

CONVERGENCE OF dG(1) IN ELASTOPLASTIC EVOLUTION*

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ABSTRACT. The discontinuous Galerkin (dG) method provides a hierarchy of time discretization schemes for evolutionary problems. A dG time discretization has been proposed for a variational inequality in the context of rate-independent inelastic material behaviour in [Alberty, Carstensen: Discontinuous Galerkin Time Discretization in Elastoplasticity: Motivation, Numerical Algorithms and Applications, *Comp. Meth. Appl. Mech. Engrg.* **191** (2002)] with the help of duality in convex analysis to justify certain jump terms. Convincing numerical experiments have already been displayed in the literature.

This paper establishes a mathematical a priori error analysis for the dG(1) scheme with discontinuous piecewise linear polynomials in the temporal and first-order finite elements for the spacial discretization. One novel key idea in the a priori convergence analysis is an optimal trace estimate under convex constraints. The numerical investigation of the empirical convergence rate in a benchmark concludes the paper.

1. INTRODUCTION AND OVERVIEW

The dG(k) time discretization of elastoplastic evolution problems has recently been proposed, implemented, and validated in [AC02]. One main motivation was to form a hierarchy of time discretization schemes for further use in adaptive time-stepping control. Figure 1.1 displays the exact and approximate stress component in tangential component in the numerical benchmark of Section 5 as a function of time t . It follows that, in fact, the performance of dG(1) appears significantly better than that of dG(0). It is known that the Crank-Nicholson (CN) scheme is of quadratic order in time [AC00] but there are counter examples that CN is superior over bE and the oscillations visible in Figure 1.1 clearly question that in practise. In comparison, dG(1) looks much better than dG(0).

This paper justifies this superiority of dG(1) over dG(0) for smooth solutions and presents an optimal mathematical analysis for the dG(1) scheme for elastoplasticity with hardening in the weak primal and dual formulation from [HR99], briefly recalled in Section 2, in the fully discrete setting when combined with a standard spatial finite element discretization of lowest order. The resulting fully discrete scheme is a variational inequality in each time-step given with all notational details below in Section 3.

The error analysis of Section 4 encounters one key difficulty: An optimal trace inequality for an approximation under constraints for an affine and hence conforming approximation. The estimate is isolated in Lemma 1.1 for an abstract framework and illustrated in Figure 1.2.

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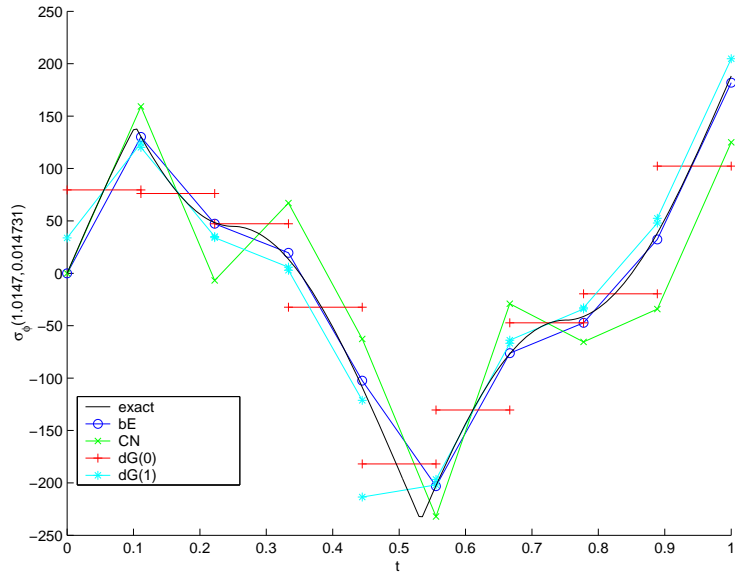


FIGURE 1.1. Time evolution for one stress component $\sigma_\phi(1.0147, 0.0147, t)$ in the benchmark example of Section 5 for the exact values and for various numerical approximations, namely the backward Euler (bE), the Crank-Nicholson (CN), and the dG(k) schemes for $k = 0$ and $k = 1$.

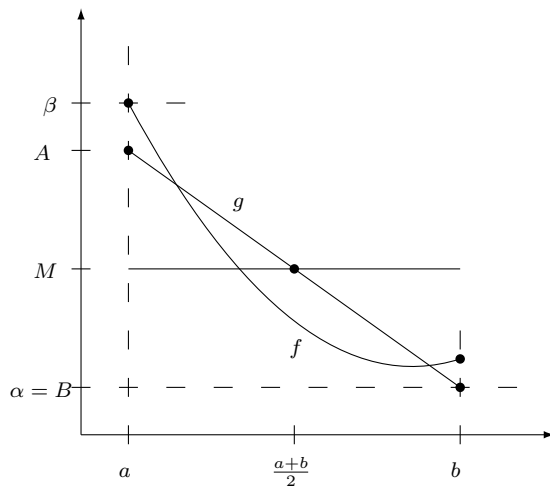


FIGURE 1.2. Illustration of Lemma 1.1 for some function $f \in C^\infty(a, b; [\alpha, \beta])$ with integral mean $M := \int_a^b f(t) dt / (b - a)$ and some line segment g through $((a + b)/2, M)$ and with endpoints (a, A) and (b, B) . The point is that $f - g$ has two zeros even if the affine g obeys the side restriction $\alpha \leq g \leq \beta$ and $\int_a^b g(x) dx = M(b - a)$.

Lemma 1.1. *Given a convex and closed subset \mathcal{K} of a (real) Hilbert space \mathcal{H} , any $f \in H^2(a, b; \mathcal{H}) \cap C(a, b; \mathcal{K})$ with integral mean $M := \int_a^b f(t) dt / (b - a)$ satisfies*

$$(1.1) \quad \min \{ \|f(a) - A\|_{\mathcal{H}} : A, B \in \mathcal{K} \text{ with } A + B = 2M \} \leq (b - a)^{3/2} \|f''\|_{L^2(a, b; \mathcal{H})}.$$

The proof of Lemma 1.1 follows from the observation in the one-dimensional setting depicted in Figure 1.2. In order to obtain the optimal-order $O((b - a)^{3/2})$, the straight line segment with endpoints (a, A) and (b, B) of Figure 1.2 hits the displayed smooth function f twice even under the constraint $A, B \in \mathcal{K} = [\alpha, \beta] \subset \mathbb{R} =: \mathcal{H}$ and $(A + B)/2 = M = \int_a^b f(t) dt / (b - a)$.

The mathematical proof of Lemma 1.1 in the general context is given in Subsection 4.2 together with the complete mathematical analysis. A numerical benchmark in Section 5 confirms the estimate empirically even with some reduced regularity property and concludes the paper with strong numerical evidence of the superiority of G(1) over G(0).

2. MATHEMATICAL MODELING OF ELASTOPLASTICITY

This section is devoted to the strong form of a model example in elastoplasticity with hardening and the weak primal and dual form [HR99]. The generalized stress and generalized plastic strains are given as

$$\Sigma = (\sigma, \chi) \quad \text{and} \quad P = (p, \xi).$$

The stress variable σ and the total (linear Green) strain,

$$\varepsilon(u) := \text{sym } Du = (u_{j,k} + u_{k,j})/2 \quad j, k = 1, 2, \dots, d,$$

are linked with the irreversible plastic strain p through an additive split

$$\varepsilon(u) = \mathbb{C}^{-1} \sigma + p$$

of small strain plasticity. The fourth-order elasticity tensor \mathbb{C} acts as

$$\mathbb{C}q = \lambda \text{tr}(q) \mathbf{1} + 2\mu q \quad \text{for all } q \in \mathbb{R}_{sym}^{d \times d}$$

with trace $\text{tr}(q) := q_{11} + \dots + q_{dd}$, the $d \times d$ unit matrix $\mathbf{1}$, and the Lamé constants $\lambda, \mu > 0$. The displacement field u is supposed to satisfy Dirichlet boundary conditions in the form

$$u = 0 \quad \text{on } \Gamma_D$$

for a fixed closed part Γ_D of $\partial\Omega = \Gamma$ of positive $d - 1$ dimensional (surface) measure. Equilibrium reads in local form

$$\sigma = \sigma^T \quad \text{and} \quad \text{div } \sigma + f = 0 \quad \text{in } \Omega$$

plus Neumann boundary conditions on the remaining part of the boundary

$$\sigma n = g \quad \text{on } \Gamma_N := \Gamma \setminus \Gamma_D.$$

The internal energy assumes the form

$$F(e, \xi) := 1/2(e : \mathbb{C}e + \xi \cdot \mathbb{H}\xi)$$

for the aforementioned fourth-order elasticity tensor \mathbb{C} and a symmetric and positive definite hardening tensor \mathbb{H} . Internal (hardening) variables ξ are written (symbolically) as m dimensional vectors (e.g., the m components of a symmetric $d \times d$ tensor for kinematic hardening or scalar, $m = 1$, for linear isotropic hardening). Hence, $\xi, \chi \in \mathbb{R}^m$ and

$$\mathbb{H} \in \mathbb{R}_{sym}^{m \times m}$$

is identified with a symmetric and positive definite $m \times m$ matrix. Recall $\varepsilon(u) = e + p$ and notice

$$\sigma = \partial F(e, \xi) / \partial e = \mathbb{C}e \quad \text{and} \quad \chi = -\partial F(e, \xi) / \partial \xi = -\mathbb{H}\xi.$$

The Prandtl-Reuß flow rule reads (\dot{P} denotes the time derivative of P)

$$\dot{P} \in N_K(\Sigma) := \{Q \in \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m : \forall T \in K, Q \star (T - \Sigma) \leq 0\}$$

for the set of admissible generalized stresses $K \subset \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m$ determined by the yield function (e.g., the von-Mises yield function) $\Phi : \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m \rightarrow \mathbb{R}$ via

$$K := \{T \in \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m : \Phi(T) \leq 0\}.$$

Throughout this paper, we distinguish between the scalar products \cdot, \cdot, \star , defined for vectors u, v , $d \times d$ matrices p, q , and generalized stresses or strains P, Q by $u \cdot v = u_1 v_1 + \dots + u_d v_d$,

$$p : q := \sum_{j,k=1}^d p_{jk} q_{jk}, \quad \text{and} \quad P \star Q := (p, \xi) \star (q, \chi) = p : q + \xi \cdot \chi.$$

Given data f and g as functions in time $[0, T]$ and space, given consistent homogeneous initial conditions (i.e., $f = g = 0$ for $t = 0$) the elastoplastic time-evolution determines u, σ, χ, p and ξ as functions on $[0, T] \times \Omega$ with

$$\sigma = \sigma^T = \mathbb{C}(\varepsilon(u) - p), \quad \text{div } \sigma + f = 0, \quad (\dot{p}, \dot{\xi}) \in N_K(\sigma, \chi) \text{ in } [0, T] \times \Omega$$

and the boundary conditions

$$u = 0 \text{ on } [0, T] \times \Gamma_D \quad \text{and} \quad \sigma n = g \text{ on } [0, T] \times \Gamma_N.$$

Following [HR99, Car97] the primal and dual formulation differ in the treatment of the elastoplastic evolution law. Convex analysis [ET76, Zei88] allows an equivalent reformulation via

$$\dot{P} \in N_K(\Sigma) \Leftrightarrow \Sigma \in \partial \text{supp}_K(\dot{P}).$$

The first inclusion is defined above and, given K via a yield function Φ , reads

$$\Phi(\Sigma) \leq 0 \quad \text{and} \quad \dot{P} \star (T - \Sigma) \leq 0 \quad \text{for all } T \in \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m \text{ with } \Phi(T) \leq 0.$$

The second inclusion involves the support function

$$\text{supp}_K(Q) := \sup_{T \in K} Q \star T = \sup_{\Phi(T) \leq 0} Q \star T$$

and its subdifferential ∂supp_K . Indeed, $\Sigma \in \partial \text{supp}_K(\dot{P})$ reads

$$\Sigma \star (Q - \dot{P}) \leq \text{supp}_K(Q) - \text{supp}_K(\dot{P}) \quad \text{for all } Q \in \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m.$$

Finally let χ_K denote the characteristic functional of K , i.e., $\chi_K(\Sigma) = 0$ if $\Phi(\Sigma) \leq 0$, while $\chi_K(\Sigma) = +\infty$ otherwise.

Throughout this paper, the combined kinematic and isotropic hardening is considered in the von Mises yield function. Then, $\mathbb{R}^m \equiv \mathbb{R} \times \mathbb{R}_{sym}^{d \times d}$ and $\Sigma = (\sigma, \chi)$ with $\chi = (a, b) = -\mathbb{H}\xi$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$ and

$$\Phi(\sigma, a, b) := |\text{dev } \sigma - \text{dev } b| - \sigma_y(1 + Ha) \text{ for } \Sigma = (\sigma, a, b) \in \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m.$$

Here and throughout this paper,

$$\mathbb{H} = \text{diag}(A, B) \in \mathbb{R}^{m \times m} \text{ for } A > 0 \text{ and } B \in \mathbb{R}^{(m-1) \times (m-1)}.$$

The material parameters $\sigma_y > 0$ and $H \geq 0$ are fixed and constant in time and space. Then,

$$K = \{(\sigma, a, b) \in \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m \mid a \geq 0 \text{ and } \Phi(\sigma, a, b) \leq 0\}$$

and the scalar product \star in \mathbb{R}^m is represented by $\text{diag}(A^{-1}, B^{-1})$. Particular cases are discussed in [HR99, AC00, Car99, Car97, ACZ99]. The hardening will be important for control of Green strain in terms of the generalized stresses.

Proposition 2.1. [Car99] *The functional supp_K is the dual of the characteristic functional χ_K of K and reads*

$$\text{supp}_K(p, \alpha, \beta) = \begin{cases} \sigma_y |p| & \text{if } \text{tr } p = 0 \text{ and } p = -\beta \text{ and } A^{-1}\alpha + \sigma_y H |p| \leq 0, \\ \infty & \text{if not.} \end{cases}$$

Moreover, the inclusion $(\sigma, \chi) \in \partial \text{supp}_K(p, \xi)$ in case $p \neq 0$ is equivalent to

$$\frac{p}{|p|} = \frac{\text{dev}(\sigma - b)}{\sigma_y(1 + Ha)} \text{ and } \alpha = -k|p| \text{ for } k := A\sigma_y H. \quad \square$$

The corresponding weak formulations of the complete model are derived by a principle of virtual displacements or by testing with a test function. The two resulting variational inequalities are summarized below; we refer to [HR99, ACZ99] for further details.

For a weak formulation, the test function spaces are based on standard Lebesgue and Sobolev spaces, namely,

$$\begin{aligned} L^2(\Omega) &:= \{v : \Omega \rightarrow \mathbb{R} : v \text{ measurable with } \int_{\Omega} |v|^2 dx < \infty\}, \\ H^1(\Omega) &:= \{v \in L^2(\Omega) : \partial v / \partial x_1, \dots, \partial v / \partial x_d \in L^2(\Omega)\} \end{aligned}$$

and powers thereof (i.e., all components belong to the respective space); $\partial v / \partial x_j$ is a weak derivative [Zei88, Eva98]. Moreover,

$$V := \{v \in H^1(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \Gamma_D\} \quad \text{and} \quad L := L^2(\Omega; \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^n).$$

Then the weak **primal formulation** reads: Seek $(u, p, \xi) : [0, T] \rightarrow W^{1,2}(0, T; \mathbb{R}^d \times \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m)$ with homogeneous initial values and the Dirichlet boundary condition $u(t) = 0$ on Γ_D such that, for almost every time $t \in (0, T)$ and, for all $v \in V$ and all $(q, \zeta) \in L$, it holds

$$\begin{aligned} &\int_{\Omega} \mathbb{C}(\varepsilon(u(t)) - p(t)) : (\varepsilon(v) - \dot{p}(t) + q) dx - \int_{\Omega} \xi(t) \cdot \mathbb{H}(\zeta - \dot{\xi}(t)) dx \\ &\leq \int_{\Omega} f(t) \cdot v dx + \int_{\Gamma_N} g(t) \cdot v ds + \int_{\Omega} \text{supp}_K(q, \zeta) dx - \int_{\Omega} \text{supp}_K(\dot{p}(t), \dot{\xi}(t)) dx. \end{aligned}$$

Hence, the **dual formulation** reads: Seek $(u, \sigma, \chi) : [0, T] \times \Omega \rightarrow \mathbb{R}^d \times \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m$ with homogeneous initial values and the Dirichlet boundary condition $u(t) = u_D$ on Γ_D , such

that, for almost every time $t \in (0, T)$ and all $v \in V$, it holds

$$(2.1) \quad \int_{\Omega} \sigma(t) : \varepsilon(v) dx = \int_{\Omega} f(t) \cdot v dx + \int_{\Gamma_T} g(t) \cdot v ds$$

and $\Phi(\sigma(t), \chi(t)) \leq 0$ such that, for all $(\tau, \psi) \in L^2(\Omega; \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m)$ with $\Phi(\tau, \psi) \leq 0$, it holds

$$(2.2) \quad \int_{\Omega} (\varepsilon(\dot{u}(t)) - \mathbb{C}^{-1} \dot{\sigma}(t)) : (\tau - \sigma(t)) dx - \int_{\Omega} \dot{\chi}(t) \star \mathbb{H}^{-1}(\psi - \chi(t)) dx \leq 0.$$

On the continuous level the dual and primal formulation are equivalent [ACZ99, HR99]. The role of the time-derivative and the choice of variables are different. The unique existence of solutions is well established [HR99, Joh76, Joh78, Suq88, Tem85] as well as regularity for the stress variables [Ser93].

3. DISCRETIZATION

This section is devoted to the motivation of discontinuous Galerkin time discretization schemes for elastoplastic evolution problems. The point of departure is the definition of a distributional derivate for discontinuous, but \mathcal{I} -piecewise smooth test functions.

The time interval $(0, T]$ is partitioned in N subintervals $I_j = (t_{j-1}, t_j]$ (open at the left and closed at the right) of length k_j , $j = 1, \dots, J$, according to

$$t_0 = 0 < t_1 < t_2 < \dots < t_J = T; \quad \mathcal{I} = \{I_1, I_2, \dots, I_J\}.$$

Define the set of \mathcal{I} -piecewise smooth functions by

$$\mathcal{C}^1(\mathcal{I}) := \{v \in L^\infty(\mathbb{R}) : v|_{I_j} \in \mathcal{C}^1[t_{j-1}, t_j] \text{ and } v|_{(-\infty, 0]} := v(0^-)\}.$$

Piecewise uniformly continuous functions u allow for one-sided limits and the definition of the jump

$$[u]_j := u(t_j^+) - u(t_j^-) \quad \text{with } u_j^\pm := u(t_j^\pm) := \lim_{t \rightarrow t_j^\pm} u(t) \quad \text{for } j = 0, \dots, J-1.$$

(Throughout, $u(0^-) := 0$ owing to homogeneous initial conditions whence $[u]_0 := u(0^+)$.) Since $u|_{(t_{j-1}, t_j]}$ is $\mathcal{C}^1(t_{j-1}, t_j)$, there exists the time derivative $u_\tau := \partial u / \partial t$ on each $(t_{j-1}, t_j]$ in the classical sense as limit of difference quotients. The distributional derivative \dot{u} is defined through

$$\int_{\mathbb{R}} \dot{u}(t)v(t)dt = - \int_{\mathbb{R}} u(t)\dot{v}(t)dt$$

for all test functions $v \in \mathcal{C}_c^\infty(\mathbb{R})$. With the delta distribution δ_{t_j} at the point t_j , it reads

$$\dot{u} = u_\tau + \sum_{j=0}^{J-1} [u]_j \delta_{t_j}.$$

This explains the action of \dot{u} on differentiable test functions. An extension to discontinuous test functions starts with globally continuous and piecewise \mathcal{C}^1 test functions v_ε which vanish outside some fixed interval I_j . The functions v_ε are defined by a regularization of v with suitable step functions χ_ε^j as depicted in Figure 3.1,

$$v_\varepsilon(t) := \chi_\varepsilon^j(t)v(t).$$

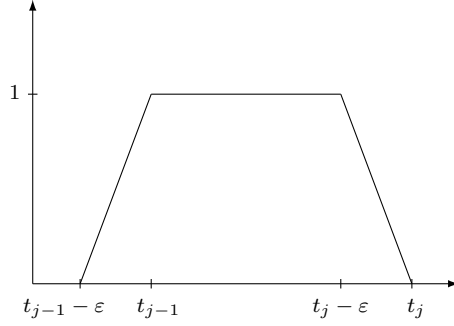


FIGURE 3.1. Test function χ_ϵ^j

Thus, we obtain a formulation of a distributional derivative for the fixed interval I_j by analyzing the equation

$$\int_{\mathbb{R}} \dot{u}(t)v_\epsilon(t) dt = \int_{t_{j-1}-\epsilon}^{t_j} u_\tau(t)v_\epsilon(t) dt + [u]_{j-1}v_\epsilon(t_{j-1})$$

in the limit $\epsilon \rightarrow 0$. One obtains

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \dot{u}(t)v_\epsilon(t) dt = \int_{I_j} u_\tau(t)v(t) dt + [u]_{j-1}v(t_{j-1}^+).$$

Extending this procedure to general $v \in \mathcal{C}^1(\mathcal{I})$ by applying test functions v_ϵ^j on each partinterval I_j , followed by a summation of the resulting terms leads to

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \dot{u} \left(\sum_{j=1}^J v_\epsilon^j \right) dt = \int_0^T u_\tau(t)v(t) dt + \sum_{j=1}^J [u]_{j-1}v(t_{j-1}^+).$$

The spline space of discontinuous Galerkin functions of order k (abbreviated as dG(k)) is defined through

$$P_k(\mathcal{I}; X) = \{u \in L^\infty([a, b], X) : \forall j = 1, \dots, J, u|_{I_j} \in P_k(I_j; X)\}.$$

The homogeneous initial data are reflected in the convention that $u(0^-) = 0$ for all discrete values below. The domain Ω is partitioned into triangles and parallelograms for 2D and tetrahedra for 3D. The resulting triangulation \mathcal{T} is supposed to be regular in the sense of Ciarlet [BS94, Cia78]. For each element T , $P_k(T)$ denotes the algebraic polynomials on T of total degree $\leq k$ if T is a triangle or tetrahedron or of partial degree $\leq k$ if T is a parallelogram. Then, the required finite element function spaces read

$$\begin{aligned} P_\ell(\mathcal{T}; \mathbb{R}^d) &:= \{v \in L^2(\Omega) : \forall T \in \mathcal{T}, v|_T \in P_\ell(T; \mathbb{R}^d)\}, \\ V_h &:= P_1(\mathcal{T}; \mathbb{R}^d) \cap V, \\ L_h &:= P_0(\mathcal{T}; \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m), \\ K_h &:= P_0(\mathcal{T}; K). \end{aligned}$$

The finite element approximation is denoted by subindices k and h (neglected for its continuous counterpart) as the underlying discretization is based on a partition \mathcal{I} in time and

a regular triangulation \mathcal{T} in space. The following abbreviations are used throughout this paper:

$$Q_j := I_j \times \Omega \quad \text{and} \quad \int_{I_j} \int_{\Omega} \dots dx dt \quad \text{reads} \quad \int_{Q_j} \dots dQ.$$

The **discrete primal problem** reads: Seek $(u_{hk}, \Sigma_{hk}, P_{hk}) \in P_1(\mathcal{I}; V_h \times L_h \times L_h)$ such that $\Sigma_{hk} = (\sigma_{hk}, \chi_{hk})$, $P_{hk} = (p_{hk}, \xi_{hk})$, $p_{hk} = \varepsilon(u_{hk}) - \mathbb{C}^{-1}\sigma_{hk}$ and $\xi_{hk} = -\mathbb{H}^{-1}\chi_{hk}$ and, for all $j = 1, \dots, J$ as well as all $v_{hk} \in P_1(I_j; V_h)$, it holds

$$(3.1) \quad \int_{Q_j} \sigma_{hk} : \varepsilon(v_{hk}) dQ = \int_{Q_j} f \cdot v_{hk} dQ + \int_{I_j} \int_{\Gamma_N} g \cdot v_{hk} ds dt$$

and, for all $Q_{hk} \in P_1(I_j; L_h)$, it holds

$$(3.2) \quad \int_{Q_j} \Sigma_{hk} \star (Q_{hk} - P_{hk,\tau}) dQ - \sum_{j=1}^J \int_{\Omega} (\Sigma_{hk})_{j-1}^+ \star [P_{hk}]_{j-1} dx \leq \sup_{R_{hk} \in P_1(I_j; K_h)} \int_{Q_j} R_{hk} \star Q_{hk} dQ \\ - \sup_{S_{hk} \in P_1(I_j; K_h)} \left(\int_{Q_j} S_{hk} \star P_{hk,\tau} dQ + \int_{\Omega} (S_{hk})_{j-1}^+ \star [P_{hk}]_{j-1} dx \right).$$

The **discrete dual problem** reads: Seek $(u_{hk}, \Sigma_{hk}, P_{hk}) \in P_1(\mathcal{I}; V_h \times K_h \times L_h)$ such that $\Sigma_{hk} = (\sigma_{hk}, \chi_{hk})$, $P_{hk} = (p_{hk}, \xi_{hk})$, $p_{hk} = \varepsilon(u_{hk}) - \mathbb{C}^{-1}\sigma_{hk}$, and $\xi_{hk} = -\mathbb{H}^{-1}\chi_{hk}$ and, for all $j = 1, \dots, J$ as well as all $v_{hk} \in P_1(I_j; V_h)$, it holds

$$(3.3) \quad \int_{Q_j} \sigma_{hk} : \varepsilon(v_{hk}) dQ = \int_{Q_j} f \cdot v_{hk} dQ + \int_{I_j} \int_{\Gamma_N} g \cdot v_{hk} ds dt$$

and, for all $T_{hk} \in P_1(I_j; K_h)$, it holds

$$(3.4) \quad \int_{Q_j} P_{hk,\tau} \star (T_{hk} - \Sigma_{hk}) dQ + \int_{\Omega} [P_{hk}]_{j-1} \star (T_{hk} - \Sigma_{hk})_{j-1}^+ dx \leq 0.$$

The jump terms in inequality (3.4) result from the aforementioned construction of the distributional derivative for the discontinuous test functions. This and the non-trivial implementation of the dG(1) discrete scheme is described in [AC02].

Theorem 3.1. *The discrete primal problem and discrete dual problem are equivalent.*

Proof. Let $(u_{hk}, \Sigma_{hk}, P_{hk})$ solve the discrete primal problem. Since $P_{\ell}(I_j; L_h)$ is a Hilbert space, the linear functional associated with $[P_{hk}]_{j-1}$ has some Riesz representation $\dot{P}_{hk} \in P_{\ell}(I_j; L_h)$ with

$$\int_{Q_j} T_{hk} \star \dot{P}_{hk} dQ = \int_{Q_j} T_{hk} \star P_{hk,\tau} dQ + \int_{\Omega} [P_{hk}]_{j-1} \star (T_{hk})_{j-1}^+ dx \quad \text{for all } T_{hk} \in P_{\ell}(I_j; L_h).$$

Then (3.2) reads, for all $Q_{hk} \in P_{\ell}(I_j; L_h)$ and all $S_{hk} \in P_{\ell}(I_j; K_h)$,

$$\int_{Q_j} \Sigma_{hk} \star (Q_{hk} - \dot{P}_{hk}) dQ \leq \sup_{R_{hk} \in P_{\ell}(I_j; K_h)} \int_{Q_j} R_{hk} \star Q_{hk} dQ - \int_{Q_j} S_{hk} \star \dot{P}_{hk} dQ.$$

Some reformulations result in

$$(3.5) \quad \int_{Q_j} (\Sigma_{hk} - S_{hk}) \star (Q_{hk} - \dot{P}_{hk}) dQ \leq \sup_{R_{hk} \in P_\ell(I_j; K_h)} \int_{Q_j} (R_{hk} - S_{hk}) \star Q_{hk} dQ.$$

The substitution of $Q_{hk} \in P_\ell(I_j; L_h)$ by $Q_h := \lambda Q_{hk}$ with $\lambda \rightarrow \infty$ in (3.5) leads to

$$(3.6) \quad 0 \leq \sup_{R_{hk} \in P_\ell(I_j; K_h)} \int_{Q_j} (R_{hk} - \Sigma_{hk}) \star Q_h dQ \quad \text{for all } Q_h \in P_\ell(I_j; L_h).$$

Given almost every $x \in \Omega$ and $\Sigma_{hk}(\cdot, x) \in P_\ell(I_j; \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m)$ let $\hat{\Sigma}_{hk}(\cdot, x)$ denote its projection onto $P_\ell(I_j; K)$ in the Hilbert space $P_\ell(I_j; \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m)$ with respect to the scalar product of $L^2(I_j; \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}^m)$. Then, $Q_{hk} := \Sigma_{hk} - \hat{\Sigma}_{hk}$ in (3.6) yields that

$$\|\hat{\Sigma}_{hk} - \Sigma_{hk}\|_{L^2(Q_j)}^2 \leq \sup_{R_{hk} \in P_\ell(I_j; K_h)} \int_{Q_j} (R_{hk} - \hat{\Sigma}_{hk}) \star (\Sigma_{hk} - \hat{\Sigma}_{hk}) dQ.$$

Since R_{hk} belongs to $P_\ell(I_j; K_h)$ and $\hat{\Sigma}_{hk}$ is the projection onto this, the scalar product is non-positive. Hence $\Sigma_{hk} = \hat{\Sigma}_{hk}$ is admissible.

The choice $Q_{hk} = 0$ in (3.5) yields, for all $S_{hk} \in P_\ell(I_j; K_h)$,

$$(3.7) \quad \int_{Q_j} (S_{hk} - \Sigma_{hk}) \star \dot{P}_{hk} dQ \leq 0.$$

Hence $\dot{P}_{hk} \in N_{K_h}(\Sigma_{hk})$ and (3.4) of the dual formulation is verified.

To prove the converse implication, suppose $\Phi(\Sigma_{hk}) \leq 0$ and (3.4) for all $T_{hk} \in P_\ell(I_j; K_h)$. This means

$$\sup_{T_{hk} \in P_\ell(I_j; K_h)} \int_{Q_j} T_{hk} \star \dot{P}_{hk} dQ_j = \int_{Q_j} \Sigma_{hk} \star \dot{P}_{hk} dQ_j.$$

This implies the discrete variational inequality (3.2) for $Q_{hk} = 0$. Finally, since $\Sigma_{hk} \in K_h$,

$$\int_{Q_j} \Sigma_{hk} \star Q_{hk} dQ \leq \sup_{R_h \in P_1(I_j; K_h)} \int_{Q_j} R_{hk} \star Q_{hk} dQ$$

for any $Q_{hk} \in P_1(I_j; L_h)$. The sum of the two displayed inequalities is (3.2). \square

4. A PRIORI CONVERGENCE ANALYSIS

4.1. A Priori Error Estimate. Given the elasticity and hardening tensor \mathbb{C} and \mathbb{H} , let $\mathbb{A} := \text{diag}(\mathbb{C}^{-1}, \mathbb{H}^{-1})$ and the (possibly discontinuous) errors $\Delta := \Sigma - \Sigma_{hk}$ with right and left limit Δ_j^\pm at t_j . The local mesh-sizes and time-step sizes $h \in P_0(\mathcal{T})$ and $k \in P_0(\mathcal{T})$ are defined by

$$h|_K := \text{diam}(K) \text{ for all } K \in \mathcal{T} \quad \text{and} \quad k_{I_j} := k_j := \text{diam}(I_j) \text{ for all } j = 1, 2, \dots, J.$$

The following is the main result and implies that the stress error converges like $O(k^{3/2} + h)$. There is no restriction on the time and space discretisation and all estimates are localised in the form of local time-step size or local mesh-size times a derivative of the exact solution in some Lebesgue norm.

Theorem 4.1. *Suppose that the exact solution and the data are smooth in the sense of*
 $u \in W^{3,1}(0, T; V) \cap W^{1,1}(0, T; H^2(\Omega; \mathbb{R}^d))$ *and* $(\Sigma, f, g) \in H^2(0, T; L \times V^* \times V^*)$.

Then it holds

$$\begin{aligned} & \|\mathbb{A}^{1/2} \Delta_J^-\|_L^2 + \sum_{j=1}^J \|\mathbb{A}^{1/2} [\Delta]_{j-1}\|_L^2 + \max_{j=1, \dots, J} \|\mathbb{A}^{1/2} \Delta_{j-1}^\pm\|_L^2 \\ & \leq C(\mathbb{A}) (\|k^2 \varepsilon(\ddot{u}), hD^2 \dot{u}\|_{L^1(0, T; L^2(\Omega) \times L^2(\Omega))}^2 + \|k^{3/2} (\mathbb{A}^{1/2} \ddot{\Sigma}, \ddot{f}, \ddot{g})\|_{L^2(0, T; L \times V^* \times V^*)}^2). \end{aligned}$$

The constant $C(\mathbb{A})$ depends on all material constants as well as the shapes of the element domains but not on their sizes.

For notational simplicity, all material constants $\mathbb{A}, A, B, \mathbb{C}, \mathbb{H}, H, \sigma_y$ are homogeneous, that means, constant in space and time. We refer to [Car99] for variable material parameters in space and a corresponding perturbation analysis.

4.2. Proof of Lemma 1.1.

Proof. Set $F := f(a) \in \mathcal{K}$ and $M := (b-a)^{-1} \int_a^b f(t) dt \in \mathcal{K}$ and define $g := f - M$ and the shifted set $\mathcal{S} := \mathcal{K} - M$ and notice $G := g(a) = F - M$.

Since $-\mathcal{S}$ is nonvoid, closed, and convex, there exists exactly one $C \in -\mathcal{S}$ with minimal distance to $G \in \mathcal{H}$,

$$\delta := \|G - C\|_{\mathcal{H}} = \|f(a) - A\|_{\mathcal{H}} = \text{dist}(G, -\mathcal{S}).$$

Since $\delta = 0$ implies that the left hand side of (1.1) must be zero, we suppose $\delta > 0$ in the sequel. Set $A = C + M \in K$, hence $G - C = F - A$, and define $N := (G - C)/\delta \in \mathcal{H}$. The function $h \in H^2(a, b; \mathbb{R})$, defined by

$$h(t) := \langle N, f(t) - \frac{b-t}{b-a} A - \frac{t-a}{b-a} (2M - A) \rangle_{\mathcal{H}} \quad \text{for } a \leq t \leq b,$$

satisfies

$$\begin{aligned} \delta &= \langle N, G - C \rangle_{\mathcal{H}} = \langle N, F - A \rangle_{\mathcal{H}} = h(a) > 0, \\ (b-a)^{-1} \int_a^b h(t) dt &= \langle N, M - \frac{1}{2} A - \frac{1}{2} (2M - A) \rangle = 0, \\ h(b) &= \langle N, f(b) - 2M + A \rangle = \langle N, g(b) + C \rangle. \end{aligned}$$

Since $g(b) = f(b) - M \in S$, $-g(b) \in -S$ and so by separation of G from $-S$ with a hyperplane through C and normal N ,

$$-h(b) = \langle N, -g(b) - C \rangle \leq 0.$$

In conclusion, the function $h \in H^2(a, b; \mathbb{R})$ satisfies

$$0 = \int_a^b h(t) dt \leq (b-a) \min\{h(a), h(b)\},$$

and so h as well as its derivative h' vanish at least once in (a, b) , i.e. $h(\xi) = 0 = h'(\eta)$ for some $a \leq \xi, \eta \leq b$. The fundamental theorem of calculus therefore proves

$$h(a) = \int_{\xi}^a (h'(t) - h'(\eta)) dt = \int_{\xi}^a \int_{\eta}^t h''(s) ds dt.$$

Hölder's inequality yields

$$\int_{\eta}^t |h''(s)| ds \leq \|h''\|_{L^2(\eta,t)} |t - \eta|^{1/2} \leq \|h''\|_{L^2(a,b)} |t - \eta|^{1/2}.$$

The combination of the previous expressions shows

$$|h(a)| \leq \int_{\xi}^a \int_{\eta}^t |h''(s)| ds dt \leq \|h''\|_{L^2(a,b)} \int_b^a |t - \eta|^{1/2} dt \leq \|h''\|_{L^2(a,b)} (b - a)^{3/2}.$$

This, $h(a) = \|f(a) - A\|$, and

$$\|h''\|_{L^2(a,b)} = \|\langle N, f'' \rangle_{\mathcal{H}}\|_{L^2(a,b)} \leq |N| \|f''\|_{L^2(a,b;\mathcal{H})} = \|f''\|_{L^2(a,b;\mathcal{H})}$$

prove the assertion. \square

4.3. Discrete Approximations $\Sigma_h^*, \Sigma_k^*, \Sigma_{hk}^*$. Given the exact solution Σ , define $\Sigma_h^* \in H^1(0, T; L_h)$ as piecewise integral mean

$$\Sigma_h^*(t)|_K := \frac{1}{|K|} \int_K \Sigma(t, x) dx \in K_h \quad \text{for all times } 0 \leq t \leq T \text{ and all } K \in \mathcal{T}.$$

A piecewise Poincaré inequality yields, at all times t , with all first-order partial derivations $D\Sigma$ of Σ and the mesh-size $h \in P_0(\mathcal{T})$,

$$\|(\Sigma - \Sigma_h^*)(t)\|_L \leq \|hD\Sigma(t)\|_{L^2(\Omega)}.$$

The approximation $\Sigma_{hk}^* \in P_1(\mathcal{T}; L_h)$ is defined \mathcal{I} -piecewise through linear interpolation

$$\Sigma_{hk}^*|_{I_j} = \frac{t_j - t}{t_j - t_{j-1}} \Sigma_{hk}^*(t_j^-) + \frac{t - t_{j-1}}{t_j - t_{j-1}} \Sigma_{hk}^*(t_{j-1}^+)$$

from a minimisation problem in $\mathbb{R}^{d \times d} \times \mathbb{R}^m$ with optimal values

$$\begin{aligned} (\Sigma_{hk}^*(t_j^-), \Sigma_{hk}^*(t_{j-1}^+)) = \arg \min \left\{ \|\mathbb{A}^{1/2}(A - \Sigma(t_{j-1}))\|_L^2 \mid (A, B) \in K_h^2 \right. \\ \left. \text{with } A + B = 2k_j^{-1} \int_{I_j} \Sigma(t) dt \right\}. \end{aligned}$$

Lemma 1.1 applies to this minimisation problem and leads to

$$(4.1) \quad \sum_{j=1}^J \|\mathbb{A}^{1/2}(\Sigma_{hk}^* - \Sigma_h^*)_{j-1}^+\|_L^2 \leq \sum_{j=1}^J k_j^3 \|\mathbb{A}^{1/2} \ddot{\Sigma}_h^*\|_{L^2(I_j; L)}^2.$$

Since time differentiation and piecewise spatial averages commute, Jensen's inequality shows

$$\|\mathbb{A}^{1/2} \ddot{\Sigma}_h^*(t)\|_L \leq \|\mathbb{A}^{1/2} \ddot{\Sigma}\|_L,$$

the previous estimate leads to

$$(4.2) \quad \sum_{j=1}^J \|\mathbb{A}^{1/2}(\Sigma_{hk}^* - \Sigma_h^*)_{j-1}^+\|_L^2 \leq \|k^{3/2} \ddot{\Sigma}\|_{L^2(0, T; L)}^2.$$

Finally, Σ_k^* will denote the piecewise affine and globally continuous nodal interpolation of the exact generalized stresses Σ in time.

4.4. Combination of Variational Inequalities. Evaluate (2.2) for $T = \Sigma_{hk}$ and (3.4) for $T_{hk} = \Sigma_{hk}^*$ and recall $P := (p, \xi)$ with $p = \varepsilon(u) - \mathbb{C}^{-1}\sigma$. The sum of the two inequalities reads

$$(4.3) \quad \int_{Q_j} \left(P_{hk,\tau} \star (\Sigma_{hk}^* - \Sigma_{hk}) + \dot{P} \star (\Sigma_{hk} - \Sigma) \right) dQ + \int_{\Omega} [P_{hk}]_{j-1} \star (\Sigma_{hk}^* - \Sigma_{hk})_{j-1}^+ dx \leq 0.$$

Since $P_{hk,\tau} \in \mathcal{P}_0(I_j; L_h)$ and Σ_{hk}^* satisfies the integral mean property on I_j ,

$$\int_{Q_j} P_{hk,\tau} \star \Sigma_{hk}^* dQ = \int_{Q_j} P_{hk,\tau} \star \Sigma_h^* dQ = \int_{Q_j} P_{hk,\tau} \star \Sigma dQ.$$

Consequently, the first integral in (4.3) reads

$$(4.4) \quad \int_{Q_j} (P_{hk,\tau} - \dot{P}) \star (\Sigma - \Sigma_{hk}) dQ.$$

Recall $P = (\varepsilon(u), 0) - \mathbb{A}\Sigma$, $P_{hk} = (\varepsilon(u_{hk}), 0) - \mathbb{A}\Sigma_{hk}$, $\Delta := \Sigma - \Sigma_{hk}$ and set $\delta := \sigma - \sigma_{hk}$, $e := u - u_{hk}$. Then (4.4) reads

$$\int_{Q_j} \Delta_\tau \star \mathbb{A}\Delta dQ - \int_{Q_j} \delta : \varepsilon(e_\tau) dQ.$$

The equilibria (2.1) and (3.3) yield

$$\int_{Q_j} \delta : \varepsilon(v_{hk}) dQ = 0 \quad \text{for all } v_{hk} \in P_1(I_j; V_h).$$

Altogether, (4.4) reads, for all $v_{hk} \in P_1(I_j; V_h)$,

$$\int_{Q_j} \Delta_\tau \star \mathbb{A}\Delta dQ - \int_{Q_j} \delta : \varepsilon(u_\tau - v_{hk}) dQ.$$

From continuity of u, σ , etc. in time, $[\Delta]_j = -[\Sigma_{hk}]_j$. Hence the second integral in (4.3) with the jumps at t_{j-1} reads

$$\int_{\Omega} [\mathbb{A}\Delta - (\varepsilon(e), 0)]_{j-1} \star (\Sigma_{hk}^* - \Sigma_{hk})_{j-1}^+ dx.$$

Since

$$2 \int_{Q_j} \Delta_\tau \star \mathbb{A}\Delta dQ = \int_{I_j} \frac{\partial}{\partial t} \|\mathbb{A}^{1/2}\Delta\|_L^2 dt = \|\mathbb{A}^{1/2}\Delta_j^-\|_L^2 - \|\mathbb{A}^{1/2}\Delta_{j-1}^+\|_L^2,$$

the sum of resulting estimates for (4.3) over $j = 1, \dots, \ell$, $\ell \leq J$, leads to

$$\begin{aligned} \sum_{j=1}^{\ell} \frac{1}{2} \int_{I_j} \frac{\partial}{\partial t} \|\mathbb{A}^{1/2}\Delta\|_L^2 dt + \sum_{j=1}^{\ell} \int_{\Omega} [\mathbb{A}\Delta - (\varepsilon(e), 0)]_{j-1} \star (\Sigma_{hk}^* - \Sigma_{hk})_{j-1}^+ dx \\ \leq \int_{(0, t_\ell) \times \Omega} \delta : \varepsilon(u_\tau - v_{hk}) dQ. \end{aligned}$$

Since $\frac{1}{2}\|\mathbb{A}\Delta_0^-\|_L^2$ vanishes with $\Delta_0^- = 0$, the first terms on the left-hand side read

$$\begin{aligned} \sum_{j=1}^{\ell} \frac{1}{2} (\|\mathbb{A}^{1/2}\Delta_j^-\|_L^2 - \|\mathbb{A}^{1/2}\Delta_{j-1}^+\|_L^2) + \sum_{j=1}^{\ell} \int_{\Omega} [\mathbb{A}\Delta]_{j-1} \star \Delta_{j-1}^+ dx \\ = \frac{1}{2} \|\mathbb{A}^{1/2}\Delta_{\ell}^-\|_L^2 + \sum_{j=1}^{\ell} \frac{1}{2} \|\mathbb{A}^{1/2}[\Delta]_{j-1}\|_L^2. \end{aligned}$$

This leads to

$$\begin{aligned} LHS(\ell) := \frac{1}{2} \|\mathbb{A}^{1/2}\Delta_{\ell}^-\|_L^2 + \sum_{j=1}^{\ell} \frac{1}{2} \|\mathbb{A}^{1/2}[\Delta]_{j-1}\|_L^2 \leq \sum_{j=1}^{\ell} \int_{\Omega} [\mathbb{A}\Delta]_{j-1} \star (\Sigma - \Sigma_{hk}^*)_{j-1}^+ dx \\ (4.5) \quad + \int_{(0,t_{\ell}) \times \Omega} \delta : \varepsilon(\dot{u} - v_{hk}) dQ + \sum_{j=1}^{\ell} \int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma_{hk}^* - \sigma_{hk})_{j-1}^+ dx. \end{aligned}$$

4.5. Approximation Errors on *RHS*. Since $[\mathbb{A}\Delta]_{j-1} = -\mathbb{A}[\Sigma_{hk}]_{j-1}$ is piecewise constant, the first term on the right-hand side $RHS(\ell)$ of (4.5) reads

$$\int_{\Omega} [\mathbb{A}\Delta]_{j-1} \star (\Sigma - \Sigma_{hk}^*)_{j-1}^+ dx = \int_{\Omega} [\mathbb{A}\Delta]_{j-1} \star (\Sigma_h^* - \Sigma_{hk}^*)_{j-1}^+ dx.$$

A Cauchy inequality plus (4.2) from Subsection 4.3 shows

$$\begin{aligned} \int_{\Omega} [\mathbb{A}\Delta]_{j-1} \star (\Sigma - \Sigma_{hk}^*)_{j-1}^+ dx \leq \frac{1}{16} \|\mathbb{A}^{1/2}[\Delta]_{j-1}\|_L^2 + 4 \|\mathbb{A}^{1/2}(\Sigma_h^* - \Sigma_{hk}^*)_{j-1}^+\|_L^2 \\ \leq \frac{1}{16} \|\mathbb{A}^{1/2}[\Delta]_{j-1}\|_L^2 + 4 k_j^3 \|\mathbb{A}^{1/2}\ddot{\Sigma}\|_{L^2(I_j;L)}^2. \end{aligned}$$

The second term of $RHS(\ell)$ from (4.5) reads

$$\int_{(0,t_{\ell}) \times \Omega} \delta : \varepsilon(\dot{u} - v_{hk}) dQ \leq \|(\mathbb{C}^{-1/2}(\varepsilon(\dot{u}) - \varepsilon(v_{hk})), 0)\|_{L^1(0,T;L)} \|\mathbb{C}^{1/2}(\sigma - \sigma_{hk})\|_{L^\infty(0,t_{\ell};L)}.$$

Let $\Sigma_k^* = (\sigma_k^*, \chi_k^*)$ denote the piecewise affine and globally continuous nodal interpolation of the exact generalized stresses Σ in time. Then, the last term is bounded from above by

$$\begin{aligned} \|\mathbb{C}^{1/2}(\varepsilon(\dot{u} - v_{hk}), 0)\|_{L^1(0,T;L)} \left(\|\mathbb{C}^{-1/2}(\sigma - \sigma_k^*)\|_{L^\infty(0,T;L)} + \|\mathbb{C}^{-1/2}(\sigma_k^* - \sigma_{hk})\|_{L^\infty(0,t_{\ell};L)} \right) \\ (4.6) \quad \leq \frac{9}{2} \|\mathbb{C}^{1/2}(\varepsilon(\dot{u}) - \varepsilon(v_{hk}))\|_{L^1(0,T;L^2(\Omega))}^2 + \frac{1}{2} \|\mathbb{C}^{-1/2}(\sigma - \sigma_k^*)\|_{L^\infty(0,T;L)}^2 \\ + \frac{1}{16} \max_{j=0,\dots,\ell} \|\mathbb{C}^{-1/2}\delta_j^\pm\|_L^2. \end{aligned}$$

Up to the factor $1/16$, the last term $\max_{j=0,\dots,\ell} \|\mathbb{C}^{-1/2}\delta_j^\pm\|_{L^2(\Omega)}$ stands for

$$\max \left\{ \|\mathbb{C}^{-1/2}\delta_0^-\|_{L^2(\Omega)}, \|\mathbb{C}^{-1/2}\delta_0^+\|_{L^2(\Omega)}, \|\mathbb{C}^{-1/2}\delta_1^-\|_{L^2(\Omega)}, \dots, \|\mathbb{C}^{-1/2}\delta_{\ell}^-\|_{L^2(\Omega)} \right\}$$

with initial error $\|\mathbb{C}^{-1/2}\delta_0^-\|_{L^2(\Omega)}$ and without the contribution $\|\mathbb{C}^{-1/2}\delta_\ell^+\| := 0$. In conclusion,

$$\begin{aligned}
(4.7) \quad 16 \text{ LHS}(\ell) &\leq \sum_{j=1}^{\ell} \|[\mathbb{A}^{1/2}\Delta]_{j-1}\|_L^2 + 64 \sum_{j=1}^J k_j^3 \|\mathbb{A}^{1/2}\ddot{\Sigma}\|_{L^2(I_j;L)}^2 \\
&+ 72 \|\mathbb{C}^{1/2}(\varepsilon(\dot{u}) - \varepsilon(v_{hk}))\|_{L^1(0,T;L^2(\Omega))}^2 + 8 \|\mathbb{A}^{1/2}(\Sigma - \Sigma_k^*)\|_{L^\infty(0,T;L)}^2 \\
&+ \max_{j=0,\dots,\ell} \|\mathbb{C}^{-1/2}\Delta_j^\pm\|_L^2 + 16 \sum_{j=1}^{\ell} \int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma_{hk}^* - \sigma_{hk})_{j-1}^+ dx.
\end{aligned}$$

4.6. Estimation of Jump Term. Let $f_k \in P_1(I_j; L^2(\Omega; \mathbb{R}^d))$ and $g_k \in P_1(I_j; L^2(\Gamma_N; \mathbb{R}^d))$ denote the L^2 -projection of $f \in H^2(I_j; L^2(\Omega; \mathbb{R}^d))$ and $g \in H^2(I_j; L^2(\Gamma_N; \mathbb{R}^d))$, onto the piecewise affine (globally continuous) functions in time such that

$$\int_{Q_j} (f - f_k) \cdot v \, dQ = 0$$

for all $v \in P_1(I_j; L^2(\Omega; \mathbb{R}^d))$ and

$$\int_{I_j} \int_{\Gamma_N} (g - g_k) \cdot v \, ds \, dt = 0$$

for all $v \in P_1(I_j; L^2(\Gamma_N; \mathbb{R}^d))$. Then, for all $t \in I_j$ and all $v_h \in V_h$, it holds

$$\int_{\Omega} f_k(t) \cdot v_h \, dx + \int_{\Gamma_N} g_k(t) \cdot v_h \, ds = \int_{\Omega} \sigma_{hk}(t) : \varepsilon(v_h) \, dx.$$

The key observation is that this holds for *all* times t . Since $[e]_{j-1} \in V_h$ and $\lim_{t \rightarrow t_{j-1}^+} \sigma_{hk}(t)$ satisfy the discrete equilibrium condition, this leads to

$$\begin{aligned}
\int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma - \sigma_{hk})_{j-1}^+ \, dx &= \int_{\Omega} (f - f_k)_{j-1}^+ \cdot [e]_{j-1} \, dx + \int_{\Gamma_N} (g - g_k)_{j-1}^+ \cdot [e]_{j-1} \, ds \\
&\leq \|(f - f_k)_{j-1}^+\|_{V^*} \| [e]_{j-1} \|_V + \|(g - g_k)_{j-1}^+\|_{V^*} \| [e]_{j-1} \|_V.
\end{aligned}$$

Standard approximation results (argue as in Lemma 1.1 without constraints) yield

$$\|(f - f_k)_{j-1}^+\|_{V^*} = \sup \left\{ \int_{\Omega} (f - f_k)_{j-1}^+ \cdot v \, dx : v \in V, \|v\|_V = 1 \right\} \leq k_j^{3/2} \|f\|_{L^2(I_j; V^*)}$$

plus an analog expression for $(g - g_k)_{j-1}^+$. Altogether, one obtains

$$\int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma - \sigma_{hk})_{j-1}^+ \, dx \leq \|k^{3/2}(\ddot{f}, \ddot{g})\|_{L^2(I_j; V^* \times V^*)} \| [e]_{j-1} \|_V.$$

Since $[\varepsilon(e)]_{j-1} = -[\varepsilon(u_{hk})]_{j-1}$ is piecewise constant,

$$\int_{\Omega} [\varepsilon(e)]_{j-1} : \sigma_h^* \, dx = \int_{\Omega} [\varepsilon(e)]_{j-1} : \sigma \, dx.$$

Consequently,

$$\begin{aligned}
& \sum_{j=1}^{\ell} \int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma_{hk}^* - \sigma_{hk})_{j-1}^+ dx \\
& \leq \sum_{j=1}^{\ell} \int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma_{hk}^* - \sigma_h^*)_{j-1}^+ dx + \sum_{j=1}^{\ell} \int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma_h^* - \sigma_{hk})_{j-1}^+ dx \\
& \leq \sum_{j=1}^{\ell} \|\mathbb{C}^{1/2}[\varepsilon(e)]_{j-1}\|_{L^2(\Omega)} \|\mathbb{C}^{-1/2}(\sigma_{hk}^* - \sigma_h^*)_{j-1}^+\|_{L^2(\Omega)} \\
& \quad + \sum_{j=1}^{\ell} k^{3/2} \|(\ddot{f}, \ddot{g})\|_{L^2(I_j; V^* \times V^*)} \|[e]_{j-1}\|_V.
\end{aligned}$$

As in (4.2) of Subsection 4.3 we have

$$\sum_{j=1}^J \|\mathbb{C}^{-1/2}(\sigma_{hk}^* - \sigma_h^*)_{j-1}^+\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^J \|\mathbb{A}^{1/2}(\Sigma_{hk}^* - \Sigma_h^*)_{j-1}^+\|_{L^2(\Omega)}^2 \leq \|k^{3/2} \mathbb{A}^{1/2} \ddot{\Sigma}\|_{L^2(0,T;L)}^2.$$

(In the last step we used that the norm of the integral mean $\ddot{\Sigma}_h$ is bounded by the norm of $\ddot{\Sigma}$ in L .) Altogether,

$$\begin{aligned}
\sum_{j=1}^{\ell} \int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma_{hk}^* - \sigma_{hk})_{j-1}^+ dx & \leq \left(\sum_{j=1}^{\ell} \|\mathbb{C}^{1/2}[\varepsilon(e)]_{j-1}\|_{L^2(\Omega)}^2 \right)^{1/2} \|k^{3/2} \mathbb{A}^{1/2} \ddot{\Sigma}\|_{L^2(0,T;L)}^2 \\
& \quad + 2 \left(\sum_{j=1}^{\ell} \|[e]_{j-1}\|_V^2 \right)^{1/2} \|k^{3/2}(\ddot{f}, \ddot{g})\|_{L^2(0,T;V^* \times V^*)}.
\end{aligned}$$

Korn's inequality and the definiteness of the elastic material tensor \mathbb{C} yield

$$\|[e]_{j-1}\|_V \leq \frac{C_1}{2} \|\mathbb{C}^{1/2}[\varepsilon(e)]_{j-1}\|_{L^2(\Omega)}$$

with some hardening-independent constant C_1 , without loss of generality $C_1 \geq 1$. Therefore,

$$\begin{aligned}
(4.8) \quad \sum_{j=1}^{\ell} \int_{\Omega} [\varepsilon(e)]_{j-1} : (\sigma_{hk}^* - \sigma_{hk})_{j-1}^+ dx & \leq C_1 \left(\sum_{j=1}^{\ell} \|\mathbb{C}^{1/2}[\varepsilon(e)]_{j-1}\|_{L^2(\Omega)}^2 \right)^{1/2} \\
& \quad \times \|k^{3/2}(\mathbb{A}^{1/2} \ddot{\Sigma}, \ddot{f}, \ddot{g})\|_{L^2(0,T;L \times V^* \times V^*)}.
\end{aligned}$$

4.7. Displacement Control through Hardening. The discrete dual formulation reveals explicitly that $\Sigma_{hk} = (\sigma_{hk}, \chi_{hk}) \in P_1(\mathcal{I}; K_h)$ is admissible almost everywhere in Q . Moreover, (3.7) in the equivalence proof reads, with appropriate interpretation,

$$\dot{P}_{hk|I_j \times T} \in N_{P_\ell(I_j; K)}(\Sigma_{hk|I_j \times T}).$$

This is *not* a pointwise version of Proposition 2.1. In fact, the $dG(1)$ discretization allows for some *averaged* discrete material evolution law. This is a new difficulty in the error analysis in comparison to backward Euler or to Crank-Nicholson schemes.

Lemma 4.2. *The discrete primal solution (p_{hk}, ξ_{hk}) with $\xi_{hk} = (\alpha_{hk}, \beta_{hk})$ satisfies*

$$tr p_{hk} = 0 \text{ and } p_{hk} = -\beta_{hk} \quad \text{almost everywhere in } Q.$$

Proof. Notice that the discrete primal solution fulfils $\text{supp}_{K_h}(\dot{p}_{hk}, \dot{\alpha}_{hk}, \dot{\beta}_{hk}) < \infty$ as the discrete dual formulation could not be satisfied otherwise. The support function reads

$$\text{supp}_{K_h}(\dot{p}_{hk}, \dot{\alpha}_{hk}, \dot{\beta}_{hk}) = \sup \left(\dot{p}_{hk} : \sigma_{hk} + a_{hk} A^{-1} \dot{\alpha}_{hk} + b_{hk} : B^{-1} \dot{\beta}_{hk} \right)$$

with the supremum taken over all $(\sigma_{hk}, a_{hk}, b_{hk}) \in \mathbb{R}_{sym}^{d \times d} \times \mathbb{R} \times \mathbb{R}_{sym}^{d \times d}$ such that $\Phi(\sigma_{hk}, a_{hk}, b_{hk}) \leq 0$. The test functions $(\sigma_{hk}, a_{hk}, b_{hk}) := (r \cdot 1_{d \times d}, 0, 0)$ for $r \in \mathbb{R}$ and $(\sigma_{hk}, a_{hk}, b_{hk}) = (M, 0, M)$ with $M := \dot{p}_{hk} + \dot{\beta}_{hk} \in \mathcal{P}_\ell(I_j; P_0(\mathcal{T}; \mathbb{R}_{sym}^{d \times d}))$ satisfy the condition $\Phi(\sigma_{hk}, a_{hk}, b_{hk}) \leq 0$ and yield

$$\text{supp}_{K_h}(\dot{p}_{hk}, \dot{\alpha}_{hk}, \dot{\beta}_{hk}) \geq r \cdot tr \dot{p}_{hk} \quad \text{and} \quad \text{supp}_{K_h}(\dot{p}_{hk}, \dot{\alpha}_{hk}, \dot{\beta}_{hk}) \geq (\dot{p}_{hk} + \dot{\beta}_{hk})^2, \quad \text{i.e.,}$$

$$tr \dot{p}_{hk} = 0 \quad \text{and} \quad \dot{p}_{hk} = -\dot{\beta}_{hk}.$$

This holds for the Riesz representation \dot{p}_{hk} etc. of the distributional time derivation of p_{hk} from the proof of Theorem 3.1. Hence, $tr[p_{hk}]_{j-1} = 0$ in Ω and $tr(p_{hk, \tau}) = 0$ in Q_j . Together with the initial conditions $tr(p_{hk}) = 0$, one concludes $tr(p_{hk}) = 0$ almost everywhere in Q . The same arguments show $p_{hk} = -\beta_{hk} = -B^{-1}b_{hk}$ almost everywhere in Q . \square

With Lemma 4.2 and the left-hand side *LHS* of (4.5), it holds

$$\begin{aligned} \|\mathbb{C}^{1/2}[\epsilon(e)]_{j-1}\|_{L^2(\Omega)}^2 &= \|\mathbb{C}^{1/2}[\epsilon(u_{hk})]_{j-1}\|_{L^2(\Omega)}^2 = \|\mathbb{C}^{1/2}[p_{hk} + \mathbb{C}^{-1}\sigma_{hk}]_{j-1}\|_{L^2(\Omega)}^2 \\ &= \|[-\mathbb{C}^{1/2}B^{-1}b_{hk} + \mathbb{C}^{-1/2}\sigma_{hk}]_{j-1}\|_{L^2(\Omega)}^2 \\ &\leq 2 \int_{\Omega} \left(|[-\mathbb{C}^{1/2}B^{-1}b_{hk}]_{j-1}|^2 + |[\mathbb{C}^{-1/2}\sigma_{hk}]_{j-1}|^2 \right) dx \\ &\leq 2 \int_{\Omega} \left(|\mathbb{C}^{1/2}B^{-1/2}[\mathbb{H}^{-1/2}\chi_{hk}]_{j-1}|^2 + |[\mathbb{C}^{-1/2}\sigma_{hk}]_{j-1}|^2 \right) dx \\ &\leq 2 \max\{1, |\mathbb{C}^{1/2}B^{-1/2}|\} \|\mathbb{A}^{1/2}[\Delta]_{j-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

This and $LHS(\ell) := \frac{1}{2} \|\mathbb{A}^{1/2}\Delta_\ell^-\|_L^2 + \sum_{j=0}^{\ell-1} \frac{1}{2} \|\mathbb{A}^{1/2}[\Delta]_j\|_L^2$ from (4.5) imply

$$\begin{aligned} (4.9) \quad \sum_{j=1}^{\ell} \|\mathbb{C}^{1/2}[\epsilon(e)]_{j-1}\|_{L^2(\Omega)}^2 &\leq 2 \max\{1, |\mathbb{C}^{1/2}B^{-1/2}|\} \sum_{j=1}^{\ell} \|\mathbb{A}^{1/2}[\Delta]_{j-1}\|_{L^2(\Omega)}^2 \\ &\leq 4 \max\{1, |\mathbb{C}^{1/2}B^{-1/2}|\} LHS(\ell). \end{aligned}$$

4.8. Finish of the Proof. The combination of (4.7)-(4.9) yield

$$\begin{aligned} 16LHS(\ell) &\leq \sum_{j=1}^{\ell} \|\mathbb{A}^{1/2}\Delta]_{j-1}\|_L^2 + 64 \|k^{3/2} \mathbb{A}^{1/2} \ddot{\Sigma}\|_{L^2(0,T;L)}^2 \\ &+ 72 \|\mathbb{C}^{1/2}\epsilon(\dot{u} - v_{hk})\|_{L^1(0,T;L)}^2 + 8 \|\mathbb{A}^{1/2}(\Sigma - \Sigma_k^*)\|_{L^\infty(0,T;L)}^2 + \max_{j=0, \dots, \ell} \|\mathbb{A}^{1/2}\Delta_\ell^\pm\|_L^2 \\ &+ 32C_1 LHS^{1/2}(\ell) \max\{1, |\mathbb{C}^{1/2}B^{-1/2}|\}^{1/2} \|k^{3/2}(\mathbb{A}^{1/2}\ddot{\Sigma}, \ddot{f}, \ddot{g})\|_{L^2(0,T;L \times V^* \times V^*)}. \end{aligned}$$

Standard interpolation error estimates on nodal interpolants in time and space like $\Sigma - \Sigma_k^*$ and $\dot{u} - v_{hk}$ and absorbing $\text{LHS}^{1/2}(\ell)$ eventually lead to some \mathbb{A} -depending constant $C(\mathbb{A}) > 0$ such that it holds, for all $\ell = 1, 2, \dots, J$,

$$8\|\mathbb{A}^{1/2}\Delta_\ell^-\|_L^2 - \max_{j=0,\dots,\ell} \|\mathbb{A}^{1/2}\Delta_j^\pm\|_L^2 + 7\sum_{j=1}^{\ell} \|\mathbb{A}^{1/2}[\Delta]_{j-1}\|_L^2 - \text{LHS}(\ell) \leq \text{RHS with}$$

$$\text{RHS} := \mathbb{C}(\mathbb{A}) \left(\|(hD^2\dot{u}, k^2\varepsilon(\ddot{u}))\|_{L^1(0,T;L^2(\Omega)\times L^2(\Omega))} + \|k^{3/2}(\mathbb{A}^{1/2}\ddot{\Sigma}, \ddot{f}, \ddot{g})\|_{L^2(0,T;L\times V^*\times V^*)} \right).$$

With the further abbreviations $M := \max_{j=0,\dots,J} \|\mathbb{A}^{1/2}\Delta_j^\pm\|_L^2$, this reads

$$(4.10) \quad 13 \text{LHS}(\ell) + \|\mathbb{A}^{1/2}\Delta_{\ell-1}^-\|_L^2 \leq M + \text{RHS}.$$

In the first case, suppose $M = \|\mathbb{A}^{1/2}\Delta_\ell^-\|_L^2$ for some index $\ell \in \{0, \dots, J-1\}$. Then, (4.10) shows $M \leq \text{RHS}$. In the second case, suppose $M = \|\mathbb{A}^{1/2}\Delta_{\ell-1}^+\|_L^2$ for some index $\ell \in \{1, \dots, J\}$. Then, (4.10) also yields, for two consecutive values of ℓ ,

$$6 \max\{\|\mathbb{A}^{1/2}\Delta_{\ell-1}^-\|_L^2, \|\mathbb{A}^{1/2}[\Delta]_{\ell-1}\|_L^2\} \leq M + \text{RHS}.$$

This and a triangle inequality lead to

$$M \leq 2\|\mathbb{A}^{1/2}\Delta_\ell^-\|_L^2 + 2\|\mathbb{A}^{1/2}[\Delta]_{\ell-1}\|_L^2 \leq 2/3(M + \text{RHS}).$$

Consequently, in all cases

$$M \leq 2 \text{RHS}.$$

This plus (4.10), for $\ell = J$, shows the final result

$$\text{LHS}(J) + M \leq 3 \text{RHS}. \quad \square$$

5. NUMERICAL EXPERIMENTS

The numerical experiments for the elastoplastic time evolution that lead to Figure 1.1 are run for the axisymmetric ring Ω of Figure 5.1 while the volume force f vanishes, the applied surface loads g_1 and g_2 describe some pure Neumann problem with a known analytical solution given in closed form in [AC00, Alb01], where further details can be found which are not recalled here for brevity.

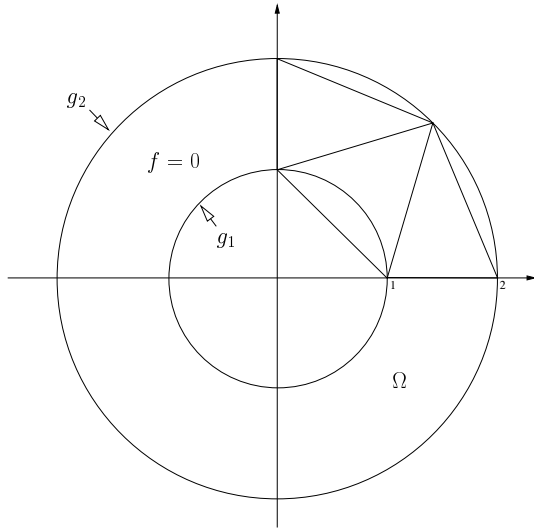
The algorithmic details on the implementation are included in [AC02] and lead to the stress evaluation of Figure 1.1. Some snap shots of the elastoplastic evolution computed with the backward Euler (bE), Crank-Nicholson (CN) and dG(0) and dG(1) are displayed in Figure 5.2. To discuss some convergence rates, the relative error

$$(5.1) \quad e_Q^2 := \frac{\int_0^1 \int_\Omega (\|\mathbb{C}^{-1/2}(\sigma(x,t) - \sigma_{hk}(x,t))\|^2 + \|\mathbb{H}^{-1/2}(\chi(x,t) - \chi_{hk}(x,t))\|^2) dx dt}{\int_0^1 \int_\Omega (\|\mathbb{C}^{-1/2}\sigma(x,t)\|^2 + \|\mathbb{H}^{-1/2}\chi(x,t)\|^2) dx dt},$$

and the stress error

$$(5.2) \quad e_\Omega^2 := \int_\Omega \|\mathbb{C}^{-1/2}(\sigma(x,1) - \sigma_{hk}(x,1))\|^2 dx$$

are computed at $t = T = 1$. The convergence history for the error e_Q (left) and e_Ω (right) is displayed in Figure 5.3 for dG(k) with $k = 0$ and 1. Several uniform spatial discretizations are fixed and give rise to different curves. Each curve displays the error as a function of the (uniform) time-step size $k = 3^{-n}$ for $n = 2, 3, 4, 5$ with both axis' in logarithmic scaling.



$$E = 70.000$$

$$\nu = 0,33$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\sigma_y = 243$$

$$\mathbb{H}_2 = 1$$

$$g_1(r, \phi, t) = 240 \sin(2\pi t)(\cos \phi, \sin \phi)$$

$$g_2(r, \phi, t) = -60 \sin(2\pi t)(\cos \phi, \sin \phi)$$

$$\kappa_1 = \frac{2\mu}{2\mu + \lambda}$$

$$\kappa_2 = \mu + \lambda$$

$$\zeta = \kappa_1 \kappa_2 + \mathbb{H}_2$$

$$\kappa_3 = \frac{\sigma_y}{\sqrt{34}}$$

$$\kappa_4 = \frac{\sigma_y \kappa_1 \kappa_2}{\sqrt{33\zeta}}$$

FIGURE 5.1. Mechanical system and material parameters for the benchmark example.

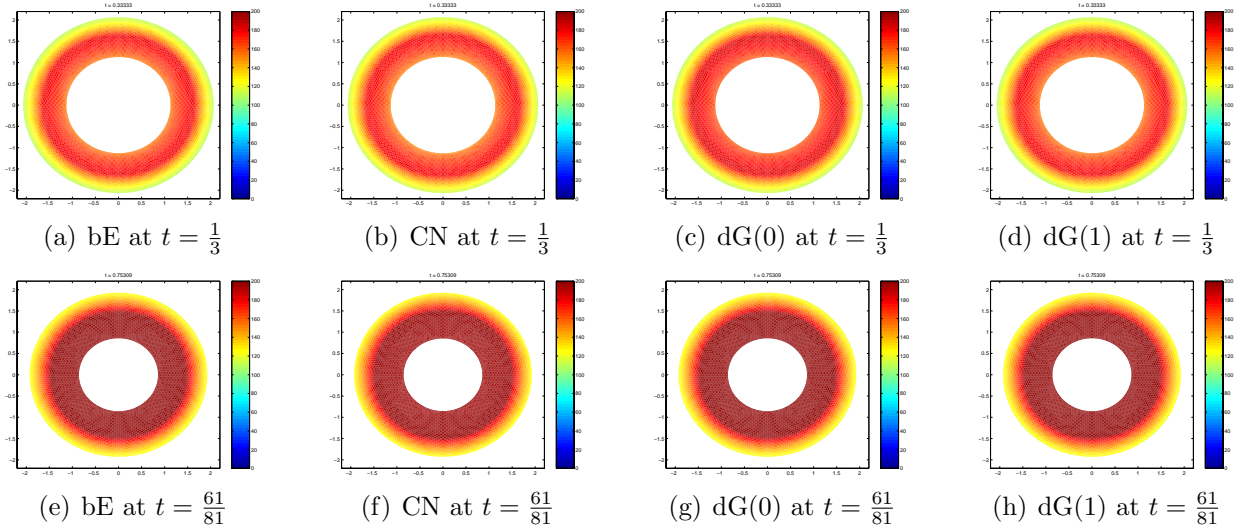


FIGURE 5.2. Elastoplastic evolution for time step size, $k = 3^{-4}$ at various times.

Notice that the convergence in time is from right to left. For a fine mesh, the empirical convergence rate in time is close to 1 for dG(0) and in fact better than 1.7 for dG(1). The value 1 and 2 for dG(ℓ) and e_Ω is no strict proof for a higher order of convergence, but an indication that the pre-asymptotic range is very large.

Notice that the spatial regularity requests of this paper are not fully met in this example. However, the elastoplastic interface is a ring with time depending radius and so affects a minority of element domains only. Hence on may expect that the lack of regularity does not dominate the convergence rates.

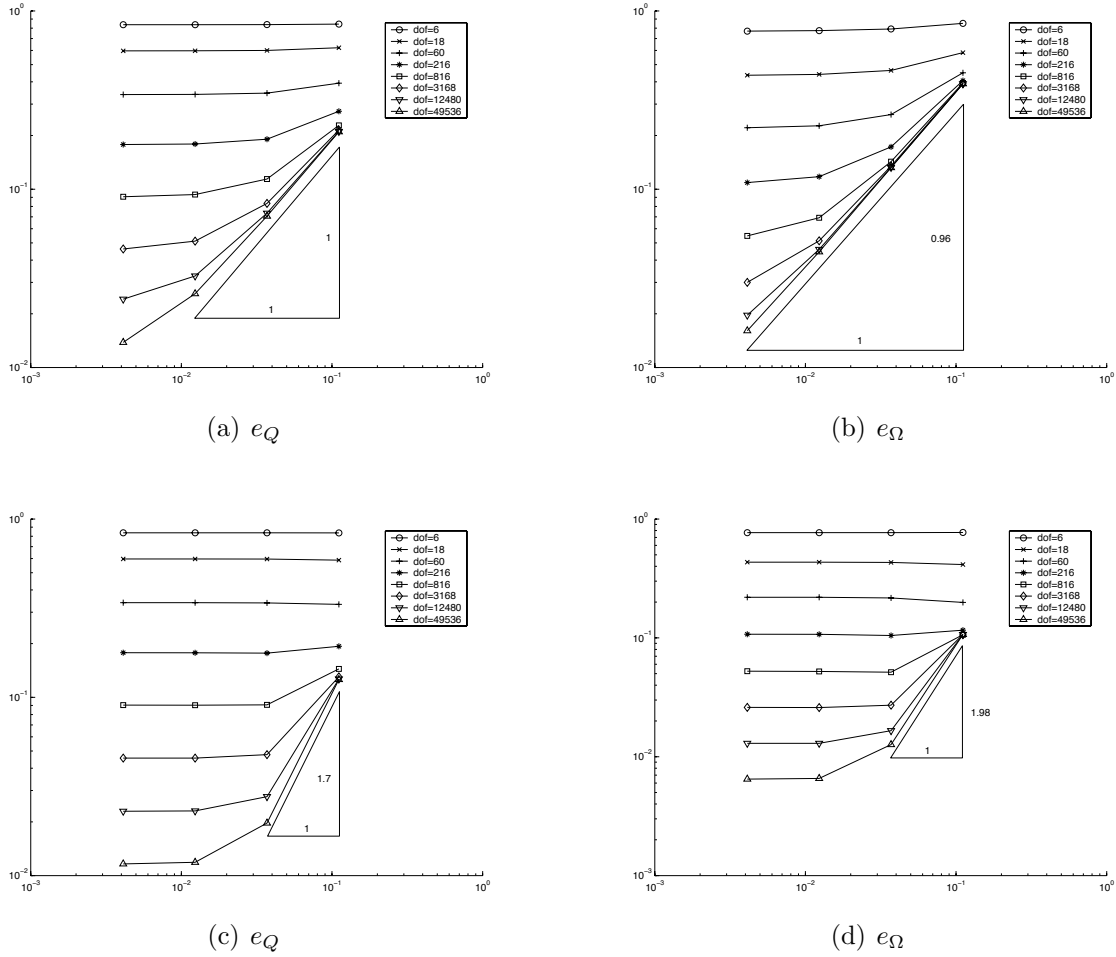


FIGURE 5.3. Convergence for the error e_Q (left) and e_Ω (right) from (5.1)-(5.2) as a function of the time-step size $k = 3^{-2}, \dots, 3^{-5}$ for various fixed uniform space discretizations with $\text{dof} = 6, \dots, 49536$ degrees of freedom for the displacements.

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