

# REGULARIZED STATE-CONSTRAINED BOUNDARY OPTIMAL CONTROL OF THE NAVIER-STOKES EQUATIONS

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ABSTRACT. The numerical solution of the Dirichlet boundary optimal control problem of the Navier-Stokes equations in presence of pointwise state constraints is investigated. Two different regularization techniques are considered. First, a Moreau-Yosida regularization of the problem is studied. Optimality conditions are derived and the convergence of the regularized solutions towards the original one is proved. A source representation of the control combined with a Lavrentiev type regularization strategy is also presented. The analysis concerning optimality conditions and convergence of the regularized solutions is carried out. In the last part of the paper numerical experiments are presented. For the numerical solution of each regularized problem a semi-smooth Newton method is applied.

## 1. INTRODUCTION

The introduction of state constraints in optimal flow control problems constitutes a recent approach in order to reach a desired fluid flow behavior or deal with multiple optimization objectives.. In particular, the reduction of flow recirculations can be reached by imposing pointwise box constraints on the state, while the energy needed is minimized. This approach has been studied analytically for distributed controls in [14, 30, 10] and numerically in [12, 13]. Despite its practical importance, the boundary control case has not been treated yet.

Optimal control problems of the Navier-Stokes equations with Dirichlet boundary controls have been studied in [16, 17, 18, 20] in absence of inequality constraints and in [11] in presence of pointwise control constraints. An important issue in this kind of problems is the choice of appropriate control spaces and correspondent cost functionals in order to get existence of an optimal solution and derive optimality conditions. In particular,  $\mathbf{L}^q$  tracking type terms in connection with Dirichlet boundary controls have been considered in [16, 18]. In those contributions a  $\mathbf{L}^q$  tracking norm, with  $q \geq 4$ , has to be chosen in order to get existence of an optimal solution. Differently from [16, 18] and thanks to the presence of pointwise state constraints, we are able to overcome this difficulty and consider a  $\mathbf{L}^2$  tracking type term in the cost functional (see Theorem 2.2). For that purpose, an appropriate a priori estimate is obtained (see Proposition 2.1), which is also needed to get approximation results.

On the other hand, the numerical treatment of pointwise state constrained optimal control problems presents important difficulties related to the lack of regularity of the Lagrange multipliers associated with the state constraints (cf. Casas [5, 6], Alibert and Raymond [1] and Bergounioux and Kunisch [3]). Moreover, the direct application of infinite dimensional semismooth Newton methods (cf. [19, 23]) is not possible in this context.

To overcome the difficulties arising from the low regularity of the multipliers associated to the state constraints, two regularization concepts were proposed in recent years. First, Ito and Kunisch [22] suggested the use of a Moreau-Yosida type regularization approach,

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which removes the pointwise state inequality constraints by adding a penalty term to the objective functional. Hereafter, the penalized problems are solved in an efficient way by using semi-smooth Newton methods. We also refer to [2, 4, 21]. Secondly, a Lavrentiev type regularization (cf. [24]) of the pointwise state inequality constraints was introduced by Meyer, Rösch and Tröltzsch in [25]. This concept is extended to Neumann boundary control problems in [28, 29] by including a source representation of control data.

In this article we investigate the application of both regularization techniques to the optimal Dirichlet boundary control of the Navier-Stokes equations. After introducing an appropriate control space and a tracking type cost functional, a comprehensive study concerning optimality conditions as well as convergence results of the regularized solutions is presented. Apart of proving convergence of global optimal solutions of the regularized problems towards a global solution of the original problem (see [22, 25]), we also show that any local optimal solution can be approximated by local optimal solutions of the regularized problems. For this result, a quadratic growth condition on the local optimal solution has to be assumed. Note that the study of both regularization strategies does not involve a numerical comparison between them. This would go beyond the scope of this paper.

Let us remark that, to the authors knowledge, no previous work on Lavrentiev regularization of Dirichlet optimal control problems has been carried out. This is a further novelty of this article.

The outline of the paper is as follows. In Section 2 the original control problem and the functional setting are stated. In Section 3 we introduce a Moreau-Yosida type regularization of the problem and afterwards the penalized problems are investigated. In Section 4 a source representation of the control and a Lavrentiev type regularization are proposed. In Section 5, a semi-smooth Newton algorithm for the solution of each Moreau-Yosida regularized problem is presented and numerical experiments are carried out.

## 2. PROBLEM STATEMENT

Consider a bounded regular domain  $\Omega \subset \mathbb{R}^2$ . Further, let  $\Gamma_1$  be an open subset of the boundary  $\Gamma$ . Our aim is to find a solution of the following optimal control problem:

$$(2.1) \quad \begin{cases} \min J(y, u) := \frac{1}{2} \|y - z_d\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\alpha}{2} \|u\|_{\mathcal{U}}^2 \\ \text{subject to} \\ -\nu \Delta y + (y \cdot \nabla) y + \nabla p = f & \text{in } \Omega \\ \operatorname{div} y = 0 & \text{in } \Omega \\ y|_{\Gamma} = g + \mathcal{B}u & \text{on } \Gamma \\ a \leq y \leq b & \text{a.e. in } \Omega, \end{cases}$$

where  $\alpha > 0$ ,  $\nu = \frac{1}{Re} > 0$  stands for the viscosity coefficient and  $Re$  for the Reynolds number of the fluid,  $z_d$  is the desired state and  $\mathcal{U}$  is the control space. The operator  $\mathcal{B}$  is the extension by zero operator, which will be specified later. The functions  $f \in \mathbf{L}^2(\Omega)$  and  $g \in \mathbf{H}^{1/2}(\Gamma)$ , with  $\int_{\Gamma} g \cdot \vec{n} \, d\Gamma = 0$  ( $\vec{n}$  the unit vector normal to the boundary), are given. Moreover, the lower and upper bounds  $a, b \in \mathbf{C}(\overline{\Omega})$  satisfy  $a(x) < b(x)$  for all  $x \in \overline{\Omega}$ . The inequalities in the last line of (2.1) have to be understood componentwise. We denote by  $(\cdot, \cdot)_X$  the inner product in the Hilbert space  $X$  and by  $\|\cdot\|_X$  the associated norm. The subindex is suppressed if the  $L^2$ -inner product or norm are meant. Hereafter, the bold notation stands for the product of spaces. Additionally, we introduce the solenoidal space  $V = \{v \in \mathbf{H}_0^1(\Omega) : \operatorname{div} v = 0\}$ , the closed subspaces  $\mathbf{H}_{\operatorname{div}}^q := \{v \in \mathbf{H}^q(\Omega) : \operatorname{div} v = 0\}$  and the trilinear form  $c : \mathbf{H}_{\operatorname{div}}^1 \times \mathbf{H}_{\operatorname{div}}^1 \times \mathbf{H}_{\operatorname{div}}^1 \rightarrow \mathbb{R}$  defined by

$$(2.2) \quad c(u, w, v) = ((u \cdot \nabla)w, v).$$

The weak formulation of the Navier-Stokes equations in the control problem (2.1) is then given by: Find  $y \in \mathbf{H}_{\text{div}}^1$  such that

$$(2.3) \quad \nu(\nabla y, \nabla v) + c(y, y, v) = (f, v), \text{ for all } v \in V,$$

$$(2.4) \quad \gamma_0 y = g + \mathcal{B}u,$$

where  $\gamma_0 : \mathbf{H}^1(\Omega) \rightarrow \mathbf{H}^{1/2}(\Gamma)$ . For later use, we define the bilinear form  $a : \mathbf{H}_{\text{div}}^1 \times \mathbf{H}_{\text{div}}^1 \rightarrow \mathbb{R}$  with  $a(y_1, y_2) = (\nabla y_1, \nabla y_2)$  and set

$$(2.5) \quad \mathcal{M}(y) := \sup_{v \in V} \frac{|c(v, y, v)|}{\|v\|_V^2} \quad \forall y \in \mathbf{H}_{\text{div}}^1.$$

Let us now consider the control space associated with (2.1) which is given by

$$(2.6) \quad \mathcal{U} = \{u \in \mathbf{H}_0^1(\Gamma_1) \mid \int_{\Gamma_1} u \cdot \vec{n} \, d\Gamma = 0\}.$$

Introducing the functional  $\Phi : \mathbf{H}_0^1(\Gamma_1) \rightarrow \mathbb{R}$  with  $\Phi(u) = \int_{\Gamma_1} u \cdot \vec{n} \, d\Gamma$ , the control space  $\mathcal{U}$  can also be written as

$$\mathcal{U} = \mathcal{N}(\Phi),$$

where  $\mathcal{N}(\Phi) \subset \mathbf{H}_0^1(\Gamma_1)$  denotes the kernel of  $\Phi$ . Notice that the continuity of  $\Phi$  immediately implies that  $\mathcal{N}(\Phi)$  is a closed subspace of  $\mathbf{H}_0^1(\Gamma_1)$ . In particular, the control space  $\mathcal{U}$  is a Hilbert space with the induced norm. Next, we formulate the operator  $\mathcal{B}$  which appears in the control problem (2.1):

$$\mathcal{B} : \mathbf{H}_0^1(\Gamma_1) \rightarrow \mathbf{H}^1(\Gamma), \quad \mathcal{B}u = \begin{cases} u & \text{in } \Gamma_1 \\ 0 & \text{in } \Gamma \setminus \Gamma_1. \end{cases}$$

In the following theorem, we summarize the main results about existence and uniqueness of the Navier-Stokes solutions.

**Theorem 2.1.** *Let  $f \in \mathbf{H}^{-1}(\Omega)$ ,  $g \in \mathbf{H}^{1/2}(\Gamma)$ , with  $\int_{\Gamma} g \cdot \vec{n} \, d\Gamma = 0$ , and  $u \in \mathcal{U}$ . There exists at least one solution  $(y, p) \in \mathbf{H}_{\text{div}}^1 \times L_0^2(\Omega)$  for the non-homogeneous stationary Navier-Stokes equations (2.3)-(2.4), that satisfies the estimate*

$$(2.7) \quad \|y - \hat{y}\|_V \leq \frac{2}{\nu} \|F\|_{V'},$$

where  $\hat{y} \in \mathbf{H}_{\text{div}}^1$  is a function such that  $\hat{y}|_{\Gamma} = g + \mathcal{B}u$  and  $F = f + \nu \Delta \hat{y} - (\hat{y} \cdot \nabla) \hat{y}$ . Moreover, if

$$|c(v, \hat{y}, v)| \leq \frac{\nu}{2} \|v\|_V^2 \text{ for all } v \in V$$

and  $\nu^2 > 4\mathcal{N} \|F\|_{V'}$ , with  $\mathcal{N} = \sup_{u, v, w \in V} \frac{|c(u, v, w)|}{\|u\|_V \|v\|_V \|w\|_V}$ , then the solution is unique.

*Proof.* For the proof we refer to [27], pp. 178-180.  $\square$

Assuming higher regularity on the functions  $f$  and  $g$ , we establish a priori estimates for the velocity field  $y$  in the space  $\mathbf{H}^{3/2}(\Omega)$ .

**Proposition 2.1.** *If additionally to the hypotheses of Theorem 2.1,  $f \in \mathbf{L}^2(\Omega)$  and  $g \in \mathbf{H}^1(\Gamma)$ , then any Navier-Stokes solution satisfies the extra regularity  $y \in \mathbf{H}^{3/2}(\Omega) \hookrightarrow \mathbf{C}(\bar{\Omega})$  and the following estimate:*

$$(2.8) \quad \|y\|_{\mathbf{H}^{3/2}(\Omega)} + \|p\|_{H^{1/2}(\Omega)/\mathbb{R}} \leq \kappa \left( \|f\|_{\mathbf{L}^2(\Omega)}^2 + \|g\|_{\mathbf{H}^1(\Gamma)} + \|y\|_{\mathbf{L}^2(\Omega)}^4 + \|y\|_{\mathbf{L}^\infty(\Omega)}^4 + \|u\|_{\mathbf{H}_0^1(\Gamma_1)}^2 \right),$$

with a constant  $\kappa > 0$  depending only on  $\nu$  and  $\Omega$ . Moreover, if  $\nu > \mathcal{M}(y)$ , then the following estimate holds:

$$(2.9) \quad \|y\|_{\mathbf{H}^{3/2}(\Omega)} + \|p\|_{H^{1/2}(\Omega)/\mathbb{R}} \leq \kappa \left( \|f\|_{\mathbf{L}^2(\Omega)}^2 + \|g\|_{\mathbf{H}^1(\Gamma)}^4 + \|y\|_{\mathbf{L}^2(\Omega)}^4 + \|u\|_{\mathbf{H}_0^1(\Gamma_1)}^4 \right).$$

*Proof.* The term  $(y \cdot \nabla)y$  can also be written as  $\sum_i y_i \partial_i y$  or, since  $\operatorname{div} y = 0$ , as  $\sum_i \partial_i (y_i y)$ . Since the continuous embedding  $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$  holds for all  $1 \leq q < \infty$ , we particularly obtain that  $y_i \partial_i y \in \mathbf{L}^{3/2}(\Omega)$ . Also from Sobolev inequalities, since the embedding  $\mathbf{H}^{1/2}(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$  is dense and continuous, it follows that  $y_i \partial_i y \in \mathbf{H}^{-1/2}(\Omega)$ . From the regularity results for the non-homogeneous Stokes equations we obtain (see [27, Ch.I, Prop.2.2 and Rem.2.6]) that  $y \in \mathbf{H}^{3/2}(\Omega)$ ,  $p \in H^{1/2}(\Omega)$  and the following estimate holds:

$$(2.10) \quad \|y\|_{\mathbf{H}^{3/2}(\Omega)} + \|p\|_{H^{1/2}(\Omega)/\mathbb{R}} \leq c_0(\|f - (y \cdot \nabla)y\|_{\mathbf{H}^{-1/2}(\Omega)} + \|g + \mathcal{B}u\|_{\mathbf{H}^1(\Gamma)}),$$

where  $c_0 = c_0(\nu, \Omega) \geq 0$ .

Let us next consider the term  $\|(y \cdot \nabla)y\|_{\mathbf{H}^{-1/2}(\Omega)}$ . From the properties of the nonlinear term (see [9, p.50]) it follows that:

$$(2.11) \quad \|(y \cdot \nabla)y\|_{\mathbf{H}^{-1/2}(\Omega)} = \sup_{z \in \mathbf{H}^{1/2}(\Omega)} \frac{|c(y, y, z)|}{\|z\|_{\mathbf{H}^{1/2}(\Omega)}} \leq c(\Omega) \|y\|_{\mathbf{H}^1(\Omega)}^2,$$

with  $c(\Omega) > 0$ . In the following,  $c(\Omega) > 0$  denotes a generic constant. For the  $\mathbf{H}^1$ -estimate of the velocity, let us consider the following auxiliary Stokes problem: Find  $y_s \in \mathbf{H}_{\operatorname{div}}^1$  such that

$$(2.12) \quad \nu(\nabla y_s, \nabla v) = 0, \text{ for all } v \in V,$$

$$(2.13) \quad \gamma_0 y_s = g + \mathcal{B}u.$$

It is well-known that there exists a unique solution  $y_s \in \mathbf{H}_{\operatorname{div}}^1$  to (2.12)-(2.13). Moreover, there exists a constant  $c(\Omega) > 0$  such that

$$(2.14) \quad \|y_s\|_{\mathbf{H}^1(\Omega)} \leq c(\Omega) \|g + \mathcal{B}u\|_{\mathbf{H}^{1/2}(\Gamma)}.$$

Further, we define  $w := y - y_s$  that satisfies the equation

$$(2.15) \quad \begin{aligned} \nu(\nabla w, \nabla v) + c(y, y, v) &= (f, v), \text{ for all } v \in V \\ \gamma_0 w &= 0. \end{aligned}$$

Taking  $v = w$  as test function, it immediately follows that

$$\nu \|w\|_{\mathbf{H}_0^1(\Omega)}^2 \leq c(\Omega) (\|f\|_{\mathbf{L}^2(\Omega)} \|w\|_{\mathbf{H}_0^1(\Omega)} + \|y\|_{\mathbf{L}^2(\Omega)} \|y\|_{\mathbf{L}^\infty(\Omega)} \|w\|_{\mathbf{H}_0^1(\Omega)}).$$

The latter inequality together with (2.14) implies that

$$(2.16) \quad \nu \|y\|_{\mathbf{H}^1(\Omega)} \leq c(\Omega) (\|f\|_{\mathbf{L}^2(\Omega)} + \nu \|g\|_{\mathbf{H}^{1/2}(\Gamma)} + \nu \|u\|_{\mathbf{H}_0^1(\Gamma_1)} + \|y\|_{\mathbf{L}^\infty(\Omega)} \|y\|_{\mathbf{L}^2(\Omega)}).$$

Using estimates (2.16) and (2.11) in (2.10) yields the existence of a constant  $\kappa_1 = \kappa_1(\nu, \Omega) > 0$  such that

$$\|y\|_{\mathbf{H}^{3/2}(\Omega)} + \|p\|_{H^{1/2}(\Omega)/\mathbb{R}} \leq \kappa_1 (\|f\|_{\mathbf{L}^2(\Omega)}^2 + \|g\|_{\mathbf{H}^1(\Gamma)}^2 + \|y\|_{\mathbf{L}^2(\Omega)}^4 + \|y\|_{\mathbf{L}^\infty(\Omega)}^4 + \|u\|_{\mathbf{H}_0^1(\Gamma_1)}^2).$$

Assume now that the solution satisfies  $\nu > \mathcal{M}(y)$ . Setting  $v = w$  in (2.15), we obtain that

$$\nu \|w\|_{\mathbf{H}_0^1(\Omega)}^2 + c(w, y, w) = (f, w) - c(y_s, y, w),$$

which implies that

$$(2.17) \quad (\nu - \mathcal{M}(y)) \|w\|_{\mathbf{H}_0^1(\Omega)} \leq c(\Omega) (\|f\|_{\mathbf{L}^2(\Omega)} + \|y_s\|_{\mathbf{L}^\infty(\Omega)} \|y\|_{\mathbf{L}^2(\Omega)}).$$

Due to the regularity of the boundary data, we get from (2.12)-(2.13) and the embedding  $\mathbf{H}^{3/2}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$  the Stokes estimate

$$(2.18) \quad \|y_s\|_{\mathbf{L}^\infty(\Omega)} \leq c(\Omega) \|y_s\|_{\mathbf{H}^{3/2}(\Omega)} \leq c(\Omega) (\|g\|_{\mathbf{H}^1(\Gamma)} + \|u\|_{\mathbf{H}_0^1(\Gamma_1)}).$$

Therefore, applying this to (2.17), it holds that

$$\|y\|_{\mathbf{H}^1(\Omega)} \leq \kappa_2 \left( \|f\|_{\mathbf{L}^2(\Omega)} + \|y\|_{\mathbf{L}^2(\Omega)} \|g\|_{\mathbf{H}^1(\Gamma)} + \|y\|_{\mathbf{L}^2(\Omega)} \|u\|_{\mathbf{H}_0^1(\Gamma_1)} + \|g\|_{\mathbf{H}^1(\Gamma)} + \|u\|_{\mathbf{H}_0^1(\Gamma_1)} \right)$$

with  $\kappa_2 = \kappa_2(\nu, \Omega) > 0$ . Combining (2.10), (2.11) and the latter inequality, we obtain (2.9).  $\square$

For the rest of the paper, let  $f \in \mathbf{L}^2(\Omega)$  and  $g \in \mathbf{H}^1(\Gamma)$ , with  $\int_{\Gamma} g \cdot \vec{n} \, d\Gamma = 0$ , be fixed. Consider further the following set:

$$(2.19) \quad \mathcal{T} := \{(y, u) \in \mathbf{H}_{\text{div}}^{3/2} \times \mathcal{U} \mid (y, u) \text{ satisfies (2.3) -- (2.4)}\}.$$

The admissible set associated with (2.1) is defined by

$$(2.20) \quad U_{ad} := \{(y, u) \in \mathcal{T} \mid a \leq y \leq b \text{ a.e. in } \Omega\}.$$

Next, we show the solvability of (2.1).

**Theorem 2.2.** *Assume that the admissible set  $U_{ad}$  is not empty. Then, (2.1) admits a solution  $(\bar{y}, \bar{u}) \in U_{ad}$ .*

*Proof.* Since the admissible set  $U_{ad} \neq \emptyset$ , there exists an infimal sequence  $\{y_n, u_n\}_{n=1}^{\infty} \subset U_{ad}$ . Since the objective functional in (2.1) is nonnegative, the infimum in (2.1) exists in  $\mathbb{R}_0^+$ . Consequently, there exists a constant  $c > 0$  such that

$$c \geq J(y_n, u_n) = \frac{1}{2} \|y_n - z_d\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\alpha}{2} \|u_n\|_{\mathbf{H}_0^1(\Gamma_1)}^2 \quad \text{for all } n \in \mathbb{N}.$$

This implies the uniform boundedness of  $\{u_n\}_{n=1}^{\infty}$  in  $\mathcal{U}$  and of  $\{y_n\}_{n=1}^{\infty}$  in  $\mathbf{L}^2(\Omega)$ . Since each  $y_n$  satisfies the state constraints, the sequence  $\{y_n\}_{n=1}^{\infty}$  is also bounded in  $\mathbf{L}^{\infty}(\Omega)$ . Therefore, from estimate (2.16), the sequence  $\{y_n\}_{n=1}^{\infty}$  is bounded in  $\mathbf{H}_{\text{div}}^{3/2}$ . Hence there exists subsequences  $\{(u_{n_k})\}_{k=1}^{\infty}$  and  $\{(y_{n_k})\}_{k=1}^{\infty}$  such that

$$u_{n_k} \rightharpoonup \bar{u}, \text{ weakly in } \mathcal{U}, \quad y_{n_k} \rightharpoonup \bar{y}, \text{ weakly in } \mathbf{H}_{\text{div}}^{3/2}.$$

It is well known that the trilinear form  $c(\cdot, \cdot, \cdot)$  in (2.3) is weakly sequentially continuous, cf. [15, p.286]. Thus, by the linearity and continuity of  $a(\cdot, \cdot)$  and of the trace operator, it follows that  $(\bar{y}, \bar{u}) \in \mathcal{T}$ . Moreover, since the set

$$\{y \in \mathbf{H}_{\text{div}}^{3/2} \mid a \leq y \leq b \text{ a.e. in } \Omega\}$$

is weakly closed, we get that  $(\bar{y}, \bar{u}) \in U_{ad}$ . Finally, since the cost functional is weakly lower semicontinuous,

$$J(\bar{y}, \bar{u}) \leq \liminf_{k \rightarrow \infty} J(y_{n_k}, u_{n_k}) = \inf_{(y, u) \in U_{ad}} J(y, u).$$

Thus,  $(\bar{y}, \bar{u}) \in \mathbf{H}_{\text{div}}^{3/2} \times \mathcal{U}$  minimizes the control problem (2.1).  $\square$

For the derivation of first order necessary optimality conditions of the regularized problems the differentiability of the control-to-state operator will be needed. This is proven in the following Lemma.

**Lemma 2.1.** *Let  $\bar{u} \in \mathcal{U}$  and let  $\bar{y}$  be a solution to the Navier-Stokes equations such that  $\nu > \mathcal{M}(\bar{y})$ . There exists a neighborhood  $B(\bar{u})$  around  $\bar{u}$  in  $\mathcal{U}$  such that the control-to-state mapping  $S : B(\bar{u}) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$ , that assigns to each  $u \in B(\bar{u})$  the unique solution  $y \in \mathbf{H}_{\text{div}}^{3/2}$  of (2.3)-(2.4), is well-defined. Furthermore,  $S$  is twice Fréchet differentiable at  $\bar{u}$  and its derivatives  $w_h := S'(\bar{u})h$  and  $w_{hh} := S''(\bar{u})[h]^2$  are given by the unique solutions of the systems:*

$$(2.21) \quad \begin{aligned} -\nu \Delta w_h + (w_h \cdot \nabla) \bar{y} + (\bar{y} \cdot \nabla) w_h + \nabla \pi &= 0 \\ \operatorname{div} w_h &= 0 \\ w_h|_{\Gamma} &= \mathcal{B}h \end{aligned}$$



This regularization approach has been utilized for state constrained control problems in [22]. Hereafter, the same penalization was applied to different constrained control problems (see [11, 26]).

**3.1. Optimality conditions.** We begin the study of the regularized optimal control problem by obtaining an optimality system that characterizes any regularized local optimal solution to  $(\mathcal{P}_\gamma)$ .

**Definition 3.1** (Local solution to (2.1)). The pair  $(y^*, u^*) \in U_{ad}$  is called a local solution of (2.1) with respect to the  $\mathcal{U}$ -topology if there exists a positive real number such that

$$(3.1) \quad J(y^*, u^*) \leq J(y, u)$$

for all  $(y, u) \in U_{ad}$  with  $\|u - u^*\|_{\mathcal{U}} \leq c$ .

Analogously, we introduce the following definition concerning local solutions to the penalized problem  $(\mathcal{P}_\gamma)$

**Definition 3.2** (Local solution to  $(\mathcal{P}_\gamma)$ ). Let  $\gamma > 0$ . Then  $(y_\gamma, u_\gamma) \in \mathcal{T}$  is called a local solution of  $(\mathcal{P}_\gamma)$  with respect to the  $\mathcal{U}$ -topology if there exists a positive real number  $c > 0$  such that

$$(3.2) \quad J_\gamma(y_\gamma, u_\gamma) \leq J_\gamma(y, u)$$

for all  $(y, u) \in \mathcal{T}$  with  $\|u - u_\gamma\|_{\mathcal{U}} \leq c$ .

Next, a result about the orthogonal decomposition of the control space is stated. The decomposition is afterwards used for the derivation of the optimality system.

**Lemma 3.1.** *The orthogonal space of  $\mathcal{U}$  can be characterized as  $\mathcal{U}^\perp = \{\sigma f_{\vec{n}} : \sigma \in \mathbb{R}\}$ , where  $f_{\vec{n}}$  denotes the Riesz representative of  $\vec{n}$  in  $\mathbf{H}_0^1(\Gamma_1)$ .*

*Proof.* As previously noted, the space  $\mathcal{U}$  is a closed subspace of  $\mathbf{H}_0^1(\Gamma_1)$ . Therefore, the space  $\mathbf{H}_0^1(\Gamma_1)$  can be decomposed as  $\mathbf{H}_0^1(\Gamma_1) = \mathcal{U}^\perp \oplus \mathcal{U}$ . Hence, taking an arbitrary but fix  $y \in \mathbf{H}_0^1(\Gamma_1)$  we can express it uniquely as  $y = y_1 + y_2$ , with  $y_1 \in \mathcal{U}$  and  $y_2 \in \mathcal{U}^\perp$ . Let us take the ansatz  $y_2 = \sigma f_{\vec{n}}$ . Then  $y - \sigma f_{\vec{n}}$  satisfies

$$(y - \sigma f_{\vec{n}}, \vec{n})_{\mathbf{L}^2(\Gamma_1)} = (y, \vec{n})_{\mathbf{L}^2(\Gamma_1)} - \sigma (f_{\vec{n}}, \vec{n})_{\mathbf{L}^2(\Gamma_1)}.$$

Since  $f_{\vec{n}} \in \mathcal{U}^\perp$ ,  $(f_{\vec{n}}, \vec{n})_{\mathbf{L}^2(\Gamma_1)} \neq 0$ . Thus, defining  $\sigma := \frac{(y, \vec{n})_{\mathbf{L}^2(\Gamma_1)}}{(f_{\vec{n}}, \vec{n})_{\mathbf{L}^2(\Gamma_1)}}$  and inserting this in the above equality, we arrive at

$$(y - \sigma f_{\vec{n}}, \vec{n})_{\mathbf{L}^2(\Gamma_1)} = 0.$$

In particular, the above equality implies that  $y - \sigma f_{\vec{n}} \in \mathcal{U}$ , see (2.6), and hence the result follows, i.e., there exists some  $\sigma \in \mathbb{R}$  such that  $y = y_1 + \sigma \vec{n}$   $\square$

In the following, we derive the first-order optimality condition associated with the regularized problem  $(\mathcal{P}_\gamma)$ .

**Theorem 3.1.** *Let  $(y_\gamma, u_\gamma) \in \mathcal{T}$  be a local optimal solution of  $(\mathcal{P}_\gamma)$  such that  $\mathcal{M}(y_\gamma) < \nu$ . Then there exist multipliers  $\mu \in \mathbf{L}^2(\Omega)$ ,  $\lambda \in V$ ,  $q \in L_0^2(\Omega)$  and  $\sigma \in \mathbb{R}$  such that*

$$(3.3) \quad \begin{aligned} -\nu \Delta y_\gamma + (y_\gamma \cdot \nabla) y_\gamma + \nabla p_\gamma &= f \\ \operatorname{div} y_\gamma &= 0 \\ y_\gamma|_\Gamma &= g + \mathcal{B}u_\gamma, \end{aligned}$$

$$(3.4) \quad \begin{aligned} -\nu \Delta \lambda - (y_\gamma \cdot \nabla) \lambda + (\nabla y_\gamma)^T \lambda + \nabla q &= z_d - y_\gamma - \mu \\ \operatorname{div} \lambda &= 0 \\ \lambda|_\Gamma &= 0, \end{aligned}$$

$$(3.5) \quad -\alpha \Delta_{\Gamma} u_{\gamma} = \mathcal{B}^* \left( -\nu \frac{\partial \lambda}{\partial \vec{n}} + q \vec{n} \right) + \sigma \vec{n} \quad \text{in } \mathbf{H}^{-1}(\Gamma_1)$$

$$(3.6) \quad \mu = \begin{cases} \gamma(y - b) & \text{in } \mathcal{A}_b \\ 0 & \text{in } \Omega \setminus (\mathcal{A}_b \cup \mathcal{A}_a) \\ \gamma(y - a) & \text{in } \mathcal{A}_a, \end{cases}$$

with  $\mathcal{A}_b := \{x \in \Omega \mid y(x) > b(x) \text{ a.e.}\}$  and  $\mathcal{A}_a := \{x \in \Omega \mid y(x) < a(x) \text{ a.e.}\}$ , hold in variational sense.

*Proof.* Since by Lemma 2.1 the control to state operator is differentiable in a neighborhood of  $(y_{\gamma}, u_{\gamma})$ , we obtain the optimality condition

$$J'_{\gamma}(y_{\gamma}(u_{\gamma}), u_{\gamma})h = 0, \quad \forall h \in \mathcal{U},$$

which implies that

$$(3.7) \quad (y_{\gamma} - z_d, S'(u_{\gamma})h) + \alpha(u_{\gamma}, h)_{\mathcal{U}} + (\mu, S'(u_{\gamma})h) = 0 \quad \forall h \in \mathcal{U}.$$

Here, by virtue of Lemma 2.1,  $S'(u_{\gamma})h = w_h$  is defined by the solution of

$$(3.8) \quad \nu(\nabla w_h, \nabla v) + c(w_h, y_{\gamma}, v) + c(y_{\gamma}, w_h, v) = 0 \quad \forall v \in V,$$

$$(3.9) \quad \gamma_0 w_h = \mathcal{B}h.$$

Moreover,  $\mu \in \mathbf{L}^2(\Omega)$  in (3.7) is given by

$$(3.10) \quad \mu = \begin{cases} \gamma(y - b) & \text{in } \mathcal{A}_b \\ 0 & \text{in } \Omega \setminus (\mathcal{A}_b \cup \mathcal{A}_a) \\ \gamma(y - a) & \text{in } \mathcal{A}_a, \end{cases}$$

where  $\mathcal{A}_b := \{x \in \Omega \mid y(x) > b(x) \text{ a.e.}\}$  and  $\mathcal{A}_a := \{x \in \Omega \mid y(x) < a(x) \text{ a.e.}\}$ .

Let us now introduce the adjoint equation

$$(3.11) \quad \begin{aligned} -\nu \Delta \lambda - (y_{\gamma} \cdot \nabla) \lambda + (\nabla y_{\gamma})^T \lambda + \nabla q &= z_d - y_{\gamma} - \mu \\ \operatorname{div} \lambda &= 0 \\ \lambda|_{\Gamma} &= 0. \end{aligned}$$

Notice that, by the assumption  $\mathcal{M}(y_{\gamma}) < \nu$ , the ellipticity of the adjoint operator can be verified by standard arguments which implies the existence of a unique adjoint state  $\lambda \in V$ . Multiplying the adjoint equation by  $w_h$ , integrating by parts and then invoking (3.7), we obtain that

$$(3.12) \quad \begin{aligned} \nu(\nabla \lambda, \nabla w_h) - \nu \left\langle \frac{\partial \lambda}{\partial \vec{n}}, \mathcal{B}h \right\rangle_{\mathbf{H}^{-1/2}(\Gamma), \mathbf{H}^{1/2}(\Gamma)} + c(y_{\gamma}, w_h, \lambda) \\ + c(w_h, y_{\gamma}, \lambda) - \alpha(u_{\gamma}, h)_{\mathcal{U}} + \langle q \vec{n}, \mathcal{B}h \rangle_{\mathbf{H}^{-1/2}(\Gamma), \mathbf{H}^{1/2}(\Gamma)} = 0. \end{aligned}$$

Now, setting  $v = \lambda$  in the variational equation (3.8) and inserting the resulting equation in the above equation, we arrive at

$$\left\langle -\nu \frac{\partial \lambda}{\partial \vec{n}} + q \vec{n}, \mathcal{B}h \right\rangle_{\mathbf{H}^{-1/2}(\Gamma), \mathbf{H}^{1/2}(\Gamma)} = \alpha(\nabla_{\Gamma} u_{\gamma}, \nabla_{\Gamma} h), \quad \text{for all } h \in \mathcal{U}.$$

Note that, from Lemma 3.1, if  $\xi \in \mathbf{H}^{-1}(\Gamma_1)$  satisfies  $\langle \xi, h \rangle_{\mathbf{H}^{-1}(\Gamma_1), \mathbf{H}_0^1(\Gamma_1)} = 0$ , for all  $h \in \mathcal{U}$ , then there exists  $\sigma \in \mathbb{R}$  such that  $\langle \xi, h \rangle_{\mathbf{H}^{-1}(\Gamma_1), \mathbf{H}_0^1(\Gamma_1)} = \langle x_{\xi}, h \rangle_{\mathbf{H}_0^1(\Gamma_1)} = \langle \sigma f_{\vec{n}}, h \rangle_{\mathbf{H}_0^1(\Gamma_1)} =$



$\langle \sigma \vec{n}, h \rangle_{\mathbf{H}^{-1}(\Gamma_1), \mathbf{H}_0^1(\Gamma_1)}$ , for all  $h \in \mathbf{H}_0^1(\Gamma_1)$ , where  $x_\xi$  denotes the Riesz representative of  $\xi$  in  $\mathbf{H}_0^1(\Gamma_1)$ . Therefore, we obtain

$$(3.13) \quad \langle \mathcal{B}^* \left( -\nu \frac{\partial \lambda}{\partial \vec{n}} + q \vec{n} \right), h \rangle_{\mathbf{H}^{-1}(\Gamma_1), \mathbf{H}_0^1(\Gamma_1)} + \langle \sigma \vec{n}, h \rangle_{\mathbf{H}^{-1}(\Gamma_1), \mathbf{H}_0^1(\Gamma_1)} \\ = -\alpha \langle \Delta_\Gamma u, h \rangle_{\mathbf{H}^{-1}(\Gamma_1), \mathbf{H}_0^1(\Gamma_1)}, \text{ for all } h \in \mathbf{H}_0^1(\Gamma_1)$$

and, consequently, equation (3.5) holds.  $\square$

**3.2. Convergence analysis.** Next, we study the convergence properties of the optimal solutions of the regularized problem  $(\mathcal{P}_\gamma)$  towards solutions of the original control problem. In particular, existence of a sequence of solutions to  $(\mathcal{P}_\gamma)$  approximating any local optimal solution of (2.1) will be shown under a quadratic growth condition hypothesis.

**Theorem 3.2.** *Let  $\{(y_\gamma, u_\gamma)\}_{\gamma>0}$  be a sequence of global solutions to  $(\mathcal{P}_\gamma)$ . Assume that  $\nu > \mathcal{M}(y_\gamma)$  holds for all  $\gamma > 0$  and  $\nu > \mathcal{M}(y^*)$  holds for every global optimal solution  $(y^*, u^*) \in U_{ad}$  of (2.1). Then, the sequence  $\{(y_\gamma, u_\gamma)\}_{\gamma>0}$  is uniformly bounded in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathbf{H}_0^1(\Gamma_1)$ . Further, every weakly converging subsequence of  $(y_\gamma, u_\gamma)_{\gamma>0}$  converges strongly in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathbf{H}_0^1(\Gamma_1)$  to a global solution of (2.1) as  $\gamma \rightarrow \infty$ .*

*Proof.* First of all, let us point out that a global solution  $(y^*, u^*)$  of (2.1) is feasible for  $(\mathcal{P}_\gamma)$  for all  $\gamma > 0$ . Hence, we find that

$$(3.14) \quad J_\gamma(y_\gamma, u_\gamma) \leq J_\gamma(y^*, u^*) = J(y^*, u^*).$$

Therefore,  $\{u_\gamma\}_{\gamma>0}$  is uniformly bounded in  $\mathbf{H}_0^1(\Gamma_1)$  and  $\{y_\gamma\}_{\gamma>0}$  is uniformly bounded in  $\mathbf{L}^2(\Omega)$ . Since by hypothesis  $\nu > \mathcal{M}(y_\gamma)$  holds for all  $\gamma > 0$ , we obtain from estimate (2.9) that  $\{y_\gamma\}_{\gamma>0}$  is uniformly bounded in  $\mathbf{H}_{\text{div}}^{3/2}$ . Consequently, there exists a subsequence, also denoted by  $\{(y_\gamma, u_\gamma)\}_{\gamma>0}$ , which converges weakly to a limit point  $(\hat{y}, \hat{u}) \in \mathbf{H}_{\text{div}}^{3/2} \times \mathbf{H}_0^1(\Gamma_1)$ . An argument analogous to the one in the proof of Theorem 2.2 implies that  $(\hat{y}, \hat{u}) \in \mathcal{T}$ .

From the penalized cost functional we also obtain that

$$(3.15) \quad \lim_{\gamma \rightarrow \infty} \|\max(0, y_\gamma - b)\| = \lim_{\gamma \rightarrow \infty} \|\min(0, y_\gamma - a)\| = 0,$$

which, by Fatou's lemma, implies that  $\hat{y} \leq b$  and  $\hat{y} \geq a$ ; cf. [11]. For this reason, the weak limit  $(\hat{y}, \hat{u})$  is feasible for (2.1) or equivalently  $(\hat{y}, \hat{u}) \in U_{ad}$ . In addition, invoking the lower semicontinuous of  $J$  and by (3.14), we infer that

$$(3.16) \quad J(\hat{y}, \hat{u}) \leq \liminf_{\gamma \rightarrow \infty} J(y_\gamma, u_\gamma) \leq \limsup_{\gamma \rightarrow \infty} J_\gamma(y_\gamma, u_\gamma) \leq J(y^*, u^*).$$

Thus, since  $(y^*, u^*)$  is a global solution to (2.1) and since  $(\hat{y}, \hat{u}) \in U_{ad}$ , it follows that

$$(3.17) \quad J(\hat{y}, \hat{u}) = J(y^*, u^*)$$

or equivalently  $(\hat{y}, \hat{u})$  is a global solution to (2.1). We show now that  $(y_\gamma, u_\gamma) \rightarrow (\hat{y}, \hat{u})$  strongly in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathbf{H}_0^1(\Gamma_1)$  as  $\gamma \rightarrow \infty$ . In view of (3.16)–(3.17), it holds that

$$(3.18) \quad \lim_{\gamma \rightarrow \infty} \frac{1}{2} \|y_\gamma - z_d\|^2 + \frac{\alpha}{2} \|u_\gamma\|_{\mathbf{H}_0^1(\Gamma_1)}^2 = \frac{1}{2} \|\hat{y} - z_d\|^2 + \frac{\alpha}{2} \|\hat{u}\|_{\mathbf{H}_0^1(\Gamma_1)}^2.$$

Invoking the compactness of the embedding  $\mathbf{H}^{3/2}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ , we obtain  $y_\gamma \rightarrow \hat{y}$  strongly in  $\mathbf{L}^2(\Omega)$  and hence

$$\lim_{\gamma \rightarrow \infty} \frac{1}{2} \|y_\gamma - z_d\|^2 = \frac{1}{2} \|\hat{y} - z_d\|^2.$$

This together with (3.18) and the weak convergence  $u_\gamma \rightharpoonup \hat{u}$  in  $\mathbf{H}_0^1(\Gamma_1)$  imply that  $u_\gamma \rightarrow \hat{u}$  strongly in  $\mathbf{H}_0^1(\Gamma_1)$ .

By the assumption, it holds that  $\nu > \mathcal{M}(\hat{y})$ . For this reason, Lemma 2.1 implies the existence of a neighborhood  $B(\hat{u})$  of  $\hat{u}$  in  $\mathcal{U}$  such that the control-to-state mapping associated with (2.3)-(2.4)

$$S : B(\hat{u}) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$$

is well-defined and twice continuously differentiable. It then follows from the convergence  $u_\gamma \rightarrow \hat{u}$  in  $\mathbf{H}_0^1(\Gamma_1)$  that  $y_\gamma \rightarrow y^*$  strongly in  $\mathbf{H}^{3/2}(\Omega)$ .  $\square$

Let us point out that in the preceding theorem, assuming existence of global solutions to  $(\mathcal{P}_\gamma)$ , the convergence of such solutions towards a global solution of the original control problem (2.1) is ensured. However, an important question that certainly deserves to be addressed is the following: if a local solution  $(y^*, u^*)$  of (2.1) is given, is it possible to find a sequence of locally optimal solutions of the penalized problems  $(\mathcal{P}_\gamma)$  converging strongly to  $(y^*, u^*)$  as  $\gamma \rightarrow \infty$ ? Such an issue is particularly important since optimization algorithms generate in general only local solutions. Under some assumption on a quadratic growth condition (Assumption 3.1), it is in fact possible to establish the existence of the local solution of  $(\mathcal{P}_\gamma)$  which converges to a local solution  $(y^*, u^*)$ .

**Assumption 3.1.** *Let  $(y^*, u^*) \in U_{ad}$  be a local solution of (2.1) such that  $\nu > \mathcal{M}(y^*)$ . Moreover, assume that the local solution  $(y^*, u^*)$  satisfies the quadratic growth condition: There exist fixed constants  $\sigma, \delta > 0$  such that*

$$(3.19) \quad J(y^*, u^*) + \frac{\sigma}{2} \|u - u^*\|_{\mathcal{U}}^2 \leq J(y, u)$$

for all  $(y, u) \in U_{ad}$  satisfying  $\|u - u^*\|_{\mathcal{U}} \leq \delta$ .

Relying on the above assumption for the local solution  $(y^*, u^*) \in U_{ad}$ , there exists in fact a sequence  $\{y_\gamma, u_\gamma\}_{\gamma>0}$  of local solutions to  $(\mathcal{P}_\gamma)$  converging strongly in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathbf{H}_0^1(\Gamma_1)$  to  $(y^*, u^*)$  as  $\gamma \rightarrow \infty$ . Notice that, in contrast to Theorem 3.2, we do not require the additional assumption that  $\nu > \mathcal{M}(y_\gamma)$  holds for all  $\gamma > 0$ .

**Theorem 3.3.** *Let  $(y^*, u^*) \in U_{ad}$  be a local solution of (2.1) satisfying Assumption 3.1. Then, there exists a sequence  $\{(y_\gamma, u_\gamma)\}_{\gamma>0}$  of local solutions of  $(\mathcal{P}_\gamma)$  converging strongly in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathbf{H}_0^1(\Gamma_1)$  to  $(y^*, u^*)$  as  $\gamma \rightarrow \infty$ .*

*Proof.* The proof is shown in the following steps:

Step 1: Since  $\nu > \mathcal{M}(y^*)$ , Lemma 2.1 implies the existence of a neighborhood  $B(u^*)$  of  $u^*$  in  $\mathcal{U}$  such that the control-to-state mapping associated with (2.3)-(2.4)

$$(3.20) \quad S : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$$

is well-defined and

$$(3.21) \quad \nu > \mathcal{M}(S(u)) \quad \forall u \in B(u^*).$$

According to Assumption 3.1, there exist fixed constants  $\sigma, \delta > 0$  such that

$$(3.22) \quad J(y^*, u^*) + \frac{\sigma}{2} \|u - u^*\|_{\mathcal{U}}^2 \leq J(y, u)$$

for all  $(y, u) \in U_{ad}$  satisfying  $\|u - u^*\|_{\mathcal{U}} \leq \delta$ . Now, consider the following auxiliary control problem

$$(\mathcal{P}_\gamma^r) \quad \begin{cases} \min J_\gamma(y, u) \\ \text{subject to} \\ (y, u) \in U_{r,\gamma} := \{(y, u) \in \mathcal{T} \mid \|u - u^*\|_{\mathcal{U}} \leq r\} \end{cases}$$

with some fixed constant  $0 < r \leq \delta$  such that  $\{u \in \mathcal{U} \mid \|u - u^*\|_{\mathcal{U}} \leq r\} \subset B(u^*)$ .

Step 2: We show that, for every  $\gamma > 0$ ,  $(\mathcal{P}_\gamma^r)$  admits an optimal solution. Let  $\gamma > 0$ . Since

$J_\gamma(y, u) \geq 0$  for all  $(y, u) \in U_{r, \gamma}$ , the infimum  $\inf_{(y, u) \in U_{r, \gamma}} J_\gamma(y, u)$  exists in  $\mathbb{R}_0^+$  and is denoted by  $j$ . Now, since the point  $(y^*, u^*)$  is feasible for  $(\mathcal{P}_\gamma^r)$ , there exists an infimal sequence  $\{y_n, u_n\}_{n=1}^\infty \subset U_{r, \gamma}$  associated with  $(\mathcal{P}_\gamma^r)$ , i.e., it holds that

$$\liminf_{n \rightarrow \infty} J(y_n, u_n) = j.$$

In particular, there exists a constant  $c > 0$  such that

$$c \geq J(y_n, u_n) = \frac{1}{2} \|y_n - z_d\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\alpha}{2} \|u_n\|_{\mathbf{H}_0^1(\Gamma_1)}^2 \quad \text{for all } n \in \mathbb{N}.$$

This implies that  $\{y_n\}_{n=1}^\infty$  and  $\{u_n\}_{n=1}^\infty$  are uniformly bounded in  $\mathbf{L}^2(\Omega)$  and  $\mathbf{H}_0^1(\Gamma_1)$ , respectively. Now, since  $\{u_n\}_{n=1}^\infty \subset B(u^*)$ , it holds that  $y_n = S(u_n)$  for all  $n$ ; cf. (3.20). Moreover, by (3.21), it satisfies  $\nu > \mathcal{M}(y_n)$  for all  $n \in \mathbb{N}$ . Therefore, invoking Proposition 2.1, we immediately obtain the uniform boundedness of  $\{y_n\}_{n=1}^\infty$  in  $\mathbf{H}_{\text{div}}^{3/2}$ . Hence, there exists subsequences  $\{(u_{n_k})\}_{k=1}^\infty$  and  $\{(y_{n_k})\}_{k=1}^\infty$  such that

$$u_{n_k} \rightharpoonup u_\gamma^r, \text{ weakly in } \mathcal{U}, \quad y_{n_k} \rightharpoonup y_\gamma^r, \text{ weakly in } \mathbf{H}_{\text{div}}^{3/2}.$$

Analogously to the proof of Theorem 2.2, it can be shown that  $(u_\gamma^r, y_\gamma^r)$  is an optimal solution to  $(\mathcal{P}_\gamma^r)$ .

Step 3: Let  $\{(u_\gamma^r, y_\gamma^r)\}_{\gamma > 0}$  be a sequence of optimal solutions to  $(\mathcal{P}_\gamma^r)$ . Since  $(y^*, u^*)$  is feasible for  $(\mathcal{P}_\gamma^r)$  for all  $\gamma > 0$ , we have

$$(3.23) \quad J_\gamma(y_\gamma^r, u_\gamma^r) \leq J_\gamma(y^*, u^*) = J(y^*, u^*) \quad \forall \gamma > 0.$$

This implies that  $\{(y_\gamma^r, u_\gamma^r)\}_{\gamma > 0}$  is uniformly bounded in  $\mathbf{L}^2(\Omega) \times \mathcal{U}$ . In addition, since  $u_\gamma^r \in B(u^*)$ ,  $\nu > \mathcal{M}(y_\gamma^r)$  holds true for all  $\gamma > 0$ . Therefore, invoking again Proposition 2.1, we obtain the uniform boundedness of  $\{y_\gamma^r\}_{\gamma > 0}$  in  $\mathbf{H}_{\text{div}}^{3/2}$ . For this reason, we may extract a subsequence of  $\{(y_\gamma^r, u_\gamma^r)\}_{\gamma > 0}$ , denoted again by  $\{(y_\gamma^r, u_\gamma^r)\}_{\gamma > 0}$ , converging weakly in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathcal{U}$  to a  $(\bar{y}, \bar{u}) \in \mathbf{H}_{\text{div}}^{3/2} \times \mathcal{U}$ . Analogously to the proof of Theorem 3.2, it holds that  $(\bar{y}, \bar{u}) \in U_{ad}$ . Further, since the set

$$\{u \in \mathcal{U} \mid \|u - u^*\|_{\mathcal{U}} \leq r\}$$

is weakly closed, we find that

$$(3.24) \quad \|\bar{u} - u^*\|_{\mathcal{U}} \leq r \leq \delta.$$

By the latter inequality, (3.22) ensures that

$$(3.25) \quad J(y^*, u^*) + \frac{\sigma}{2} \|\bar{u} - u^*\|_{\mathcal{U}}^2 \leq J(\bar{y}, \bar{u}).$$

In addition, from the lower semicontinuity of  $J$  together with (3.23), it follows that

$$(3.26) \quad J(\bar{y}, \bar{u}) \leq \liminf_{\gamma \rightarrow \infty} J(y_\gamma^r, u_\gamma^r) \leq \limsup_{\gamma \rightarrow \infty} J(y_\gamma^r, u_\gamma^r) \leq J(y^*, u^*).$$

Hence, collecting (3.25)-(3.26), we come to the conclusion that

$$J(y^*, u^*) + \frac{\sigma}{2} \|\bar{u} - u^*\|_{\mathcal{U}}^2 \leq J(y^*, u^*)$$

which implies that  $u^* = \bar{u}$ . An argument analogously to the one in the proof of Theorem 3.2 implies that

$$u_\gamma^r \rightarrow u^* \quad \text{strongly in } \mathcal{U}.$$

and hence by the continuity of  $S : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$

$$y_\gamma^r = S(u_\gamma^r) \rightarrow S(u^*) = y^* \quad \text{strongly in } \mathbf{H}_{\text{div}}^{3/2}.$$

Now, the assertion of the theorem is verified once we show that  $(y_\gamma^r, u_\gamma^r)$  is a local solution to  $(\mathcal{P}_\gamma)$  for almost all  $\gamma$ . For this, let  $(y, u) \in \mathcal{T}$  be any pair with  $\|u - u_\gamma^r\|_{\mathcal{U}} \leq \frac{r}{2}$ . Since  $u_\gamma^r \rightarrow u^*$  strongly in  $\mathcal{U}$ , there exists  $\bar{\gamma} > 0$  such that

$$\|u - u^*\|_{\mathcal{U}} \leq \|u - u_\gamma^r\|_{\mathcal{U}} + \|u_\gamma^r - u^*\|_{\mathcal{U}} \leq \frac{r}{2} + \frac{r}{2} = r \quad \forall \gamma > \bar{\gamma}.$$

This implies that  $(y, u)$  is feasible for  $(\mathcal{P}_\gamma^r)$  for all  $\gamma > \bar{\gamma}$ . Consequently, for  $\gamma > \bar{\gamma}$ , we have

$$J_\gamma(y_\gamma^r, u_\gamma^r) \leq J_\gamma(y, u).$$

Therefore, we arrive at the conclusion that for  $\gamma > \bar{\gamma}$ ,  $(y_\gamma^r, u_\gamma^r)$  is a local solution to  $(\mathcal{P}_\gamma)$ .  $\square$

**Remark 3.1.** *Considering the auxiliary control problem  $(\mathcal{P}_\gamma^r)$  follows the idea of Casas and Tröltzsch [8]. In a similar context [26], this idea was also used.*

#### 4. SOURCE REPRESENTATION AND LAVRENTIEV'S REGULARIZATION STRATEGY

In this section we consider an alternative regularization technique for solving (2.1). We utilize a source representation of the control combined with a Lavrentiev type regularization for the pointwise state constraints of (2.1). More precisely, we consider the following source representation of the boundary control as the image of a "distributed" control  $v \in L^2(\Omega)$ :

$$(4.1) \quad u = T(v).$$

**Assumption 4.1.** *On the operator  $T$  in (4.1), we impose the following assumption:*

- (1) *The operator  $T : \mathbf{L}^2(\Omega) \rightarrow \mathcal{U}$  is surjective or the range  $T(\mathbf{L}^2(\Omega))$  is dense in  $\mathcal{U}$ .*
- (2)  *$T$  is twice continuously differentiable from  $\mathbf{L}^2(\Omega)$  to  $\mathcal{U}$ .*

**Remark 4.1.** *The operator  $T$  can be chosen according to the specific problem considered. However, apart of being surjective, it should be easy to compute numerically. In linear problems the choice  $T = S^*$  has been used (see [28]).*

Hereafter, we convert the state constraints in (2.1) into

$$y_a \leq \varepsilon v + y(T(v)) \leq y_b \text{ a.e. in } \Omega, \quad \varepsilon > 0,$$

where we used the new auxiliary control  $v$  instead of  $u$ . Thus, we regularize (2.1) in the following way:

$$(4.2) \quad \begin{cases} \min_{(v,y) \in \mathbf{L}^2(\Omega) \times \mathbf{H}} \frac{1}{2} \|y - z_d\|^2 + \frac{\alpha}{2} \|T(v)\|_{\mathcal{U}}^2 \\ \text{subject to} \\ -\nu \Delta y + (y \cdot \nabla) y + \nabla p = f & \text{in } \Omega \\ \text{div } y = 0 & \text{in } \Omega \\ y|_\Gamma = g + \mathcal{B}T(v) & \text{on } \Gamma \\ a \leq \varepsilon v + y \leq b & \text{a.e. in } \Omega. \end{cases}$$

To gain coercivity of the cost functional with respect to the new control  $v$ , we add to the objective functional in (4.2) the term  $\frac{\beta}{2} \|v\|^2$ , with  $\beta > 0$ . Finally, we arrive at the

following Lavrentiev type regularized problem:

$$(4.3) \quad \begin{cases} \min_{(v,y) \in \mathbf{L}^2(\Omega) \times \mathbf{H}} \mathcal{J}(y,v) := \frac{1}{2} \|y - z_d\|^2 + \frac{\alpha}{2} \|T(v)\|_{\mathcal{U}}^2 + \frac{\beta}{2} \|v\|^2 \\ \text{subject to} \\ -\nu \Delta y + (y \cdot \nabla) y + \nabla p = f & \text{in } \Omega \\ \operatorname{div} y = 0 & \text{in } \Omega \\ y|_{\Gamma} = g + \mathcal{B}T(v) & \text{on } \Gamma \\ a \leq \varepsilon v + y \leq b & \text{a.e. in } \Omega. \end{cases}$$

The admissible set associated with (4.3) is given by

$$(4.4) \quad U_{ad}^{\varepsilon} := \{(y,v) \in \mathbf{H}_{\operatorname{div}}^1 \times \mathbf{L}^2(\Omega) \mid (y,v) \text{ satisfies the constraints in (4.3)}\}.$$

Analogously to  $(\mathcal{P}_{\gamma})$ , the regularization allows the consideration of the constraints in  $\mathbf{L}^2(\Omega)$  and the direct derivation of first order optimality conditions.

**4.1. Optimality conditions.** Our aim in this section is to present the first order necessary optimality condition for the regularized problem (4.3). We follow basically the lines of [13]. The idea consists of transforming (4.3) locally around an optimal solution  $v_{\varepsilon}$  into a problem with pure control-constraints. Thus, the optimality conditions can be derived in a standard way. First of all, let us introduce the notion of local solutions to the regularized problem (4.2).

**Definition 4.1.** Let  $\varepsilon > 0$ . We say that  $(y_{\varepsilon}, v_{\varepsilon}) \in U_{ad}^{\varepsilon}$  a local solution to the regularized problem (4.3) with respect to the  $\mathbf{L}^2(\Omega)$ -topology if there exists a positive real number  $c > 0$  such that

$$\mathcal{J}(y_{\varepsilon}, v_{\varepsilon}) \leq \mathcal{J}(y, v)$$

for all feasible  $(y, v) \in U_{ad}^{\varepsilon}$  satisfying  $\|v - v_{\varepsilon}\| \leq c$ .

Assume in the following that  $v_{\varepsilon} \in \mathbf{L}^2(\Omega)$  is a local optimal solution of (4.3) with the corresponding state  $y_{\varepsilon} = y_{\varepsilon}(T(v_{\varepsilon}))$  satisfying  $\nu > \mathcal{M}(y_{\varepsilon})$ . By Lemma 2.1, there exists an open neighborhood  $B_0$  of  $T(v_{\varepsilon})$  in  $\mathcal{U}$  such that the control-to-state mapping

$$S : B_0 \rightarrow \mathbf{H}$$

is well-define and twice continuously differentiable. Since  $T$  is continuous, we find further an open neighborhood  $B_1$  of  $v_{\varepsilon}$  in  $\mathbf{L}^2(\Omega)$  such that  $T(B_1) \subset B_0$ . Let us consider a new "control function"  $z := \varepsilon v + S(T(v))$ . We will show that the mapping  $v \mapsto z$  is invertible in a  $\mathbf{L}^2$  neighborhood of  $v_{\varepsilon}$ . To this aim, we define an operator  $F : \mathbf{L}^2(\Omega) \times B_1 \rightarrow \mathbf{L}^2(\Omega)$  by

$$(4.5) \quad F(z, v) = \varepsilon v + S(T(v)) - z,$$

and analyze the solvability of  $F(z, v) = 0$ . Since by definition  $F(z_{\varepsilon}, v_{\varepsilon}) = 0$ , with  $z_{\varepsilon} = \varepsilon v_{\varepsilon} + S(T(v_{\varepsilon}))$ , the solvability can then be concluded from the properties of  $F_v(z_{\varepsilon}, v_{\varepsilon})$  by using the implicit function theorem.

In order to show the existence of the continuous inverse operator  $F_v^{-1}(z_{\varepsilon}, v_{\varepsilon}) : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$ , we use Fredholm's theorem. Owing to the continuous differentiability of  $T$ , we have

$$F_v(z_{\varepsilon}, v_{\varepsilon})h = (\varepsilon I + S'(T(v_{\varepsilon}))T'(v_{\varepsilon}))h$$

for all  $h \in \mathbf{L}^2(\Omega)$ . Let us now define the operator  $\mathcal{S}(v_{\varepsilon}) : \mathbf{L}^2(\Omega) \rightarrow \mathbf{H}_{\operatorname{div}}^{3/2}$  by

$$(4.6) \quad \mathcal{S}(v_{\varepsilon}) := S'(T(v_{\varepsilon}))T'(v_{\varepsilon}).$$

We consider further  $\mathcal{S}(v_{\varepsilon})$  as a mapping with range in  $\mathbf{L}^2(\Omega)$  and hence due to the compactness of the embedding of  $\mathbf{H}^{3/2}(\Omega)$  to  $\mathbf{L}^2(\Omega)$ ,  $\mathcal{S}(v_{\varepsilon}) : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$  is compact. In the following, we impose a further assumption on the operator  $\mathcal{S}(v_{\varepsilon})$ .

**Assumption 4.2.** *The operator  $\mathcal{S}(v_\varepsilon) = S'(T(v_\varepsilon))T'(v_\varepsilon) : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$  does not admit an eigenvalue  $-\varepsilon$ , i.e., the equation*

$$(\varepsilon I + \mathcal{S}(v_\varepsilon))v = 0$$

*admits only the trivial solution  $v = 0$ .*

Based on Assumption 4.2, Fredholm's theorem ensures that for each  $\phi \in \mathbf{L}^2(\Omega)$ , the equation

$$(\varepsilon I + \mathcal{S}(v_\varepsilon))v = \phi$$

has a unique solution  $v \in \mathbf{L}^2(\Omega)$ . This implies the bijectivity of  $F_v(z_\varepsilon, v_\varepsilon)$  and hence, by the Banach inverse mapping theorem,  $F_v(z_\varepsilon, v_\varepsilon)$  is continuously invertible.

By the implicit function theorem, there exist  $B_{r_1}(z_\varepsilon) := \{z \in \mathbf{L}^2(\Omega) \mid \|z - z_\varepsilon\| \leq r_1\}$ ,  $r_1 > 0$ , and  $B_{r_2}(v_\varepsilon) := \{v \in \mathbf{L}^2(\Omega) \mid \|v - v_\varepsilon\| \leq r_2\}$ , with  $r_2 > 0$ , such that  $B_{r_2}(v_\varepsilon) \subset B_1$  and for every  $z \in B_{r_1}(z_\varepsilon)$  there exists a unique  $K(z) = v \in B_{r_2}(v_\varepsilon)$  solving the equation

$$F(z, K(z)) = 0.$$

Notice that by the implicit function theorem, the twice continuous Fréchet differentiability of  $F$  ensures the twice continuous Fréchet differentiability of the operator

$$K : B_{r_1}(z_\varepsilon) \rightarrow B_{r_2}(v_\varepsilon).$$

Therefore, locally around  $v_\varepsilon$ , problem (4.3), is equivalent to the following optimal control problem with box constraints:

$$\begin{cases} \min_{z \in B_{r_1}(z_\varepsilon)} \tilde{\mathcal{J}}(z) := \mathcal{J}(S(T(K(z))), K(z)) \\ \text{subject to:} \\ a \leq z \leq b \quad \text{a.e. in } \Omega. \end{cases}$$

Next, we obtain the first order optimality conditions for (4.3).

**Theorem 4.1.** *Let  $v_\varepsilon \in \mathbf{L}^2(\Omega)$  be a local optimal solution of (4.3) with the associated state  $y_\varepsilon = S(T(v_\varepsilon)) \in \mathbf{H}_{\text{div}}^{3/2}$  such that  $\nu > \mathcal{M}(y_\varepsilon)$ . Then under Assumption 4.2, there exist Lagrange multipliers  $\lambda \in V$ ,  $q \in L_0^2(\Omega)$  and  $\mu_a, \mu_b \in \mathbf{L}^2(\Omega)$  such that*

$$(4.7) \quad \begin{aligned} -\nu \Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) y_\varepsilon + \nabla p_\varepsilon &= f \\ \text{div } y_\varepsilon &= 0 \\ y_\varepsilon|_\Gamma &= g + \mathcal{B}T(v_\varepsilon), \end{aligned}$$

$$(4.8) \quad \begin{aligned} -\nu \Delta \lambda - (y_\varepsilon \cdot \nabla) \lambda + (\nabla y_\varepsilon)^T \lambda + \nabla q &= z_d - y_\varepsilon + \mu_a - \mu_b \\ \text{div } \lambda &= 0 \\ \lambda|_\Gamma &= 0, \end{aligned}$$

$$(4.9) \quad \beta v_\varepsilon + (T'(v_\varepsilon))^* (-\alpha \Delta_\Gamma T(v_\varepsilon) - \varphi) = \varepsilon(\mu_a - \mu_b) \quad \text{in } \mathbf{L}^2(\Omega),$$

$$(4.10) \quad \varphi = \mathcal{B}^* (-\nu \partial_n \lambda + q \vec{n}) \quad \text{in } \mathbf{H}^{-1}(\Gamma_1),$$

$$(4.11) \quad \begin{aligned} a &\leq \varepsilon v_\varepsilon + y_\varepsilon \leq b, \\ \mu_a &\geq 0, \quad \mu_b \geq 0, \\ (\mu_{a_i}, a_i - \varepsilon v_{\varepsilon,i} - y_{\varepsilon,i}) &= (\mu_{b_i}, b_i - \varepsilon v_{\varepsilon,i} - y_{\varepsilon,i}) = 0, \quad \text{for } i = 1, 2, \end{aligned}$$

*hold in variational sense.*

*Proof.* Due to the local optimality of  $v_\varepsilon$ , we obtain for some  $r > 0$  that

$$\mathcal{J}(v_\varepsilon) \leq \mathcal{J}(v),$$

for all  $v \in B_r(v_\varepsilon) \subset B_1$  with  $a \leq \varepsilon v + S(T(v)) \leq b$ . Equivalently, since  $v = K(z)$  holds locally,

$$\tilde{\mathcal{J}}(z_\varepsilon) \leq \tilde{\mathcal{J}}(z),$$

for all  $z \in B_{r_0}(z_\varepsilon)$  with  $a \leq z \leq b$ , and for an appropriate constant  $r_0 > 0$ .

Thus,  $z_\varepsilon$  satisfies the following first order necessary condition

$$(4.12) \quad \tilde{\mathcal{J}}'(z_\varepsilon)(z - z_\varepsilon) \geq 0, \text{ for all } a \leq z \leq b.$$

Using the chain rule, the derivative of  $\tilde{\mathcal{J}}(z_\varepsilon)$  in any direction  $\zeta \in \mathbf{L}^2(\Omega)$  is given by

$$(4.13) \quad (\tilde{\mathcal{J}}'(z_\varepsilon), \zeta) = (y_\varepsilon - z_d, \mathcal{S}(v_\varepsilon)K'(z_\varepsilon)\zeta) \\ + \alpha(u_\varepsilon, T'(v_\varepsilon)K'(z_\varepsilon)\zeta)_{\mathbf{H}_0^1(\Gamma_1)} + \beta(v_\varepsilon, K'(z_\varepsilon)\zeta),$$

where  $\mathcal{S}(v_\varepsilon)$  is as defined in (4.6). Denoting by  $h := K'(z_\varepsilon)\zeta$ , the latter equality yields

$$(4.14) \quad (\tilde{\mathcal{J}}'(z_\varepsilon), \zeta) = (y_\varepsilon - z_d, \mathcal{S}(v_\varepsilon)h) + \alpha(\nabla u_\varepsilon, \nabla(T'(v_\varepsilon)h))_\Gamma + \beta(v_\varepsilon, h).$$

Denoting by  $\mu \in \mathbf{L}^2(\Omega)$  the Riesz representative of  $-\tilde{\mathcal{J}}'(z_\varepsilon)$  and using explicitly the derivative of  $K$  we obtain

$$(\mu, \zeta) = (\mu, (\varepsilon + \mathcal{S}(v_\varepsilon))h) = \varepsilon(\mu, h) + (\mu, \mathcal{S}(v_\varepsilon)h).$$

Therefore, equation (4.14) is equivalent to

$$(4.15) \quad (y_\varepsilon - z_d, \mathcal{S}(v_\varepsilon)h) + (\mu, \mathcal{S}(v_\varepsilon)h) + \alpha(\nabla u_\varepsilon, \nabla(T'(v_\varepsilon)h))_\Gamma + \beta(v_\varepsilon, h) + (\varepsilon\mu, h) = 0,$$

which, integrating by parts, yields

$$(4.16) \quad (y_\varepsilon - z_d, \mathcal{S}(v_\varepsilon)h) + (\mu, \mathcal{S}(v_\varepsilon)h) - \alpha \langle \Delta_\Gamma u_\varepsilon, T'(v_\varepsilon)h \rangle_{\mathcal{U}^*, \mathcal{U}} + \beta(v_\varepsilon, h) + (\varepsilon\mu, h) = 0,$$

Let us now introduce the adjoint system of equations

$$(4.17) \quad \begin{aligned} -\nu\Delta\lambda - (y_\varepsilon \cdot \nabla)\lambda + (\nabla y_\varepsilon)^T \lambda + \nabla q &= z_d - y_\varepsilon - \mu \\ \operatorname{div} \lambda &= 0 \\ \lambda|_\Gamma &= 0. \end{aligned}$$

Since, by hypothesis  $\nu > \mathcal{M}(y_\varepsilon)$ , the adjoint operator is bijective and, therefore, for  $z_d - y_\varepsilon - \mu \in \mathbf{L}^2(\Omega)$ , there exists a unique solution  $\lambda \in V$  for system (4.17).

Using the adjoint equations and introducing  $\phi := \mathcal{S}(v_\varepsilon)h = S'(T(v_\varepsilon))T'(v_\varepsilon)h$ , we obtain that

$$(4.18) \quad (z_d - y_\varepsilon - \mu, \phi) = -(\nu\Delta\lambda, \phi) - c(y_\varepsilon, \lambda, \phi) + c(\phi, y_\varepsilon, \lambda) + (\nabla q, \phi),$$

which by applying integration by parts yields

$$(4.19) \quad (z_d - y_\varepsilon - \mu, \phi) = -(\nu\nabla\phi, \nabla\lambda) + c(y_\varepsilon, \phi, \lambda) + \\ c(\phi, y_\varepsilon, \lambda) - \langle \nu\partial_n\lambda, \phi \rangle_{\mathbf{H}^{-1/2}(\Gamma), \mathbf{H}^{1/2}(\Gamma)} + \langle q\vec{n}, \phi \rangle_{\mathbf{H}^{-1/2}(\Gamma), \mathbf{H}^{1/2}(\Gamma)} - (q, \operatorname{div} \phi).$$

Since  $S'(T(v_\varepsilon))T'(v_\varepsilon)h = \phi$  is, according to Lemma 2.1, given by the solution of

$$\begin{aligned} -\nu(\nabla\phi, \nabla v) + c(\phi, y_\varepsilon, v) + c(y_\varepsilon, \phi, v) &= 0 \quad \forall v \in V \\ \operatorname{div} \phi &= 0 \\ \phi|_\Gamma &= \mathcal{B}T'(v_\varepsilon)h, \end{aligned}$$

equation (4.19) immediately implies

$$(4.20) \quad (z_d - y_\varepsilon - \mu, \phi) = \langle -\nu\partial_n\lambda + q\vec{n}, \mathcal{B}T'(v_\varepsilon)h \rangle_{\mathbf{H}^{-1/2}(\Gamma), \mathbf{H}^{1/2}(\Gamma)}.$$

Introducing the variable  $\varphi := \mathcal{B}^*(-\nu\partial_n\lambda + q\vec{n})$  and inserting the latter equality in (4.16), we arrive at

$$(4.21) \quad (T'(v_\varepsilon))^*(-\alpha\Delta_\Gamma u_\varepsilon - \varphi) + \beta v_\varepsilon = -\varepsilon\mu.$$

Utilizing the decomposition  $\mu = \mu_b - \mu_a$ , with

$$\begin{aligned} \mu_b &:= \mu_+ = \frac{1}{2}(\mu + |\mu|) \\ \mu_a &:= \mu_- = \frac{1}{2}(-\mu + |\mu|), \end{aligned}$$

where  $|\mu| = (|\mu_1|, |\mu_2|)^T$ , the optimality condition (4.12) can be rewritten as

$$(\tilde{\mathcal{J}}'(z_\varepsilon), z_\varepsilon) = \min_{a \leq z \leq b} (\mu_a - \mu_b, z) = \min_{a \leq z \leq b} \{(\mu_{a,1}, z_1) - (\mu_{b,1}, z_1) + (\mu_{a,2}, z_2) - (\mu_{b,2}, z_2)\}.$$

By fixing the second component of the new control variable  $z_2 = z_{\varepsilon,2}$  and considering the mutual disjoint sets  $\{x : \mu_{a,1}(x) > 0\}$  and  $\{x : \mu_{b,1}(x) > 0\}$ , we obtain that

$$(\tilde{\mathcal{J}}'(z_\varepsilon), z_\varepsilon) = (\mu_{a,1}, a_1) - (\mu_{b,1}, b_1) + (\mu_{a,2}, z_{\varepsilon,2}) - (\mu_{b,2}, z_{\varepsilon,2})$$

and, consequently,

$$(\mu_{a,1}, a_1 - \varepsilon u_{\varepsilon,1} - y_{\varepsilon,1}) - (\mu_{b,1}, b_1 - \varepsilon u_{\varepsilon,1} - y_{\varepsilon,1}) = 0.$$

Fixing now the first component of  $z$  and proceeding in a similar manner we get that

$$(\mu_{a,2}, a_2 - \varepsilon v_{\varepsilon,2} - y_{\varepsilon,2}) - (\mu_{b,2}, b_2 - \varepsilon v_{\varepsilon,2} - y_{\varepsilon,2}) = 0.$$

Taking into account that, by definition,  $\mu_a, \mu_b \geq 0$  componentwise, the complementarity system (4.11) follows.  $\square$

**4.2. Convergence analysis.** Our focus now is set on the convergence of the regularized solutions of (4.3) in the case of a vanishing regularization parameter  $\varepsilon \downarrow 0$ . Analogously to Theorem (3.3), we address the existence of a solution of (4.3) converging strongly to a given local solution  $(y^*, u^*)$  of (2.1) which will be established by invoking the assumption on the quadratic growth condition for  $(y^*, u^*)$ . However, compared to Theorem 3.3, the mixing of the state and control variables within the explicit inequality constraints of (4.3) raises some additional difficulties in the analysis. To show the existence result associated with (4.3), we need some Slater-type assumption which is referred to as *linearized Slater condition*.

**Definition 4.2.** Let  $(y^*, u^*) \in \mathcal{T}$  such that  $\nu > \mathcal{M}(y^*)$ . Then, we say that  $(y^*, u^*)$  satisfies the linearized Slater condition if there exists an interior (Slater) point  $v_0 \in \mathbf{L}^\infty(\Omega)$  such that

$$(4.22) \quad a(x) + \delta \leq S(u^*)(x) + (S'(u^*)T'(u^*)v_0)(x) \leq b(x) - \delta \quad \forall x \in \overline{\Omega},$$

for some fixed  $\delta > 0$ .

Notice that since  $\nu > \mathcal{M}(y^*)$ , Lemma 2.1 implies that there exists a neighborhood  $B(u^*)$  of  $u^*$  in  $\mathcal{U}$  such that the control-to-state mapping associated with (2.3)-(2.4)

$$S : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$$

is well define and twice continuously differentiable. Hence, thanks to the continuous embedding  $\mathbf{H}_{\text{div}}^{3/2} \hookrightarrow \mathbf{C}(\overline{\Omega})$ , Definition 4.2 makes sense. Notice also that the  $L^2$ - $L^\infty$ -norm gap involved in the definition above is necessitated by the pointwise state constraints. We require this later for the proof of Theorem 4.2 below.

**Assumption 4.3.** *We rely on the following assumptions:*



- 1) The cost parameter in the objective functional of the regularized problem (4.3)  $\beta = \beta(\varepsilon)$  satisfies

$$\beta = \sigma_0 \varepsilon^{1+\sigma_1}$$

with some constants  $\sigma_0 > 0$  and  $0 \leq \sigma_1 < 1$ .

- 2) The operator  $T : \mathbf{L}^2(\Omega) \rightarrow \mathcal{U}$  is linear and continuous.

In order to make the dependence of the regularized problem on  $\varepsilon$  transparent, we refer to the regularized problem (4.3) as  $(P_\varepsilon)$ . We start by verifying the following feasibility property.

**Lemma 4.1** (Feasibility property). *Let Assumption 4.3 be satisfied. Further, suppose that  $(y^*, u^*) \in \mathcal{T}$  with  $\nu > \mathcal{M}(y^*)$  satisfying the linearized Slater condition. Then, there exists a sequence  $\{v_k^0\}_{k=1}^\infty \subset \mathbf{L}^\infty(\Omega)$  with the following properties:*

- (i) *It holds that  $\lim_{k \rightarrow \infty} \|Tv_k^0 - u^*\|_{\mathcal{U}} = 0$ . Moreover, there exists some  $\tilde{k} > 0$  and an open neighborhood  $B(u^*)$  of  $u^*$  in  $\mathcal{U}$  such that  $\{Tv_k^0\}_{k=\tilde{k}}^\infty \subset B(u^*)$  and the control-to-state mapping associated with (2.3)-(2.4)*

$$S : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$$

*is well-defined and twice continuously differentiable.*

- (ii) *For every  $k > \tilde{k}$ , there is a real number  $\varepsilon_k$  such that  $(S(Tv_k^0), v_k^0)$  is feasible for  $(P_\varepsilon)$  for all  $\varepsilon \leq \varepsilon_k$ . In other words, it holds that*

$$a \leq \varepsilon v_k^0 + S(Tv_k^0) \leq b \quad \forall \varepsilon \leq \varepsilon_k.$$

*Proof.* Since the range  $T(\mathbf{L}^2(\Omega))$  is dense in  $\mathcal{U}$  and  $\mathbf{C}(\bar{\Omega})$  is dense in  $\mathbf{L}^2(\Omega)$ , we can find a sequence  $\{a_k\}_{k=1}^\infty$  in  $\mathbf{C}(\bar{\Omega})$  such that

$$(4.23) \quad \|u^* - Ta_k\|_{\mathcal{U}} \leq \frac{1}{k} \quad \forall k \in \mathbb{N}.$$

Since the associated state  $y^*$  of  $u^*$  satisfies  $\nu > \mathcal{M}(y^*)$ , Lemma 2.1 implies the existence of an open neighborhood  $B(u^*)$  of  $u^*$  in  $\mathcal{U}$  such that the control-to-state mapping associated with (2.3)-(2.4) is well-defined and twice continuously differentiable. Therefore, there exists a constant  $c_0 > 0$  such that

$$\|S'(u^*)(u^* - Ta_k)\|_{\mathbf{H}^{3/2}} \leq c_0 \|u^* - Ta_k\|_{\mathcal{U}} \quad \forall k \in \mathbb{N}.$$

Thus, setting (4.23) in the inequality above, we obtain

$$(4.24) \quad \|S'(u^*)(u^* - Ta_k)\|_{\mathbf{H}^{3/2}} \leq \frac{c_0}{k} \quad \forall k \in \mathbb{N}.$$

Let us now define the sequence  $\{v_k^0\}_{k=1}^\infty$  in  $\mathbf{L}^\infty(\Omega)$  by

$$(4.25) \quad v_k^0 := a_k + \frac{3c_0}{\delta k} v_0 = a_k + \frac{c_1}{k} v_0,$$

where  $v_0 \in \mathbf{L}^\infty(\Omega)$  is the Slater point, see Assumption 4.3, and  $c_1 := \frac{3c_0}{\delta}$ . By (4.25) and (4.23), we obtain that

$$(4.26) \quad \|Tv_k^0 - u^*\|_{\mathcal{U}} \leq \|Ta_k - u^*\|_{\mathcal{U}} + \frac{c_1}{k} \|Tv_0\|_{\mathcal{U}} \leq \frac{1 + c_1 \|Tv_0\|_{\mathcal{U}}}{k},$$

which implies that

$$(4.27) \quad \lim_{k \rightarrow \infty} \|Tv_k^0 - u^*\|_{\mathcal{U}} = 0.$$

Hence, there exists an index number  $\tilde{k} \in \mathbb{N}$  such that

$$(4.28) \quad Tv_k^0 \in B(u^*) \quad \forall k \geq \tilde{k}.$$

Since  $S : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$  is continuously differentiable, the Taylor expansion of  $S$  at  $u^*$  implies that

$$(4.29) \quad S(Tv_k^0) = S(u^*) + S'(u^*)(Tv_k^0 - u^*) + R(Tv_k^0) \quad \forall k \geq \tilde{k}$$

where the remainder term  $R : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$  satisfies

$$(4.30) \quad \lim_{k \rightarrow \infty} \frac{\|R(Tv_k^0)\|_{\mathbf{H}^{3/2}}}{\|Tv_k^0 - u^*\|_{\mathcal{U}}} = 0.$$

Further, in view of (4.26)

$$\begin{aligned} \|R(Tv_k^0)\|_{\mathbf{H}^{3/2}} &= \frac{\|R(Tv_k^0)\|_{\mathbf{H}^{3/2}}}{\|Tv_k^0 - u^*\|_{\mathcal{U}}} \|Tv_k^0 - u^*\|_{\mathcal{U}} \\ &\leq \frac{\|R(Tv_k^0)\|_{\mathbf{H}^{3/2}}}{\|Tv_k^0 - u^*\|_{\mathcal{U}}} (1 + c_1 \|Tv_0\|_{\mathcal{U}}) \frac{1}{k} \quad \forall k \geq \tilde{k}. \end{aligned}$$

Thus, (4.30) implies the existence of an index number  $k_0 \geq \tilde{k}$  such that

$$(4.31) \quad \|R(Tv_k^0)\|_{\mathbf{H}^{3/2}} \leq \frac{c_0}{k} \quad \forall k \geq k_0.$$

Next, let  $k \in \mathbb{N}$  be arbitrarily fixed with  $k \geq \max\{c_1, k_0\}$  and we rewrite (4.29) as follows:

$$\begin{aligned} S(Tv_k^0) &= S(u^*) + S'(u^*)(Tv_k^0 - u^*) + R(Tv_k^0) \\ &= S(u^*) + S'(u^*)(Ta_k + \frac{c_1}{k}Tv_0 - u^*) + R(Tv_k^0) \\ &= (1 - \frac{c_1}{k})S(u^*) + S'(u^*)(Ta_k - u^*) + \frac{c_1}{k}(S(u^*) + S'(u^*)Tv_0) + R(Tv_k^0). \end{aligned}$$

Since  $S(u^*)$  satisfies the inequality constraints in (2.1), the last equality together with (4.24), (4.22) and (4.31) imply

$$(4.32) \quad S(Tv_k^0) \leq (1 - \frac{c_1}{k})b + \frac{c_0}{k} + \frac{c_1}{k}(b - \delta) + \frac{c_0}{k} = b - \frac{c_0}{k},$$

where we used  $c_1 = 3c_0\delta^{-1}$ . Thus

$$\varepsilon v_k^0 + S(Tv_k^0) \leq \varepsilon \|v_k^0\|_{\mathbf{L}^\infty(\Omega)} + b - \frac{c_0}{k} \quad \text{a.e. in } \Omega.$$

We choose now  $\varepsilon_k > 0$  such that

$$\varepsilon \|v_k^0\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{c_0}{k} \quad \forall \varepsilon \leq \varepsilon_k.$$

Consequently

$$\varepsilon v_k^0 + S(Tv_k^0) \leq b \quad \text{a.e. in } \Omega, \quad \forall \varepsilon \leq \varepsilon_k.$$

By analogous arguments, for all sufficiently small  $\varepsilon$

$$\varepsilon v_k^0 + S(Tv_k^0) \geq a \quad \text{a.e. in } \Omega.$$

This completes the proof.  $\square$

**Theorem 4.2.** *Let  $(y^*, u^*) \in U_{ad}$  be a local solution of the original control problem (2.1) satisfying Assumption 3.1 and the linearized Slater condition. Then, there exists a sequence  $\{(y_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0}$  of local solutions of  $(P_\varepsilon)$  such that*

$$Tv_\varepsilon \rightarrow u^* \quad \text{strongly in } \mathcal{U} \quad \text{and} \quad y_\varepsilon = S(Tv_\varepsilon) \rightarrow y^* \quad \text{strongly in } \mathbf{H}_{\text{div}}^{3/2}$$

as  $\varepsilon \rightarrow 0$ .

*Proof.* The proof is partially analogous to the one of Theorem 3.3 which is given by the following steps:

Step 1: According to Assumption 3.1, it holds that  $\nu > \mathcal{M}(y^*)$  and hence Lemma 2.1 implies the existence of a neighborhood  $B(u^*)$  of  $u^*$  in  $\mathcal{U}$  such that the control-to-state mapping associated with (2.3)-(2.4):

$$(4.33) \quad S : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$$

is well-defined and

$$(4.34) \quad \nu > \mathcal{M}(S(u)) \quad \forall u \in B(u^*).$$

Further Assumption 3.1 ensures also the existence of fixed constants  $\sigma, \delta > 0$  such that

$$(4.35) \quad J(y^*, u^*) + \frac{\sigma}{2} \|u - u^*\|_{\mathcal{U}}^2 \leq J(y, u)$$

for all  $(y, u) \in U_{ad}$  satisfying  $\|u - u^*\|_{\mathcal{U}} \leq \delta$ . Now, consider the following auxiliary control problem

$$(P_\varepsilon^r) \quad \begin{cases} \min \mathcal{J}(y, v) \\ \text{subject to} \\ (y, v) \in U_{r, \varepsilon} := \{(y, v) \in U_{ad}^\varepsilon \mid \|Tv - u^*\|_{\mathcal{U}} \leq r\} \end{cases}$$

with some fixed constant  $0 < r \leq \delta$  such that  $\{Tv \in \mathcal{U} \mid \|Tv - u^*\|_{\mathcal{U}} \leq r\} \subset B(u^*)$ .

Step 2: We demonstrate that the auxiliary problem  $(P_\varepsilon^r)$  admits a solution for all sufficiently small  $\varepsilon > 0$ . First of all, since  $\mathcal{J}(y, v) \geq 0$  for all  $(y, v) \in U_{r, \varepsilon}$ , the infimum  $\inf_{(y, v) \in U_{r, \varepsilon}} \mathcal{J}(y, v)$  exists in  $\mathbb{R}_0^+$  and is denoted by  $j$ . Next, thanks to Lemma 4.1, there exist  $\hat{\varepsilon} > 0$  and  $\hat{v} \in \mathbf{L}^2(\Omega)$  such that  $(S(T\hat{v}), \hat{v})$  is feasible for  $(P_\varepsilon^r)$  all  $\varepsilon \leq \hat{\varepsilon}$ . In the sequel, let  $\varepsilon \leq \hat{\varepsilon}$ . Therefore, there exists an infimal sequence  $\{y_n, v_n\} \subset U_{r, \varepsilon}$  associated with  $(P_\varepsilon^r)$ , i.e., it satisfies:

$$\lim_{n \rightarrow \infty} \mathcal{J}(y_n, v_n) = j$$

In particular, there exists a constant  $c > 0$  such that

$$c \geq \mathcal{J}(y_n, v_n) = \frac{1}{2} \|y_n - z_d\|^2 + \frac{\alpha}{2} \|Tv_n\|_{\mathbf{H}_0^1(\Gamma_1)}^2 + \frac{\beta(\varepsilon)}{2} \|v_n\|^2 \quad \text{for all } n \in \mathbb{N}.$$

This implies that the sequences  $\{y_n\}_{n=1}^\infty$ ,  $\{Tv_n\}_{n=1}^\infty$  and  $\{v_n\}_{n=1}^\infty$  are uniformly bounded in  $\mathbf{L}^2(\Omega)$ ,  $\mathbf{H}_0^1(\Gamma_1)$  and  $\mathbf{L}^2(\Omega)$ , respectively. Now, since  $\{Tv_n\}_{n=1}^\infty \subset B(u^*)$ , it holds that  $y_n = S(Tv_n)$  for all  $n$ ; cf. (4.33). Moreover, by (4.34), it satisfies  $\nu > \mathcal{M}(y_n)$  for all  $n \in \mathbb{N}$ . Therefore, invoking Proposition 2.1, we immediately obtain the uniform boundedness of  $\{y_n\}_{n=1}^\infty$  in  $\mathbf{H}_{\text{div}}^{3/2}$ . Hence, there exists subsequences  $\{v_{n_k}\}_{k=1}^\infty$  and  $\{y_{n_k}\}_{k=1}^\infty$  such that

$$v_{n_k} \rightharpoonup v_\varepsilon^r, \text{ weakly in } \mathbf{L}^2(\Omega), \quad y_{n_k} \rightharpoonup y_\varepsilon^r, \text{ weakly in } \mathbf{H}_{\text{div}}^{3/2}.$$

Analogously to the proof of Theorem 2.2, it can be shown that  $(v_\varepsilon^r, y_\varepsilon^r)$  is an optimal solution to  $(P_\varepsilon^r)$ .

Step 3: Let  $\{\varepsilon_n\}_{n=1}^\infty$  be a sequence of positive real numbers converging to zero and assume that  $\varepsilon_n \leq \hat{\varepsilon}$  for all  $n \in \mathbb{N}$ . According to the claim verified in Step 2, there exists a sequence  $\{(y_n^r, v_n^r)\}_{n=1}^\infty$  of optimal solutions to  $(P_{\varepsilon_n}^r)$ . Moreover, we have already mentioned in Step 2 that  $(\hat{y}, \hat{v})$  is feasible for  $(P_{\varepsilon_n}^r)$  for all  $n \in \mathbb{N}$ . Consequently

$$(4.36) \quad \mathcal{J}(y_n^r, v_n^r) \leq \mathcal{J}(\hat{y}, \hat{v}) \quad \forall n \in \mathbb{N}.$$

This implies that

$$(4.37) \quad \mathcal{J}(\hat{y}, \hat{v}) \geq \mathcal{J}(y_n^r, v_n^r) = \frac{1}{2} \|y_n - z_d\|^2 + \frac{\alpha}{2} \|Tv_n^r\|_{\mathcal{U}}^2 + \frac{\beta(\varepsilon_n)}{2} \|v_n^r\|^2 \quad \forall n \in \mathbb{N}.$$

Thus,  $\{y_n\}_{n=1}^\infty$  and  $\{Tv_n^r\}_{n=1}^\infty$  are uniformly bounded in  $\mathbf{L}^2(\Omega)$  and  $\mathcal{U}$ , respectively. In addition, since  $\{Tv_n^r\}_{n=1}^\infty \subset B(u^*)$ ,  $\nu > \mathcal{M}(y_n^r)$  holds true for all  $n \in \mathbb{N}$ . Therefore, invoking again Proposition 2.1, we obtain the uniform boundedness of  $\{y_n^r\}_{n=1}^\infty$  in  $\mathbf{H}_{\text{div}}^{3/2}$ . For this we may extract a subsequence of  $\{(y_n^r, Tv_n^r)\}_{n=1}^\infty$  denoted again by  $\{(y_n^r, Tv_n^r)\}_{n=1}^\infty$  which converges weakly in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathcal{U}$  to a  $(\bar{y}, \bar{u}) \in \mathbf{H}_{\text{div}}^{3/2} \times \mathcal{U}$ . Let us demonstrate now that the weak limit  $(\bar{y}, \bar{u})$  is feasible for the original control problem (2.1). We have already mentioned that the trilinear form  $c : \mathbf{H}_{\text{div}}^1 \times \mathbf{H}_{\text{div}}^1 \times \mathbf{H}_{\text{div}}^1 \rightarrow \mathbb{R}$  in (2.3) is weakly sequentially continuous. Hence, an argument analogously to the proof of Theorem 2.2 implies that  $(\bar{y}, \bar{u}) \in \mathcal{T}$  ( $\mathcal{T}$  is defined in (2.19)). Moreover, due to the compactness of the  $\mathbf{L}^2(\Omega)$ , it holds that  $y_n^r \rightarrow \bar{y}$  strongly in  $\mathbf{L}^2(\Omega)$ . On the other hand, the pair  $(y_n^r, v_n^r)$  is feasible for  $(P_{\varepsilon_n}^r)$  and consequently it holds that

$$a \leq \varepsilon_n v_n^r + y_n^r \leq b \text{ a.e. in } \Omega \quad \forall n \in \mathbb{N}.$$

Therefore, it suffices to verify that  $\varepsilon_n v_n^r$  converges strongly in  $\mathbf{L}^2(\Omega)$  to zero as  $n \rightarrow \infty$ . By virtue of Assumption 4.3

$$\frac{\beta(\varepsilon_n)}{2} \|v_n^r\|^2 = \frac{\sigma_0 \varepsilon_n^{1+\sigma_1}}{2\varepsilon_n^2} \|\varepsilon_n v_n^r\|^2 = \frac{\sigma_0 \varepsilon_n^{\sigma_1-1}}{2} \|\varepsilon_n v_n^r\|^2.$$

Setting this in (4.37)

$$\frac{\sigma_0 \varepsilon_n^{\sigma_1-1}}{2} \|\varepsilon_n v_n^r\|^2 \leq \mathcal{J}(\hat{y}, \hat{v})$$

and hence

$$\|\varepsilon_n v_n^r\|^2 \leq \varepsilon_n^{1-\sigma_1} \frac{2}{\sigma_0} \mathcal{J}(\hat{y}, \hat{v}),$$

which implies that  $\varepsilon_n v_n^r \rightarrow 0$  strongly in  $\mathbf{L}^2(\Omega)$ . Thus, we come to the conclusion that the weak limit of  $\{y_n^r, Tv_n^r\}_{n=1}^\infty$  is feasible, i.e.,  $(\bar{y}, \bar{u}) \in U_{ad}$ .

**Step 4:** We demonstrate that  $y_n^r \rightarrow y^*$  strongly in  $\mathbf{H}_{\text{div}}^{3/2}$  and  $Tv_n^r \rightarrow u^*$  strongly in  $\mathcal{U}$ . Since the set

$$\{u \in \mathcal{U} \mid \|u - u^*\|_{\mathcal{U}} \leq r\}$$

is weakly closed, we find that

$$(4.38) \quad \|\bar{u} - u^*\|_{\mathcal{U}} \leq r \leq \delta.$$

By the latter inequality, (3.22) ensures that

$$(4.39) \quad J(y^*, u^*) + \frac{\sigma}{2} \|\bar{u} - u^*\|_{\mathcal{U}}^2 \leq J(\bar{y}, \bar{u}).$$

In addition, according to Lemma 4.1, there exists a sequence  $\{v_k^0\}_{k=1}^\infty \subset \mathbf{L}^\infty(\Omega)$  with the following properties:

(A<sub>1</sub>) It holds that  $\lim_{k \rightarrow \infty} \|Tv_k^0 - u^*\|_{\mathcal{U}} = 0$ . Moreover,  $\|Tv_k^0 - u^*\| \leq r$  for all  $n \in \mathbb{N}$ .

(A<sub>2</sub>) For every  $k \in \mathbb{N}$ , there is an index number  $n_k$  such that

$$a \leq \varepsilon_n v_k^0 + S(Tv_k^0) \leq b \quad \forall n \geq n_k.$$

Hence, (A<sub>1</sub>) and (A<sub>2</sub>) particularly implies that the pair  $(S(Tv_k^0), v_k^0)$  is feasible for  $(P_{\varepsilon_n}^r)$  for all  $n \geq n_k$ . For this reason, since  $(y_n^r, v_n^r)$  is an optimal solution for  $(P_{\varepsilon_n}^r)$ , it follows that

$$(4.40) \quad J(y_n^r, Tv_n^r) \leq J(y_n^r, Tv_n^r) + \frac{\beta(\varepsilon_n)}{2} \|v_n^r\|^2 \leq J(S(Tv_k^0), Tv_k^0) + \frac{\beta(\varepsilon_n)}{2} \|v_k^0\|^2,$$

for all  $n \geq n_k$ . Moreover, due to the lower semicontinuity of  $J$ , we obtain

$$(4.41) \quad J(\bar{y}, \bar{u}) \leq \liminf_{n \rightarrow \infty} J(y_n^r, Tv_n^r) \leq \limsup_{n \rightarrow \infty} J(y_n^r, Tv_n^r) \leq J(S(Tv_k^0), v_k^0).$$

On account of the continuity of  $J(S(T\cdot), \cdot)$ , by passing to the limit  $k \rightarrow \infty$ , (A<sub>1</sub>) and (4.41) imply that

$$(4.42) \quad J(\bar{y}, \bar{u}) \leq \liminf_{n \rightarrow \infty} J(y_n^r, T v_n^r) \leq \limsup_{n \rightarrow \infty} J(y_n^r, T v_n^r) \leq J(S(u^*), u^*) = J(y^*, u^*).$$

Applying (4.39) to the latter inequality

$$J(y^*, u^*) + \frac{\sigma}{2} \|\bar{u} - u^*\|_{\mathcal{U}}^2 \leq J(\bar{y}, \bar{u}) \leq J(y^*, u^*)$$

which leads to

$$(4.43) \quad T v_n^r \rightharpoonup \bar{u} = u^* \text{ weakly in } \mathcal{U} \quad \text{and} \quad J(y^*, u^*) = J(\bar{y}, \bar{u}).$$

The latter equality together with (4.42) implies that

$$\lim_{n \rightarrow \infty} J(y_n^r, T v_n^r) = J(y^*, u^*),$$

which is equivalent to

$$(4.44) \quad \lim_{n \rightarrow \infty} \left( \frac{1}{2} \|y_n^r - z_d\|^2 + \frac{\alpha}{2} \|T v_n^r\|_{\mathcal{U}}^2 \right) = \frac{1}{2} \|y^* - z_d\|^2 + \frac{\alpha}{2} \|u^*\|_{\mathcal{U}}^2.$$

Invoking the compactness of the embedding  $\mathbf{H}^{3/2}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ , the weak convergence  $y_n^r \rightharpoonup \bar{y}$  in  $\mathbf{H}^{3/2}(\Omega)$  as  $n \rightarrow \infty$  ensures that

$$\lim_{n \rightarrow \infty} \frac{1}{2} \|y_n^r - z_d\|^2 = \frac{1}{2} \|\bar{y} - z_d\|^2.$$

Consequently

$$\lim_{n \rightarrow \infty} \frac{\alpha}{2} \|T v_n^r\|_{\mathcal{U}}^2 = \frac{\alpha}{2} \|u^*\|_{\mathcal{U}}^2,$$

and hence from the weak convergence (4.43), we obtain

$$(4.45) \quad \lim_{n \rightarrow \infty} T v_n^r = u^* \quad \text{in } \mathcal{U}.$$

The strong convergence of  $y_n^r$  to  $y^*$  follows then from the continuity of  $S : B(u^*) \rightarrow \mathbf{H}_{\text{div}}^{3/2}$ :

$$\lim_{n \rightarrow \infty} y_n^r = \lim_{n \rightarrow \infty} S(T v_n^r) = S(u^*) = y^* \quad \text{in } \mathbf{H}^{3/2}(\Omega).$$

In this way, we have just shown that  $(y_n^r, T v_n^r)$  converges strongly in  $\mathbf{H}_{\text{div}}^{3/2} \times \mathcal{U}$  to the local solution  $(y^*, u^*)$  of (2.1) as  $n \rightarrow \infty$ .

Step 5: Lastly, we complete the proof by verifying that  $(y_n^r, v_n^r)$  is a local solution to  $(P_{\varepsilon_n})$  for all sufficiently large  $n \in \mathbb{N}$ . For this, let  $(y, v) \in U_{ad}^\varepsilon$  be any pair satisfying  $\|v - v_n^r\| \leq \frac{r}{2s}$  where  $s := \|T\|_{L^2(\Omega) \rightarrow \mathcal{U}}$ . Owing to the linearity and continuity of  $T$

$$\|T v - u^*\|_{\mathcal{U}} \leq \|T(v - v_n^r)\|_{\mathcal{U}} + \|T v_n^r - u^*\|_{\mathcal{U}} \leq \frac{r}{2} + \|T v_n^r - u^*\|_{\mathcal{U}}$$

Hence, since  $T v_n^r \rightarrow u^*$  strongly in  $\mathcal{U}$ , there exists  $\bar{n} > 0$  such that

$$\|T v - u^*\|_{\mathcal{U}} \leq \frac{r}{2} + \|T v_n^r - u^*\|_{\mathcal{U}} \leq r \quad \forall n \geq \bar{n}.$$

This implies that  $(y, v)$  is feasible for  $(P_{\varepsilon_n}^r)$  for all  $n \geq \bar{n}$ . Consequently, for  $n \geq \bar{n}$ , we have

$$\mathcal{J}(y_n^r, v_n^r) \leq \mathcal{J}(y, v).$$

Therefore,  $(y_n^r, v_n^r)$  is a local solution to  $(P_{\varepsilon_n}^r)$  for all  $n \geq \bar{n}$ .  $\square$

## 5. NUMERICAL RESULTS

In this section we present some numerical experiments which illustrate the performance of the Moreau-Yosida regularization technique applied to the boundary optimal control of the Navier-Stokes equations with pointwise state constraints. The regularized problems are solved by means of a semi-smooth Newton method (SSN) as developed in [19]. The algorithm is based on a reformulation of the complementarity problem as an operator equation involving the *max* and *min* functions. A main feature of this type of algorithms is its local superlinear convergent behavior (cf. [19]).

The algorithm for the regularized Dirichlet control problem is stated next.

### Algorithm 5.1.

(1) *Initialization: choose  $(u_0, y_0, \lambda_0) \in \mathcal{U} \times \mathbf{H}_{\text{div}}^{3/2} \times \mathbf{L}^2(\Omega)$  and set  $n = 1$ .*

(2) *Until a stopping criteria is satisfied, set*

$$\mathcal{A}_n^b = \{x : \gamma(y_{n-1} - y_b) \geq 0\} \quad \mathcal{A}_n^a = \{x : \gamma(y_{n-1} - y_a) \leq 0\}$$

and

$$\mathcal{I}_n = \{x : \gamma(y_{n-1} - y_b) < 0 < \gamma(y_{n-1} - y_a)\}.$$

*Find the solution  $(y_n, p_n, u_n, \lambda_n, q_n, \mu_n)$  of:*

$$(5.1) \quad \begin{aligned} -\nu \Delta y_n + (y_{n-1} \cdot \nabla) y_n + (y_n \cdot \nabla) y_{n-1} + \nabla p_n &= f + (y_{n-1} \cdot \nabla) y_{n-1} \\ \operatorname{div} y_n &= 0 \\ y_n|_{\Gamma} &= g + \mathcal{B}u_n, \end{aligned}$$

$$(5.2) \quad \begin{aligned} -\nu \Delta \lambda_n - (y_n \cdot \nabla) \lambda_{n-1} - (y_{n-1} \cdot \nabla) \lambda_n + (\nabla y_{n-1})^T \lambda_n + (\nabla y_n)^T \lambda_{n-1} \\ + \nabla q_n &= z_d - y_n - \mu_n - (y_{n-1} \cdot \nabla) \lambda_{n-1} + (\nabla y_{n-1})^T \lambda_{n-1} \\ \operatorname{div} \lambda_n &= 0 \\ \lambda_n|_{\Gamma} &= 0, \end{aligned}$$

$$(5.3) \quad -\alpha \Delta_{\Gamma} u_n = \mathcal{B}^* \left( -\nu \frac{\partial \lambda_n}{\partial \vec{n}} + q_n \vec{n} \right) + \sigma_n \vec{n},$$

$$(5.4) \quad \mu_n = \begin{cases} \gamma(y_n - y_b) & \text{in } \mathcal{A}_n^b \\ 0 & \text{in } \mathcal{I}_n \\ \gamma(y_n - y_a) & \text{in } \mathcal{A}_n^a, \end{cases}$$

and set  $n = n + 1$ .

The semi-smooth Newton algorithm is terminated when the norm of the increments reaches the precision *tol*, whose value is typically set equal to  $tol = 10^{-5}$ . The resulting linear systems in each semi-smooth Newton iteration are solved exactly using MATLAB's sparse solver.

For the numerical tests we utilize a forward facing step channel (see Figure 1). The fluid flows from left to right with inflow boundary condition of parabolic type and outflow stress free condition. The domain is discretized using an homogeneous staggered grid with step  $h$ . Also a first order upwind scheme is used for the approximation of the convective term.

The target of the control problem is to drive the fluid to an almost linear behavior given by the Navier-Stokes flow with Reynolds number equal to 1 and, through the presence of pointwise state constraints, reduce recirculations after the step. In that sense, the  $Re = 1$  flow is chosen as desired state  $z_d$ . The uncontrolled flow with  $Re = 800$  depicted in Figure 2, illustrates the main recirculation zones in the channel.

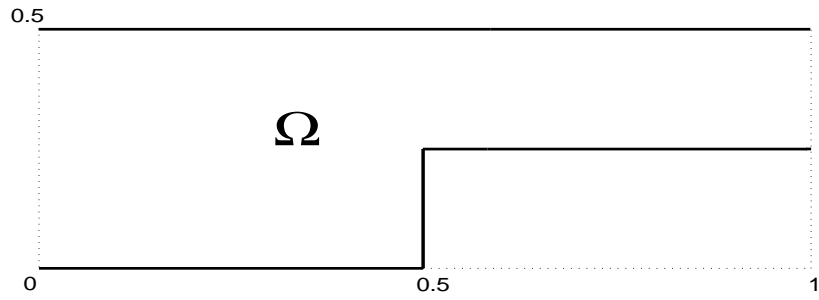


FIGURE 1. Forward facing step channel

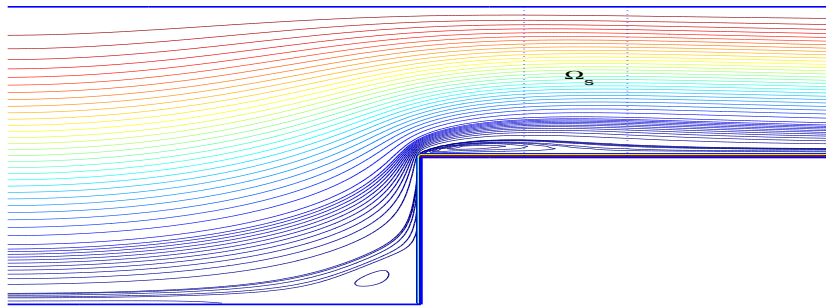


FIGURE 2. Uncontrolled flow with  $Re = 800$

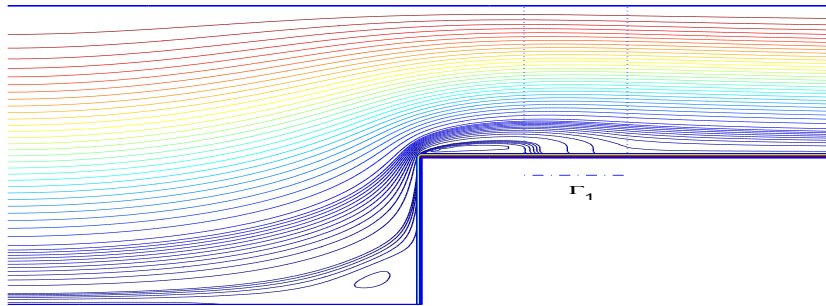


FIGURE 3. Example 1: optimal state

5.1. **Example 1.** In this example we impose a state constraint over the backward fluid flow in sector  $\Omega_S$  depicted in Figure 2. In this manner a substantial reduction of the recirculation after the step is expected. Specifically, the state constraint is given by  $y_1 \geq -10^{-7}$ . The boundary part where the control acts consists of the lower wall after the step between 0.625 and 0.75. This boundary sector is depicted in Figure 3 together with the resulting optimal state.

With the parameter values  $\alpha = 0.01$ ,  $Re = 800$ ,  $\gamma = 10^7$ , the semi-smooth Newton algorithm stops after 15 iterations with the mesh step size  $h = 1/240$ . The control action consists of the suction of fluid trough the boundary sector  $\Gamma_1$ , This can be observed from the zoom plot of the flow field given in Figure 4.

In Table 1 the behavior of the semi-smooth Newton method for different  $\gamma$  values is presented. The remaining parameters are  $\alpha = 0.01$ ,  $h = 1/160$  and  $Re = 800$ . As can be

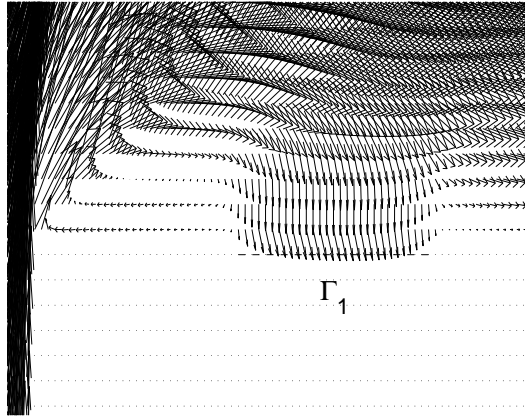


FIGURE 4. Example 1: optimal control

$\gamma$	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$	$10^7$
iter.	6	6	7	8	9	14
active	24	21	16	12	10	6
$J(y, u)$	0.0057783	0.0057948	0.0060249	0.0072535	0.011384	0.019233

TABLE 1. Example 1,  $\alpha = 0.01$ , 160 mesh nodes,  $Re=800$ .

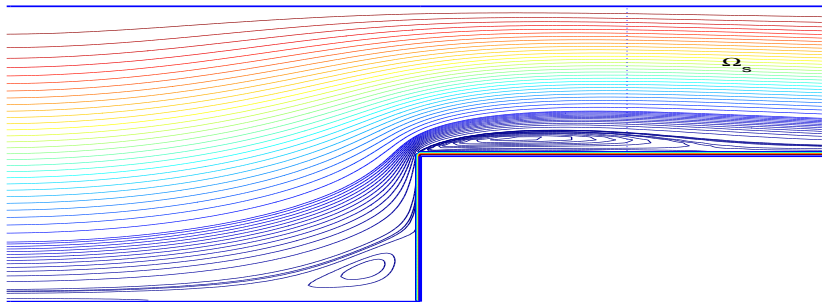


FIGURE 5. Example 2: uncontrolled state,  $Re=1500$

inferred from the data, the number of iterations increases together with the values of  $\gamma$ . On the other hand, the size of the active set decreases as  $\gamma$  increases.

**5.2. Example 2.** In this example a  $Re = 1500$  flow is controlled by means of a Dirichlet boundary condition. The control acts on the same boundary sector as in Example 1. The uncontrolled velocity field is depicted in Figure 5, where the larger size of the bubble can be observed. The constraint  $y_1 \geq -10^{-7}$  is imposed in the subdomain  $\Omega_s$  also shown in Figure 5.

The resulting optimal control, with  $\gamma = 10^5$ , is shown in detail in Figure 6. Differently to Example 1, the optimal strategy in this case consists in injecting fluid on  $\Gamma_1$ . The semi-smooth Newton algorithm takes 11 iterations to converge.

In Table 2 the data for the semi-smooth Newton method with  $\gamma = 10^4$ ,  $\alpha = 0.1$ ,  $Re = 1500$ , and  $h = 1/160$  is given. The superlinear convergence rate of the method can be inferred from the data.



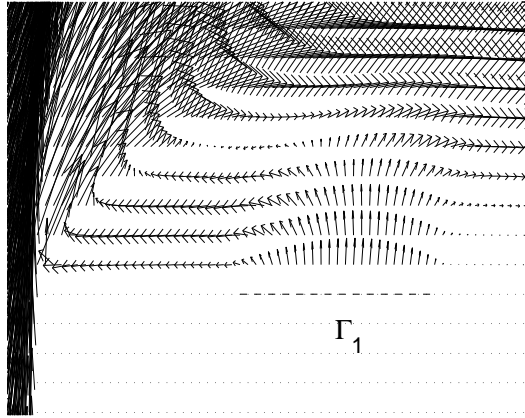


FIGURE 6. Example 2: zoom of the optimal control

It.	$ \mathcal{A}_n^\alpha $	$J(y, u)$	$\ y_n - y_{n-1}\ $	NCP	rate
1	0	$1.17495 \times 10^{-4}$	$1.058 \times 10^2$	0	-
2	0	0.010345	22.25250	0	-
3	0	0.009259	8.991736	$2.31616 \times 10^3$	22.2525
4	39	0.028324	2.424757	$3.49444 \times 10^2$	0.404077
5	34	0.0365871	0.18168	32.60469	0.269665
6	32	0.0365832	0.002243	0	0.074929
7	32	0.0365846	$3.2785 \times 10^{-5}$	0	0.012350

TABLE 2. Example 2,  $\alpha = 0.1$ ,  $\gamma = 10^4$ , 160 mesh nodes, Re=1500.

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