrix}$. Hence all eigenvectors of $AA^\top$, corresponding to positive eigenvalues are of the form $(\mathbf{x}^\top, \mathbf{0}^\top)^\top, \mathbf{x} \in \mathbb{R}^p$. Thus $U_r^\top = [U_{r1}^\top\ 0_{r\times(n-p)}]$ where $U_{r1} \in \mathbb{R}^{p\times r}$. Recall that $A^\dagger = V_r\Sigma_r^{-1}U_r^\top$. The above form of U_r establishes the lemma. ☐

In the following example we permute some rows and columns of A, in order to find the Schur complement of the right lower block.

Example 3.2. *Consider a nonnegative idempotent matrix in the block form*

$$B = \left[\begin{array}{ccc|cc} \mathbf{u}_1\mathbf{v}_1^\top & 0 & \mathbf{u}_1\mathbf{s}_1^\top & \mathbf{u}_1\mathbf{t}_1^\top & 0 \\ 0 & \mathbf{a}_2\mathbf{b}_2^\top & \mathbf{a}_2\mathbf{s}_2^\top & \mathbf{a}_1\mathbf{t}_2^\top & \mathbf{a}_2\mathbf{y}_2^\top \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x}_2\mathbf{b}_2^\top & \mathbf{x}_2\mathbf{s}_2^\top & \mathbf{x}_1\mathbf{t}_2^\top & \mathbf{x}_2\mathbf{y}_2^\top \end{array}\right].$$

Then,

$$B[\alpha,\alpha] = \begin{bmatrix} 0 & 0 \\ \mathbf{x}_1\mathbf{t}_2^\top & \mathbf{x}_2\mathbf{y}_2^\top \end{bmatrix},\ B[\alpha,\alpha]^\dagger = \begin{bmatrix} 0 & \frac{\mathbf{t}_2\mathbf{x}_2^\top}{(\mathbf{x}_2^\top\mathbf{x}_2)(\mathbf{t}_2^\top\mathbf{t}_2 + y_2^T y_2)} \\ 0 & \frac{\mathbf{y}_2\mathbf{x}_2^\top}{(\mathbf{x}_2^\top\mathbf{x}_2)(\mathbf{t}_2^\top\mathbf{t}_2 + \mathbf{y}_2^\top y_2)} \end{bmatrix},$$

and

$$B[\alpha^c,\alpha]B[\alpha,\alpha]^\dagger B[\alpha,\alpha^c] = \begin{bmatrix} 0 & \frac{\mathbf{t}_1^\top\mathbf{t}_2\mathbf{u}_1\mathbf{b}_2^\top}{\mathbf{t}_2^\top\mathbf{t}_2 + \mathbf{y}_2^\top\mathbf{y}_2} & \frac{\mathbf{t}_1^\top\mathbf{t}_2\mathbf{u}_1\mathbf{s}_2^\top}{\mathbf{t}_2^\top\mathbf{t}_2 + \mathbf{y}_2^\top\mathbf{y}_2} \\ 0 & \mathbf{a}_2\mathbf{b}_2^\top & \mathbf{a}_2\mathbf{s}_2^\top \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $B_\dagger(\alpha)_{11} > 0$, $B_\dagger(\alpha)_{12} \le 0$ and the Moore-Penrose inverse Schur complement is neither nonnegative nor nonpositive if $t_1^T t_2 > 0$.

3.2. Nonnegativity of semi-inverse Schur complement. In this section we extend the results of Section 2 for idempotent matrices of the form (2.2) for some Schur complements with zero diagonal entries. We start with the following simple observation.

Proposition 3.3. *Let the assumptions of Lemma 3.1 hold. Suppose that*

$$A_{11}(A_{11})^\dagger A_{12} = A_{12}.$$

Then $A^{\mathrm{ginv}} = \begin{bmatrix} (A_{11})^\dagger & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{bmatrix}$ is a semi-inverse of A. In particular any principle submatrix of an idempotent matrix as in (3.1) with at least one zero diagonal element has a semi-inverse of this form.

Proof. The proposition follows by checking the conditions in (2.1). ☐

Note that condition $A_{11}(A_{11})^\dagger A_{12} = A_{12}$ holds in general for idempotent matrices A of the form as in (3.1).

The following Theorem states the general result of this subsection.

Theorem 3.4. *Let $A \in \mathbb{R}_+^{n\times n}$ be of the form (2.2), where $n_2 + n_3 \ge 1$ and the condition in (2.3) holds. Furthermore, let $\alpha_1 \subset \langle n\rangle$ be of the following form*

$$\begin{array}{ll} \textit{either} & \alpha_1 = \alpha \cup \beta, \emptyset \ne \beta \subseteq n_1 + \langle n_2\rangle, \\ \textit{or} & \alpha_1 = \alpha \cup \gamma, \emptyset \ne \gamma \subseteq n_1 + n_2 + \langle n_3\rangle, \end{array} \tag{3.2}$$

where $\alpha \subseteq \langle n_1 \rangle$. *Then, there exists a semi-inverse* $A^{\mathrm{ginv}}[\alpha_1, \alpha_1]$ *of* $A[\alpha_1, \alpha_1]$ *such that* $A_{\mathrm{ginv}}(\alpha_1)$ *as defined in (1.2) is a nonnegative idempotent matrix. The rank of* $A_{ginv}(\alpha_1)$ *is equal to the multiplicity of the eigenvalue* 1 *in* $A[\alpha', \alpha']$, *where* $\alpha' = \langle n_1 \rangle \backslash \alpha$. *In particular, if* 1 *is not an eigenvalue of* $A[\alpha', \alpha']$, *then* $A_{\mathrm{ginv}}(\alpha) = 0$.

Proof. First we consider the case that $\alpha_1 = \alpha \cup \beta$. If $\alpha = \emptyset$, then $A[\alpha_1, \alpha_1]$ and $A[\alpha_1, \alpha_1]^{\mathrm{ginv}}$ are zero matrices for any semi-inverse and $A_{\mathrm{ginv}}(\alpha_1) = A[\alpha_1^c, \alpha_1^c]$. Using the proof of Theorem 2.2 we obtain that $A_{\mathrm{ginv}}(\alpha_1)$ is a nonnegative idempotent matrix of rank k.

Assuming now that $\alpha \neq \emptyset$, we observe that $A[\alpha_1, \alpha_1]$ satisfies the assumption of Proposition 3.3. Defining $A[\alpha_1, \alpha_1]^{\mathrm{ginv}}$ as in Proposition 3.3 and following the arguments of the proof of Theorem 2.2 we deduce the theorem in this case.

We assume now that $\alpha_1 = \alpha \cup \gamma$. If $\alpha = \emptyset$ we obtain that $A_{\mathrm{ginv}}(\alpha_1)$ is a nonnegative idempotent matrix of rank k as above. Assuming finally that $\alpha \neq \emptyset$, we have that $A[\alpha_1, \alpha_1]^\top$ satisfies the assumption of Proposition 3.3. Define $(A[\alpha_1, \alpha_1]^\top)^{\mathrm{ginv}}$ as in Proposition 3.3 and let $A[\alpha_1, \alpha_1]^{\mathrm{ginv}} := ((A[\alpha_1, \alpha_1]^\top)^{\mathrm{ginv}})^\top$. Repeating the arguments of the proof of Theorem 2.2 we deduce the theorem in this case. $\square$

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