Solitary wave solutions for few-cycle optical pulses

Sh. Amiranashvili, A.G. Vladimirov, U. Bandelow
Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D10117 Berlin, Germany
(Dated: March 12, 2008)

Propagation of short optical pulses in a nonlinear dispersive medium is considered without the use of slow envelope and unidirectional propagation approximations. The existence of uniformly moving solitary solutions is predicted in the anomalous dispersion domain. A four-parametric family of such solutions is found that contains the classical envelope soliton in the limit of large pulse durations. In the opposite limit we get another family member, which in contrast to the envelope soliton strongly depends on nonlinearity model and represents the shortest and the most intense pulse which can propagate in a stationary manner.

PACS numbers: 42.65.Tg, 42.81.Dp, 05.45.Yv

I. INTRODUCTION

Ultra-short optical pulses have numerous applications which include measurements of fast relaxation times, processing of materials, testing of high-speed devices, tracing of chemical reactions, investigations of light-matter interactions [1, 2]. In the present paper we consider the most extreme representatives of ultra-short pulses – the few-cycle pulses – containing only several oscillations of the electromagnetic field. We investigate propagation of such pulses in a transparent medium assuming that the electric field effectively depends on time and one space variable like, e.g., in a single-mode fiber.

Description of ultra-short processes requires a modification of standard slow varying envelope (SVE) models based on the nonlinear Schrödinger equation (NSE) [3, 4]. Apart from straightforward numeric solution of Maxwell equations, two theories of ultra-short pulses are common in the literature. The first one adds several higher-order dispersion terms to NSE (higher-order NSE [5, 6]). These terms are especially important near the so-called zero dispersion frequency (ZDF) where the second-order dispersion vanishes. Note, that higher-order NSEs still assume that the pulse spectral width $\Delta\omega$ is much smaller than the carrier frequency ω_0, so that the SVE approximation can be used. On the contrary, when considering few-cycle pulses with $\Delta\omega \sim \omega_0$ the SVE approximation is not valid. In the latter case an analog of a higher-order NSE can be derived [7]. Several improvements of this model were suggested later [8–13]. These models are also referred to as higher-order NSEs and are now routinely used to model short pulses and continuum generation [14].

The second approach to the description of few-cycle pulses, which does not make use of the pulse envelope concept, is to derive a simplified model directly for the wave fields. The first-order evolution equations are obtained under the so-called unidirectional approximation. In the most simple form it states that

$$\partial_t^2 - \partial_z^2 \approx 2\partial_t(\partial_t + \partial_z) \tag{1}$$

for the normalized space-time variables and pulses propagating along the z-axis. Following this approach several non-envelope models were proposed [15–17]. A relation between NSE and unidirectional models was discussed in [18, 19].

Another characteristic feature of ultrashort optical pulses is that the traditional representation of either response function $\epsilon(\omega)$ or dispersion relation $k(\omega)$ as a Taylor series near ω_0 can become invalid. This happens if $\Delta\omega$ becomes comparable with the spectral width of the transparency window. Since, due to resonances, $\epsilon(\omega)$ always has singularity points in the complex plane, the convergence radius for any Taylor expansion is finite and determined by the singularity which is nearest to ω_0. Therefore, an adequate description of material dispersion cannot be given by the higher-order NSE within the whole frequency range covered by the ultra-short pulse spectrum, whatever the number of terms in the dispersion operator is [20].

A possible way to overcome this difficulty is to deal with the full medium description in terms of Bloch equations. Thereby a modified Korteweg de-Vries (KdV) equation was derived in the optical transparency limit and a sine-Gordon type equation in the opposite case (see [21–26] and references cited therein). In addition, several phenomenological non-envelope models were developed where medium polarization was described by introducing artificial equations for nonlinear oscillators [27–30].

Another possibility to avoid Taylor expansions is to construct a suitable fit to $\epsilon(\omega)$ in the desired spectral range. For instance, a simple expression with three fit parameters A, B and $\bar{\epsilon}$

$$\epsilon(\omega) \approx \bar{\epsilon}\left(1 + A\omega^2 - \frac{B}{\omega^2}\right) \tag{2}$$

provides an accurate approximation to the dispersion function in the transparency window between two resonances [31]. Using Eq. (2) together with the unidirectional approximation (1) the following reduced model for the normalized electric field $E(z,\tau)$

$$\partial_z E - a\partial_\tau^3 E + b\int_{-\infty}^{\tau} E d\tau \pm E^2\partial_\tau E = 0 \tag{3}$$

was derived for a pulse propagating in a Kerr medium [32]. Here $\tau = t - \beta_1 z$ corresponds to the coordinate

frame moving with the group velocity $1/\beta_1$, parameters a and b are proportional to A and B respectively, the sign of the cubic term is determined by the sign of $\chi^{(3)}$.

In the situation when the pulse spectrum is located above the ZDF the last term in Eq. (2) can be neglected. In this case $b = 0$ and Eq. (3) reduces to a modified KdV model which is completely integrable by the inverse scattering technique [33]. If, on the other hand, the pulse spectrum is located below the ZDF the A-term in Eq. (2) can be dropped and therefore one has $a = 0$ in Eq. (3). This particular case of Eq. (3) was introduced recently [34, 35] and it is referred to as a short pulse equation. The later is integrable in a full analogy with the standard NSE, KdV, and sine-Gordon equations [36, 37]. The model (3) can be considered as a possible replacement of the higher-order NSE for few-cycle pulses [31, 32].

The most interesting class of solutions of above mentioned models corresponds to solitary pulses propagating uniformly without changing of their average shape as long as the dissipative effects can be neglected. Several such solutions are known for the higher-order NSE [38–43]. With a special choice of model parameters even a general solution with an arbitrary number of solitons can be constructed. On the other hand, to our knowledge, solitary solutions are not known for the modifications of NSE accounting for the fast envelope evolution.

Solitary solutions of non-envelope equations were reported for the integrable unidirectional models as cited above and were further obtained for more complex unidirectional models [44]. Many of them do not contain any internal field oscillations (video pulses). On the other hand, any localized solution of Eq. (3) with $b \neq 0$ must satisfy the area condition $\int_{-\infty}^{-\infty} E(z,\tau)d\tau = 0$ and therefore it contains at least one oscillation. Although oscillating solitary solutions are usually non-stationary even in the co-moving frame of reference and therefore are quite difficult to find, they are natural representatives of extremely short electromagnetic waves we are interested in. The oscillating pulses can coexist with the video pulses. For instance, both breathers and ordinary solitons are solutions of the modified KdV equation [33]. An oscillating localized solution of the short pulse equation (Eq. (3) with $a = 0$) was reported in [45]. Similar solution for a circularly polarized short pulse (a two-component generalization of Eq. (3) with $a = 0$) was found in [46]. All reported solutions have an important common property with breathers. Namely, as the number of oscillation increases they become similar to the usual envelope solitons of NSE thus providing a link between two theories of short pulses.

The goal of our paper is to investigate the transition between the few-cycle pulses and envelope solitons. The main result is that both the classical envelope and the non-envelope solitons reported recently are representatives of the same family of localized solutions. To obtain this family neither the SVE approximation nor the unidirectional approximation are necessary. Therefore, by construction the obtained solitons satisfy the above mentioned modifications of NSE. Such solitons exist under three essential assumptions: (i) anomalous dispersion, (ii) instantaneous Kerr-like or saturable nonlinearity, and (iii) negligible dissipation effect for the time-scales of interest. Our solution method uses ideas of the recent paper [46] which is devoted to unidirectional equations.

The paper is organized as follows. The assumptions on dispersion and nonlinearity are quantified and the model equations are introduced in the next section. Soliton tails are analyzed in Section III, where the solution Ansatz is explained. The full nonlinear problem is posed and solved in Section IV. Finally, in the Conclusions section, we summarize the results of our analysis.

II. MODEL EQUATIONS

We assume that the radial dependence of the electric filed $\mathbf{E} = (E_x, E_y)$ is integrated out so that the field can be described in the $1+1$ dimensional approximation $\mathbf{E} = \mathbf{E}(z,t)$. The approximation is adequate for single-mode guiding structures for which we write

$$\partial_t^2 \mathbf{D} - c^2 \partial_z^2 \mathbf{E} = 0. \tag{4}$$

The electric displacement vector $\mathbf{D}(\mathbf{E})$ contains both linear and nonlinear parts. The linear part is given in a frequency domain as $\mathbf{D}_\omega^{\rm lin} = \epsilon(\omega)\mathbf{E}_\omega$. Below we use the following fit for the responce function

$$\epsilon(\omega) \approx \bar{\epsilon}\left(1 - \mu^2 \frac{\omega_0^2}{\omega^2}\right) \tag{5}$$

in the anomalous dispersion range. Here ω_0 is some reference frequency within the pulse spectrum. Both $\bar{\epsilon}$ and μ^2 are dimensionless fit parameters. An exemplary fit (5) for a fluoride glass is shown in Fig. 1 for a frequency interval where $\Re[\epsilon(\omega)]$ is concave. We have chosen $\omega_0 = 1\,\mathrm{PHz}$, which is close to the ZDF $= 0.9825\,\mathrm{PHz}$, and obtained

$$\bar{\epsilon} = 1.552 \qquad \text{and} \qquad \mu^2 = 0.0121. \tag{6}$$

Approximation (5) provides a better agreement with the experimentally measured response function than the eight order Taylor expansion around the central frequency ω_0 (see Fig. 1). More details on the approximation accuracy can be found in [34]. In the coordinate space it corresponds to

$$\partial_t^2 \mathbf{D}^{\rm lin} = \bar{\epsilon}(\partial_t^2 \mathbf{E} + \mu^2 \omega_0^2 \mathbf{E}).$$

The physical origin of the dispersion relation (5) was recently discussed in [46].

We assume that the nonlinear part of electric displacement vector is described by an instantaneous self-focusing Kerr expression $\mathbf{D}^{\rm nonl} = 4\pi\chi^{(3)}|\mathbf{E}|^2\mathbf{E}$ with positive frequency independent $\chi^{(3)}$. Inserting $\mathbf{D} = \mathbf{D}^{\rm lin} + \mathbf{D}^{\rm nonl}$ into Eq. (4) we obtain the following basic evolution equation

$$\bar{\epsilon}(\partial_t^2 \mathbf{E} + \mu^2 \omega_0^2 \mathbf{E}) - c^2 \partial_z^2 \mathbf{E} + 4\pi\chi^{(3)} \partial_t^2 (|\mathbf{E}|^2 \mathbf{E}) = 0.$$

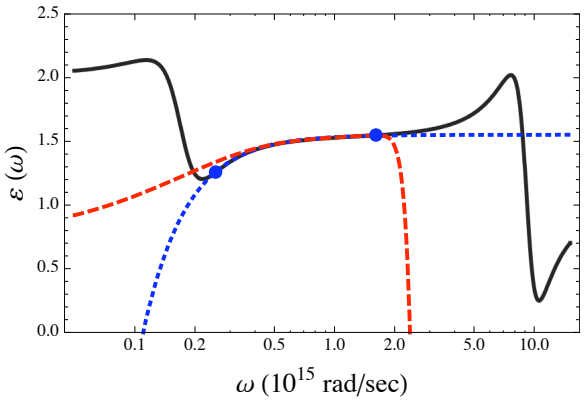

FIG. 1: Real part of dispersion function $\epsilon(\omega)$ for fluoride glass (solid line) and an approximation function (5) (dotted line). The approximation parameters $\bar{\epsilon}$ and μ^2 were chosen to provide the best fit between two inflection frequencies $\omega_{\min} = 0.2536\,\mathrm{PHz}$ and $\omega_{\max} = 1.6127\,\mathrm{PHz}$ (thick points). For comparison the 8th order Taylor expansion of $\epsilon(\omega)$ with the central frequency $\omega_0 = 1\,\mathrm{PHz}$ is also shown (dashed line).

Introducing a wave vector k_0 such that $\omega_0/k_0 = c/\sqrt{\bar{\epsilon}}$ and normalized variables

$$\bar{t} = \omega_0 t, \qquad \bar{z} = k_0 z, \qquad \mathbf{u} = \sqrt{\frac{4\pi\chi^{(3)}}{\bar{\epsilon}\omega_0^2}}\mathbf{E},$$

we transform our basic equation to the dimensionless form

$$\mathbf{u}_{tt} - \mathbf{u}_{zz} + \mu^2\mathbf{u} + (|\mathbf{u}|^2\mathbf{u})_{tt} = 0, \tag{7}$$

where derivatives are denoted by indices and bars are omitted. Note, that we do not use the unidirectional approximation in (7). Moreover, the expression $|\mathbf{u}|^2\mathbf{u}$ in Eq. (7) can be replaced by a more general instantaneous nonlinearity $f(|\mathbf{u}|^2)\mathbf{u}$, accounting for, e.g., saturation effects. Finally, we introduce a complex quantity $\psi = u_x + iu_y$ and rewrite the basic model as

$$\psi_{tt} - \psi_{zz} + \mu^2\psi + [f(|\psi|^2)\psi]_{tt} = 0, \tag{8}$$

where

$$f(\zeta) = \zeta + O(\zeta^2) \qquad \text{as} \qquad \zeta \to 0. \tag{9}$$

In what follows we study solitary solutions of Eq. (8). Note, that ψ as opposed by a complex amplitude in NSE represents the components of the electric field directly. In the next sections we use also the auxiliary function

$$F(\zeta) = \frac{1}{\zeta}\int_0^\zeta f(\zeta)d\zeta,$$

where $F(\zeta) = \zeta/2 + O(\zeta^2)$ as $\zeta \to 0$.

Finally, we stress that the hyperbolic operator in Eq. (8) and its unidirectional approximation (1) act almost identically on the simplest travelling wave solutions that are stationary in the co-moving coordinate frame. Breathers described in this paper are different: they depend on both space and time variables in any frame.

III. SOLITON TAILS

Before proceeding with the nonlinear case let us first discuss a small amplitude limit of Eq. (8)

$$\psi_{tt} - \psi_{zz} + \mu^2\psi = 0, \tag{10}$$

which is the Klein-Gordon equation. This equation describes two decaying tails of any localized solution of the full model. Let us show that such solutions can exist only if some necessary conditions are satisfied. First, introducing field amplitude a and phase φ

$$\psi = u_x + iu_y = ae^{i\varphi}, \tag{11}$$

we replace Eq. (10) with the following system

$$a_{tt} - a_{zz} + (\mu^2 - \varphi_t^2 + \varphi_z^2)a = 0, \tag{12}$$

$$(a^2\varphi_t)_t - (a^2\varphi_z)_z = 0. \tag{13}$$

Next, substituting a usual travelling-wave Ansatz for the field phase

$$\varphi = \Omega\left(t - \frac{z}{\lambda}\right) \tag{14}$$

with Ω and λ being free parameters into Eq. (13) we get $\lambda(a^2)_t + (a^2)_z = 0$. This equation immediately suggests the following Ansatz for the wave amplitude

$$a = a(\xi), \qquad \xi = t - \lambda z. \tag{15}$$

Note, that the normalized velocities in Eqs. (14) and (15) are different indicating that the solution is non-stationary in any moving frame of reference. Finally, due to Eq. (15) the phase Eq. (13) is satisfied automatically and the amplitude Eq. (12) reduces to the form

$$\ddot{a} - \left(\frac{\mu^2}{\lambda^2 - 1} - \frac{\Omega^2}{\lambda^2}\right)a = 0, \tag{16}$$

where an overdot denotes a derivative with respect to ξ. Decaying soliton tails exist only if

$$s^2 = \frac{\mu^2}{\lambda^2 - 1} - \frac{\Omega^2}{\lambda^2} > 0, \tag{17}$$

which means that the following necessary conditions

$$\lambda^2 > 1, \qquad \mu^2 > \frac{\lambda^2 - 1}{\lambda^2}\Omega^2 \tag{18}$$

must be satisfied.

Equations (14) and (15) correspond to the circular polarization of electromagnetic field. The scaling parameter λ determines the normalized soliton velocity, while the second parameter Ω is the carrier frequency normalized by ω_0. Since our dispersion approximation (5) is valid for $\Omega \simeq 1$ and μ^2 in Eq. (6) is small, it follows from (18) that λ^2 is only slightly greater than 1. As we will see, the solitary solution exists in a parameter range $\lambda^2_{\min} < \lambda^2 < \lambda^2_{\max}$. Finally, we note that the first inequality in (18) has a simple interpretation: the dispersion curve $\omega^2 = \mu^2 + k^2$ of Eq. (10) on (ω, k)-plane does not intersect the curve $\omega^2 = k^2/\lambda^2$ corresponding to (15). This agrees with the general statement obtained for any soliton in [47].

IV. NONLINEAR SOLUTION

Now we turn to the solitary solution of the full nonlinear model (8). Using the amplitude-phase representation (11) we obtain an amplitude equation

$$[a_{tt} - a\varphi_t^2] - [a_{zz} - a\varphi_z^2] + \mu^2 a + [f(a^2)a]_{tt} - f(a^2)a\varphi_t^2 = 0 \quad (19)$$

instead of Eq. (12) and a phase equation

$$[2a_t\varphi_t + a\varphi_{tt}] - [2a_z\varphi_z + a\varphi_{zz}] + 2[f(a^2)a]_t\varphi_t + f(a^2)a\varphi_{tt} = 0 \quad (20)$$

instead of Eq. (13). The simplest expression (14) for the phase should be generalized to account for the last two terms in Eq. (20). Following [48] we introduce an Ansatz

$$\varphi = \Omega\left(t - \frac{z}{\lambda} + \int_{-\infty}^{\xi} g(a)d\xi\right), \quad (21)$$

where the function $g(a)$ accounting for the nonlinear phase correction will be specified later. We require $g(0) = 0$ such that Eq. (14) from the previous section can be recovered for the linear case. Inserting Eq. (21) and Eq. (15) into Eq. (20) we rewrite the phase equation as an ordinary differential equation for $g(a)$

$$[a - f_\lambda(a^2)a]g' + 2[a - f_\lambda(a^2)a]'g = 2[f_\lambda(a^2)a]'. \quad (22)$$

Here, a derivative with respect to a is denoted by prime and the notations

$$f_\lambda(a^2) = \frac{f(a^2)}{\lambda^2 - 1}, \qquad F_\lambda(a^2) = \frac{F(a^2)}{\lambda^2 - 1} \quad (23)$$

are introduced. Integrating Eq. (22) and using the condition $g(0) = 0$ we obtain

$$g(a) = \frac{f_\lambda(a^2) - F_\lambda(a^2)}{[1 - f_\lambda(a^2)]^2} + \frac{f_\lambda(a^2)}{1 - f_\lambda(a^2)}. \quad (24)$$

Next, we derive an equation for $a(\xi)$ by inserting Eq. (15) and Eq. (21) into Eq. (19)

$$[\ddot{a} - \Omega^2(1+g)^2 a] - [\lambda^2\ddot{a} - \Omega^2(\lambda^{-1} + \lambda g)^2 a] + \mu^2 a + \frac{d^2}{d\xi^2}[f(a^2)a] - \Omega^2(1+g)^2 f(a^2)a = 0$$

and rewriting it in the form

$$\frac{d^2}{d\xi^2}[a - f_\lambda(a^2)a] - s^2 a + \Omega^2[(1+g)^2 f_\lambda(a^2) - g^2]a = 0, \quad (25)$$

where s^2 is defined by Eq. (17).

The amplitude $a(\xi)$ is completely defined by Eqs. (24) and (25). The linear part of Eq. (25) is identical to Eq. (16) and therefore ensures correct soliton asymptotics at $\xi \to \pm\infty$. Equation (25) can be multiplied with $(d/d\xi)[a - f_\lambda(a^2)a]$ and integrated once. The result can be transformed into the form

$$\frac{\dot{a}^2}{2} + U(a) = \text{const} \quad (26)$$

with an effective potential function $U(a)$, which can be derived by a straightforward integration for any nonlinearity function $f(a^2)$. The solitary solution of Eq. (8) corresponds to a homoclinic orbit of Eq. (26). If any, the orbit must satisfy the equation

$$\dot{a} = \pm\sqrt{2[U(0) - U(a)]}.$$

After $U(a)$ is calculated, it is straightforward to check if the desired homoclinic orbit exists. Moreover, such an orbit always exists for the self-focusing nonlinearity (9) at least for small values of the soliton amplitude. The solution is equivalent to the classical envelope soliton of NSE. As the amplitude increases the pulse is shortened and at some critical amplitude the shortest soliton is obtained. The latter has a non-envelope nature. These issues are discussed in the reminder of the section.

A. Envelope solitons

In this subsection Eqs. (24) and (25) are considered in the limiting case

$$f_\lambda(a^2) = \frac{f(a^2)}{\lambda^2 - 1} \ll 1 \quad (27)$$

corresponding to a small but finite solution amplitude. Since λ^2 is usually only slightly greater than 1, inequality (27) implies a strong restriction on $a(\xi)$. Let us show that the soliton always exists in this limiting case and has an universal shape of the NSE soliton. First, omitting the high order terms in Eq. (24) we get

$$g(a) = 2f_\lambda(a^2) - F_\lambda(a^2).$$

Next, we reduce Eq. (25) to the form

$$\ddot{a} - s^2 a + \Omega^2 f_\lambda(a^2)a = 0.$$

Finally, we simplify $f_\lambda(a^2)$ using Eq. (9) and obtain

$$\ddot{a} - s^2 a + \frac{\Omega^2}{\lambda^2 - 1}a^3 = 0. \quad (28)$$

Equation (28) has a standard solitary solution

$$a(\xi) = \frac{\sqrt{\lambda^2 - 1}}{\Omega}\frac{s\sqrt{2}}{\cosh(s\xi)} \quad (29)$$

similar to that of the NSE. It follows from Eq. (29) that in order to satisfy the condition (27) we need $s \ll 1$. Therefore, in accord with Eq. (16), the soliton tails are slowly decaying and contain many oscillations of the carrier wave. That is, Eq. (29) describes an envelope soliton.

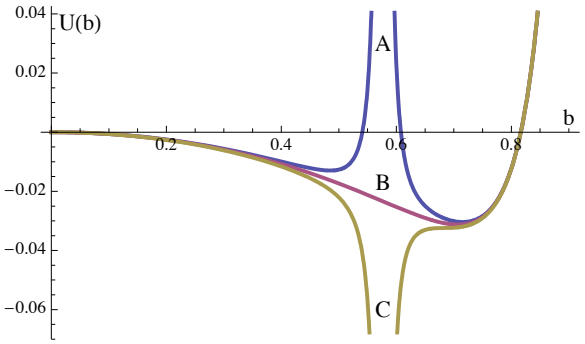

FIG. 2: The effective potential $U(b)$ from Eq. (30) for $\Omega = 1$ and three values of the parameter s: (A) $8s^2 = 0.97$; (B) $8s^2 = 1$; (C) $8s^2 = 1.03$.

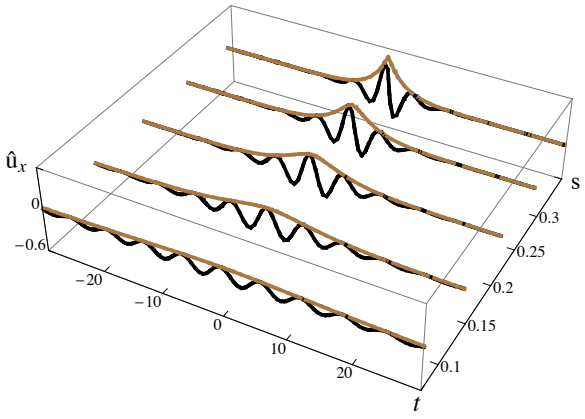

FIG. 3: Normalized electric field $\hat{u}_x = u_x(\xi)/\sqrt{\lambda^2-1}|_{z=0}$ and pulse envelope $b(\xi)|_{z=0}$ computed from Eq. (30) for $\Omega = 1$ and different values of the parameter s.

B. Non-envelope solitons

In this subsection we consider a solitary solution of the full system (24) and (25) with the simplest nonlinear function $f(\zeta) = \zeta$ corresponding to the self-focusing Kerr medium. In accord with Eq. (23) it is helpful to introduce a new amplitude variable

$$b(\xi) = \frac{a(\xi)}{\sqrt{\lambda^2 - 1}}$$

such that $f_\lambda(a^2) = b^2$ and $F_\lambda(a^2) = b^2/2$. Then Eq. (24) reduces to the form

$$g(b) = \frac{b^2(3-2b^2)}{2(1-b^2)^2}.$$

Now, the amplitude Eq. (25) takes the form

$$\frac{d^2}{d\xi^2}(b - b^3) - s^2 b + \Omega^2[(1+g)^2 b^2 - g^2]b = 0$$

or

$$\frac{d^2}{d\xi^2}(b - b^3) - s^2 b + \Omega^2 \frac{4 - 9b^2 + 4b^4}{4(1-b^2)^3} b^3 = 0,$$

which is a generalization of Eq. (28). After one integration we obtain

$$\frac{\dot{b}^2}{2} - s^2 \frac{b^2(2-3b^2)}{4(1-3b^2)^2} + \Omega^2 \frac{b^4(2-7b^2+6b^4)}{8(1-b^2)^2(1-3b^2)^2} = 0 \quad (30)$$

with the integration constant being zero for a localized solution.

Equation (30) belongs to the general class (26) with a rather complicated effective potential $U(b)$. A similar equation was investigated in [48]. Solitary solutions exist as long as $s^2 \le \Omega^2/8$. For $s^2 \to 0$ they reduce to the envelope solitons described in the previous section. In the limit $s^2 \to \Omega^2/8$ we get non-envelope short-pulse solutions. To understand the behavior of the solution in this limit, we plot $U(b)$ from Eq. (30) for three different situations: (A) s^2 is slightly below $\Omega^2/8$, (B) $s^2 = \Omega^2/8$, and (C) s^2 is slightly above $\Omega^2/8$ (see Fig. 2). In the last case a regular solution for $b(\xi)$ does not exist. Exactly for $s^2 = \Omega^2/8$ two singular terms in $U(\xi)$ cancel each other. This is also evident from the expansion of the potential

$$U(b) \to \frac{\Omega^2}{96(1-3b^2)^2} - \frac{s^2}{12(1-3b^2)^2} \quad (31)$$

at $b^2 \to 1/3$. The critical value of s^2 was actually found from the expansion (31). At $s^2 = \Omega^2/8$ Eq. (30) becomes

$$\dot{b}^2 = \Omega^2 \frac{b^2(2-3b^2)}{16(1-b^2)^2} \quad (32)$$

and can be solved in quadratures

$$\frac{1}{3}\sqrt{2-3b^2} - \frac{1}{\sqrt{2}} \operatorname{arccosh}\left(\frac{\sqrt{2}}{\sqrt{3}b}\right) = \pm\frac{\Omega\xi}{4}. \quad (33)$$

Here $b^2(0) = 2/3$ is the normalized pulse peak intensity.

For $s^2 < \Omega^2/8$ we have a continuous family of pulses. Figure 3 illustrates how the soliton shape evolves as s^2 approaches the critical value. It is interesting to note that the limiting pulse for $s^2 \nearrow \Omega^2/8$ (the last curve in Fig. 3) has a cusp-like shape and differs from the exact solution (33) obtained for $s^2 = \Omega^2/8$. At the cusp point we have $b^2(0) = 1/3$.

Thus, the soliton exists if the condition

$$0 < \frac{(\mu/\Omega)^2}{\lambda^2 - 1} - \frac{1}{\lambda^2} < \frac{1}{8}$$

is fulfilled. This condition defines the range $\lambda^2_{\min} < \lambda^2 < \lambda^2_{\max}$ of the soliton velocities. Note, that similarly to the envelope solitons the soliton family found in this paper depends on four parameters: the velocity $1/\lambda$, carrier frequency Ω, an arbitrary initial phase and position.

V. CONCLUSIONS

To summarize, generalizing the results of [46], we have found a family of ultra-short optical solitons for Eq. (7) which does not assume slow-envelope and unidirectional approximations. Instead of a local Taylor expansion, a global rational fit of the material response function was used. In addition to neglecting dissipation effects only two assumptions were made: instantaneous self-focusing nonlinearity and an anomalous dispersion in the frequency range of interest.

Our mathematical procedure leads to a qualitatively simple Eq. (26), which allows for a straightforward analysis of the solitary solutions. All solitons have a following universal property: with the increase of the pulse duration they reduce to a standard envelope soliton. Such behavior was first predicted in [47]. On the other hand, when the pulse duration decreases the soliton evolves to a limiting shape representing the shortest pulse for which dispersive spreading is still compensated by nonlinearity (Fig. 3). The half-width of this pulse is of order of the carrier wave period (single-cycle solution). The appearance of the cusp-like singularity which prevents the existence of shorter pulses is discussed.

Acknowledgments

This work was supported by the DFG Research Center MATHEON under project D 14. The authors would like to thank Prof. A. Mielke for helpful discussions.

[1] K. Akimoto, J. Phys. Soc. Jpn. **65**, 2020 (1996).
[2] T. Brabec and F. Krausz, Rev. Mod. Phys. **72**, 545 (2000).
[3] J. E. Rothenberg, Opt. Lett. **17**, 1340 (1992).
[4] J. K. Ranka and A. L. Gaeta, Opt. Lett. **23**, 534 (1998).
[5] G. Agrawal, *Nonlinear Fiber Optics* (Academic Press, New York, 1989).
[6] A. Hasegawa, *Optical Solitons in Fibers* (Springer, Heidel-berg, 1980).
[7] T. Brabec and F. Krausz, Phys. Rev. Lett. **78**, 3282 (1997).
[8] A. A. Zozulya, S. A. Diddams, and T. S. Clement, Phys. Rev. A **58**, 3303 (1998).
[9] A. A. Zozulya, S. A. Diddams, A. G. Van Engen, and T. S. Clement, Phys. Rev. Lett. **82**, 1430 (1999).
[10] N. Aközbek, M. Scalora, C. M. Bowden, and S. L. Chin, Opt. Commun. **191**, 353 (2001).
[11] M. A. Porras, Phys. Rev. E **65**, 026606 (2002).
[12] P. Kinsler and G. H. C. New, Phys. Rev. A **67**, 023813 (2003).
[13] P. Kinsler and G. H. C. New, Phys. Rev. A **69**, 013805 (2004).
[14] J. M. Dudley, G. Genty, and S. Coen, Rev. Mod. Phys. **78**, 1135 (2006).
[15] E. M. Belenov and A. V. Nazarkin, JETP Lett. **53**, 200 (1991).
[16] M. Geissler, G. Tempea, A. Scrinzi, M. Schnürer, F. Krausz, and T. Brabec, Phys. Rev. Lett. **83**, 2930 (1999).
[17] A. V. Husakou and J. Herrmann, Phys. Rev. Lett. **87**, 203901 (2001).
[18] M. Kolesik and J. V. Moloney, Phys. Rev. E **70**, 036604 (2004).
[19] M. Kolesik, J. V. Moloney, and M. Mlejnek, Phys. Rev. Lett. **89**, 283902 (2002).
[20] K. E. Oughstun and H. Xiao, Phys. Rev. Lett. **78**, 642 (1997).
[21] E. M. Belenov, P. G. Kryukov, A. V. Nazarkin, A. N. Oraevskii, and A. V. Uskov, JETP Lett. **47**, 523 (1988).
[22] E. M. Belenov and A. V. Nazarkin, JETP Lett. **51**, 288 (1990).
[23] S. V. Sazonov, JETP Lett. **53**, 420 (1991).
[24] H. Leblond and F. Sanchez, Phys. Rev. A **67**, 013804 (2003).
[25] S. V. Sazonov and N. V. Ustinov, JETP **103**, 561 (2006).
[26] S. V. Sazonov and N. V. Ustinov, JETP Lett. **83**, 483 (2006).
[27] A. I. Maimistov and S. O. Elyutin, J. Mod. Opt. **39**, 2201 (1992).
[28] E. V. Kazantseva and A. I. Maimistov, Phys. Lett. A **263**, 434 (1999).
[29] A. I. Maimistov, Opt. Spectrosc. **94**, 251 (2003).
[30] E. V. Kazantseva, A. I. Maimistov, and J.-G. Caputo, Phys. Rev. E **71**, 056622 (2005).
[31] V. G. Bespalov, S. A. Kozlov, Y. A. Shpolyanskiy, and I. A. Walmsley, Phys. Rev. A **66**, 013811 (2002).
[32] S. A. Kozlov and S. V. Sazonov, JETP **84**, 221 (1997).
[33] M. Wadati, J. Phys. Soc. Jpn. **34**, 1289 (1973).
[34] T. Schäfer and C. E. Wayne, Physica D **196**, 90 (2004).
[35] Y. Chung, C. K. R. T. Jones, T. Schäfer, and C. E. Wayne, Nonlinearity **18**, 1351 (2005).
[36] A. Sakovich and S. Sakovich, J. Phys. Soc. Jpn. **74**, 239 (2005).
[37] J. C. Brunelli, Phys. Lett. A **353**, 475 (2006).
[38] E. M. Gromov and V. I. Talanov, Radiophys. Quantum Electron. **39**, 486 (1996).
[39] K. Porsezian and K. Nakkeeran, Phys. Rev. Lett. **76**, 3955 (1996).
[40] M. Gedalin, T. C. Scott, and Y. B. Band, Phys. Rev. Lett. **78**, 448 (1997).
[41] E. M. Gromov and V. I. Talanov, Radiophys. Quant. Electron. **41**, 143 (1998).
[42] E. M. Gromov and V. I. Talanov, Chaos **10**, 551 (2000).
[43] C. E. Zaspel, Phys. Rev. Lett. **82**, 723 (1999).
[44] A. I. Maimistov, Quantum Electron. **30**, 287 (2000).
[45] A. Sakovich and S. Sakovich, J. Phys. A **39**, L361 (2006).
[46] S. A. Skobelev, D. V. Kartashov, and A. V. Kim, Phys. Rev. Lett. **99**, 203902 (2007).
[47] V. E. Zakharov and E. A. Kuznetsov, JETP **86**, 1035 (1998).
[48] D. V. Kartashov, A. V. Kim, and S. A. Skobelev, JETP Lett. **78**, 276 (2003).