# On algebraic multi-level methods for non-symmetric systems - comparison results

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March 13, 2008

Dedicated to Richard S. Varga on the occasion of his 80th birthday

**Abstract**

We establish theoretical comparison results for algebraic multi-level methods applied to nonsingular non-symmetric M-matrices. We consider two types of multi-level approximate block factorizations or AMG methods, the AMLI and the MAMLI method. We compare the spectral radii of the iteration matrices of these methods. This comparison shows, that the spectral radius of the MAMLI method is less than or equal to the spectral radius of the AMLI method. Moreover, we establish how the quality of the approximations in the block factorization effects the spectral radii of the iteration matrices. We prove comparisons results for different approximation of the fine grid block as well as for the used Schur complement. We also establish a theoretical comparison between the AMG methods and the classical block Jacobi and block Gauss-Seidel methods.

**Keywords.** Algebraic multi-level methods, multi-level approximate block factorizations, algebraic multigrid methods, AMLI method

**AMS subject classifications.** 65F10, 65F50, 65N22

## 1 Introduction

In the last decade, algebraic multigrid methods became a powerful tool for solving nonsingular linear systems of equations

$$Ax = b.$$

The idea of algebraic multigrid methods is to use only information on the matrix structure and the matrix entries [22]. Although these algebraic multi-level methods work very well in practice for many problems, there is not that much known

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about their theoretical convergence properties, especially for non-symmetric problems. Recently, a theoretical comparison of different algebraic multigrid methods applied to symmetric positive definite systems was given by Notay in [19].

In [16] two types of algebraic multi-level methods resulting from approximative block factorizations are considered, namely additive and multiplicative block factorizations. The first type includes an abstract and basic version of the AMLI method (algebraic multi-level iteration) introduced by Axelsson and Vassilewski in [2, 3]. The other type, the multiplicative type, includes the so-called MAMLI method. This method can be seen as an AMG method with fine grid smoothing only and special restriction and prolongation operators [15]. A symmetrized version of the MAMLI method leads to the so-called SMAMLI method [16]. The AMLI and MAMLI methods can be formulated also as additive and multiplicative Schwarz methods [5, 23, 16].

In [16] convergence results are established for these AMG methods applied to non-symmetric matrices. It was shown, that the AMLI and MAMLI method converge for non-symmetric M-matrices.

M-matrices are used in various fields of applied mathematics such as numerical analysis, probability, economics and operations research [7]. Moreover, Markov chain modeling became relevant in several applications from computer science such as information retrieval [14]. In Markov chain modeling iterative solvers, like algebraic multigrid methods are used to compute the steady state solution of a Markov chain, i.e. algebraic multigrid methods are used to find the solution of a system with non-symmetric M-matrix structure. But, Markov chain problems usually lead to singular M-matrices. However, the Google matrix, introduced by Brin and Page (see e.g. [14]) is a non-singular M-matrix, based on a modified Markov chain model. The size of the Google matrix is equal to the number of webpages. Hence, solving a linear system with this matrix is a non-trivial task. Therefore theoretical comparisons of the speed of convergence of iterative methods help to choose between different methods.

In [16] upper bounds for the spectral radii of the iteration matrices of the AMLI and MAMLI methods are compared. These upper bounds are given in terms of weighted max norms.

In this paper, we continue such a comparison in detail. We compare the spectral radii of the iteration matrices of the abstract AMLI, MAMLI, and SMAMLI methods. The comparison shows, that the spectral radius of the MAMLI method is less than or equal to the spectral radius of the AMLI method. This result is one of few theoretical results that gives a direct comparison of the asymptotic convergence rates of a multiplicative Schwarz (MAMLI) method compared with that of the additive Schwarz (AMLI) method [18, 13].

Moreover, we will show theoretically how the quality of the approximations in the block factorization effects the spectral radii of the iteration matrices of both the AMLI and the MAMLI method. We will give comparisons for different approximations of the fine grid block as well as for the used Schur complement. Our comparison theorems are mainly based on matrix splitting theory introduced by R.S. Varga in the sixties (see [24]). We explain in detail how the AMG methods are induced by splittings of the system matrix $A$. The analysis of these splittings is then the key to our comparison results, since we are using comparison theorems for basic stationary iterative methods or in other words for splittings. The first comparison theorem for so called regular splittings was

given by R.S. Varga in [24]. With the help of the splitting theory and the comparison results for splittings we are then able to give short and elegant proofs for the comparisons of the AMG methods described above.
Of course a comparison of spectral radii alone does not determine the advantage of one method over another. A comparison of the work per iteration is also needed, but combining both aspects in theory is hard. Nevertheless, we will show that the work per iteration of the AMLI and MAMLI method is about the same. Moreover, we will show some numerical results that also compare the overall performance of the methods we consider.
The paper is organized as follows. In the next section we list some notation and give some well-known results. Section 3 gives a short introduction into the multi-level approximate block factorization (AMLI) method and describes the AMLI, MAMLI, and SMAMLI method. The comparison of the spectral radii of the iteration matrices of these methods is given in section 4. In Section 5 it is shown how the quality of the approximations effects the spectral radii. The comparison between the AMG methods and the classical methods is given in section 6. Section 7 contains some numerical results.

## 2 Notation and well-known results

The property of nonnegativity is the major tool in this paper. A matrix $T$ is nonnegative (positive), denoted $T \geq 0$ ($T > 0$), if its entries are nonnegative (positive). We say that $T \geq S$ if $T - S \geq 0$, and similarly with the strict inequality. By $T \lneq S$ we denote that the matrix $T - S \leq 0$ but $T \neq S$. These definitions carry over to vectors. A matrix $A$ is a non-singular $M$-matrix if its off-diagonal elements are non-positive, and it is monotone, i.e., $A^{-1} \geq 0$. It follows that if $A$ and $B$ are non-singular $M$-matrices and $A \geq B$, then $A^{-1} \leq B^{-1}$ [7, 24]. By $\rho(B)$ we denote the spectral radius of the matrix $B$.

**Definition 2.1.** *[7, 24, 25] The matrix pair $(M, N)$ is a splitting of $A$ if $A = M - N$ and $M$ is non-singular. A splitting is called*

- *regular if $M^{-1} \geq 0$ and $N \geq 0$;*
- *weak regular of the first type if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;*
- *weak regular of the second type if $M^{-1} \geq 0$ and $NM^{-1} \geq 0$;*
- *nonnegative if $M^{-1} \geq 0$, $M^{-1}N \geq 0$, and $NM^{-1} \geq 0$.*

Here, we consider stationary iterative methods to solve $Ax = b$. These methods start with a vector $x^{(0)}$ and build a sequence of vectors $x^{(i+1)}$ such that

$$x^{(i+1)} = Tx^{(i)} + c \quad \text{for} \quad i = 1, 2, \ldots. \tag{1}$$

The matrix $T$ is called iteration matrix. If $\rho(T) < 1$ then, there exists a unique splitting $(M, N)$ such that $T = M^{-1}N$. This splitting is given by $M = A(I - T)^{-1}$ and $N = M - A$, see e.g. [6]. We say that $T$ is induced by this splitting $(M, N)$.
If one wants to compare two different stationary iteration methods in terms of their speed of convergence, one usually compares the spectral radii of the iteration matrices, i.e. one compares the different induced splittings.

There are many comparison theorems known for two splittings. The first and famous result was proved by R.S. Varga (see [24] page 97).

**Theorem 2.2.** *Let $A$ be a non-singular M-matrix. Let $(M_1, N_1)$ and $(M_2, N_2)$ be two regular splittings of $A$ with $N_1 \leq N_2$. Then*

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2).$$

This result was generalized in many directions during the last decades, see e.g. [9, 10, 8, 25]. In this paper we use the following generalization which can be found in [25].

**Theorem 2.3.** *Let $A$ be a non-singular M-matrix. Let $(M_1, N_1)$ and $(M_2, N_2)$ be two splittings of different types of $A$ with $M_1^{-1} \geq M_2^{-1}$. Then*

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2).$$

There are also several theorems known that give strict inequality comparisons. Here we summarize some of them in the next theorem, see also [11].

**Theorem 2.4.** *Assume that $A$ is a nonsingular matrix such that $A^{-1} > O$ and let $A = M_1 - N_1$, $A = M_2 - N_2$ be two splittings of $A$. Then $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2)$ in the following cases:*
(i) *both splittings are regular, $N_1 \neq 0$ and $M_1 \lneq M_2$ or, equivalently, $N_1 \lneq N_2$ ([24]).*
(ii) *both splittings are regular and $M_1^{-1} > M_2^{-1}$ ([25], see also [9]).*
(iii) *both splittings are weak regular splittings of different type and $M_1^{-1} > M_2^{-1}$ ([8]).*

However, we will see in the next sections that the assumptions of the above theorem are difficult to fulfill in our comparison of algebraic multigrid methods. Either, the considered methods do not induce regular splittings or a comparison as used in (iii) is not realistic since sparse approximations are usually involved in algebraic multigrid methods. Thus $M_1^{-1} > M_2^{-1}$ can not be fulfilled for practical applications.
Next we recall the definition of the weighted max-norm. Given a positive vector $w \in \mathbb{R}^n$, denoted $w > 0$, the weighted max-norm is defined for any $y \in \mathbb{R}^n$ as $\|y\|_w = \max_{j=1,\cdots,n} |\frac{1}{w_j} y_j|$. The corresponding matrix norm is defined as $\|T\|_w = \sup_{\|x\|_w=1} \|Tx\|_w$.
With this norm we can compare two weak regular splittings of the same type.

**Theorem 2.5.** *Let $A$ be a non-singular M-matrix. Let $(M_1, N_1)$ and $(M_2, N_2)$ be two splittings of the same type of $A$ with $M_1^{-1} \geq M_2^{-1}$. Then*

$$\|M_1^{-1}N_1\|_w \leq \|M_2^{-1}N_2\|_w$$

*with $w = A^{-1}e$ for a positive vector $e$.*

A proof of Theorem 2.5 can be found in [12].

## 3 The AMLI and the MAMLI approaches

In algebraic multigrid methods a so-called coarsing process is performed before the iteration starts. During this process, the set $\{1, \ldots, n\}$ is split into two disjoint sets $F$ and $C$. Having done this, there is a permutation $P$ such that

$$PAP^T = \begin{bmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{bmatrix}.$$

Note, that in practice this permutation is only implicitly used. In this paper we assume that the system matrix $A$ is already partitioned in block $2 \times 2$ form, i.e. we assume that

$$A = \begin{bmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{bmatrix}. \tag{2}$$

In the AMG language, $F$ denotes the set of fine grid unknowns, and $C$ denotes the set of coarse grid unknowns with $|F| = n_F$ and $|C| = n_C$. The sets $F$ and $C$ are determined by a so-called coarsing process.
We will denote by $I$ the $n \times n$ identity matrix and with $I_F$ and $I_C$ the $n_F \times n_F$ and $n_C \times n_C$ the identity matrix, respectively.
Moreover, we assume that $A_{FF}$ is non-singular. Then $A$ can be factorized as

$$A = \begin{bmatrix} I_F & 0 \\ A_{CF}A_{FF}^{-1} & I_C \end{bmatrix} \begin{bmatrix} A_{FF} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_F & A_{FF}^{-1}A_{FC} \\ 0 & I_C \end{bmatrix},$$

where

$$S := (A/A_{FF}) := A_{CC} - A_{CF}A_{FF}^{-1}A_{FC}$$

is the Schur complement. If we now use an approximation $\widetilde{A}_{FF}$ of $A_{FF}$ and an approximation $\widetilde{S}$ of $S$, or approximations of the inverses of these matrices, we obtain the matrix $M$ with

$$M = \begin{bmatrix} I_F & 0 \\ A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} \begin{bmatrix} \widetilde{A}_{FF} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \begin{bmatrix} I_F & \widetilde{A}_{FF}^{-1}A_{FC} \\ 0 & I_C \end{bmatrix}.$$

This factorization is known as an approximate two-level (multi-level) block factorization [19]. Many multi-level methods use this two-level block approximate factorization as a major tool (see e.g. [2, 3, 1, 4, 21] and references in [19]). One of these methods is the AMLI method by Axelsson and Vassilevski [2, 3]. The AMLI method, in its basic form, can be described as the stationary iteration with the iteration matrix

$$T_{AMLI} = I - M^{-1}A.$$

If the AMLI method is used as a preconditioner for a Krylov subspace method, the preconditioner is $M^{-1}$. For the iteration matrix we obtain

$$
\begin{aligned}
T_{AMLI} &= (I - M^{-1}A) \\
&= I - \left( \begin{bmatrix} I_F & 0 \\ A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} \begin{bmatrix} \widetilde{A}_{FF} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \begin{bmatrix} I_F & \widetilde{A}_{FF}^{-1}A_{FC} \\ 0 & I_C \end{bmatrix} \right)^{-1} A \\
&= I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A \qquad (3) \\
&\quad - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A.
\end{aligned}
$$

Using the following operators

$$
\widetilde{R} := [-A_{CF}\widetilde{A}_{FF}^{-1} \;\; I_C], \qquad \widetilde{P}^T := \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix}, \qquad \widehat{R}^T := \begin{bmatrix} I_F \\ 0 \end{bmatrix} \qquad (4)
$$

we obtain

$$
T_{AMLI} = I - \widetilde{P}^T\widetilde{S}^{-1}\widetilde{R}A - \widehat{R}^T\widetilde{A}_{FF}^{-1}\widehat{R}A. \qquad (5)
$$

The operators $\widetilde{R}$ and $\widetilde{P}^T$ are known as restriction and prolongation operators in algebraic multigrid methods [19]. The term $\widetilde{P}^T\widetilde{S}^{-1}\widetilde{R}A$ acts as a coarse grid correction while $\widehat{R}^T\widetilde{A}_{FF}^{-1}\widehat{R}A$ can be seen as a smoother. Both parts are combined in an additive way in (5).
The multiplicative version, which is called the MAMLI method, is given by:

$$
\begin{aligned}
T_{MAMLI} &= \left( I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A \right) \\
&\quad \cdot \left( I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A \right) \qquad (6) \\
&= (I - \widetilde{P}^T\widetilde{S}^{-1}\widetilde{R}A)(I - \widehat{R}^T\widetilde{A}_{FF}^{-1}\widehat{R}A).
\end{aligned}
$$

Comparing the iteration matrices $T_{AMLI}$ and $T_{MAMLI}$ in (5) and (6) we observe that the amount of work for one step in the iteration (1) is about the same for both methods. Hence, we expect that the time needed for one iteration step is the same for the AMLI and MAMLI method. This is confirmed by the numerical results given in Section 7.
Closely related to the MAMLI method is the symmetrized MAMLI method, the SMAMLI method, given by

$$
\begin{aligned}
T_{SMAMLI} &= \left( I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A \right) \\
&\quad \cdot \left( I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A \right) \\
&\quad \cdot \left( I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A \right) \qquad (7) \\
&= (I - \widehat{R}^T\widetilde{A}_{FF}^{-1}\widehat{R}A)(I - \widetilde{P}^T\widetilde{S}^{-1}\widetilde{R}A)(I - \widehat{R}^T\widetilde{A}_{FF}^{-1}\widehat{R}A).
\end{aligned}
$$

The multiplicative AMLI techniques are closely related to certain geometric and algebraic multigrid methods. As mentioned above, the second factor can be seen as a relaxation or smoothing step, while the first factor in (6) is a coarse grid correction. In particular, the MAMLI method can be viewed as a two-level V(1,0) cycle. The SMAMLI method is a two-level V(1,1) cycle. From a more abstract point of view, the above methods are subspace correction methods, i.e. methods that adds different corrections, that act only on a subspace, to the actual approximation vector (see e.g. [26]). In this terminology, the AMLI method is a parallel (or additive) subspace correction method, the MAMLI method is a successive (or multiplicative) subspace correction method. More details about the MAMLI and SMAMLI method can be found in [16].
There are different choices for the approximations $\widetilde{S}$ of the Schur complement $S$. One choice is to simply use the matrix $(A/\widetilde{A}_{FF}) := A_{CC} - A_{CF}\widetilde{A}_{FF}^{-1}A_{FC}$ or use an approximation of $(A/\widetilde{A}_{FF})$, see [19]. Another choice, is to use the coarse grid matrix or Galerkin matrix $\widetilde{R}A\widetilde{P}^T$ or approximate $\widetilde{R}A\widetilde{P}^T$, see [20]. In order to differ between these approaches, we call the later variant, a Galerkin type approach or Galerkin type method.

Of course the quality of the approximations $\tilde{A}_{FF}$ of $A_{FF}$ and $\tilde{S}$ of $S$ will be important for the convergence behavior of all these methods, see Section 5.
In [16] the following assumptions on the approximations are used to prove convergence of the AMLI method.

**Assumption 3.1.** *Let $A$ be a non-singular (non-symmetric) $M$-matrix and let $A$ be partitioned as in* (2).
*Furthermore, let $\tilde{A}_{FF}$ and $\tilde{S}$ be chosen such that the splittings $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ and $\left(\widetilde{S}, \widetilde{S} - (A/\widetilde{A}_{FF})\right)$ are weak regular of the first type.*

For the multiplicative versions a slightly modified set of approximations is used in [16] to study also the Galerkin type MAMLI method.

**Assumption 3.2.** *Let $A$ be a non-singular (non-symmetric) $M$-matrix and let $A$ be partitioned as in* (2).
*Furthermore, let $\tilde{A}_{FF}$ and $\tilde{S}$ be chosen such that the splittings $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ and $\left(\widetilde{S}, \widetilde{S} - \widetilde{R}A\widetilde{P}^T\right)$ are weak regular of the first type.*

Note that

$$\widetilde{R}A\widetilde{P}^T = A_{CC} - A_{CF}\left(2\widetilde{A}_{FF}^{-1} - \widetilde{A}_{FF}^{-1}A_{FF}\widetilde{A}_{FF}^{-1}\right)A_{FC}\ . \tag{8}$$

If we compare these two Assumptions 3.1 and 3.2, we see that the only difference is on the approximation $\tilde{S}$.
Using equation (8), it follows that Assumption 3.1 implies Assumption 3.2, in other words Assumption 3.2 is weaker or more general than Assumption 3.1.
In [16] the following convergence results for the two-level methods are established.

**Theorem 3.3.** *Let Assumption 3.1 be satisfied. Then*

$$\begin{aligned} T_{AMLI} &\geq 0, \\ \rho\left(T_{AMLI}\right) \leq \|T_{AMLI}\|_w &< 1, \end{aligned}$$

*where* $w = A^{-1}e$ *for an arbitrary positive vector* $e$. *Moreover,*

$$T_{AMLI} = I - C_{AMLI}A,$$

*where* $(C_{AMLI}^{-1}, C_{AMLI}^{-1} - A)$ *is a weak regular splitting of first type of* $A$.

The weaker Assumption 3.2 allows a convergence proof for the Galerkin type MAMLI method also.

**Theorem 3.4.** *Let Assumption 3.2 be satisfied. Then*

$$\begin{aligned} T_{MAMLI} &\geq 0, \\ \rho\left(T_{MAMLI}\right) \leq \|T_{MAMLI}\|_w &< 1, \end{aligned}$$

*where* $w = A^{-1}e$ *for an arbitrary positive vector* $e$. *Moreover,*

$$T_{MAMLI} = I - C_{MAMLI}A,$$

*where* $(C_{MAMLI}^{-1}, C_{MAMLI}^{-1} - A)$ *is a weak regular splitting of first type of* $A$.

In order to obtain comparison results for the AMLI and MAMLI method we need to change the assumptions on the approximations a little bit. In Assumptions 3.1 and 3.2 weak regular splittings of first type are used to specify the approximations. Moreover, by Theorems 3.3 and 3.4 $T_{AMLI}$ and $T_{MAMLI}$ are induced by weak regular splittings. However, Elsner showed in [10], that a comparison theorem like Theorem 2.2 or Theorem 2.3 does not hold for just two weak regular splittings. Nevertheless, we are able to prove a comparison theorem for the AMLI and MAMLI method. But therefore we have to use nonnegative splittings for the approximations. Of course, using weak regular splittings is more general than using nonnegative splittings. However, in practice, the use of nonnegative splittings is not a drawback. All relevant approximations or splittings, like the Jacobi, the Gauss-Seidel, the ILU and others, are nonnegative splittings or even more, are regular splittings for $M$-matrices.

**Assumption 3.5.** *Let* $A$ *be a non-singular (non-symmetric)* $M$*-matrix and let* $A$ *be partitioned as in* (2). *Furthermore, let* $\tilde{A}_{FF}$ *and* $\tilde{S}$ *be chosen such that the splittings* $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ *and* $\left(\widetilde{S}, \widetilde{S} - (A/\widetilde{A}_{FF})\right)$ *are nonnegative, i.e.*

$$\begin{aligned} \widetilde{A}_{FF}^{-1} &\geq 0, \\ I_F - \widetilde{A}_{FF}^{-1}A_{FF} &\geq 0, \\ I_F - A_{FF}\widetilde{A}_{FF}^{-1} &\geq 0, \end{aligned}$$

*and*

$$\begin{aligned} \widetilde{S}^{-1} &\geq 0, \\ I_C - \widetilde{S}^{-1}(A/\widetilde{A}_{FF}) &\geq 0, \\ I_C - (A/\widetilde{A}_{FF})\widetilde{S}^{-1} &\geq 0. \end{aligned}$$

**Assumption 3.6.** *Let* $A$ *be a non-singular (non-symmetric)* $M$*-matrix and let* $A$ *be partitioned as in* (2). *Furthermore, let* $\tilde{A}_{FF}$ *and* $\tilde{S}$ *be chosen such that the* *splittings* $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ *and* $\left(\widetilde{S}, \widetilde{S} - \widetilde{R}A\widetilde{P}^T\right)$ *are nonnegative, i.e.*

$$\begin{aligned} \widetilde{A}_{FF}^{-1} &\geq 0, \\ I_F - \widetilde{A}_{FF}^{-1} A_{FF} &\geq 0, \\ I_F - A_{FF}\widetilde{A}_{FF}^{-1} &\geq 0, \end{aligned}$$

*and*

$$\begin{aligned} \widetilde{S}^{-1} &\geq 0 \\ I_C - \widetilde{S}^{-1}(\widetilde{R}A\widetilde{P}^T) &\geq 0, \\ I_C - (\widetilde{R}A\widetilde{P}^T)\widetilde{S}^{-1} &\geq 0. \end{aligned}$$

As in the cases considered before, Assumption 3.5 implies Assumption 3.6, hence Assumption 3.6 is weaker than Assumption 3.5. Moreover, the convergence results for the AMLI and MAMLI method still hold, since nonnegative splittings are weak regular splittings. However, now the iteration matrices $T_{AMLI}$ and $T_{SMAMLI}$ are induced by nonnegative splittings, see Theorems 3.7 and 3.11 below.

**Theorem 3.7.** *Let Assumption 3.5 be satisfied. Then*

$$T_{AMLI} = I - C_{AMLI}A,$$

*and* $(C_{AMLI}^{-1}, C_{AMLI}^{-1} - A)$ *is a nonnegative splitting of* $A$.

**Proof:**
Since each nonnegative splitting is also a weak regular splitting of first type, we can use the results in [16, Theorem 4.3] to get, that

$$C_{AMLI} = \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix}$$

is nonsingular and $(C_{AMLI}^{-1}, C_{AMLI}^{-1} - A)$ is a weak regular splitting of first type of $A$, i.e.

$$\begin{aligned} C_{AMLI} &\geq 0, \\ I - C_{AMLI}A &\geq 0. \end{aligned}$$

To show that $(C_{AMLI}^{-1}, C_{AMLI}^{-1} - A)$ is also a nonnegative splitting of $A$ it is sufficient to check that $I - AC_{AMLI}$ is also nonnegative.
But

$$\begin{aligned} & I - AC_{AMLI} \\ = & \begin{bmatrix} I_F - A_{FF}\widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} \left(I_F - A_{FF}\widetilde{A}_{FF}^{-1}\right) A_{FC}\widetilde{S}^{-1}A_{CF}\widetilde{A}_{FF}^{-1} & 0 \\ -\left(I_C - (A/\widetilde{A}_{FF})\widetilde{S}^{-1}\right) A_{CF}\widetilde{A}_{FF}^{-1} & 0 \end{bmatrix} \\ & + \begin{bmatrix} 0 & -\left(I_F - A_{FF}\widetilde{A}_{FF}^{-1}\right) A_{FC}\widetilde{S}^{-1} \\ 0 & I_C - (A/\widetilde{A}_{FF})\widetilde{S}^{-1} \end{bmatrix} \end{aligned}$$

is also nonnegative due to the splitting properties of $\widetilde{A}_{FF}$ and $\widetilde{S}$ and the M-matrix property of $A$.
Thus, we get that $(C_{AMLI}^{-1}, C_{AMLI}^{-1} - A)$ is a nonnegative splitting of $A$.

□

For the MAMLI method we have the following result which follows directly from Theorem 3.4.

**Theorem 3.8.** *Let Assumption 3.6 be satisfied. Then*

$$T_{MAMLI} = I - C_{MAMLI}A,$$

*and* $(C_{MAMLI}^{-1}, C_{MAMLI}^{-1} - A)$ *is a weak regular splitting of first type of* $A$.

Observe, that although we used nonnegative splittings for the approximations, the MAMLI iteration matrix $T_{MAMLI}$ is not induced by a nonnegative splitting, i.e. the splitting $(C_{MAMLI}^{-1}, C_{MAMLI}^{-1} - A)$ of $A$ satisfying $T_{MAMLI} = I - C_{MAMLI}A$ is in general not a nonnegative splitting. This is demonstrated in the following example.

**Example 3.9.** *We consider the M-matrix*

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

*and take the upper* $2 \times 2$ *diagonal block as block* $A_{FF} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. *Moreover, we use the approximations*

$$\begin{aligned} \widetilde{A}_{FF} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ \widetilde{S} &= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

*Due to this choice we get the nonnegative splittings* $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ *and* $\left(\widetilde{S}, \widetilde{S} - (A/\widetilde{A_{FF}})\right)$.
*Using these approximations we get the matrices*

$$C_{MAMLI} = \frac{1}{12}\begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 8 & 4 & 0 \\ 2 & 4 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

*and*

$$I - C_{MAMLI}A = \frac{1}{12}\begin{bmatrix} 0 & 6 & 0 & 0 \\ 6 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 6 & 0 \end{bmatrix} \text{ and } I - AC_{MAMLI} = \frac{1}{12}\begin{bmatrix} 1 & 8 & 4 & 0 \\ 6 & 0 & 0 & 0 \\ -3 & 0 & 0 & 6 \\ 2 & 4 & 8 & 0 \end{bmatrix}.$$

*So the splitting* $\left(C_{MAMLI}^{-1}, C_{MAMLI}^{-1} - A\right)$ *is just weak regular of the first type but not nonnegative.*

In Theorem 3.7 the Assumption 3.5 is required. As mentioned above, Assumption 3.6 is weaker than Assumption 3.5. But the next example shows that Theorem 3.7 does not hold under Assumption 3.6.

**Example 3.10.** *We consider the following M-matrix*

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ -1 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}.$$

*The first two unknowns are chosen to be fine, i.e.* $A_{FF}^{(1)} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$. *As an approximation of* $A_{FF}^{(1)}$ *we take*

$$\widetilde{A}_{FF}^{(1)} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

*By this choice the splitting* $\left(\widetilde{A}_{FF}^{(1)}, \widetilde{A}_{FF}^{(1)} - A_{FF}^{(1)}\right)$ *is weak regular of first type and also nonnegative. Moreover, we get*

$$\left(A/\widetilde{A}_{FF}\right) = A_{CC} - A_{CF}\widetilde{A}_{FF}^{-1}A_{FC} = \begin{bmatrix} \frac{15}{4} & -1 \\ -1 & 4 \end{bmatrix},$$

$$\widetilde{R}A\widetilde{P}^T = \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I \end{bmatrix} A \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I \end{bmatrix} = \begin{bmatrix} \frac{59}{16} & -1 \\ -1 & 4 \end{bmatrix}.$$

*As an approximation* $\widetilde{S}$ *we choose* $\widetilde{S} = \widetilde{R}A\widetilde{P}^T$.
*So the splitting* $\left(\widetilde{S}, \widetilde{S} - \widetilde{R}A\widetilde{P}^T\right)$ *is nonnegative and weak regular of first type, but the splitting* $\left(\widetilde{S}, \widetilde{S} - \left(A/\widetilde{A}_{FF}\right)\right)$ *is neither nonnegative nor weak regular of first type.*
*Hence, Assumption 3.5 is fulfilled but not Assumption 3.6. Moreover, we obtain*

$$T_{AMLI} = \frac{1}{220}\begin{bmatrix} 0 & 55 & 0 & 0 \\ 59 & 4 & -1 & 0 \\ 16 & 16 & -4 & 0 \\ 4 & 4 & -1 & 0 \end{bmatrix}.$$

*Thus, Theorem 3.7 does not hold using Assumption 3.6.*

Next we consider the SMAMLI method.

**Theorem 3.11.** *Let Assumption 3.6 be satisfied. Then*

$$T_{SMAMLI} = I - C_{SMAMLI}A,$$

*and* $(C_{SMAMLI}^{-1}, C_{SMAMLI}^{-1} - A)$ *is a nonnegative splitting of* $A$.

**Proof:**
Due to the definition of the SMAMLI-method it is clear that

$$\begin{aligned} T_{SMAMLI} &= \left(I - \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} A\right)(I - C_{MAMLI}A) \\ &= I - \left(\begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \left(I - \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} A\right) C_{MAMLI}\right) A \\ &= I - C_{SMAMLI}A \end{aligned}$$

with

$$C_{SMAMLI} := M_S + (I - M_S A)\, C_{MAMLI}, \tag{9}$$

$$M_S := \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

A simple computation leads to

$$C_{SMAMLI} = \begin{bmatrix} \widetilde{B}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\widetilde{B}_{FF}^{-1} A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{B}_{FF}^{-1} & I_C \end{bmatrix} \tag{11}$$

with $\widetilde{B}_{FF}^{-1} = 2\widetilde{A}_{FF}^{-1} - \widetilde{A}_{FF}^{-1} A_{FF} \widetilde{A}_{FF}^{-1}$.
Since each nonnegative splitting is also a weak regular splitting of first type we get with [16, Lemma 5.1] that $\left(\widetilde{B}_{FF}, \widetilde{B}_{FF} - A_{FF}\right)$ is a weak regular splitting of first type. Furthermore

$$\begin{aligned} I - A_{FF}\widetilde{B}_{FF}^{-1} &= I - 2A_{FF}\widetilde{A}_{FF}^{-1} - A_{FF}\widetilde{A}_{FF}^{-1} A_{FF} \widetilde{A}_{FF}^{-1} \\ &= \left(I - A_{FF}\widetilde{A}_{FF}^{-1}\right)\left(I - A_{FF}\widetilde{A}_{FF}^{-1}\right) \\ &\geq 0\ . \end{aligned}$$

So $\left(\widetilde{B}_{FF}, \widetilde{B}_{FF} - A_{FF}\right)$ is also a nonnegative splitting.
Since $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ and $\left(\widetilde{S}, \widetilde{S} - (A/\widetilde{A}_{FF})\right)$ are nonnegative splittings and $A$ is an M-matrix we get

$$\begin{aligned} & I - \widetilde{S}^{-1}(A/\widetilde{B}_{FF}) \\ = \ & I - \widetilde{S}^{-1}\left(A_{CC} - A_{CF}\widetilde{B}_{FF}^{-1} A_{FC}\right) \\ = \ & \underbrace{\left(I - \widetilde{S}^{-1}\left(A_{CC} - A_{CF}\widetilde{A}_{FF}^{-1} A_{FC}\right)\right)}_{\geq 0} + \underbrace{A_{CF}}_{\leq 0} \underbrace{\left(I - \widetilde{A}_{FF}^{-1} A_{FF}\right)}_{\geq 0} \underbrace{\widetilde{A}_{FF}^{-1}}_{\geq 0} \underbrace{A_{FC}}_{\leq 0} \\ \geq \ & 0. \end{aligned}$$

Similarly we get

$$I - (A/\widetilde{B}_{FF})\widetilde{S}^{-1} \geq 0.$$

Thus $\left(\widetilde{S}, \widetilde{S} - (A/\widetilde{B}_{FF})\right)$ is also a nonnegative splitting.
Therefore $C_{SMAMLI}$ has the same structure as $C_{AMLI}$ , with the only difference being that $\widetilde{A}_{FF}$ is replaced by $\widetilde{B}_{FF}$. So we can use Theorem 3.7 to get that $(C_{SMAMLI}^{-1}, C_{SMAMLI}^{-1} - A)$ is a nonnegative splitting of $A$. $\square$

# 4 Comparison of AMLI, MAMLI and SMAMLI

In this section we will compare the convergence rates of the AMLI and MAMLI methods. We will prove that the spectral radius of the iteration matrix of the MAMLI method is always less than or equal to the spectral radius of the iteration matrix of the AMLI method.
As mentioned above, the iteration matrices are induced by splittings. The matrices $C_{AMLI}$, $C_{MAMLI}$ and $C_{SMAMLI}$ can be given explicitely, as seen in the proofs of Theorem 3.7 and 3.11 and in Theorem 4.7 of [16]. We have

$$
\begin{aligned}
C_{AMLI} &= M_S + M_{CG}\ , &\text{(12a)}\\
C_{MAMLI} &= M_S + M_{CG} - M_{CG} A M_S\ , &\text{(12b)}\\
C_{SMAMLI} &= M_S + (I - M_S A)\, C_{MAMLI} &\text{(12c)}
\end{aligned}
$$

where $M_S$ as in (10) and

$$
M_{CG} \;=\; \begin{bmatrix} -\widetilde{A}_{FF}^{-1} A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix}.
$$

Moreover, $C_{AMLI}$ and $C_{SMAMLI}$ are given explicitly in (9) and (11). For $C_{MAMLI}$ we obtain

$$
C_{MAMLI} \;=\; \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\widetilde{A}_{FF}^{-1} A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{B}_{FF}^{-1} & I_C \end{bmatrix}\ ,
$$

with $\widetilde{B}_{FF}^{-1} = 2\widetilde{A}_{FF}^{-1} - \widetilde{A}_{FF}^{-1} A_{FF} \widetilde{A}_{FF}^{-1}$.
These explicit formulas of the matrices, allows us to prove a comparison theorem for the AMLI, the MAMLI and the SMAMLI method.

**Theorem 4.1.** *Consider the AMLI, MAMLI and SMAMLI method and let Assumption 3.5 hold. Then*

$$
\rho(T_{SMAMLI}) \le \rho(T_{MAMLI}) \le \rho(T_{AMLI}).
$$

**Proof:**
Since $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ is a nonnegative splitting we obtain for $\widetilde{B}_{FF}^{-1} = 2\widetilde{A}_{FF}^{-1} - \widetilde{A}_{FF}^{-1} A_{FF} \widetilde{A}_{FF}^{-1}$,

$$
\widetilde{B}_{FF}^{-1} - \widetilde{A}_{FF}^{-1} = \widetilde{A}_{FF}^{-1} - \widetilde{A}_{FF}^{-1} A_{FF} \widetilde{A}_{FF}^{-1} = \underbrace{\left(I - \widetilde{A}_{FF}^{-1} A_{FF}\right)}_{\ge 0} \underbrace{\widetilde{A}_{FF}^{-1}}_{\ge 0} \ge 0.
$$

Furthermore, due to the fact that $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ and $\left(\widetilde{S}, \widetilde{S} - (A/\widetilde{A}_{FF})\right)$ are nonnegative splittings and $A$ is an M-matrix we get

$$
\begin{aligned}
& C_{MAMLI} - C_{AMLI} \\
= & \begin{bmatrix} -\underbrace{\widetilde{A}_{FF}^{-1}}_{\ge 0}\underbrace{A_{FC}}_{\le 0} \\ \underbrace{I_C}_{\ge 0} \end{bmatrix} \underbrace{\widetilde{S}^{-1}}_{\ge 0} \begin{bmatrix} -\underbrace{A_{CF}}_{\ge 0}\underbrace{\left(\widetilde{B}_{FF}^{-1} - \widetilde{A}_{FF}^{-1}\right)}_{\ge 0} & \underbrace{I_C}_{\ge 0} \end{bmatrix} \\
\ge & \;\; 0.
\end{aligned}
$$

Similarly

$$
C_{SMAMLI} - C_{MAMLI} \ge 0\ .
$$

With Theorem 3.7 and Theorem 3.11 we have that $(C_{AMLI}^{-1}, C_{AMLI}^{-1} - A)$ and $(C_{SMAMLI}^{-1}, C_{SMAMLI}^{-1} - A)$ are nonnegative splittings. So these splittings are also weak regular of second type. Furthermore due to Theorem 3.4

$$
(C_{MAMLI}^{-1}, C_{MAMLI}^{-1} - A)
$$

is a weak regular splitting of first type. Hence, by using Theorem 2.3, we obtain

$$\rho(T_{SMAMLI}) \leq \rho(T_{MAMLI}) \leq \rho(T_{AMLI}).$$

$\square$

Thus, the SMAMLI method has the smallest spectral radius among these methods, if the spectral radii are not the same. But this is not surprising, since the SMAMLI method uses a pre and a post smoothing step in contrast to the MAMLI method. However, the spectral radius of the MAMLI iteration matrix is less than or equal to that of the AMLI method. But both methods use about the same amount of work. Hence, beside some parallel implementation aspects, the MAMLI method seems to outperform the AMLI method. Note, that Theorem 4.1 is also one of the few Theorems that clearly states that a multiplicative combination of subspace corrections do not converges slower (in general faster) than an additive combination, see also [18, 13].
In Section 7, this theoritical result will be illustrated by numerical examples.

# 5 Quality of the approximations

Next we establish, how the asymptotic convergence rate behave if the quality of the approximations varies. To do so, we consider different approximations $\widehat{A}_{FF}$ and $\widehat{S}$ of $A_{FF}$ and the Schur complement $S$.
With these approximations the new iteration matrices $\widehat{T}_{AMLI}$ and $\widehat{T}_{MAMLI}$ are build as in (3) and (6) but using $\widehat{A}_{FF}$ and $\widehat{S}$ instead of $\widetilde{A}_{FF}$ and $\widetilde{S}$.

**Theorem 5.1.** *Let Assumption 3.5 be satisfied. Let $\widehat{A}_{FF}$ and $\widehat{S}$ be approximations of $A_{FF}$ and $(A/A_{FF})$ such that*

$$\widetilde{A}_{FF} \leq \widehat{A}_{FF}. \tag{13}$$

$$\widetilde{S} \leq \widehat{S}. \tag{14}$$

*Moreover, let $\left(\widehat{A}_{FF}, \widehat{A}_{FF} - A_{FF}\right)$ and $(\widehat{S}, \widehat{S} - (A/\widehat{A}_{FF}))$ be nonnegative splittings.*
*Let $\widehat{T}_{AMLI}$ be constructed as $T_{AMLI}$ in (3) by using $\widehat{A}_{FF}$ and $\widehat{S}$. Then*

$$\rho(T_{AMLI}) \leq \rho(\widehat{T}_{AMLI}).$$

**Proof:** Since $\widehat{A}_{FF}$ and $\widehat{S}$ induce nonnegative splittings, we immediately obtain by (13) and (14) that

$$\begin{aligned}
\widetilde{A}_{FF}^{-1} - \widehat{A}_{FF}^{-1} &= \underbrace{\widehat{A}_{FF}^{-1}}_{\geq 0} \underbrace{\left(\widehat{A}_{FF} - \widetilde{A}_{FF}\right)}_{\geq 0} \underbrace{\widetilde{A}_{FF}^{-1}}_{\geq 0} \geq 0 \ , \\
\widetilde{S}^{-1} - \widehat{S}^{-1} &= \underbrace{\widetilde{S}^{-1}}_{\geq 0} \underbrace{\left(\widehat{S} - \widetilde{S}\right)}_{\geq 0} \underbrace{\widehat{S}^{-1}}_{\geq 0} \geq 0 \ .
\end{aligned}$$

Moreover $\widehat{A}_{FF}$ and $\widehat{S}$ fulfill the conditions of Assumption 3.5 with $\widetilde{A}_{FF}$ replaced by $\widehat{A}_{FF}$.

Hence, by Theorem 3.7, $\widehat{T}_{AMLI}$ is induced by a nonnegative splitting $(\widehat{C}_{AMLI}^{-1}, \widehat{C}_{AMLI}^{-1} - A)$. Moreover,

$$\widehat{C}_{AMLI}^{-1} = \widehat{M}_S + \widehat{M}_{CG} \tag{15}$$

where

$$\begin{aligned} \widehat{M}_S &= \begin{bmatrix} \widehat{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\ \widehat{M}_{CG} &= \begin{bmatrix} -\widehat{A}_{FF}^{-1} A_{FC} \\ I_C \end{bmatrix} \widehat{S}^{-1} \begin{bmatrix} -A_{CF}\widehat{A}_{FF}^{-1} & I_C \end{bmatrix}. \end{aligned}$$

Thus, since $A$ is an $M$-matrix, we obtain by comparing $C_{AMLI}^{-1}$ in (12a) and $\widehat{C}_{AMLI}^{-1}$ in (15) that

$$C_{AMLI}^{-1} \geq \widehat{C}_{AMLI}^{-1}.$$

Hence, by Theorem 2.3,

$$\rho(T_{AMLI}) \leq \rho(\widehat{T}_{AMLI}).$$

□

In Theorem 5.1 it was proved that, the better the approximations are, the smaller are the spectral radii of the iteration matrices. Note that Theorem 5.1 gives a direct comparison of the spectral radii, not only a comparison of bounds for the spectral radii.

If one only wants to modify the approximation of the Schur complement, the statement of Theorem 5.1 can be expressed in the following way. Let $S_1$ and $S_2$ be two approximations of $\left(A/\widetilde{A}_{FF}\right)$. Now denote by $T_{AMLI}(S_i)$ the AMLI iteration matrices as in (3), using the approximation $S_i$, $i = 1, 2$.

**Corollary 5.2.** *Let Assumption 3.5 be satisfied. Let $S_1$ and $S_2$ be approximations of $(A/A_{FF})$ such that*

$$\widetilde{S} \leq S_1 \leq S_2. \tag{16}$$

*Moreover, let $(S_2, S_2 - (A/\widetilde{A}_{FF}))$ and $(S_2, S_2 - (A/\widetilde{A}_{FF}))$ be nonnegative splittings. Then*

$$\rho(T_{AMLI}) \leq \rho(T_{AMLI}(S_1)) \leq \rho(T_{AMLI}(S_2)).$$

Next we consider the MAMLI method. As mentioned above, Elsner showed in [10] that there is no comparison theorem for weak regular splittings similar to Theorems 2.2 and 2.3. Since the MAMLI method is induced by a weak regular spitting of first type, comparison theorems like for the AMLI and SMAMLI method (see below) can not be proved in that way. But we are able to establish the following results.

**Theorem 5.3.** *Let Assumption 3.6 be satisfied. Let $\widehat{A}_{FF}$ and $\widehat{S}$ be other approximations of $A_{FF}$ and $\widetilde{R}A\widetilde{P}^T$ such that*

$$\begin{aligned} \widetilde{A}_{FF} &\leq \widehat{A}_{FF}, & (17) \\ \widetilde{S} &\leq \widehat{S}. & (18) \end{aligned}$$

*Moreover, let* $\left(\widehat{A}_{FF}, \widehat{A}_{FF} - A_{FF}\right)$ *and* $(\widehat{S}, \widehat{S} - \widetilde{R}A\widetilde{P}^T)$ *be nonnegative splittings. Let* $\widehat{T}_{MAMLI}$ *be constructed as* $T_{MAMLI}$ *in* (6) *by using the approximations* $\widehat{A}_{FF}$ *and* $\widehat{S}$*. Then*

$$\|T_{MAMLI}\|_w \leq \|\widehat{T}_{MAMLI}\|_w$$

*with* $w = A^{-1}e$ *for a positive vector* $e$*.*

**Proof:** Using the same arguments as in the proof of Theorem 5.1 we obtain that $\widehat{A}_{FF}$ and $\widehat{S}$ fulfill the conditions of Assumption 3.6. Hence, by Theorem 3.4, $\widehat{T}_{MAMLI}$ is induced by a weak regular splitting of first type.
In order to use Theorem 2.5 we need to compare the induced splittings of $T_{MAMLI}$ and $\widehat{T}_{MAMLI}$. In detail, we want to compare $C_{MAMLI}$ and $\widehat{C}_{MAMLI}$, where

$$\widehat{C}_{MAMLI} = \widehat{M}_{CG} + \widehat{M}_S - \widehat{M}_{CG}A\widehat{M}_S,$$

and

$$\begin{aligned}
\widehat{M}_S &= \begin{bmatrix} \widehat{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\
\widehat{M}_{CG} &= \begin{bmatrix} -\widehat{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix}, \widehat{S}^{-1} \begin{bmatrix} -A_{CF}\widehat{A}_{FF}^{-1} & I_C \end{bmatrix},
\end{aligned}$$

and

$$-\widehat{M}_{CG}A\widehat{M}_S = \begin{bmatrix} \widehat{A}_{FF}^{-1}A_{FC}\widehat{S}^{-1}A_{CF}\left(I_F - \widehat{A}_{FF}^{-1}A_{FF}\right)\widehat{A}_{FF}^{-1} & 0 \\ -\widehat{S}^{-1}A_{CF}\left(I_F - \widehat{A}_{FF}^{-1}A_{FF}\right)\widehat{A}_{FF}^{-1} & 0 \end{bmatrix}. \tag{19}$$

It was established in the proof of Theorem 5.1 that

$$M_{CG} + M_S \geq \widehat{M}_{CG} + \widehat{M}_S.$$

By using the M-matrix properties of $A$, using (17) and (18), and using the structure (19) we get that

$$-M_{CG}AM_S \geq -\widehat{M}_{CG}A\widehat{M}_S.$$

Hence,

$$C_{MAMLI} \geq \widehat{C}_{MAMLI}.$$

Therefore, by Theorem 2.5 we get the desired result.

□

As for the AMLI method we obtain the following corollary that measures the quality of the approximation of the Garlerkin product. Therefore, let $S_1$ and $S_2$ be two approximations of $\widetilde{R}A\widetilde{P}^T$ . By $T_{MAMLI}(S_i)$ and $T_{SMAMLI}(S_i)$ we then denote the MAMLI and SMAMLI iteration matrices as in (6) and (7) using the approximation $S_i$, $i = 1, 2$.

**Corollary 5.4.** *Let Assumption 3.6 be satisfied. Let $S_1$ and $S_2$ be approximations of $\widetilde{R}A\widetilde{P}^T$ such that*

$$\widetilde{S} \leq S_1 \leq S_2, \tag{20}$$

*and $(S_1, S_1 - \widetilde{R}A\widetilde{P}^T)$ and $(S_2, S_2 - \widetilde{R}A\widetilde{P}^T)$ are nonnegative splittings. Then*

$$\|T_{MAMLI}\|_w \leq \|\bar{T}_{MAMLI}(S_1)\|_w \leq \|\bar{T}_{MAMLI}(S_2)\|_w.$$

The Corollaries 5.2 and 5.4 provide some relevant information for the practical use of AMG methods. Having a good approximation of the block $A_{FF}$, the approximation of the Schur complement $S$ or the Galerkin product is somewhat more difficult. Such an approximation should not be to far away from the original but it should be also sparse. The above Theorems or Corollaries indicate, how the spectral radii or the bounds for them depend on the quality of such approximations.
Moreover, theoretical statements can now be made between AMG methods that use approximations of the Schur complement

$$\left(A/\widetilde{A}_{FF}\right)$$

or approximations of the Galerkin product

$$\widetilde{R}A\widetilde{P}^T = \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I \end{bmatrix} A \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I \end{bmatrix}.$$

Let

$$\begin{aligned} S_1 &:= A_{CC}, \\ S_2 &:= \left(A/\widetilde{A}_{FF}\right), \\ S_3 &:= \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I \end{bmatrix} A \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I \end{bmatrix}. \end{aligned}$$

It is shown in [16] that $S_1, S_2$ and $S_3$ are nonsingular $M$-matrices and that

$$A_{CC} \geq \left(A/\widetilde{A}_{FF}\right) \geq \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I \end{bmatrix} A \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I \end{bmatrix} \geq (A/A_{FF}) \tag{21}$$

holds. Due to these inequalities the approximations $S_1$ and $S_2$ fulfill the conditions of Assumption 3.5 and the approximations $S_1$, $S_2$ and $S_3$ fulfill the conditions of Assumption 3.6.
Hence, we obtain the following comparison.

**Corollary 5.5.** *Let Assumption 3.5 be satisfied. Then*

$$\rho(T_{AMLI}(S_2)) \leq \rho(T_{AMLI}(S_1)).$$

**Corollary 5.6.** *Let Assumption 3.6 be satisfied. Then*

$$\|T_{MAMLI}(S_3)\|_w \leq \|T_{MAMLI}(S_2)\|_w \leq \|T_{MAMLI}(S_1)\|_w$$

*with $w = A^{-1}e$ for a positive vector $e$.*

Hence, the Galerkin product approach yields the smallest bound for the spectral radius. Therefore, we expect that the Galerkin type MAMLI method converges faster than the other methods.
At the end of this section we return to the SMAMLI method. It is now not hard to see, that we can get the same comparison theorems for the SMAMLI method as we proved for the AMLI method. Both methods are induced by nonnegative splittings. We mention these comparison results but we omit the proofs here.

**Theorem 5.7.** *Let Assumption 3.6 be satisfied. Let $\widehat{A}_{FF}$ and $\widehat{S}$ be other approximation of $A_{FF}$ and $S$ such that $\widehat{A}_{FF}$ and $\widehat{S}$ are $M$-matrices and such that*

$$\begin{aligned} \widetilde{A}_{FF} &\leq \widehat{A}_{FF}, \\ \widetilde{S} &\leq \widehat{S}. \end{aligned}$$

*Moreover, let $(\widehat{S}, \widehat{S} - \widetilde{R}A\widetilde{P}^T)$ be a nonnegative splitting of $\widetilde{R}A\widetilde{P}^T$.*
*Let $\widehat{T}_{SMAMLI}$ be constructed as $T_{SMAMLI}$ in (7) by using the approximations $\widehat{A}_{FF}$ and $\widehat{S}$. Then*

$$\rho(T_{SMAMLI}) \leq \rho(\widehat{T}_{SMAMLI}).$$

**Corollary 5.8.** *Let Assumption 3.6 be satisfied. Let $S_1$ and $S_2$ be approximations of $\widetilde{R}A\widetilde{P}^T$ such that*

$$\widetilde{S} \leq S_1 \leq S_2, \tag{22}$$

*and $(S_1, S_1 - \widetilde{R}A\widetilde{P}^T)$ and $(S_2, S_2 - \widetilde{R}A\widetilde{P}^T)$ are nonnegative splittings. Then*

$$\rho(T_{SMAMLI}) \leq \rho(T_{SMAMLI}(S_1)) \leq \rho(T_{SMAMLI}(S_2)).$$

Note, that the work per iteration of the SMAMLI method is larger than the work for the MAMLI and AMLI methods.
As mentioned in Section 2, strict inequality comparison results can be proved for the algebraic multigrid methods. Therefore we have to use comparison theorems like Theorem 2.4. But to fulfill the assumptions of this theorem we need to assume restrictive properties of the approximations $\widetilde{A}$ and $\widetilde{S}$. This can be seen by following the proofs of our comparison theorems.

# 6 Comparison with classical iterative methods

In this section we compare the multigrid methods with classical iterative methods like the block Jacobi and block Gauss-Seidel method.
Starting with the block partitioning (2), the exact block Jacobi and block Gauss-Seidel methods (see e.g. [24]) are given by the following iteration matrices

$$\begin{aligned} T_{BJ} &= I - C_{BJ}A, \\ T_{BGS} &= I - C_{BGS}A, \end{aligned}$$

where

$$
\begin{aligned}
C_{BJ} &= \begin{bmatrix} A_{FF}^{-1} & 0 \\ 0 & A_{CC}^{-1} \end{bmatrix}, \\
C_{MS} &= \begin{bmatrix} A_{FF}^{-1} & 0 \\ A_{CC}^{-1} A_{CF} A_{FF}^{-1} & A_{CC}^{-1} \end{bmatrix}.
\end{aligned}
$$

By using nonsingular approximations $\widetilde{A}_{FF}$ for $A_{FF}$ and $\widetilde{S}$ for $A_{CC}$ we obtain the corresponding inexact methods and the iteration matrices

$$
\begin{aligned}
T_{AS} &= I - C_{AS} A, \\
T_{MS} &= I - C_{MS} A,
\end{aligned}
$$

where

$$
\begin{aligned}
C_{AS} &= \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & \widetilde{S}^{-1} \end{bmatrix}, \\
C_{MS} &= \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ \widetilde{S}^{-1} A_{CF} \widetilde{A}_{FF}^{-1} & \widetilde{S}^{-1} \end{bmatrix}.
\end{aligned}
$$

We easily obtain

$$
\begin{aligned}
T_{AS} = I \; & - \begin{bmatrix} 0 \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} 0 & I_C \end{bmatrix} A \\
& - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A,
\end{aligned}
$$

and

$$
\begin{aligned}
T_{MS} &= \left( I - \begin{bmatrix} 0 \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} 0 & I_C \end{bmatrix} \right) \\
& \quad \cdot \left( I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A \right),
\end{aligned}
$$

Note, that $T_{AS}$ and $T_{MS}$ can be seen as abstract inexact non-overlapping additive and multiplicative Schwarz methods for two domains (see eq. [23, 5]).

**Theorem 6.1.** *Let $A$ be a nonsingular $M$-matrix partitioned as in (2). Let $\widetilde{A}_{FF}$ and $\widetilde{S}$ be approximations of $A_{FF}$ and $A_{CC}$ respectively, such that $\widetilde{A}_{FF}$ and $\widetilde{S}$ are $M$-matrices with*

$$
\widetilde{A}_{FF} \geq A_{FF}, \tag{23}
$$

$$
\widetilde{S} \geq A_{CC}. \tag{24}
$$

*Then*

$$
\begin{aligned}
\rho(T_{AMLI}) &\leq \rho(T_{AS}), \\
\rho(T_{MAMLI}) &\leq \rho(T_{MS}).
\end{aligned}
$$

**Proof:** It was shown in [5] that if $\widetilde{A}_{FF}$ and $\widetilde{S}$ are $M$-matrices and (23) and (24) hold then $T_{AS}$ and $T_{MS}$ are induced by nonnegative splittings, so these splittings are also weak regular of second type
But with these approximations, Assumptions 3.1 and 3.2 are fulfilled. To see this, one has to use (21). Thus, by Theorems 3.3 and 3.4 the AMLI and the MAMLI methods using the above approximations, are induced by at least weak regular splittings of first type.
In order to get the desired result, we will use Theorem 2.3 and the explicit form of $C_{AMLI}$ and $C_{MAMLI}$ given in (12).
It was shown in the proof of Theorem 4.1 that $-M_{CG}AM_S \geq 0$.
Hence, using the $M$-matrix property and (23) and (24) we easily obtain that

$$\begin{aligned} C_{AMLI} &\geq C_{AS}, \\ C_{MAMLI} &\geq C_{MS}. \end{aligned}$$

□

Theorem 6.1 clearly states, that a multigrid grid approach gives in general better asymptotic convergence rates than just using the classical methods.

# 7 Numerical examples

As a numerical example we consider the problem of calculating the Google$^{TM}$'s PageRank$^{TM}$. Calculating the PageRank can be explained in the following way. The PageRank is determined entirely by the link structure of the World Wide Web. Consider surfing the Web, i.e. going from page to page by randomly choosing an outgoing link from one page to get to the next page. However, this can lead to dead ends at pages with no outgoing links, or cycles around cliques of interconnected pages. So, with a certain probability one assume that a surfer choose a random page from the Web. This model results in a theoretical random walk that is a Markov chain or Markov process. The state vector $x$ of this Markov chain is then used for a ranking of the visited webpage. For more details see e.g. [14, 17].
In detail, the PageRank can then be calculated in the following manner.
Let $G$ be the $n \times n$ connectivity matrix of a portion of the Web, that is $g_{ij} = 1$ if there is a hyperlink to page $i$ from page $j$ and zero otherwise. Then the state vector $x$ containing the PageRank is the solution of the linear system

$$(I - pGD)x = \delta e \ , \tag{25}$$

where $p$ is the probability that the random walk follows a link, $D$ the diagonal matrix with $d_{jj} = \sum_i g_{ij}$, $e$ is a vector with all ones, and $\delta$ is the probability of jumping from one page to another without following a link, i.e. $\delta = (1-p)/n$. One can see that as long as $p$ is strictly less than one, the coefficient matrix $I - pGD$ is a nonsingular M-matrix.
In this paper we used to set up the linear system (25) the algorithm described in [17, Chapter 2]. We started the random walk on one of the authors homepage[1] and restricted us to 1024 webpages. The probability $p$ following a link was set to 0.85.

[1] www.math.tu-berlin.de/~mense

To solve the resulting linear system we used the described AMLI, MAMLI and SMAMLI-methods, using different approximations and coarse grid matrices. There are several algorithms known to define the fine and coarse unknowns [20, ?, ?, ?, ?]. Here, we use a coarsing algorithm that simply uses the strongest neighborhood coupling. The coarser is given in Algorithm 1. The zero vector was used as a starting vector and we stopped the iteration if the residuum was smaller than $10^{-6}$.

$(\mathbf{F}, \mathbf{U}) \leftarrow$ **SimpleCoarsing(A)**
**begin**
  $F = \emptyset$, $C = \emptyset$, $U = \{1, \ldots, n\}$
  **while** $U \neq \emptyset$ **do**
    $(i, j) \leftarrow \arg \max_{i,j \in U, i \neq j} |a_{ij}|$
    $F \leftarrow F \cup \{i\}$
    $C \leftarrow C \cup \{j\}$
    $U \leftarrow U \setminus \{i, j\}$
  **end**
**end**

**Algorithm 1**: Simple Coarsing

For the results given in Table 1 we used as approximations $\widetilde{A}_{FF}$ and $\widetilde{S}$ the diagonal of $A_{FF}$ and $A_{CC}$, respectively. The numerical results confirm the theoretical comparison of the different methods. The spectral radius of the MAMLI method is between the radii of the AMLI and the SMAMLI methods. Note that the time per iteration of the AMLI and MAMLI method is about the same.

| | AMLI | MAMLI | SMAMLI |
|---|---|---|---|
| spectral radius | 0.5348 | 0.5013 | 0.4803 |
| iterations | 27 | 24 | 23 |
| iteration time | 1.9806 | 1.7619 | 1.7044 |
| time per iteration step | 0.0734 | 0.0734 | 0.0741 |

Table 1: Comparison of the methods

For the experiments given in Table 2 we used different sets of approximations. The results of the first column related to each method are obtained by using $\widetilde{A}_{FF}$ and $\widetilde{S}$ as the diagonal of $A_{FF}$ and $\left(A/\widetilde{A}_{FF}\right)$, respectively. The data of each second column are obtained by using the lower triangular part of $A_{FF}$ and $\left(A/\widetilde{A}_{FF}\right)$, respectively. The numerical results confirm the theoretical results given in first part of Section 5. Better approximations lead to better spectral radii for each method.

| | AMLI | | MAMLI | | SMAMLI | |
|---|---|---|---|---|---|---|
| spectral radius | 0.5284 | 0.3308 | 0.4932 | 0.3044 | 0.4702 | 0.2998 |
| iterations | 26 | 16 | 24 | 15 | 22 | 14 |
| iteration time | 1.9102 | 1.2829 | 1.7619 | 1.2074 | 1.6403 | 1.1494 |
| time per iter. step | 0.0735 | 0.0802 | 0.0734 | 0.0805 | 0.0746 | 0.0821 |

Table 2: Comparison of the quality of the approximation

In Table 3 we compare approximations of the Schur complement with approximations of the Galerkin matrix. We used as approximations of $\widetilde{A}_{FF}$ and $\widetilde{S}$ the lower triangular part of $A_{FF}$ and $\left(A/\widetilde{A}_{FF}\right)$, respectively. The results are given in the first column of each method. Each second column gives the data obtained by using the lower triangular part of $\widetilde{R}A\widetilde{P}^T$. Again, the theoretical results are confirmed.

| | MAMLI | | SMAMLI | |
|---|---|---|---|---|
| spectral radius | 0.3044 | 0.2711 | 0.2998 | 0.2640 |
| iterations | 15 | 14 | 14 | 13 |
| iteration time | 1.2074 | 1.1289 | 1.1494 | 1.0690 |
| time per iter. step | 0.0805 | 0.0806 | 0.0821 | 0.0822 |

Table 3: Comparison of the approximated Schur complement with approximated Galerkin matrix

# 8 Conclusion

We established theoretical comparison results for algebraic multi-level methods applied to non-symmetric M-matrices. These comparisons show, that the spectral radius of the MAMLI method is less than or equal to the spectral radius of the AMLI method. Moreover, we established how the quality of the approximations within the AMG methods effects the spectral radii or bounds for spectral radii of the iteration matrices. We proved comparisons results for different approximation of the fine grid block as well as for the used Schur complement or Galerkin type approximations. We also established a theoretical comparison between the AMG methods and classical Schwarz iterations.

# Aknowledgement

We would like to thank the referee for helpful comments.

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