Transformation formulas for Ginzburg-Landau and applications to regularity

Christof Melcher

Abstract

Building on a transformation formula that we previously found in connection with Landau-Lifshitz-Gilbert equations, we present a strategy for utilizing compensated compactness methods in the context of Ginzburg-Landau approximation for harmonic maps and related problems in magnetism and superconductivity. Its applicability is illustrated by a number of new examples.

1 Introduction

In recent analytical developments for harmonic maps and related evolution equations (like e.g. harmonic map heat flow, wave maps, Landau-Lifshitz equations), methods of compensated compactness play a crucial role. In exploiting the symmetries of the target manifold, the geometric nonlinearity reveals its special antisymmetric structure necessary for such methods. This has first been observed and used by Hélein [9] for proving regularity of harmonic maps in two space dimensions. Indeed, for maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{S}^{N-1}$ of finite Dirichlet energy the nonlinearity $|\nabla u|^2 \, u$ can equivalently by written as the product

$$b(u, \nabla u)\nabla u = (u \otimes \nabla u - \nabla u \otimes u) \; \nabla u \tag{1.1}$$

Obviously, the factors have L^2 bounds in terms of the Dirichlet energy. If in addition u is weakly harmonic, i.e. u is a weak solution of

$$\Delta u + |\nabla u|^2 \, u = 0$$

then the product (1.1) exhibits the so-called *div-curl structure*, i.e.

$$\operatorname{div} b(u, \nabla u) = 0 \quad \text{and} \quad \operatorname{curl} \nabla u = 0$$

in the sense of distributions. This structure is accompanied by improved product estimates related to compensation phenomena of Jacobians and the $\mathcal{H}^1$-BMO duality, see [13, 5] and [9] for a comprehensive discussion. Regualrity is then a consequnce of *higher integrability* that now follows from linear theory.
The standard approximation for harmonic maps with spherical target is based upon replacing the Dirichlet energy by the Ginzburg-Landau energy with a small penalization parameter $\varepsilon > 0$

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{4\varepsilon^2}\int_\Omega (1 - |u|^2)^2 \, dx.$$

This approach is particularly useful if one is concerend with energy divergence in the presence of topological singularities that are enforced for instance by boundary conditions. The corresponding Euler-Lagrange equations reads

$$\Delta u + V_\varepsilon(u) \, u = 0 \quad \text{where} \quad V_\varepsilon(u) = \frac{1}{\varepsilon^2}(1 - |u|^2).$$

It clear from standard argumets that weak solutions $u_\varepsilon \in H^1(\Omega;\mathbb{R}^N)$ are smooth in the interior. Investigating ε-uniform estimates, however, one observes that there is no immediate relation between the nonlinearity $V_\varepsilon(u)\,u$ and the energy $E_\varepsilon(u)$, at least if one tries to bound $V_\varepsilon(u)$ in a pointwise fashion. Aim of this note is show that after suitable transformations $\mathcal{L}_\varepsilon$ near the identity the approximate PDE exhibits a *div-curl structure* as well, i.e. for weak solutions $u = u_\varepsilon$ of the Euler-Lagrange equation

$$\mathcal{L}_\varepsilon\Big(V_\varepsilon(u)\,u\Big) = b(u,\nabla u)\nabla u + o(1) \quad \text{where} \quad \mathcal{L}_\varepsilon = \mathbf{1} + o(1) \quad \text{as} \quad \varepsilon \to 0,$$

making compensated compactness methods applicable in the approximate context as well. We underline the strength of this observation by a compactness result for approximate harmonic maps (Theorem 1), a transparent global proof of a well-known result and a new bound in the regularity theory for complex Ginzburg-Landau equations with vortices (Theorem 4), and a recent existence and regularity result in the context of Landau-Lifshitz equations [11] where this approach has sucessfully been used for the first time.

2 A transformation formula

Suppose that $\Omega \subset \mathbb{R}^n$ is a smooth open set. For $\varepsilon > 0$ and functions $f \in L^2(\Omega;\mathbb{R}^n)$ we consider critical points $u \in H^1(\Omega;\mathbb{R}^n)$ of the functional

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2\,dx + \int_\Omega f\cdot u\,dx,$$

that is we consider weak solutions of the Euler-Lagrange equation

$$\Delta u + V_\varepsilon(u)\,u = f \quad \text{with} \quad V_\varepsilon(u) = \frac{1}{\varepsilon^2}\big(1-|u|^2\big). \tag{2.1}$$

In doing so we approximate harmonic maps with spherical targets and with a right-hand-side f. In case $\mathcal{F}_\varepsilon(u_\varepsilon)$ is uniformly bounded for a family of solutions it is well-known and easy to show that for some null-sequence ε_k the corresponding sequnce u_{ε_k} weakly converges to a solution of

$$\Delta u + |\nabla u|^2 u = (1 - u\otimes u)\,f.$$

Moraly, an anti-symmetric multiplication of the equation by u makes the penalty term disappear, so that one has for $u = u_\varepsilon$ and any $\varepsilon > 0$

$$\nabla\cdot(u\wedge\nabla u) = u\wedge f \tag{2.2}$$

that is equivalent to the harmonic map equation if $|u| = 1$ is attained. In search for a more direct view point our goal is to represent the nonlinearity $V_\varepsilon(u)\,u$ in terms of the product $V_\varepsilon(u)\,u \sim b(u,\nabla u)\,\nabla u$ where the anti-symmetric bilinear mapping $b(u,\nabla u)$ is defined by

$$b^{ij}_\alpha(u,p) = u^i p^j_\alpha - u^j p^i_\alpha \quad \text{for} \quad 1\le i,j\le N \quad \text{and} \quad 1\le\alpha\le n. \tag{2.3}$$

Observe that for solutions $u = u_\varepsilon$ the div-curl structure encountered in the context of harmonic maps is preserved on the ε-level:

$$\operatorname{div} b(u,\nabla u) = u\otimes f - f\otimes u \quad \text{and} \quad \operatorname{curl}\nabla u = 0$$

It is clear that such a correspondence between $V_\varepsilon(u)\,u$ and $b(u,\nabla u)\,\nabla u$ has to be nonlocal, and it turns out that a suitable transformation is given by the singularly perturbed differential operator

$$\mathcal{L}_\varepsilon = \mathbf{1} - \frac{1}{2}\,\varepsilon^2\,\Delta.$$

Basic manipulations yield the following transformation formula for $V_\varepsilon(u)\,u$:

Proposition 2.1. *For a sufficiently regular solution $u = u^{(\varepsilon)}$ of (2.1) the identity*

$$\mathcal{L}_\varepsilon\Big(V_\varepsilon(u)\,u\Big) = b(u,\nabla u)\nabla u - \nabla\cdot\left(\frac{3}{2}(1-|u|^2)\nabla u\right) + A(u)\,f(x) \tag{2.4}$$

$$\textit{where}\quad \mathcal{L}_\varepsilon = \mathbf{1} - \frac{1}{2}\,\varepsilon^2\,\Delta \quad \textit{and}\quad A(u) = \big(1-|u|^2\big)\,\mathbf{1} + u\otimes u,$$

holds true in the sense of distributions.

Proof. Since (2.4) is scale invariant we may assume that $\varepsilon = 1$. We have

$$\Delta\big(V(u)\,u\big) = V(u)\,\Delta u - 4(u\cdot\nabla u)\nabla u - 2|\nabla u|^2 u - 2(u\cdot\Delta u)\,u,$$

where $V(u) = V_1(u)$. On the other hand, invoking the equation, we have

$$\begin{aligned}
2(u\cdot\nabla u)\,\nabla u &= -\nabla\cdot\big[\big(1-|u|^2\big)\nabla u\big] - |V(u)|^2 u + V(u)\,f(x),\\
(u\cdot\Delta u)\,u &= |V(u)|^2 u - V(u)\,u + (u\cdot f(x))\,u,\\
V(u)\,\Delta u &= V(u)\,f(x) - |V(u)|^2 u,\\
|\nabla u|^2 u &= b(u,\nabla u)\nabla u + (u\cdot\nabla u)\nabla u.
\end{aligned}$$

With the last identity the above expansion for $\Delta\big(V(u)\,u\big)$ becomes

$$V(u)\,\Delta u - 2\,b(u,\nabla u)\nabla u - 6\,(u\cdot\nabla u)\nabla u - 2\,(u\cdot\Delta u)\,u.$$

Substituting the other identities yields the formula to be proved. □

The formula permits to directly pass to the limit in (2.1) after applying the operator $\mathcal{L}_\varepsilon$ to the equation. Indeed, for solutions $u = u_\varepsilon$ with uniformly bounded energy $\mathcal{F}_\varepsilon(u_\varepsilon) \le c$

$$\mathcal{L}_\varepsilon\Big(f - \Delta u\Big) = b(u,\nabla u)\,\nabla u + (f\cdot u)\,u - \frac{3}{2}\nabla\cdot\Big(\big(1-|u|^2\big)\nabla u\Big) + (1-|u|^2)\,f(x).$$

For a subsequence $\varepsilon_k \to 0$ we can assume that $u_k := u_{\varepsilon_k} \to u$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ for some $u \in H^1(\Omega;\mathbb{R}^N)$ with $|u| = 1$. But then if $f \in L^p(\Omega)$ for some $p > n/2$ we find that

$$\nabla\cdot b(u_k,\nabla u_k) = u_k\otimes f - f\otimes u_k \in H^{-1}(\Omega)$$

is precompact so that by the compensated compactness principle (cf. e.g. [14])

$$b(u_k,\nabla u_k)\,\nabla u_k \overset{*}{\rightharpoonup} b(u,\nabla u)\,\nabla u \quad\text{as}\quad k\to\infty.$$

Passing to the limit we find the following limit equation for maps $u \in H^1(\Omega;\mathbb{S}^{N-1})$

$$\Delta u + b(u,\nabla u)\,\nabla u = f^{\mathrm{Tan}} \quad\text{where}\quad f^{\mathrm{Tan}} = (1 - u\otimes u)f.$$

The latter argument only uses that $\mathcal{L}_\varepsilon$ is formally self-adjoint and that $\mathcal{L}_\varepsilon\phi \to \phi$ in say $C^2(\Omega;\mathbb{R}^N)$ for any test function $\phi \in C_0^\infty(\Omega;\mathbb{R}^N)$. It is well-know that the asymptotic equation can be derived indirectly from (2.2). The indirect argument, however, does not permit to deduce strong convergence properties of the gradient in certain L^p norms. But this can be deduced from the transformation formula, as will be shown in Theorem 1 below. In order to improve the strength of convergence, we need to take into account uniform regularity properties of the inverse operator. We will provide some selected estimates within the next section.

The transformation formula extends to evolution problems like e.g. the Ginzburg-Landau approximation for the Landau-Lifshitz equation. In [11] a parabolic variant of the transformation formula has been used for the first time in order to proof an existence and regularity result. For spin fields $u(t) : \mathbb{R}^3 \to \mathbb{S}^2$ the Landau-Lifshitz equation reads

$$\alpha\, \partial_t u + \beta u \wedge \partial_t u = \Delta u + |\nabla u|^2\, u \quad \text{where} \quad \alpha > 0.$$

For this hybrid heat and Schrödinger flow for harmonic maps we have the following approximation scheme has been used by Alouges and Soyeur [1] for the construction of global weak solutions for initial data $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{S}^2)$

$$\alpha\, \partial_t u + \beta u \wedge \partial_t u = \Delta u + V_\varepsilon(u)\, u. \tag{2.5}$$

That Ginzburg-Landau approximation yields indeed partially regular (smooth away from a closed subset of lower parabolic Hausdorff dimension in space-time) solutions has been proved in [11] on by means of a perturbation argument and special trilinear Sobolev inequalities (cf. [12]) that were applicable on the basis of the following transformation formula:

Proposition 2.2. *For sufficiently smooth solutions of (2.5) the formula*

$$\mathcal{P}_\varepsilon\Big(V_\varepsilon(u)\, u\Big) = b(u, \nabla u)\nabla u - \frac{3}{2}\nabla \cdot \Big(\big(1-|u|^2\big)\nabla u\Big) + \big(1-|u|^2\big)\, \Big(R(u) + \frac{\alpha}{2}\Big)\, \partial_t u$$

where

$$\mathcal{P}_\varepsilon = \mathbf{1} + \frac{\varepsilon^2}{2}(\alpha\, \partial_t - \Delta) \quad \textit{and} \quad R(u)\, \xi = \alpha\, \xi + \beta\, u \wedge \xi$$

holds true in the sense of distributions.

2.1 Uniform invertability of $\mathcal{L}_\varepsilon$

In case $\Omega = \mathbb{R}^n$ the unbounded operator $\mathcal{L}_\varepsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ admits, as a consequence of Plancherel's theorem, a uniformly L^2-bounded inverse $\mathcal{K}_\varepsilon$. In real space $\mathcal{K}_\varepsilon$ is given by the convolution with a rescaled Bessel type potential $\Gamma_\varepsilon = \Gamma_\varepsilon(|\cdot|) \in L^1(\mathbb{R}^n)$. More precisely, with

$$\Gamma(r) = \mathcal{F}^{-1}\left(\xi \mapsto 1 + \frac{1}{2}|\xi|^2\right)(x) \quad \text{where} \quad r = |x| \quad \text{and} \quad \Gamma_\varepsilon(r) = \frac{1}{\varepsilon^n}\Gamma\left(\frac{r}{\varepsilon}\right)$$

we have

$$(\mathcal{K}_\varepsilon F)(x) = \int_{\mathbb{R}^n} \Gamma_\varepsilon(|x-y|)\, F(y)\, dy.$$

This implies in particular the boundedness of $\mathcal{K}_\varepsilon$ in $L^p(\mathbb{R}^n)$ for any $1 \le p \le \infty$. The same holds true for any bounded smooth domain $\Omega \subset \mathbb{R}^n$ when imposing homogeneous Dirichlet boundary conditions. Indeed, if

$$k_\varepsilon(x, y) = k_\varepsilon(y, x) \quad \text{for} \quad \varepsilon > 0 \quad \text{and} \quad x, y \in \Omega$$

denotes the *Green's function* of $\mathcal{L}_\varepsilon$ in Ω so that $k_\varepsilon(x, \cdot) = 0$ on $\partial\Omega$ then the following bounds hold true independently of $\varepsilon > 0$:

Proposition 2.3. *Suppose that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Then the Green's function admits the pointwise estimate*

$$0 \le k_\varepsilon(x,y) \le \Gamma_\varepsilon(|x-y|) \quad \text{for any} \quad \varepsilon > 0 \ \text{ and } \ x,y \in \Omega.$$

For the operator $(\mathcal{K}_\varepsilon f)(x) = \int_\Omega k_\varepsilon(x,y)F(y)\,dy$ we have in particular

$$\|\mathcal{K}_\varepsilon f\|_{L^p} \le C\,\|f\|_{L^p} \quad \text{for any} \quad 1 \le p \le \infty$$

for some constant that is independent of $\varepsilon > 0$. For the gradient we have moreover

$$\varepsilon \left| \frac{\partial k_\varepsilon}{\partial y}(x,y) \right| \le C \left(1 + \frac{\varepsilon}{|x-y|}\right) \Gamma_\varepsilon \left(\frac{|x-y|}{2} \right) \tag{2.6}$$

for some universal constant $C > 0$ if $\varepsilon > 0$ is sufficiently small. In particular we have

$$\varepsilon\,\|\mathcal{K}_\varepsilon(\nabla \cdot F)\|_{L^p} \le C\,\|F\|_{L^p} \quad \text{for any} \quad 1 \le p \le \infty$$

for some constant that is independent for $0 < \varepsilon < \varepsilon_0$.

Proof. The upper and lower bound for $k_\varepsilon = k_\varepsilon(x,y)$ follows from the positivity of the fundamental solution Γ_ε and the maximum principle. The bound for the associated operator $\mathcal{K}_\varepsilon$ follows from the Hausdorff-Young inequality. Regarding the gradinet estimate, we have the following basic elliptic estimate (cf. [8] section 3.4)

$$|\nabla_y k_\varepsilon(x,y)| \le \frac{c}{r} \sup_{B_r(y)} |k_\varepsilon(x,\cdot)| \tag{2.7}$$

that is merely a consequence of the maximum principle. We distinguish three cases. If $\operatorname{dist}(y,\partial\Omega) > \frac{|x-y|}{2}$ we set $r = \frac{|x-y|}{2}$ then we deduce from (2.7)

$$\left| \frac{\partial k_\varepsilon}{\partial y}(x,y) \right| \le \frac{c}{|x-y|} \sup_{B_r(y)} |k_\varepsilon(x,\cdot)| \le \frac{c}{|x-y|}\, \Gamma_\varepsilon \left(\frac{|x-y|}{2} \right).$$

If otherwise $\operatorname{dist}(y,\partial\Omega) \le \frac{|x-y|}{2}$ but $r = \operatorname{dist}(y,\partial\Omega) \ge \varepsilon$ then (2.7) implies

$$\left| \frac{\partial k_\varepsilon}{\partial y}(x,y) \right| \le \frac{c}{\varepsilon} \sup_{B_r(y)} |k_\varepsilon(x,\cdot)| \le \frac{c}{\varepsilon}\, \Gamma_\varepsilon \left(\frac{|x-y|}{2} \right).$$

If finally $\operatorname{dist}(y,\partial\Omega) < \varepsilon < \frac{|x-y|}{2}$ and ε is sufficiently small we can assume that $\partial\Omega$ is flat and $y \in B_\varepsilon^+(0)$. Standard boundary estimates (cf. [8] section 6.2) imply after rescaling

$$\sup_{B_\varepsilon^+(0)} \left| \frac{\partial k_\varepsilon}{\partial y}(x,\cdot) \right| \le \frac{c}{\varepsilon} \sup_{B_{2\varepsilon}^+(0)} |k_\varepsilon(x,\cdot)| \le \frac{c}{\varepsilon}\, \Gamma_\varepsilon \left(\frac{|x-y|}{2} \right).$$

This implies the bound in (2.6). Finally the bound $\mathcal{K}_\varepsilon(\nabla \cdot F)$ follows from

$$\int_\Omega k_\varepsilon(x,y)(\nabla \cdot F)(y)\,dy = -\int_\Omega \frac{\partial k_\varepsilon}{\partial y}(x,y) \cdot F(y)\,dy,$$

(2.6) and the Hausdorff-Young inequality. $\square$

Proposition 2.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $f_\varepsilon : \Omega \to \mathbb{C}^n$ is a family of vector fields of class L^p for some $p \in (1,\infty)$. Suppose that $u \in H^2(\Omega;\mathbb{C})$ is a weak solution of*

$$\mathcal{L}_\varepsilon \Delta u = \nabla \cdot f_\varepsilon \quad \text{in} \ \Omega$$

subjected to boundary conditions

$$u - g_\varepsilon \in W_0^{1,p}(\Omega) \quad \text{and} \quad \Delta u \in W_0^{1,p}(\Omega)$$

for functions g_ε so that $\mathcal{L}_\varepsilon g_\varepsilon$ in $W^{1,p}(\Omega)$. Then $u \in W^{2,p}(\Omega)$ with an estimate

$$\varepsilon \|\Delta u\|_{L^p} + \|\nabla u\|_{L^p} \leq C \left[\|f_\varepsilon\|_{L^p} + \|\mathcal{L}_\varepsilon g_\varepsilon\|_{W^{1,p}} \right]$$

that holds true for some universal constant $C = C(\Omega, p)$ independent of ε.

Proof. We only sketch the argument which is a straight foreward generalization of well-known Schauder estimtes for second order equations in divergence form (cf. e.g. [7]). We assume that $g_\varepsilon = 0$ and consider the linear operator $T : f \mapsto \nabla u$. Then T extends to a bounded self-adjoint operator on $L^2(\Omega)$. More specifically

$$\varepsilon^2 \|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq \|f\|_{L^2}^2.$$

By duality it is sufficient to prove uniform boundedness in L^p for $2 < p < \infty$. For this purpose we prove, for bounded $f \in L^\infty$, a BMO estimate for ∇u. In view of an interpolation argument by Stampacchia the claim follows. We only sketch the argument for the local BMO estimate. Let B_R be a ball that is strictly inside Ω and $r \in (0, R)$. Then we let

$$\Phi(u, r) = \varepsilon^2 \int_{B_r} |\Delta u - (\Delta u)_{B_r}|^2 \, dx + \int_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx$$

where $(\cdot)_{B_r}$ denotes the average, and claim the following decay estimate

$$\Phi(u, r) \leq C \left((r/R)^{n+2} \Phi(u, R) + \|f\|_{L^\infty}^2 r^n \right) \tag{2.8}$$

for some universal constant $C > 0$, i.e. a unifrom estimate for the mean oscillation as claimed. Set $u = v + w$ where v solves $\mathcal{L}_\varepsilon \Delta v = 0$ in B_R and $u = v$ on ∂B_R so that

$$\Phi(u, r) \leq \Phi(v, r) + 2\varepsilon^2 \int_{B_R} |\Delta w|^2 \, dx + 2 \int_{B_R} |\nabla w|^2 \, dx.$$

Standard Hilbert space regularity implies $\Phi(v, r) \leq c(r/R)^{n+2} \Phi(v, R)$ and with Hölder's inequality

$$\varepsilon^2 \int_{B_R} |\Delta w|^2 \, dx + \int_{B_R} |\nabla w|^2 \, dx \leq R^n \sup |f|^2$$

Utilizing a standard iteration procedure we deduce (2.8). The global estimate follows after flattening the boundary that yields a system with smooth coefficients on half-balls where analog estimates apply. □

2.2 $W^{1,p}$ compactness for approximate harmonic maps

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and*

$$g_\varepsilon \in W^{2,1}(\Omega; \mathbb{R}^N) \cap H^{1/2}(\partial\Omega; \mathbb{S}^{N-1})$$

with uniform bounds. Suppose that $u_\varepsilon \in g_\varepsilon + H_0^1(\Omega; \mathbb{R}^N)$ satisfies

$$\Delta u_\varepsilon + V_\varepsilon(u_\varepsilon)\, u_\varepsilon = 0 \quad \text{and} \quad E_\varepsilon(u_\varepsilon) \leq c$$

for a constant $c > 0$. Then $(u_\varepsilon) \subset W^{1,p}(\Omega; \mathbb{R}^N)$ is relatively compact for any $p < 2$.

Proof. First we infer from the weak maximum principle that $|u_\varepsilon| \le 1$. It is enough to show that

$$\Delta(u_\varepsilon - g) = f_\varepsilon \quad \text{where} \quad \|f_\varepsilon\|_{L^1} \le c\, E_\varepsilon(u_\varepsilon).$$

Then the claim follows by Murat's Lemma (cf. e.g. [14], Theorem 3.4). But in view of Proposition 2.1 we find that f_ε is given by

$$f_\varepsilon = \mathcal{K}_\varepsilon\Big(b(u_\varepsilon, \nabla u_\varepsilon)\,\nabla u_\varepsilon\Big) - \varepsilon\big(\mathcal{K}_\varepsilon \nabla \cdot\,\big)\left(\frac{3}{2}\frac{(1-|u_\varepsilon|^2)}{\varepsilon}\,\nabla u\right) - \Delta g_\varepsilon.$$

In view of Proposition 2.3 and Hölder's inequality uniform L^1 bounds follow. $\square$

3 Application to complex Ginzburg-Landau

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply connected domain. Considering the associated harmonic map problem with the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ as target manifold, i.e. looking for stationary points of $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$ in $H^1(\Omega;\mathbb{S}^1)$ one is faced with topological obstructions if homotopically non-trivial boundary conditions are imposed. Indeed, if $g : \partial\Omega \cong \mathbb{S}^1 \to \mathbb{S}^1$ is smooth but with non-trivial degree, then the set $\{u \in H^1(\Omega;\mathbb{S}^1) : u|_{\partial\Omega} = g\}$ is empty, since otherwise Hélein's regularity result would conflict with Brouwer's degree lemma. Accordingly, Brézis, Bethuel, and Hélein proposed to study the corresponding (singular) Ginzburg-Landau problem

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\,dx \to \min$$

(cf. the monograph [4]) and more generally corresponding solutions $u_\varepsilon : \Omega \to \mathbb{C}$ of the Ginzburg-Landau equation with smooth Dirichlet data $g_\varepsilon : \partial\Omega \to \mathbb{S}^1$

$$\Delta u_\varepsilon + V(u_\varepsilon)\,u_\varepsilon = 0 \quad \text{in} \quad \Omega \quad \text{with} \quad u_\varepsilon = g_\varepsilon \quad \text{on} \quad \partial\Omega. \tag{3.1}$$

The main observation is that as $\varepsilon \to 0$ point singularities (vortices) develop at an energy cost of the order $\ln(1/\varepsilon)$, that arrange according to a certain renormalized energy only depending upon their position within Ω. Considering (3.1) this motivates the following assumption: there are positive constants M_0 and M_1 so that, as ε tends to zero

$$E_\varepsilon(u_\varepsilon) \le M_0\,\ln(1/\varepsilon) \quad \text{and} \quad \|g_\varepsilon\|_{H^{1/2}(\partial\Omega)} \le M_1. \tag{3.2}$$

It turns out that below the critical exponent, uniform gradient bounds in case $p < n/(n-1)$ follow from the equation and the logarithmic energy bound, a regularity result, that plays a crucial role in the asymptotic analysis of solutions as ε tends to zero. The following fundamental estimate has first been proved for $n = 2$ and smooth boundary data by Bethuel, Brezis, and Hélein in the monograph [4]. The following version holds true in any dimension under the assumptions (3.2):

Theorem 2 (Bethuel et al.[2]). *Let $1 \le p < \frac{n}{n-1}$, then there is a constant $C = C(p)$ that only depends on M_0, M_1, Ω and p so that*

$$\int_\Omega |\nabla u_\varepsilon|^p\,dx \le C(p).$$

The theorem is optimal in dimension $n = 2$. For $n \ge 3$, however, the estimate extends up to the end-point $p = n/(n-1)$, cf. [3, 6]. In addition, local gradient for every $p < 2$ estimtes hold true in any dimension, cf. [6]. Since this rather aim for a global view point for the boundary value problem, we oncentrate on the above version. Then the main analytic ingredient in the proof from [2] stems form a variant of the important estimate for the jacobian by Jerrard and Soner. The proof in [2] starts with a suitable decomposition of the domain Ω, according to the deviation form the target. It turns out that our representation formula can circumvent this decomposition and provides a transformation to a global elliptic standard problem. Indeed, we show that any solution u_ε satisfies an equation of the form:

Theorem 3. *Suppose that $u = u_\varepsilon$ is a solution of (3.1) with bounds as in (3.2). Then there is a complex vector field $f_\varepsilon \in L^p(\Omega;\mathbb{C}^n)$ for any $1 \leq p < \frac{n}{n-1}$ with bounds independent of ε and so that*

$$\begin{aligned} \mathcal{L}_\varepsilon \Delta u_\varepsilon &= \nabla\cdot f_\varepsilon \quad \text{in } \Omega && (3.3)\\ \Delta u_\varepsilon &= 0 \qquad \text{on } \partial\Omega && (3.4)\\ u_\varepsilon &= g_\varepsilon \qquad \text{on } \partial\Omega. && (3.5) \end{aligned}$$

Indeed we will determine f_ε explicitly in (3.11) in terms of the energy components and the *current density* $j(u_\varepsilon) = \det(u|\nabla u)$. Observe that the boundary condition (3.4) results from the equation and the assumption that $|g_\varepsilon| = 1$. It identifies $u = u_\varepsilon$ as the unique solution of the problem. The result then follows from linear elliptic regularity theory as in Propsition 2.4:

Theorem 4. *Suppose that $u = u_\varepsilon$ is a solution of (3.1) with bounds as in (3.2). Assume in addition that for some $1 < p < n/(n-1)$ and a constant $M_2 = M_2(p)$*

$$\varepsilon\,\|\Delta g_\varepsilon\|_{L^p} + \|\nabla g_\varepsilon\|_{L^p} \leq M_2.$$

Then we have

$$\varepsilon\,\|\nabla^2 u_\varepsilon\|_{L^p} + \|\nabla u_\varepsilon\|_{L^p} \leq C(p)$$

where the constant $C(p)$ only depends on M_0, M_1, M_2, and p.

3.1 Relation with Jacobians and the Jerrard-Soner result

In the section we conclude the proof of Theorem 3. For complex functions $u \in H^1(\Omega;\mathbb{C})$, the current $j(u)$ is the 1-form defined by

$$j(u) = j_\alpha(u)\,dx_\alpha = \det\big(u\,\big|\,\partial_\alpha u\big)\,dx_\alpha. \tag{3.6}$$

The main connection with Ginzburg-Landau equations reflects in the following fact: If u_ε is a solutions of (3.1), then the current satisfies the continuity equation

$$\operatorname{div} j(u_\varepsilon) = \partial_\alpha\, j_\alpha(u_\varepsilon) = \det\big(u\,\big|\,\Delta u\big) = 0 \tag{3.7}$$

in the sense of distributions. The latter continuity property can be expressed in term of the quantity

$$b(u,\nabla u) = (u\otimes\nabla u - \nabla u\otimes u)$$

from our previous discussion, that is essentially a scalar object for the target $\mathbb{S}^1$.

Lemma 3.1. *For any $\xi \in \mathbb{C} \cong \mathbb{R}^2$ we have*

$$b(u,\nabla u)\,\xi = j(u)\,\xi^\perp \quad \textit{where} \quad \xi^\perp = (-\xi_2,\xi_1). \tag{3.8}$$

In order to deduces an estimate for $b(u,\nabla u)$ from the continuity equation (3.7) we would need a bound for the vorticity: The jacobian $J(u)$ is the 2-form defined by

$$J(u) = \frac{1}{2}\operatorname{curl} j(u) = \frac{1}{2}\,\det\big(\partial_\alpha u\,\big|\,\partial_\beta u\big)\,dx_\alpha\wedge dx_\beta. \tag{3.9}$$

Theorem 5 (Jerrard&Soner [10]). *Let $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded smooth domain. Suppose that (u_ε) admits the above energy bound. Then we have*

$$\|J(u_\varepsilon)\|_{[C_0^\alpha(\Omega)]^*} \leq C(\alpha)$$

for a constant independent of ε.

A extension of this result necessary for application to regularity has been proved in [2], where global estimates for $J(u_\varepsilon)$ in $C^\alpha(\overline{\Omega})^*$ are provided. By the Sobolev embedding they imply in particular an estimate in $W^{-1,p}(\Omega) = [W^{1,p'}(\Omega)]^*$ for any $p' > n$ so that $p < n/(n-1)$. In order to apply these arguments to (3.1) we make use of the transformation formula that in view of (3.8) and the continuity equation immediately tells

$$\mathcal{L}_\varepsilon\Big(V_\varepsilon(u)\,u\Big) = \nabla \cdot \left[j(u)\,u^\perp + \frac{3}{2}\,V(u)\,\nabla u\right]. \tag{3.10}$$

Thus $\mathcal{L}_\varepsilon \Delta u = \nabla \cdot \big(j(u)\,u^\perp + \frac{3}{2}\,V(u)\,\nabla u\big)$. Accordingly, we define for $u = u_\varepsilon$ the function

$$f_\varepsilon = j(u)\,u^\perp + \frac{3}{2}\,V(u)\,\nabla u. \tag{3.11}$$

The aim is to give a uniform L^p estimate for f_ε. Notice that from the energy bound

$$\|V(u)\,\nabla u\|_{L^p}^p \leq \|\nabla u\|_{L^2}\|V(u)\|_{L^{2p/(2-p)}} \leq \varepsilon^{2-p}\ln(1/\varepsilon)^p = o(1)$$

as $\varepsilon \to 0$. The estimate for the first term in (3.11) is more involved and based on a global version of Theorem 5 (cf.[2], Theorem 2Bis) that only involves $H^{1/2}$ bounds of boundary data in addition. We breifly sketch the argument from [2] and, for simplicity, carry it out only in dimension $n = 2$. Recall that the maximum prinziple implies that $|u| \leq 1$, so we only need to derive a uniform L^p estimate for $j(u)$. The 1-form $j(u)$ solves the Hodge system

$$\begin{aligned} \operatorname{div} j(u) &= 0 \\ \operatorname{curl} j(u) &= 2\,J(u) \in W^{-1,p}(\Omega) \end{aligned}$$

with boundary conditions in terms of g. This is a linear elliptic system of first order, and standard elliptic regularity theory implies that, for $p < 2$, $\|j(u_\varepsilon)\|_{L^p} \leq C(p)$ uniformly as $\varepsilon \to 0$. Going back to (3.3) the fundamental estimate follows.

References

[1] F. Alouges and A. Soyeur, On global weak solutions for Landau-Lifshitz equations : existence and nonuniqueness, *Nonlinear analysis* **18** (1992)

[2] F. Bethuel, J. Bourgain, H. Brezis, and G. Orlandi, $W^{1,p}$ estimates for solutions to the Ginzburg-Landau equation with boundary data in $H^{1/2}$, *C. R. Acad. Sci. Paris* **333** (2001)

[3] J. Bourgain, H. Brezis, and P. Mironescu, $H^{1/2}$-maps with values into the circle: minimal connections, lifting and the Ginzburg-Landau equation. *Publ. Math. Inst. Hautes Etudes Sci.* (to appear)

[4] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser (1994)

[5] R. Coifman, P.L. Lions, Y. Meyer, and S. Semmes, Compensated compactness and Hardy spaces, *J. math. pures appl.* **72** (1993)

[6] F. Bethuel, G. Orlandi, and D. Smets, Improved estimates for the Ginzburg-Landau equation: the elliptic case, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **5** (2005) 319–355

[7] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Birkhäuser (1993)

[8] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer (1983)

[9] F. Hélein, *Applications harmoniques, lois de conservations et repéres mobiles*, Diderot Paris (1996)

[10] R. Jerrard and M. Soner, The jacobian and the Ginzburg-Landau energy, *Calc. Var. & PDE* **14** (2002)

[11] C. Melcher, Existence of partially regular solutions for Landau-Lifshitz equations in $\mathbf{R^3}$, *Commun. PDE* **30** (4) (2005)

[12] R. Moser, Partial regularity for Landau-Lifshitz equations in small dimensions, preprint

[13] S. Müller, A surprising higher integrability of determinants, *Bull. AMS* **21** (1989)

[14] M. Struwe, *Variational Methods*, 2nd edition, Springer (1996)