POD A-POSTERIORI ERROR ESTIMATES FOR LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS

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Abstract. The main focus of this paper is on an a-posteriori analysis for the method of proper orthogonal decomposition (POD) applied to optimal control problems governed by parabolic and elliptic PDEs. Based on a perturbation method it is deduced how far the suboptimal control, computed on the basis of the POD model, is from the (unknown) exact one. Numerical examples illustrate the realization of the proposed approach for linear-quadratic problems governed by parabolic and elliptic partial differential equations.

1 Introduction

Optimal control problems for partial differential equation are often hard to tackle numerically because their discretization leads to very large scale optimization problems. Therefore, different techniques of model reduction were developed to approximate these problems by smaller ones that are tractable with less effort. Among them, the method of proper orthogonal decomposition (POD) and the balanced truncation method seem to be most widely used.

Recently, both approaches have received increasing attention; we refer, e.g., to [2, 3, 10, 20, 23] for proper orthogonal decomposition and to [4, 18, 25, 29] for balanced truncation.

Proper orthogonal decomposition is based on projecting the dynamical system onto subspaces of basis elements that express characteristics of the expected solution. This is in contrast to, e.g., finite element techniques, where the elements are not correlated to the physical properties of the system they approximate.

In our present work, POD is applied to linear-quadratic optimal control problems. Linear-quadratic problems are interesting in several respects; in particular, they occur in each level of sequential quadratic programming (SQP) methods; see, e.g., [22].

In contrast to methods of balanced truncation type, the POD method is somehow lacking a reliable a-priori error analysis. Unless its snapshots are generating a sufficiently rich state space, it is not a-priorily clear how far the optimal solution of the POD problem is from the exact one. On the other hand, the POD method is a universal tool that is applicable also to problems with time-dependent coefficients or to nonlinear equations. Moreover, by generating snapshots from the real (large)

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model, a space is constructed that inhibits the main and relevant physical properties of the state system. This, and its ease of use makes POD very competitive in practical use, despite of a certain heuristic flavor.

In this paper, we again address the problem of error analysis. Our main focus is on an a-posteriori analysis. We use a fairly standard perturbation method to deduce how far the suboptimal control, computed on the basis of the POD model, is from the (unknown) exact one. This idea turned out to be very efficient in our examples. It is able to compensate for the lack of a priori analysis for POD methods. We also briefly discuss a priori error estimates. This analysis needs certain strong assumptions. Nevertheless, we include these results to show that there is a real chance to decrease the error up to zero by taking more snapshots, provided the assumptions are fulfilled.

In contrast to [15] the POD basis will be fixed during the numerical algorithm. Only the number of the utilized POD ansatz functions is increased, if necessary.

The paper is organized as follows: In Section 2, we introduce the linear-quadratic optimal control problem of parabolic type and review first-order necessary optimality conditions. The a-posteriori error analysis is carried out in Section 3, and the POD method is explained in Section 4. Moreover, an associated convergence analysis is carried out there. In Section 5, numerical test examples are presented.

2 The linear-quadratic parabolic optimal control problem

In this section, we introduce a class of linear-quadratic parabolic optimal control problems and recall the associated first-order necessary optimality conditions.

2.1 Problem formulation. Let V and H be real, separable Hilbert spaces and suppose that V is dense in H with compact embedding. By $\langle \cdot\,,\cdot\rangle_H$ we denote the inner product in H. The inner product in V is given by a symmetric bounded, coercive, bilinear form $a : V \times V \to \mathbb{R}$:

$$\langle \varphi, \psi\rangle_V = a(\varphi,\psi) \quad \text{for all } \varphi,\psi \in V \tag{2.1}$$

with associated norm $\|\cdot\|_V = \sqrt{a(\cdot\,,\cdot)}$. By identifying H and its dual H' it follows that $V \hookrightarrow H = H' \hookrightarrow V'$, each embedding being continuous and dense.

Recall that for $T > 0$ the space $W(0,T)$

$$W(0,T) = \big\{\varphi \in L^2(0,T;V) : \varphi_t \in L^2(0,T;V')\big\}$$

is a Hilbert space endowed with the common inner product (see, for example, [6, p. 473]). It is well-known that $W(0,T)$ is continuously embedded into $C([0,T];H)$, the space of continuous functions from $[0,T]$ to H.

Let $\mathcal{I}$ be an open and bounded subset of $\mathbb{R}^d$ with $d \in \mathbb{N}$. By $U_{\mathsf{ad}} \subset L^2(\mathcal{I})$ we define the closed, convex and bounded subset

$$U_{\mathsf{ad}} = \big\{u \in L^2(\mathcal{I}) \,|\, u_a(s) \le u(s) \le u_b(s) \text{ for almost all (f.a.a.) } s \in \mathcal{I}\big\}$$

with $u_a, u_b \in L^2(\mathcal{I})$ satisfying $u_a \le u_b$ almost everywhere (a.e.) in $\mathcal{I}$. For $y_0 \in H$, $r \in L^2(0,T;V')$ and $u \in U_{\mathsf{ad}}$ we consider the linear evolution problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y(t),\varphi\rangle_H + a(y(t),\varphi) = \langle (r+\mathcal{B}u)(t),\varphi\rangle_{V',V} \quad \text{f.a.a. } t \in [0,T],\ \forall \varphi \in V, \tag{2.2a}$$

$$\langle y(0),\varphi\rangle_H = \langle y_0,\varphi\rangle_H \qquad \forall \varphi \in V, \tag{2.2b}$$

where $\mathcal{B} : L^2(\mathcal{I}) \to L^2(0,T;V')$ is a continuous, linear operator.

Example 2.1. Let us present an example for (2.2). Suppose that $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, is an open and bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$. The boundary is split into two measurable disjoint parts $\Gamma_C \subset \Gamma$ and $\Gamma_N = \Gamma \setminus \Gamma_C$. For $T > 0$ we set $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$, $\Sigma_C = (0,T) \times \Gamma_C$ and $\Sigma_N = (0,T) \times \Gamma_N$. Let $H = L^2(\Omega)$, $V = H^1(\Omega)$ and $\mathcal{I} = (0,T)$. Then, for given control $u \in L^2(0,T)$ we consider the linear heat equation

$$y_t(t,\mathbf{x}) - \Delta y(t,\mathbf{x}) + y(t,\mathbf{x}) = f(t,\mathbf{x}) \quad \text{f.a.a. } (t,\mathbf{x}) \in Q \tag{2.3a}$$

together with the inhomogeneous Neumann boundary condition

$$\begin{aligned} \frac{\partial y}{\partial n}(t,\mathbf{s}) &= u(t)b(t,\mathbf{s}) && \text{f.a.a. } (t,\mathbf{s}) \in \Sigma_C,\\ \frac{\partial y}{\partial n}(t,\mathbf{s}) &= g(t,\mathbf{x}) && \text{f.a.a. } (t,\mathbf{s}) \in \Sigma_N \end{aligned} \tag{2.3b}$$

and with the initial condition

$$y(0,\mathbf{x}) = y_0(\mathbf{x}) \quad \text{f.a.a. } \mathbf{x} \in \Omega, \tag{2.3c}$$

where $y_0 \in H$ is given. In (2.3) we suppose that $f \in L^2(0,T;V')$, $g \in L^2(\Sigma_C)$, $b \in L^\infty(0,T;L^2(\Gamma_C))$, and $y_0 \in H$. Introducing the bilinear form $a : V \times V \to \mathbb{R}$ by

$$a(\varphi,\psi) = \int_\Omega \nabla\varphi \cdot \nabla\psi + \varphi\psi \,\mathrm{d}\mathbf{x} \quad \text{for } \varphi,\psi \in V,$$

the linear, bounded functional $r \in L^2(0,T;V')$ by

$$\langle r(t),\phi\rangle_{V',V} = \int_\Omega f(t,\cdot)\phi \,\mathrm{d}\mathbf{x} + \int_{\Gamma_N} g(t,\cdot)\phi \,\mathrm{d}\mathbf{s} \quad \text{for } \phi \in V,\ t \in (0,T) \text{ a.e.}$$

and $\mathcal{B} : L^2(0,T) \to L^2(0,T;V')$ as

$$\langle(\mathcal{B}u)(t),\phi\rangle_{V',V} = u(t)\int_{\Gamma_C} b(t,\cdot)\,\mathrm{d}\mathbf{s} \quad \text{for } \phi \in V,\ t \in (0,T) \text{ a.e.}$$

it follows that the weak formulation of (2.3) can be expressed in the form (2.2). $\Diamond$

It is well-known (see, e.g., [6]) that for every $r \in L^2(0,T;V')$, $u \in L^2(\mathcal{I})$ and $y_0 \in H$ there exists a unique weak solution $y \in W(0,T)$ satisfying (2.2) and

$$\|y\|_{W(0,T)} \le C\big(\|u\|_{L^2(\mathcal{I})} + \|y_0\|_H + \|r\|_{L^2(0,T;V')}\big) \tag{2.4}$$

with a constant $C > 0$ independent of y.

Remark 2.2. Let $\hat{y}_0 \in W(0,T)$ be the unique solution to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{y}_0(t),\varphi\rangle_H + a(\hat{y}_0(t),\varphi) &= \langle r(t),\varphi\rangle_{V',V} && \text{f.a.a. } t \in [0,T],\ \forall\varphi \in V,\\ \langle\hat{y}_0(0),\varphi\rangle_H &= \langle y_0,\varphi\rangle_H && \forall\varphi \in V. \end{aligned}$$

Moreover, we introduce the linear and bounded operator $\mathcal{S} : L^2(\mathcal{I}) \to W(0,T)$ as follows: $\tilde{y} = \mathcal{S}u \in W(0,T)$ is the unique solution to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle\tilde{y}(t),\varphi\rangle_H + a(\tilde{y}(t),\varphi) &= \langle(\mathcal{B}u)(t),\varphi\rangle_{V',V} && \text{f.a.a. } t \in [0,T],\ \forall\varphi \in V,\\ \langle\tilde{y}(0),\varphi\rangle_H &= 0 && \forall\varphi \in V. \end{aligned}$$

Then, $y = \hat{y}_0 + \mathcal{S}u$ is the weak solution to (2.2). $\Diamond$

Next we introduce the cost functional $J : W(0,T) \times L^2(\mathfrak{I}) \to \mathbb{R}$ by

$$J(y,u) = \frac{\alpha_1}{2} \, \|\mathcal{C}y - z_1\|_{W_1}^2 + \frac{\alpha_2}{2} \, \|\mathcal{D}y(T) - z_2\|_{W_2}^2 + \frac{\sigma}{2} \, \|u\|_{L^2(\mathfrak{I})}^2, \tag{2.5}$$

where W_1, W_2 are Hilbert spaces, $\mathcal{C} : L^2(0,T;H) \to W_1$ and $\mathcal{D} : H \to W_2$ are bounded linear operators, and $(z_1, z_2) \in W_1 \times W_2$ holds. Furthermore, α_1, α_2 are nonnegative parameters and $\sigma > 0$.

Remark 2.3. In the context of Example 2.3 we choose $\alpha_1 = 0$, $\alpha_2 = 1$, $W_2 = L^2(\Omega)$, $z_2 \in L^2(\Omega)$, $\mathcal{D} = \mathrm{id}$ auf $L^2(\Omega)$, and $\mathfrak{I} = (0,T)$. Then, (2.5) yields the cost functional

$$J(y,u) = \int_\Omega \big(y(T,\mathbf{x}) - z_2(\mathbf{x})\big)^2 \, \mathrm{d}\mathbf{x} + \frac{\sigma}{2} \int_0^T \big(u(t)\big)^2 \, \mathrm{d}t$$

for $(y,u) \in W(0,T) \times L^2(\mathfrak{I})$. $\Diamond$

The optimal control problem is given by

$$\min J(y,u) \quad \text{s.t.} \quad (y,u) \in W(0,T) \times U_{\mathsf{ad}} \text{ solves (2.2).} \tag{$\mathbf{P}$}$$

Applying standard arguments (see [19], for instance) one can prove that there exists a unique optimal solution $\bar{x} = (\bar{y}, \bar{u})$ to $(\mathbf{P})$.

2.2 First-order optimality conditions. First-order necessary optimality conditions for our parabolic optimal control problem are well known. We briefly recall them, because they are needed for our subsequent error analysis.

Suppose that $\bar{x} = (\bar{y}, \bar{u})$ is the optimal solution to $(\mathbf{P})$ (in the paper, a bar indicates optimality). Then there exists a unique Lagrange-multiplier $\bar{p} \in W(0,T)$ satisfying together with $\bar{x}$ the *first-order necessary optimality conditions*, which consist of the *state equations* (2.2), the *adjoint equations* in $[0,T]$

$$-\frac{\mathrm{d}}{\mathrm{d}t} \langle \bar{p}(t), \varphi \rangle_H + a(\bar{p}(t), \varphi) = \alpha_1 \, \langle z_1 - \mathcal{C}\bar{y}, \mathcal{C}\varphi \rangle_{W_1} \text{ f.a.a. } t \in [0,T], \; \forall \varphi \in V, \tag{2.6a}$$

$$\langle \bar{p}(T), \varphi \rangle_H = \alpha_2 \langle z_2 - \mathcal{D}\bar{y}(T), \mathcal{D}\varphi \rangle_{W_2} \quad \forall \varphi \in V, \tag{2.6b}$$

and of the *variational inequality*

$$\langle \sigma \bar{u} - \mathcal{B}^\star \bar{p}, u - \bar{u} \rangle_{L^2(\mathfrak{I})} \geq 0 \quad \forall u \in \mathcal{U}_{\mathsf{ad}}. \tag{2.7}$$

Here, the linear and bounded operator $\mathcal{B}^\star : L^2(0,T;V) \to L^2(\mathfrak{I})' \sim L^2(\mathfrak{I})$ stands for the dual operator of $\mathcal{B}$ satisfying

$$\langle \mathcal{B}u, \varphi \rangle_{L^2(0,T;V'), L^2(0,T;V)} = \langle \mathcal{B}^\star \varphi, u \rangle_{L^2(\mathfrak{I})} \quad \forall (u, \varphi) \in L^2(\mathfrak{I}) \times L^2(0,T;V).$$

In Remark 2.2 the linear and bounded operator $\mathcal{S}$ has been defined. The associated dual $\mathcal{S}^\star : W(0,T)' \to L^2(\mathfrak{I})$ is defined as

$$\langle \mathcal{S}^\star f, u \rangle_{L^2(\mathfrak{I})} = \langle f, \mathcal{S}u \rangle_{W(0,T)', W(0,T)} \quad \forall (f, u) \in W(0,T)' \times L^2(\mathfrak{I}). \tag{2.8}$$

We will make use of the following lemma that is a variant of Lemma 4.1 in [12]. For its proof we refer the reader to the Appendix.

Lemma 2.4. *Suppose that $z_1 \in L^2(0,T;H)$, $z_2 \in H$ and $\bar{y} = \hat{y}_0 + \mathcal{S}\bar{u} \in W(0,T)$ with given optimal control $\bar{u} \in L^2(\mathfrak{I})$, where $\hat{y}_0$ has been defined in Remark* 2.2. *Moreover, let $\bar{p} \in W(0,T)$ denote the unique solution to the adjoint system* (2.6).

Furthermore, $\Theta : W(0,T) \to W(0,T)'$ *and* $\Xi : L^2(0,T;H) \times H \to W(0,T)'$ *are defined by*

$$\langle \Theta(\chi), \phi \rangle_{W(0,T)',W(0,T)} = \alpha_1 \langle \mathcal{C}\chi, \mathcal{C}\phi \rangle_{W_1} + \alpha_2 \langle \mathcal{D}\chi(T), \mathcal{D}\phi(T) \rangle_{W_2}, \tag{2.9}$$

$$\langle \Xi(\chi_1,\chi_2), \varphi \rangle_{W(0,T)',W(0,T)} = \alpha_1 \langle \chi_1, \mathcal{C}\varphi \rangle_{W_1} + \alpha_2 \langle \chi_2, \mathcal{D}\varphi(T) \rangle_{W_2} \tag{2.10}$$

for $\chi, \phi, \varphi \in W(0,T)$ *and* $(\chi_1, \chi_2) \in W_1 \times W_2$. *Then it follows that*

$$\mathcal{B}^\star \bar p = \mathcal{S}^\star \Big(\Xi(z_1,z_2) - \Theta(\bar y) \Big) \in L^2(\mathfrak{I}).$$

Remark 2.5. We continue the discussion of Example 2.3 and Remark 2.3. The adjoint equations (2.6) are given by

$$\begin{aligned} -\bar p_t(t,\mathbf{x}) - \Delta \bar p(t,\mathbf{x}) + \bar p(t,\mathbf{x}) &= 0 && \text{f.a.a. } (t,\mathbf{x}) \in Q,\\ \frac{\partial \bar p}{\partial n}(t,\mathbf{s}) &= 0 && \text{f.a.a. } (t,\mathbf{s}) \in \Sigma,\\ p(T,\mathbf{x}) &= z_2(\mathbf{x}) - \bar y(T,\mathbf{x}) && \text{f.a.a. } \mathbf{x} \in \Omega. \end{aligned}$$

Moreover, the variational inequality (2.7) has the form

$$\int_0^T \bigg(\sigma \bar u(t) - \int_{\Gamma_C} b(t,\mathbf{s}) \bar p(t,\mathbf{s}) \,\mathrm{d}\mathbf{s} \bigg) \big(u(t) - \bar u(t) \big) \,\mathrm{d}t \geq 0 \quad \text{for all } u \in U_{\mathsf{ad}}$$

and $\mathcal{B}^\star \bar p \in L^2(\mathfrak{I})$ is given by $(\mathcal{B}^\star \bar p)(t) = \int_{\Gamma_C} b(t,\mathbf{s}) \bar p(t,\mathbf{s}) \,\mathrm{d}\mathbf{s}$ f.a.a. $t \in [0,T]$. ◊

2.3 The reduced control problem. Utilizing the solution operator $\mathcal{S}$ (see Remark 2.2) we introduce the so-called reduced cost functional as

$$\hat J(u) = J(\hat y_0 + \mathcal{S}u, u).$$

Then, we can express (**P**) as the reduced problem

$$\min \hat J(u) \quad \text{s.t.} \quad u \in U_{\mathsf{ad}}. \tag{$\hat{\mathbf{P}}$}$$

It follows that $\hat J'(\bar u) = \sigma \bar u - \mathcal{B}^\star \bar p \in L^2(\mathfrak{I})$ is the gradient of $\hat J$ at $\bar u$, where $\bar p$ solves the dual sytem (2.6) for $\bar y = \hat y_0 + \mathcal{S}\bar u$. Moreover, the variational inequality (2.7) is equivalent to

$$\bar u(s) = \mathcal{P}_{[u_a(s),u_b(s)]} \bigg(\frac{1}{\sigma} \big(\mathcal{B}^\star \bar p \big)(s) \bigg) \quad \text{f.a.a. } s \in \mathfrak{I}, \tag{2.11}$$

where $\mathcal{P}_{[a,b]} : \mathbb{R} \to [a,b]$ denotes the projection operator onto the convex interval $[a,b] \subset \mathbb{R}$.

3 A-posteriori error analysis

In principle, this section contains the main idea underlying our a-posteriori error analysis. Suppose that u_p is an arbitrary control of U_{ad}. Our goal is to estimate the difference

$$\|\bar u - u_p\|_{L^2(\mathfrak{I})}$$

without the knowledge of the optimal solution $\bar u$. The associated idea is not new. For instance, it was used by Malanowski et al. [21] in the context of error estimates for the optimal control of ODEs. It was extended later to elliptic optimal control problems in [1] and [5]. Let us explain this basic idea here.

If $u_p \neq \bar{u}$ then u_p does not satisfy the necessary (and by convexity sufficient) optimality conditions (2.7) respectively (2.11). However, there exists a function $\zeta \in L^2(\mathfrak{I})$ such that

$$\langle \sigma u_p - \mathcal{B}^\star p_p + \zeta, u - u_p \rangle_{L^2(\mathfrak{I})} \geq 0 \quad \forall u \in U_{ad}, \tag{3.1}$$

where $p_p \in W(0,T)$ solves the adjoint equation associated with u_p

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\langle p_p(t), \varphi \rangle_H + a(p_p(t), \varphi) &= \alpha_1 \langle z_1 - \mathcal{C} y_p, \mathcal{C}\varphi \rangle_{W_1} \text{ f.a.a. } t \in [0,T], \forall \varphi \in V, \\ \langle p_p(T), \varphi \rangle_H &= \alpha_2 \langle z_2 - \mathcal{D} y_p(T), \mathcal{D}\varphi \rangle_{W_2} \quad \forall \varphi \in V, \end{aligned} \tag{3.2}$$

and $y_p = \hat{y} + \mathcal{S} u_p$ is the state corresponding to u_p. Therefore, u_p satisfies the optimality condition of a perturbed parabolic optimal control problem with "perturbation" ζ. We refer, e.g., to [1]. The smaller ζ is, the closer u_p is to $\bar{u}$.

The computation of ζ is possible on the basis of the known data u_p, y_p, and p_p. We carry out this construction below in Proposition 3.2. First, however, we estimate $\|\bar{u} - u_p\|_{L^2(\mathfrak{I})}$ in terms of $\|\zeta\|_{L^2(\mathfrak{I})}$. Choosing $u = u_p$ in (2.7) and $u = \bar{u}$ in (3.1) we obtain

$$\begin{aligned} 0 &\leq \langle \sigma \bar{u} - \mathcal{B}^\star \bar{p}, u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} + \langle \sigma u_p - \mathcal{B}^\star p_p + \zeta, \bar{u} - u_p \rangle_{L^2(\mathfrak{I})} \\ &= \langle \sigma \bar{u} - \mathcal{B}^\star \bar{p}, u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} - \langle \sigma u_p - \mathcal{B}^\star p_p + \zeta, u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} \\ &= \sigma \langle \bar{u} - u_p, u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} - \langle \mathcal{B}^\star (\bar{p} - p_p), u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} - \langle \zeta, u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} \\ &= -\sigma \|\bar{u} - u_p\|^2_{L^2(\mathfrak{I})} + \langle \mathcal{B}^\star (p_p - \bar{p}), u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} - \langle \zeta, u_p - \bar{u} \rangle_{L^2(\mathfrak{I})}. \end{aligned} \tag{3.3}$$

Lemma 2.4 yields

$$\mathcal{B}^\star \bar{p} = \mathcal{S}^\star \Big(\Xi(z_1, z_2) - \Theta(\bar{y}) \Big) \quad \text{with } \bar{y} = \hat{y}_0 + \mathcal{S}\bar{u}.$$

Analogously, we obtain

$$\mathcal{B}^\star p_p = \mathcal{S}^\star \Big(\Xi(z_1, z_2) - \Theta(y_p) \Big) \quad \text{with } y_p = \hat{y}_0 + \mathcal{S} u_p.$$

Thus,

$$\begin{aligned} &\langle \mathcal{B}^\star (p_p - \bar{p}), u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} \\ &\quad = \Big\langle \mathcal{S}^\star \Big(\Xi(z_1, z_2) - \Theta(y_p) \Big) - \mathcal{S}^\star \Big(\Xi(z_1, z_2) - \Theta(\bar{y}) \Big), u_p - \bar{u} \Big\rangle_{L^2(\mathfrak{I})} \\ &\quad = \Big\langle \Theta(\bar{y}) - \Theta(y_p), \mathcal{S}(u_p - \bar{u}) \Big\rangle_{L^2(\mathfrak{I})} = -\Big\langle \Theta(y_p - \bar{y}), y_p - \bar{y} \Big\rangle_{L^2(\mathfrak{I})} \\ &\quad = -\alpha_1 \|\mathcal{C}(y_p - \bar{y})\|^2_{W_1} - \alpha_2 \|\mathcal{D}(y_p - \bar{y})\|^2_{W_2} \leq 0. \end{aligned}$$

Therefore we conclude from (3.3) that

$$\sigma \|\bar{u} - u_p\|^2_{L^2(\mathfrak{I})} \leq -\langle \zeta, u_p - \bar{u} \rangle_{L^2(\mathfrak{I})} \leq \|\zeta\|_{L^2(\mathfrak{I})} \|\bar{u} - u_p\|_{L^2(\mathfrak{I})},$$

which gives easily

$$\|\bar{u} - u_p\|_{L^2(\mathfrak{I})} \leq \frac{1}{\sigma} \|\zeta\|_{L^2(\mathfrak{I})}.$$

Summarizing we have proved the following theorem.

Theorem 3.1. *Let $\bar{u}$ be the optimal solution to* (**P**)*, $\bar{y}$ the associated optimal state, and $\bar{p}$ the associated Lagrange multiplier. Suppose that $u_p \in U_{\mathsf{ad}}$ is chosen arbitrarily, $y_p = \hat{y} + \mathcal{S} u_p$, and p_p is the solution to* (3.2)*. Then it follows that*

$$\|\bar{u} - u_p\|_{L^2(\mathfrak{I})} \leq \frac{1}{\sigma} \|\zeta\|_{L^2(\mathfrak{I})},$$

where ζ is chosen such that (3.1) *holds.*

We proceed by constructing the function ζ. Suppose that we have u_p and the associated adjoint state p_p solving to (3.2). The goal is to determine $\zeta \in L^2(\mathfrak{I})$ satisfying (3.1). We distinguish between three different cases.

1) Case $u_p(s) = u_a(s)$ for fixed $s \in \mathfrak{I}$: Then, $u(s) - u_p(s) = u(s) - u_a(s) \geq 0$ for all $u \in U_{\mathsf{ad}}$. Hence, $\zeta(s)$ has to satisfy
$$\big(\sigma u_p - \mathcal{B}^\star p_p\big)(s) + \zeta(s) \geq 0. \tag{3.4}$$
Setting
$$\begin{aligned}\zeta(s) &= [(\sigma u_p - \mathcal{B}^\star p_p)(s)]_- = -\min\big(0, (\sigma u_p - \mathcal{B}^\star p_p)(s)\big)\\ &= \frac{1}{2}\big(|(\sigma u_p - \mathcal{B}^\star p_p)(s)| - (\sigma u_p - \mathcal{B}^\star p_p)(s)\big)\end{aligned}$$
the value $\zeta(s)$ satisfies (3.4). Here, $[s]_- = -\min(0, s)$ denotes the negative part function.
2) Case $u_p(s) = u_b(s)$ for fixed $s \in \mathfrak{I}$: Now, $u(s) - u_p(s) = u(s) - u_b(s) \leq 0$ for all $u \in U_{\mathsf{ad}}$. Analogously to the first case we define
$$\begin{aligned}\zeta(s) &= [(\sigma u_p - \mathcal{B}^\star p_p)(s)]_+ = \max\big(0, (\sigma u_p - \mathcal{B}^\star p_p)(s)\big)\\ &= \frac{1}{2}\big((\sigma u_p - \mathcal{B}^\star p_p)(s) + |(\sigma u_p - \mathcal{B}^\star p_p)(s)|\big)\end{aligned}$$
to ensure (3.4), where, $[s]_+ = \max(0, s)$ denotes the positive part function.
3) Case $u_a(s) < u_p(s) < u_b(s)$ for fixed $s \in \mathfrak{I}$: Consequently, $(\sigma u_p - \mathcal{B}^\star p_p)(s) + \zeta(s) = 0$ holds so that $\zeta(s) = -(\sigma u_p - \mathcal{B}^\star p_p)(s)$ guarantees (3.4).

Clearly, $\zeta \equiv 0$ holds in the case, where u_p satisfies the first-order necessary optimality conditions.

Proposition 3.2. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied. Define* $\zeta \in L^2(\mathfrak{I})$ *as follows:*
$$\zeta(s) = \begin{cases} \big[(\sigma u_p - \mathcal{B}^\star p_p)(s)\big]_- & \text{on } \mathcal{A}_- = \big\{s \in \mathfrak{I}\,\big|\, u_p(s) = u_a(s)\big\},\\ \big[(\sigma u_p - \mathcal{B}^\star p_p)(s)\big]_+ & \text{on } \mathcal{A}_+ = \big\{s \in \mathfrak{I}\,\big|\, u_p(s) = u_b(s)\big\},\\ -(\sigma u_p - \mathcal{B}^\star p_p)(s) & \text{on } \mathfrak{J} = \mathfrak{I} \setminus \big(\mathcal{A}_- \cup \mathcal{A}_+\big).\end{cases} \tag{3.5}$$
Then, the estimate
$$\|\bar{u} - u_p\|_{L^2(\mathfrak{I})} \leq \frac{1}{\sigma}\,\|\zeta\|_{L^2(\mathfrak{I})} \tag{3.6}$$
is satisfied.

We will call (3.6) an a-posteriori error estimate, since, in the next section, we shall apply it to suboptimal controls u_p that have already been computed from a POD model. After having computed u_p, we determine the associated state y_p and adjoint state (Lagrange multiplier) p_p. Then we can determine ζ and its L^2-norm and (3.6) gives an upper bound for the distance of u_p to $\bar{u}$. In this way, the error caused by the POD method can be estimated a-posteriorily. If the error is too large, then we have to include more POD basis functions in our Galerkin ansatz; see Section 4.6. This approach compensates the lack of a-priori error estimates for the POD method.

Remark 3.3. Similar arguments can be used to derive an analogous error estimate (as in (3.6)) for linear-quadratic optimal control problems governed by linear elliptic problems; see Section 5, Run 2. $\Diamond$

4 The POD Galerkin discretization

In this section we briefly introduce the POD method and derive the reduced-order model. To keep the notation simple, we apply only a spatial discretization with POD basis functions, but no time integration by, e.g., an implicit Euler method. Therefore, in the analysis we utilize a continuous POD. Let us mention the work [8], where convergence of POD Galerkin approximations for evolution problems is analyzed using also a continuous version of POD.

4.1 The POD method. Let an arbitrary $u \in L^2(\mathcal{I})$ be chosen such that the corresponding state variable $y = \hat{y}_0 + \mathcal{S}u \in W(0,T)$ belongs to $C([0,T];V)$. Then,

$$\mathcal{V} = \operatorname{span}\big\{y(t)\,|\,t \in [0,T]\big\} \subseteq V. \tag{4.1}$$

If $y_0 \neq 0$ holds, then $\operatorname{span}\{y_0\} \subset \mathcal{V}$ and $d = \dim \mathcal{V} \geq 1$, but $\mathcal{V}$ may have infinite dimension. We define a bounded linear operator $\mathcal{Y} : L^2(0,T) \to V$ by

$$\mathcal{Y}\varphi = \int_0^T \varphi(t)y(t)\,\mathrm{d}t \quad \text{for } \varphi \in L^2(0,T).$$

Its Hilbert space adjoint $\mathcal{Y}^\star : V \to L^2(0,T)$ satisfying

$$\langle \mathcal{Y}\varphi, z\rangle_V = \langle \varphi, \mathcal{Y}^\star z\rangle_{L^2(0,T)} \quad \text{for } (\varphi, z) \in L^2(0,T) \times V$$

is given by

$$\big(\mathcal{Y}^\star z\big)(t) = \langle z, y(t)\rangle_V \quad \text{for } z \in V \text{ and f.a.a. } t \in [0,T].$$

The bounded linear operator $\mathcal{R} = \mathcal{Y}\mathcal{Y}^\star : V \to \mathcal{V} \subseteq V$ has the form

$$\mathcal{R}z = \int_0^T \langle z, y(t)\rangle_V \, y(t)\,\mathrm{d}t \quad \text{for } z \in V. \tag{4.2}$$

Moreover, let $\mathcal{K} = \mathcal{Y}^\star\mathcal{Y} : L^2(0,T) \to L^2(0,T)$ be defined by

$$\big(\mathcal{K}\varphi\big)(t) = \int_0^T \langle y(s), y(t)\rangle_V \, \varphi(s)\,\mathrm{d}s \quad \text{for } \varphi \in L^2(0,T).$$

First we observe that the linear and bounded operator $\mathcal{R}$ is self-adjoint. Since $y \in W(0,T) \subset L^2(0,T;V)$ the kernel of $\mathcal{K}$ is square integrable over $(0,T) \times (0,T)$, so that the integral operator is Hilbert-Schmidt and therefore compact. This implies that $\mathcal{R}$ is compact as well. Moreover, $\mathcal{R}$ is non-negative. From the Hilbert-Schmidt theorem [24, p. 203] it follows that there exists a complete orthonormal basis $\{\psi_i\}_{i=1}^d$ for $\mathcal{V} = \operatorname{range}(\mathcal{R})$ and a sequence $\{\lambda_i\}_{i=1}^d$ of real numbers such that

$$\mathcal{R}\psi_i = \lambda_i\psi_i \text{ for } i = 1, \ldots, d \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \geq 0. \tag{4.3}$$

Remark 4.1.
1) By the Riesz-Schauder theorem the spectrum of $\mathcal{R}$ is a pure point spectrum except for possibly 0; see [24, p. 203].
2) To obtain a complete orthonormal basis in the separable Hilbert space V we need an orthonormal basis for $(\operatorname{range}(\mathcal{R}))^\perp$. This can be done by the Gram-Schmidt procedure. Hence, we suppose in the following that $\{\psi_i\}_{i=1}^\infty$ is a complete orthonormal basis for V.

3) If $1 \leq d = \dim \mathcal{V} < \infty$ holds and $\{\psi_i\}_{i=1}^\infty$ is as described in Part 2), it follows that $\lambda_i > 0$ for $1 \leq i \leq d$ and $\mathcal{R}\psi_i = 0$ for all $i > d$.
4) Analogously to the theory of singular value decompositions for matrices, we find that the linear, bounded, compact and self-adjoint operator $\mathcal{K}$ has the same eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ as the operator $\mathcal{R}$. For all $\lambda_i > 0$ the corresponding eigenfunctions of $\mathcal{K}$ are given by

$$v_i(t) = \frac{1}{\sqrt{\lambda_i}} \big(\mathcal{Y}^*\psi_i\big)(t) = \frac{1}{\sqrt{\lambda_i}} \,\langle \psi_i, y(t)\rangle_V \text{ f.a.a. } t \in [0,T] \text{ and } 1 \leq i \leq \ell.$$

$\Diamond$

In the following proposition we formulate properties of the eigenvalues and eigenfunctions of $\mathcal{R}$. Therefore, for given $\ell \in \mathbb{N}$ we introduce the mapping

$$\mathfrak{J} : \underbrace{V \times \ldots \times V}_{\ell-\text{times}} \to \mathbb{R}, \quad \mathfrak{J}(\psi_1,\ldots,\psi_\ell) := \int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i\rangle_V \,\psi_i \Big\|_V^2 \,\mathrm{d}t.$$

Note that

$$\mathfrak{J}(\psi_1,\ldots,\psi_\ell) = \int_0^T \Big\| y(t) - \mathcal{P}^\ell y(t) \Big\|_V^2 \,\mathrm{d}t. \tag{4.4}$$

Proposition 4.2. *Suppose that V is a separable Hilbert space, $y \in C([0,T];V)$ holds and $\mathcal{V}$ is given as in* (4.1). *Let the linear operator $\mathcal{R} : V \to V$ be defined as in* (4.2). *Then, $\mathcal{R}$ is bounded, self-adjoint, compact and non-negative, and there exists $\{\lambda_i\}_{i\in\mathbb{N}}$ and $\{\psi_i\}_{i\in\mathbb{N}}$ satisfying* (4.3). *Moreover, for any $\ell \leq d = \dim \mathcal{V}$ the elements $\{\psi_i\}_{i=1}^\ell$ solve the minimization problem*

$$\min \mathfrak{J}(\tilde\psi_1,\ldots,\tilde\psi_\ell) \quad \text{s.t.} \quad \langle \tilde\psi_j, \tilde\psi_i\rangle_V = \delta_{ij} \quad \text{for } 1 \leq i,j \leq \ell \tag{4.5}$$

and

$$\mathfrak{J}(\psi_1,\ldots,\psi_\ell) = \sum_{i=\ell+1}^{\infty} \lambda_i. \tag{4.6}$$

For a proof we refer to [13, Section 3], [24, Sections II and VI] and [27], for instance.

4.2 The discrete POD method. In real computations, we do not have the whole trajectory $y(t)$ for all $t \in [0,T]$. For that purpose let $0 = t_1 < t_2 < \ldots < t_n = T$ be a given grid in $[0,T]$ and let $y_j = y(t_j)$ denote approximations for y at time instance t_j, $j = 1,\ldots,n$. We set $\mathcal{V}^n = \operatorname{span}\{y_1,\ldots,y_n\}$ with $d^n = \dim \mathcal{V}^n \leq n$. Then, for given $\ell \leq n$ we consider the minimization problem

$$\min \sum_{j=1}^{n} \alpha_j \Big\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i^n\rangle_V \,\psi_i^n \Big\|_V^2 \text{ s.t. } \langle \psi_i^n, \psi_j^n\rangle_V = \delta_{ij} \text{ for } 1 \leq i,j \leq \ell \tag{4.7}$$

instead of (4.5). In (4.7) the α_j's stand for the trapezoidal weights

$$\alpha_1 = \frac{t_2 - t_1}{2}, \quad \alpha_j = \frac{t_{j+1} - t_{j-1}}{2} \text{ for } 2 \leq j \leq n-1, \quad \alpha_n = \frac{t_n - t_{n-1}}{2}.$$

The solution to (4.7) is given by the solution to the eigenvalue problem

$$\mathcal{R}^n \psi_i^n = \sum_{j=1}^{n} \alpha_j \,\langle y_j, \psi_i^n\rangle_V \, y_j = \lambda_i^n \psi_i^n, \quad i = 1,\ldots,\ell,$$

where $\mathcal{R}^n : V \to \mathcal{V}^n \subset V$ is a linear, bounded, compact, self-adjoint and non-negative operator. Thus, there exists an orthonormal set $\{\psi_i^n\}_{i=1}^{d^n}$ of eigenfunctions and corresponding non-negative eigenvalues $\{\lambda_i^n\}_{i=1}^{d^n}$ satisfying

$$\mathcal{R}^n \psi_i^n = \lambda_i^n \psi_i^n, \quad \lambda_1^n \geq \lambda_2^n \geq \ldots \geq \lambda_{d^n}^n > 0. \tag{4.8}$$

We refer to [17], where the relationship between (4.3) and (4.8) is investigated.

4.3 POD Galerkin scheme for the state equation. The analysis worked out in Sections 4.3-4.6 is not needed to understand the main principle of our a-posteriori error estimation. This has already been explained in the preceding section and the reader might proceed with Theorem 4.11. However, this analysis shows that there is a real chance to decrease the error by increasing the number of snapshots used by the POD method, provided that some natural assumptions are satisfied. First, we derive an error estimate for the state equation, where the control u is fixed.

Let $y = \hat{y}_0 + \mathcal{S}u$ be the state associated with some control $u \in L^2(\mathfrak{I})$, and let $\mathcal{V}$ be given as in (4.1). We fix ℓ with $\ell \leq \dim \mathcal{V}$ and compute the first ℓ POD basis functions $\psi_1, \ldots, \psi_\ell \in V$ by solving either $\mathcal{R}\psi_i = \lambda_i \psi_i$ or $\mathcal{K}v_i = \lambda v_i$ for $i = 1, \ldots, \ell$ (see Remark 4.1). Then we define the finite dimensional linear space

$$V^\ell = \operatorname{span}\big\{\psi_1, \ldots, \psi_\ell\big\} \subset V.$$

Endowed with the topology in V it follows that V^ℓ is a Hilbert space. Let $\mathcal{P}^\ell$ denote the orthogonal projection $\mathcal{P}^\ell$ of V onto V^ℓ defined by

$$\mathcal{P}^\ell \varphi = \sum_{i=1}^{\ell} \langle \varphi, \psi_i \rangle_V \, \psi_i \quad \text{for } \varphi \in V. \tag{4.9}$$

Combining (4.4) and (4.5) we obtain that

$$\mathfrak{J}(\psi_1, \ldots, \psi_\ell) = \int_0^T \big\| y(t) - \mathcal{P}^\ell y(t) \big\|_V^2 \, \mathrm{d}t = \big\| y - \mathcal{P}^\ell y \big\|_{L^2(0,T;V)}^2 = \sum_{i=\ell+1}^{\infty} \lambda_i. \tag{4.10}$$

The POD Galerkin scheme for the state equation (2.2) leads to the following linear problem: determine a function $y^\ell = \sum_{i=1}^{\ell} y_i(t)\psi_i$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y^\ell(t), \psi \rangle_H + a(y^\ell(t), \psi) = \langle (r + \mathcal{B}u)(t), \psi \rangle_{V',V} \text{ f.a.a. } t \in [0,T], \forall \psi \in V^\ell, \tag{4.11a}$$

$$\langle y^\ell(0), \psi \rangle_H = \langle y_0, \psi \rangle_H \quad \forall \psi \in V^\ell. \tag{4.11b}$$

For every $r \in L^2(0,T;V')$, $u \in L^2(\mathfrak{I})$, $y_0 \in H$ and for every $\ell \in \mathbb{N}$ problem (4.11) admits a unique solution $y^\ell \in H^1(0,T;V^\ell)$; see [12, Proposition 3.4]. From $V^\ell \hookrightarrow V$ it follows that $y^\ell \in W(0,T)$ holds.

Let $\hat{y}_0^\ell \in H^1(0,T;V^\ell)$ be the solution to (4.11) for $u \equiv 0$. Analogously to Remark 2.2 we introduce the linear operator $\mathcal{S}^\ell : L^2(\mathfrak{I}) \to H^1(0,T;V^\ell)$ for fixed ℓ: For given $u \in L^2(\mathfrak{I})$ the element $\tilde{y}^\ell = \mathcal{S}^\ell u$ solves (4.11) with $r \equiv 0$ and $y_0 \equiv 0$. Thus, y^ℓ is given by $y^\ell = \hat{y}_0^\ell + \tilde{y}^\ell$. It follows from [12, Proposition 3.4] that the operator $\mathcal{S}^\ell$ is bounded independently of ℓ.

Proposition 4.3. *For given $r \in L^2(0,T;V')$, $u \in L^2(\mathfrak{I})$, and $y_0 \in H$ we suppose that $y = \hat{y} + \mathcal{S}u$ belongs to $y \in C([0,T];V)$. Suppose that, for $\ell \leq \dim \mathcal{V}$, the* *elements* $\{\psi_i\}_{i=1}^{\ell}$ *solve* (4.5)*. Then, there exists a constant* $C > 0$ *such that*

$$\|y - y^\ell\|^2_{W(0,T)} \leq C\bigg(\big\|y^\ell(0) - \mathcal{P}^\ell y_0\big\|^2_H + \big\|y_t - \mathcal{P}^\ell y_t\big\|^2_{L^2(0,T;V')} + \sum_{i=\ell+1}^{\infty} \lambda_i\bigg),$$

where the linear projector $\mathcal{P}^\ell : V \to V^\ell$ *is given by* (4.9) *and* $y^\ell = \hat{y}_0^\ell + \mathcal{S}^\ell u$ *denotes the unique solution to* (4.11)*.*

Proof. Proceeding similarly as in the proof of Proposition 4.7 in [12] it follows that

$$\big\|y - y^\ell\big\|^2_{W(0,T)} \leq C\left(\big\|y^\ell(0) - \mathcal{P}^\ell y_0\big\|^2_H + \big\|y - \mathcal{P}^\ell y\big\|^2_{W(0,T)}\right), \tag{4.12}$$

see in the Appendix. Utilizing

$$\big\|y - \mathcal{P}^\ell y\big\|^2_{W(0,T)} = \big\|y - \mathcal{P}^\ell y\big\|^2_{L^2(0,T;V)} + \big\|y_t - \mathcal{P}^\ell y_t\big\|^2_{L^2(0,T;V')}$$

and (4.10) we obtain the claim. □

Proposition 4.3 permits to show that the POD approximations y^ℓ converge to y in the $W(0,T)$-norm:

Proposition 4.4. *For given* $r \in L^2(0,T;V')$*,* $u \in L^2(\mathfrak{I})$*, and* $y_0 \in V$ *we suppose that* $y = \hat{y} + \mathcal{S}u$ *belongs to* $y \in H^1(0,T;V)$*. Suppose that, for* $\ell \leq \dim \mathcal{V}$*, the elements* $\{\psi_i\}_{i=1}^{\ell}$ *solve* (4.5)*. Then, it follows that*

$$\lim_{\ell\to\infty} \big\|y - y^\ell\big\|_{W(0,T)} = 0,$$

where $y^\ell = \hat{y}_0^\ell + \mathcal{S}^\ell u$ *denotes the unique solution to* (4.11)*.*

Proof. By assumption, $y_t(t) \in V$ holds for almost all $t \in [0,T]$. As $\{\psi_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis in the separable Hilbert space V, we have

$$y_t(t) = \sum_{i=1}^{\infty} \langle y_t(t), \psi_i\rangle_V\, \psi_i \quad \text{f.a.a. } t \in [0,T]$$

and

$$\int_0^T \sum_{i=1}^{\infty} \big|\langle y_t(t), \psi_i\rangle_V\big|^2\, \mathrm{d}t = \int_0^T \|y_t(t)\|^2_V\, \mathrm{d}t. \tag{4.13}$$

This follows from the Lebesgue dominated convergence theorem [24, p. 24], since $\|y_t(\cdot)\|^2_V \in L^1(0,T)$ holds. Moreover, V is continuously embedded into V' (via the identification $H = H'$). Thus, there exists a constant $C > 0$ satisfying

$$\begin{aligned}\big\|y_t - \mathcal{P}^\ell y_t\big\|^2_{L^2(0,T;V')} &\leq C\int_0^T \big\|y_t(t) - \mathcal{P}^\ell y_t(t)\big\|^2_V\, \mathrm{d}t\\ &= C\int_0^T \sum_{i=\ell+1}^{\infty} \big|\langle y_t(t), \psi_i\rangle_V\big|^2\, \mathrm{d}t.\end{aligned} \tag{4.14}$$

We proceed by proving that the right hand side in (4.14) tends to zero as $\ell \to \infty$: For any $\ell \in \mathbb{N}$, define the mapping $F_\ell : [0,T] \to \mathbb{R}$ by

$$F_\ell(t) = \sum_{i=\ell+1}^{\infty} \big|\langle y_t(t), \psi_i\rangle_V\big|^2 \quad \text{f.a.a. } t \in [0,T]$$

From (4.13) it follows that $F_\ell \in L^1(0,T)$ for all $\ell \in \mathbb{N}$. Moreover,

$$\lim_{\ell\to\infty} F_\ell(t) = 0 \quad \text{and} \quad \big|F_\ell(t)\big| \leq \|y_t(t)\|^2_V \quad \text{f.a.a. } t \in [0,T]$$

and

$$F_\ell(t) \le \|y_t(t)\|_V^2 \text{ f.a.a. } t \in [0,T] \quad \text{with} \quad \|y_t(\cdot)\|_V^2 \in L^1(0,T).$$

Now it follows from (4.14) and the Lebesgue dominated convergence theorem [24, p. 24] that

$$\left\|y_t - \mathcal{P}^\ell y_t\right\|^2_{L^2(0,T;V')} \to 0 \quad \text{as } \ell \to \infty. \tag{4.15}$$

From (4.11b) we obtain $\langle y^\ell(0) - y_0, \psi\rangle_H = 0$ for all $\psi \in V^\ell$. As $y_0 \in V$ by assumption, we have $\mathcal{P}^\ell y_0 \in V^\ell$ and

$$y_0 = \sum_{i=1}^\infty \langle y_0, \psi_i\rangle_V\, \psi_i. \tag{4.16}$$

Choosing $\psi = y^\ell(0) - \mathcal{P}^\ell y_0 \in V^\ell$ it follows that

$$\begin{aligned} 0 &= \langle y^\ell(0) - y_0, y^\ell(0) - \mathcal{P}^\ell y_0\rangle_H = \langle y^\ell(0) - \mathcal{P}^\ell y_0 + \mathcal{P}^\ell y_0 - y_0, y^\ell(0) - \mathcal{P}^\ell y_0\rangle_H \\ &= \left\|y^\ell(0) - \mathcal{P}^\ell y_0\right\|_H^2 + \langle \mathcal{P}^\ell y_0 - y_0, y^\ell(0) - \mathcal{P}^\ell y_0\rangle_H. \end{aligned}$$

Hence,

$$\left\|y^\ell(0) - \mathcal{P}^\ell y_0\right\|_H^2 = \langle y_0 - \mathcal{P}^\ell y_0, y^\ell(0) - \mathcal{P}^\ell y_0\rangle_H \le \left\|y_0 - \mathcal{P}^\ell y_0\right\|_H \left\|y^\ell(0) - \mathcal{P}^\ell y_0\right\|_H,$$

which gives

$$\left\|y^\ell(0) - \mathcal{P}^\ell y_0\right\|_H \le \left\|y_0 - \mathcal{P}^\ell y_0\right\|_H. \tag{4.17}$$

Thus, using (4.16) we arrive at

$$\begin{aligned} 0 &\le \lim_{\ell\to\infty} \left\|y^\ell(0) - \mathcal{P}^\ell y_0\right\|_H \le \lim_{\ell\to\infty} \left\|y_0 - \mathcal{P}^\ell y_0\right\|_H \\ &= \lim_{\ell\to\infty} \Bigg\| \sum_{i=\ell+1}^\infty \langle y_0, \psi_i\rangle_H\, \psi_i \Bigg\|_H = 0. \end{aligned} \tag{4.18}$$

From Proposition 4.3, (4.14) and (4.18) the claim follows. □

Remark 4.5.
1) Due to the continuous embedding of $W(0,T)$ into the space $C([0,T];H)$, Proposition 4.4 implies $y^\ell \to y$ in $C([0,T];H)$ as $\ell \to \infty$. In particular, $y^\ell(T)$ converges to $y(T)$ in H as ℓ tends to ∞.
2) Let us mention that the convergence result in Proposition 4.4 is true for any fixed u provided that the system $\{\psi_i\}_{i=1}^\infty$ computed from the snapshots associated with u is complete. ◊

4.4 POD Galerkin scheme for the adjoint equation. It is more or less clear that the convergence result $y^\ell \to y$ implies an associated one for the adjoint states, i.e. $p^\ell \to p$ as $\ell \to \infty$. This is expressed in the next result.

We turn to the POD Galerkin scheme for the adjoint system (2.6a). For that purpose let $u \in L^2(\mathfrak{I})$ be arbitrarily given, $\{\psi_1, \ldots, \psi_\ell\}$ the associated POD basis of rank ℓ, and let $y^\ell \in H^1(0,T;V^\ell)$ denote the unique solution to (4.11). Then, $p^\ell = \sum_{i=1}^\ell p_i(t)\psi_i$ satisfies the linear system

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle p^\ell(t), \psi\rangle_H + a(p^\ell(t), \psi) = \alpha_1 \langle z_1 - \mathcal{C}y^\ell, \mathcal{C}\psi\rangle_{W_1} \text{ f.a.a. } t \in [0,T], \forall \psi \in V^\ell, \tag{4.19a}$$

$$\langle p^\ell(T), \psi\rangle_H = \alpha_2 \langle z_2 - \mathcal{D}y^\ell(T), \mathcal{D}\psi\rangle_{W_2} \quad \forall \psi \in V^\ell. \tag{4.19b}$$

Analogously to the arguments for the solvability of (4.11) it follows that for any $(z_1, z_2) \in W_1 \times W_2$ there exists a unique solution $p^\ell \in H^1(0,T;V^\ell)$ to (4.19); see [12, Proposition 3.5]. Furthermore, we find that

$$\mathcal{B}^\star p^\ell = \mathcal{S}_\ell^\star\big(\Xi(z_1,z_2) - \Theta(y^\ell)\big), \tag{4.20}$$

compare Lemma 4.3 in [12] and Lemma 2.4 above.

Proposition 4.6. *For given $r \in L^2(0,T;V')$, $u \in L^2(\mathfrak{I})$, $y_0 \in H$ suppose that $y = \hat{y} + \mathcal{S}u$ belongs to $H^1(0,T;V)$. Suppose that for $\ell \leq \dim \mathcal{V}$ the elements $\{\psi_i\}_{i=1}^\ell$ solve (4.5). Let $y^\ell = \hat{y}_0^\ell + \mathcal{S}^\ell u$, p, and p^ℓ be the solutions to (4.11), (2.6) and (4.19), respectively. Then there exists a constant $C > 0$ depending on α_1, α_2, $\mathcal{C}$, and $\mathcal{D}$*

$$\begin{aligned}\big\|p - p^\ell\big\|_{L^2(0,T;V)} &\leq C\Big(\big\|p(T) - \mathcal{P}^\ell p(T)\big\|_H + \|p - \mathcal{P}^\ell p\|_{W(0,T)}\Big)\\ &\quad + C\Big(\big\|y(T) - y^\ell(T)\big\|_H + \big\|y - y^\ell\big\|_{L^2(0,T;H)}\Big).\end{aligned} \tag{4.21}$$

where the linear projector $\mathcal{P}^\ell : V \to V^\ell$ is given by (4.9). If, in addition, $y_0 \in V$ and $p \in H^1(0,T;V)$ hold, then $\lim_{\ell\to\infty} \|p - p^\ell\|_{L^2(0,T;V)} = 0$ holds.

Proof. Proceeding as in the proof of Proposition 4.7 in [12] we find (4.21). Since $W(0,T)$ and $L^2(0,T;H)$ are continuously embedded into $C([0,T];H)$ there exists a constant $C_E > 0$ such that

$$\big\|y(T) - y^\ell(T)\big\|_H + \big\|y - y^\ell\big\|_{L^2(0,T;H)} \leq C_E \big\|y - y^\ell\big\|_{W(0,T)}.$$

Thus, if $y_0 \in V$ holds, we infer $\lim_{\ell\to 0} \|y - y^\ell\|_{W(0,T)} = 0$ from Propositions 4.3 and 4.8. The reminder of the proof is a variant of the proof of Proposition 4.8. For the details we refer the reader to the Appendix. □

Remark 4.7. Arguing as in Remark 4.5-2) we derive that the convergence result of Proposition 4.6 remains true if the POD basis is computed using an input $\tilde{u} \in L^2(\mathfrak{I})$ that differs from u. Of course, the convergence rate of p^ℓ to p as $\ell \to \infty$ depends on the approximation properties of the POD basis for the adjoint variable p; see [7, 12]. ♢

4.5 POD approximation of $(\hat{\mathbf{P}})$. The Galerkin projection of $(\hat{\mathbf{P}})$ leads to the discretized optimal control problem

$$\min \hat{J}^\ell(u) \quad \text{s.t.} \quad u \in U_{\mathsf{ad}}, \tag{$\hat{\mathbf{P}}^\ell$}$$

where $\hat{J}^\ell(u) = J(y^\ell(u), u)$ is the reduced objective function and $y^\ell(u)$ denotes the solution to (4.11) associated with $u \in U_{\mathsf{ad}}$. We call $(\hat{\mathbf{P}}^\ell)$ a reduced-order model for $(\hat{\mathbf{P}})$.

Problem $(\hat{\mathbf{P}}^\ell)$ admits a unique optimal solution $\bar{u}^\ell$ that is interpreted as a suboptimal solution to $(\hat{\mathbf{P}})$. First-order necessary optimality conditions for $(\hat{\mathbf{P}}^\ell)$ are given by

$$\langle \sigma\bar{u}^\ell - \mathcal{B}^\star \bar{p}^\ell, u - \bar{u}^\ell\rangle_{L^2(\mathfrak{I})} \geq 0 \quad \text{for all } u \in U_{\mathsf{ad}}, \tag{4.22}$$

where, $\bar{y}^\ell \in H^1(0,T;V^\ell)$ denotes the optimal state solving (4.11) with $u = \bar{u}$ and $\bar{p}^\ell \in H^1(0,T;V^\ell)$ is the adjoint state for the POD model.

4.6 Convergence of the suboptimal controls. We proceed similarly as in [12, Section 4]. However, an essential difference is that we derive convergence results utilizing a POD basis of rank ℓ that is not necessarily related to the optimal control $\bar{u}$ as an input function for the generation of the snapshots.

Proposition 4.8. *Suppose that the POD basis of rank ℓ is computed using an arbitrarily chosen $u \in L^2(\mathfrak{I})$. Let $\bar{u}$ and $\bar{u}^\ell$ be the optimal solutions to $(\hat{\mathbf{P}})$ and $(\hat{\mathbf{P}}^\ell)$, respectively. Moreover, $\bar{p} \in W(0,T)$ denotes the adjoint state associated with $\bar{u}$. Then,*

$$\|\bar{u} - \bar{u}^\ell\|_{L^2(\mathfrak{I})} \leq c\,\|\bar{p} - \hat{p}^\ell\|_{L^2(0,T;V)}, \tag{4.23}$$

where $\hat{p}^\ell$ solves

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{p}^\ell(t), \psi\rangle_H + a(\hat{p}^\ell(t), \psi) &= \alpha_1 \langle z_1 - \mathcal{C}\hat{y}^\ell, \mathcal{C}\psi\rangle_{W_1} \text{ f.a.a. } t \in [0,T], \forall \psi \in V^\ell, \\ \langle \hat{p}^\ell(T), \psi\rangle_H &= \alpha_2 \langle z_2 - \mathcal{D}\hat{y}^\ell(T), \mathcal{D}\psi\rangle_{W_2} \quad \forall \psi \in V^\ell \end{aligned} \tag{4.24}$$

and $\hat{y}^\ell$ is the solution to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{y}^\ell(t), \psi\rangle_H + a(\hat{y}^\ell(t), \psi) &= \langle (r + \mathcal{B}\bar{u})(t), \psi\rangle_{V',V} \text{ f.a.a. } t \in [0,T], \forall \psi \in V^\ell, \\ \langle \hat{y}^\ell(0), \psi\rangle_H &= \langle y_0, \psi\rangle_H \quad \forall \psi \in V^\ell. \end{aligned} \tag{4.25}$$

Proof. The proof is a variant of the proof of Theorem 4.5 in [12]. For more details we refer to the Appendix. ☐

Notice that $\hat{p}^\ell$ is the POD-approximate associated with $\hat{y}^\ell$ and $\hat{y}^\ell = \hat{y}_0^\ell + \mathcal{S}^\ell \bar{u}$. Therefore, both $\hat{y}^\ell$ and $\hat{p}^\ell$ are associated with the same optimal control $\bar{u}$ so that we can apply Proposition 4.3 and Proposition 4.6 to estimate the difference $\bar{y} - \hat{y}^\ell$ and $\bar{p} - \hat{p}^\ell$, respectively. In contrast to this, $\bar{y}^\ell = \hat{y}_0^\ell + \mathcal{S}^\ell \bar{u}^\ell$ corresponds to the suboptimal control $\bar{u}^\ell$, which we estimate in the next theorem.

Theorem 4.9. *Suppose that the POD basis of rank ℓ is computed using an arbitrarily chosen $u \in L^2(\mathfrak{I})$. Let $\bar{u}$ and $\bar{u}^\ell$ be the optimal solutions to $(\hat{\mathbf{P}})$ and $(\hat{\mathbf{P}}^\ell)$, respectively. Moreover, let $\bar{y}$ and $\bar{p}$ denote the optimal state and adjoint, respectively, associated with $\bar{u}$. Then there exists a constant $C > 0$ not depending on ℓ such that*

$$\begin{aligned} &\big\|\bar{u} - \bar{u}^\ell\big\|_{L^2(\mathfrak{I})} \\ &\quad \leq C\Big(\big\|\bar{y} - \mathcal{P}^\ell \bar{y}\big\|_{W(0,T)} + \big\|\bar{y}^\ell(0) - \mathcal{P}^\ell y_0\big\|_H + \big\|\bar{p} - \mathcal{P}^\ell \bar{p}\big\|_{W(0,T)}\Big), \end{aligned} \tag{4.26}$$

where the linear projector $\mathcal{P}^\ell : V \to V^\ell$ is given in (4.9).

If, in addition, $y_\circ \in V$ and $\bar{y}$, $\bar{p} \in H^1(0,T;V)$ hold and $\{\psi_i\}_{i=1}^\infty$ is a complete orthonormal basis for V, then

$$\lim_{\ell \to \infty} \big\|\bar{u} - \bar{u}^\ell\big\|_{L^2(\mathfrak{I})} = 0.$$

Proof. Combining (4.21) and (4.23) we find

$$\begin{aligned} \|\bar{u} - \bar{u}^\ell\|_{L^2(\mathfrak{I})} \leq\; & cC\Big(\big\|\bar{p}(T) - \mathcal{P}^\ell \bar{p}(T)\big\|_H + \big\|\bar{p} - \mathcal{P}^\ell \bar{p}\|_{W(0,T)}\Big) \\ & + C\Big(\big\|\bar{y}(T) - y^\ell(T)\big\|_H + \big\|\bar{y} - y^\ell\big\|_{L^2(0,T;H)}\Big). \end{aligned}$$

Since $W(0,T)$ is contiunuously embedded into $C([0,T];H)$ we derive

$$\|\bar u-\bar u^\ell\|_{L^2(\mathfrak I)}\le\tilde C\Big(\|\bar p-\mathcal P^\ell\bar p\|_{W(0,T)}+\big\|\bar y-y^\ell\big\|_{W(0,T)}\Big)$$

for a constant $\tilde C>0$. Now the claim follows from Propositions 4.3, 4.4, and 4.6. □

Remark 4.10. Let us consider the following idealized situation [12]: Let $\bar u$ be the optimal solution to $(\hat{\mathbf P})$. Moreover, let $\bar y$, $\bar p\in H^1(0,T;V)$ denote the optimal state and adjoint state, respectively, associated with $\bar u$ and let $y_0\in V$. Then we consider the minimization problem

$$\min_{\psi_1,\dots,\psi_\ell}\|\bar y-\mathcal P^\ell\bar y\|^2_{H^1(0,T;V)}+\|\bar p-\mathcal P^\ell\bar p\|^2_{H^1(0,T;V)}\text{ s.t. }\langle\psi_i,\psi_j\rangle_V=\delta_{ij},\ 1\le i,j\le\ell.$$

Its solution $\{\bar\psi_i\}_{i=1}^\ell$ of rank ℓ satisfies the eigenvalue problem

$$\bar{\mathcal R}\bar\psi_i=\bar\lambda_i\bar\psi_i,\quad 1\le i\le\ell,$$

where the linear, bounded, non-negative and self-adjoint operator $\bar{\mathcal R}$ is defined as

$$\bar{\mathcal R}z=\int_0^T\langle\bar y(t),z\rangle_V\,y(t)+\langle\bar y_t(t),z\rangle_V\,\bar y_t(t)+\langle\bar p(t),z\rangle_V\,\bar p(t)+\langle\bar p_t(t),z\rangle_V\,\bar p_t(t)\,\mathrm dt$$

for $z\in V$. Then, (4.26) can be replaced by

$$\big\|\bar u-\bar u^\ell\big\|^2_{L^2(\mathfrak I)}\le\bar C\bigg(\big\|\bar y^\ell(0)-\mathcal P^\ell y_0\big\|^2_H+\sum_{i=\ell+1}^\infty\bar\lambda_i\bigg)$$

with a constant $\bar C>0$. Now we can estimate the decay of the norms $\|\bar y-\mathcal P^\ell\bar y\|_{W(0,T)}$ and $\|\bar p-\mathcal P^\ell\bar p\|_{W(0,T)}$ in (4.26) in terms of the eigenvalues $\bar\lambda_i$ and obtain an error estimate with respect to the remainder $\sum_{i=\ell+1}^\infty\bar\lambda_i$. In contrast to this, the decay of the eigenvalues λ_i can only be used to bound $\|\bar y-\mathcal P^\ell\bar y\|_{L^2(0,T;V)}$ from above, but not the expression $\|\bar y_t-\mathcal P^\ell\bar y_t\|_{L^2(0,T;V')}+\|\bar p-\mathcal P^\ell\bar p\|_{W(0,T)}$. ♢

4.7 A-posteriori error estimate for the POD approximation. In this subsection, we complete the discussion of the a-posteriori estimate by combining Theorem 4.9 and Proposition 3.2. The proposition permits to estimate $\|\bar u-\bar u^\ell\|$ by the norm of an appropriate ζ, while Theorem 4.9 will be used to show that ζ tends to zero as $\ell\to\infty$, since it ensures convergence of $\bar u^\ell$ to the optimal solution $\bar u$ of $(\hat{\mathbf P})$.

For any ℓ let $\bar u^\ell\in U_{\mathsf{ad}}$ be the optimal solution to $(\hat{\mathbf P}^\ell)$. This optimal $\bar u^\ell$ is taken as a suboptimal u_p for $(\hat{\mathbf P})$, i.e. in Proposition 3.2 we take $u_p:=\bar u^\ell$.

Theorem 4.11. 1) *Let $\ell\le d$ be arbitrarily given and $\bar u^\ell\in U_{\mathsf{ad}}$ be the optimal solution to $(\hat{\mathbf P}^\ell)$. Denote by $\tilde y=\tilde y(\bar u^\ell)=\hat y_0+\mathcal S\bar u^\ell$ the solution to (2.2) with $u=\bar u^\ell$ and let $\tilde p=\tilde p(\bar u^\ell)$ solve the associated adjoint equation*

$$\begin{aligned}-\frac{\mathrm d}{\mathrm dt}\langle\tilde p(t),\varphi\rangle_H+a(\tilde p(t),\varphi)&=\alpha_1\,\langle z_1-\mathcal C\tilde y,\mathcal C\varphi\rangle_{W_1}\ \textit{f.a.a. } t\in[0,T],\ \forall\varphi\in V,\\ \langle\tilde p(T),\varphi\rangle_H&=\alpha_2\langle z_2-\mathcal D\tilde y(T),\mathcal D\varphi\rangle_{W_2}\quad\forall\varphi\in V.\end{aligned}\tag{4.27}$$

Define, according to (3.5), the function $\zeta^\ell\in L^2(\mathfrak I)$ by

$$\zeta^\ell(s)=\begin{cases}\big[(\sigma\bar u^\ell-\mathcal B^\star\tilde p(\bar u^\ell))(s)\big]_- & \textit{on } \mathcal A^\ell_-=\big\{s\in\mathfrak I\,\big|\,\bar u^\ell(s)=u_a(s)\big\},\\ \big[(\sigma\bar u^\ell-\mathcal B^\star\tilde p(\bar u^\ell))(s)\big]_+ & \textit{on } \mathcal A^\ell_+=\big\{s\in\mathfrak I\,\big|\,\bar u^\ell(s)=u_b(s)\big\},\\ -(\sigma\bar u^\ell-\mathcal B^\star\tilde p^\ell(\bar u^\ell))(s) & \textit{on } \mathcal J^\ell=\mathfrak I\setminus(\mathcal A^\ell_-\cup\mathcal A^\ell_+).\end{cases}\tag{4.28}$$

Then

$$\|\bar{u} - \bar{u}^\ell\|_{L^2(\mathfrak{I})} \le \frac{1}{\sigma}\, \|\zeta^\ell\|_{L^2(\mathfrak{I})}.$$

2) *If all hypotheses of Proposition* 4.6 *and Theorem* 4.9 *are satisfied, in particular* $\{\psi_i\}_{i=1}^\infty$ *is a complete orthonormal basis for* V*, then the sequences* $\{\bar{u}^\ell\}_{\ell\in\mathbb{N}}$ *and* $\{\mathcal{B}^\star\hat{p}^\ell\}_{\ell\in\mathbb{N}}$ *converge to* $\bar{u}$ *respectively* $\mathcal{B}^\star\bar{p}$ *in* $L^2(\mathfrak{I})$ *as* $\ell\to\infty$ *and*

$$\|\zeta^\ell\|_{L^2(\mathfrak{I})} \to 0.$$

Proof. By Theorem 4.9, the sequences $\{\bar{u}^\ell\}_{\ell\in\mathbb{N}}$ and $\{\mathcal{B}^\star\bar{p}^\ell\}_{\ell\in\mathbb{N}}$ converge to $\bar{u}$ respectively $\mathcal{B}^\star\bar{p}$ in $L^2(\mathfrak{I})$. There exist subsequences $\{\bar{u}^{\ell_k}\}_{k\in\mathbb{N}}$ and $\{\mathcal{B}^\star\bar{p}^{\ell_k}\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} \bar{u}^{\ell_k}(s) = \bar{u}(s) \quad \text{and} \quad \lim_{k\to\infty} \mathcal{B}^\star\bar{p}^{\ell_k}(s) = \mathcal{B}^\star\bar{p}(s) \quad \text{f.a.a. } s\in\mathfrak{I}.$$

Next we consider the active and inactive sets for $\bar{u}$.

1) Let $s\in\mathcal{J} = \{s\in\mathfrak{I}\,|\,u_a(s) < \bar{u}(s) < u_b(s)\}$. For $k_\circ = k_\circ(s)\in\mathbb{N}$ sufficiently large, $\bar{u}^{\ell_k}(s)\in(u_a(s),u_b(s))$ for all $k\ge k_\circ$ and f.a.a. $s\in\mathcal{J}$. Thus, $\sigma\bar{u}^{\ell_k}(s) - \mathcal{B}^\star\bar{p}^{\ell_k}(s) = 0$ for all $k\ge k_\circ(s)$ in $\mathcal{J}$ a.e. This implies

$$\zeta^{\ell_k}(s) = 0 \quad \forall k\ge k_\circ(s) \text{ and f.a.a. } s\in\mathcal{J} \tag{4.29}$$

2) Suppose that $s\in\mathcal{A}_- = \{s\in\mathfrak{I}\,|\,u_a(s) = \bar{u}(s)\}$. From $\sigma\bar{u}(s) - \mathcal{B}^\star\bar{p}(s)\ge 0$ in $\mathcal{A}_-$ a.e. we deduce

$$\lim_{k\to\infty} \zeta^{\ell_k}(s) = \big[(\sigma\bar{u}^\ell - \mathcal{B}^\star\bar{p}^\ell)(s)\big]_- = 0 \quad \text{f.a.a. } s\in\mathcal{A}_-. \tag{4.30}$$

3) Suppose that $s\in\mathcal{A}_+ = \{s\in\mathfrak{I}\,|\,u_b(s) = \bar{u}(s)\}$. Analogously to part 2) we find

$$\lim_{k\to\infty} \zeta^{\ell_k}(s) = \big[(\sigma\bar{u}^\ell - \mathcal{B}^\star\bar{p}^\ell)(s)\big]_+ = 0 \quad \text{f.a.a. } s\in\mathcal{A}_-. \tag{4.31}$$

Combining (4.29)-(4.31) we conclude that $\lim_{k\to\infty}\zeta^{\ell_k} = 0$ a.e. in $(0,T)$. Utilizing the dominated convergence theorem [24, p. 24] we have

$$\lim_{k\to\infty} \left\|\zeta^{\ell_k}\right\|_{L^2(\mathfrak{I})} = 0.$$

Since all subsequences contain a subsequence converging to zero, the claim follows from a standard argument. □

Remark 4.12. 1) Notice that $\tilde{y}$ and $\tilde{p}$ must be taken as the solutions to the (full) state and adjoint equation, respectively, not of their POD-approximations.

2) Part 2) of Theorem 4.11 shows that $\|\zeta^\ell\|_{L^2(\mathfrak{I})}$ can be expected smaller than any $\varepsilon > 0$ provided that ℓ is taken sufficiently large. Motivated by this result, we set up the Algorithm 1. ◊

Remark 4.13. In the numerical realization of Algorithm 1, Step 6 requires the solution of the state as well as of the adjoint equation by, e.g., a finite element or finite differerence scheme. In Section 5, Run 1, we will see that the main part of the CPU time for Algorithm 1 is consumed by step 6. ◊

Algorithm 1 POD reduced-order method with a-posteriori estimator.

```
Choose an input u ∈ U_ad, an initial number ℓ for POD ansatz functions, a
   maximal number ℓ^max > ℓ of POD ansatz functions, and a stopping tolerance
   ε > 0; compute y = ŷ_0 + Su.
Determine a POD basis of rank ℓ utilizing the state y = ŷ_0 + Su and derive the
   reduced-order model (P̂^ℓ).
repeat
   Establish the discretized optimal control problem (P̂^ℓ).
   Calculate the optimal solution ū^ℓ of (P̂^ℓ).
   Evaluate ỹ(ū^ℓ) = ŷ_0 + Sū^ℓ and compute the solution p̃(ū^ℓ) to (4.27) as well
      as ζ^ℓ from (4.28).
   if ‖ζ^ℓ‖_{L²(J)} < ε or ℓ = ℓ^max then
      Return ℓ and suboptimal control ū^ℓ and STOP.
   else
      Set ℓ = ℓ + 1.
   end if
until ℓ > ℓ^max
```

5 Numerical experiments

In this section we present two numerical test examples, where the first one is a parabolic problem as in Example 2.3, while the second one is of elliptic type and demonstrates that our method applies also to other types of problems (see Remark 3.3-2). All coding is done in MATLAB using routines from the FEMLAB 2.2 package for the finite element (FE) implementation.

Run 1 (Parabolic example)**.** Let $\Omega = (0,1) \times (0,1)$ be the open unit square, $T = 1$ and $Q = (0,T) \times \Omega$. We set

$$\Gamma_N = \big\{(x_1,x_2)\,|\,x_1 \in \{0,1\} \text{ and } x_2 \in [0,1]\big\} \quad \text{and} \quad \Gamma_C = \Gamma \setminus \Gamma_N.$$

We consider the minimization problem

$$\min \frac{1}{2}\int_\Omega \big(y(T,\mathbf{x}) - 20\big)^2 \,\mathrm{d}\mathbf{x} + \frac{1}{400}\int_0^T u(t)^2 \,\mathrm{d}t$$

subject to the heat equation

$$\begin{aligned}
y_t(t,\mathbf{x}) - \Delta y(t,\mathbf{x}) &= 0 && \text{f.a.a. } (t,\mathbf{x}) \in Q,\\
\frac{\partial y}{\partial n}(t,\mathbf{x}) &= 0 && \text{f.a.a. } (t,\mathbf{x}) \in \Sigma_N = (0,T) \times \Gamma_N,\\
\frac{\partial y}{\partial n}(t,\mathbf{x}) &= u(t) && \text{f.a.a. } (t,\mathbf{x}) \in \Sigma_C = (0,T) \times \Gamma_C,\\
y(0,\mathbf{x}) &= 30 && \text{f.a.a. } \mathbf{x} \in \Omega
\end{aligned}$$

and to the bilateral control constraints

$$-6 \le u(t) \le 1 \quad \text{f.a.a. } t \in (0,T).$$

We discretize the domain Ω by a uniform rectangular triangulation with mesh-size $h = 1/39$; see Figure 1. For the time integration we apply an implicit Euler method with step size $\tau = 1/300$. Since we do not know the exact optimal control $\bar u$, we take the FE solution $\bar u^{h,\tau}$ to $(\hat{\mathbf{P}})$ as a substitute, considering the mesh with $h = 1/39$

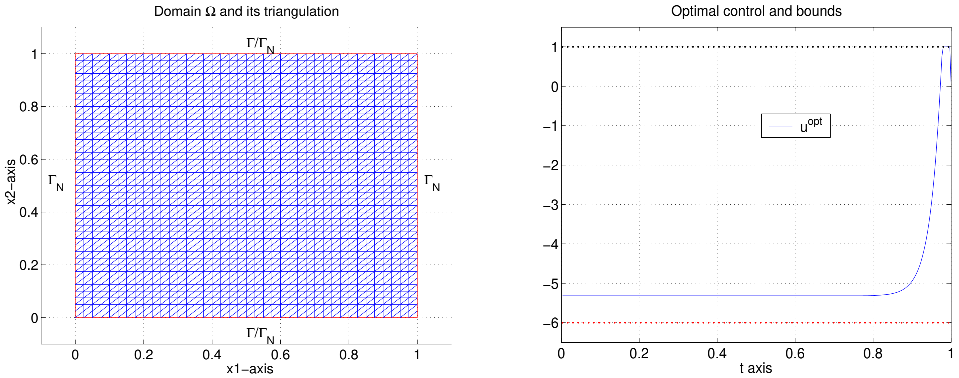

FIGURE 1. Run 1: Domain Ω (left plot) and optimal FE control with lower and upper bound(right plot).

as sufficiently fine; see Figure 1. The reduced optimal control problem $(\hat{\mathbf{P}})$ as well as its low-order approximation $(\hat{\mathbf{P}}^\ell)$ are solved by a primal-dual active set strategy, cf. [11], where the linear systems in each level of the algorithm are treated by the preconditioned conjugate gradient method. According to Theorem 4.11, we compare the error $\bar{u}^{h,\tau} - \bar{u}^\ell$ with the norm of ζ^ℓ for different values of ℓ in Table 1. It turns out that the norms decay with increasing ℓ. Moreover, $\sigma^{-1}\|\zeta^\ell\|_{L^2(0,T)}$ is

ℓ	$\\|\bar{u}^{h,\tau} - \bar{u}^\ell\\|_{L^2(0,T)}$	$\frac{1}{\sigma}\\|\zeta^\ell\\|_{L^2(0,T)}$
2	0.079670	3.361904
3	0.016831	0.065327
4	0.001876	0.002951
5	0.000943	0.002229

Computation part	CPU time
FE optimizer	1611 s
Step 1	8 s
Step 2 with $\ell^{\max} = 10$	5 s
Step 4 for $\ell = 6$	$\ll 1$ s
Step 5 for $\ell = 6$	3 s
Step 6 for $\ell = 6$	18 s

TABLE 1. Run 1: Norms $\|\bar{u}^{h,\tau} - \bar{u}^\ell\|_{L^2(0,T)}$ and $\sigma^{-1}\|\zeta^\ell\|_{L^2(0,T)}$ for different ℓ (left table); CPU times for different parts needed to carry out Algorithm 1.

an upper bound for $\|\bar{u}^{h,\tau} - \bar{u}^\ell\|_{L^2(0,T)}$ as stated in Theorem 3.1. For the CPU times refer to Table 1. Note that for $h = 1/39$ and $\tau = 1/300$ we expect that $\|\bar{u}^{h,\tau} - \bar{u}\|_{L^2(0,T)} \sim c \cdot 9\text{e-}5$. Starting Algorithm 1 with $u = 1 \in U_{\mathsf{ad}}$, $\ell = 2$, $\ell^{\max} = 10$, and choosing the tolerance $\varepsilon = 10^{-2}$ the method stops after 50 seconds — compared to 1611 seconds needed for the FE optimization solver. $\Diamond$

Run 2 (Elliptic example)**.** In this numerical example we consider a problem motivated by acoustic applications in vehicle simulations [9, 10, 28]. Furthermore, this example is constructed in such a way that the exact optimal control is known. Suppose that the interior of the car is simplified by the two-dimensional domain Ω plotted in Figure 2. The boundary $\Gamma = \partial\Omega$ is split into two measurable disjunct

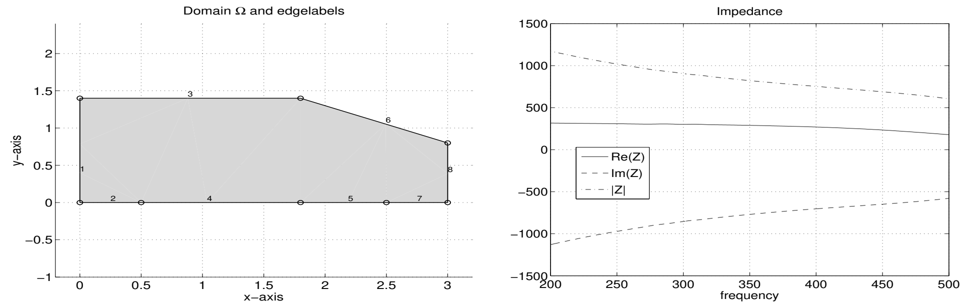

FIGURE 2. Run 2: Interior Ω of the vehicle, where the boundary part Γ_R consists in parts 4 and 5 (left plot); impedance values $Z = Z_{\Re} + \jmath Z_{\Im}$ for Melamin with $50mm$ width in the frequency range from 200 to $400Hz$ (right plot).

parts Γ_R and Γ_N. For given complex impedance $Z \neq 0$ (see Figure 2) the associated sound pressure $p : \Omega \to \mathbb{C}$ is governed by the Helmholtz equation

$$\Delta p(\mathbf{x}) + k^2 p(\mathbf{x}) = ub(\mathbf{x}) \quad \text{for all } \mathbf{x} = (x, y) \in \Omega, \tag{5.1a}$$

together with the boundary conditions

$$\frac{\jmath}{\varrho_\circ \omega} \frac{\partial p(\mathbf{x})}{\partial n} = \frac{p(\mathbf{x})}{Z} \qquad \text{for all } \mathbf{x} \in \Gamma_R, \tag{5.1b}$$

$$\frac{\jmath}{\varrho_\circ \omega} \frac{\partial p(\mathbf{x})}{\partial n} = 0 \qquad \text{for all } \mathbf{x} \in \Gamma_N. \tag{5.1c}$$

In (5.1a) the constant $c = 343.799\left[\frac{m}{s}\right]$ denotes the sound of speed, $\varrho_\circ = 1.19985\left[\frac{kg}{m^3}\right]$ is an ambient density, f stands for the frequency, $\omega = 2\pi f$ is the circle frequency and $k = \omega/c$ is the wave number. The right-hand side is a simplified model for a source at $\mathbf{x}_q = (0.21, 1.28)$ (e.g., a loudspeaker) with the intensity $|u|$, $u \in \mathbb{C}$, and shape function

$$b(\mathbf{x}) = \frac{1}{10} \exp\bigg(-\frac{1}{0.02}\big((x - 0.21)^2 + (y - 1.28)^2\big)\bigg) \quad \text{for } \mathbf{x} = (x, y) \in \Omega.$$

For the normal impedance boundary condition (5.1b) let $\jmath$ be the imaginary unit and $\frac{\partial}{\partial n}$ denote the derivative in the outward normal direction. All other parts on the boundary are assumed to be perfectly rigid, see (5.1c). We suppose that for all values of $Z \in \mathbb{C}$, plotted in Figure 2, and for all f in the frequency range from 200 to 400 $[Hz]$, problem (5.1) admits a unique solution. Due to the Fredholm theory, [24], we can ensure existence of a solution provided k^2 is not an eigenvalue of $-\Delta$ considered on Ω with Neumann and Robin boundary conditions on Γ_N respectively Γ_R.

Now we define the data such that the optimal solution is known in advance. To this aim, let $u_\circ(f) = 2\cos(\pi(f-200)/50) + 2\jmath\sin(\pi(f-200)/50)$ (see Figure 3) and let $p_\circ = p_\circ(f)$ be the unique solution to (5.1) for the choice $u = u_\circ(f)$ in (5.1a). We set $p_i^{\mathrm{m}} = p_\circ(\mathbf{x}_i)$, $i = 1, \ldots, 10$, with 10 different observation points $\mathbf{x}_i \in \Omega \cup \Gamma_N$,

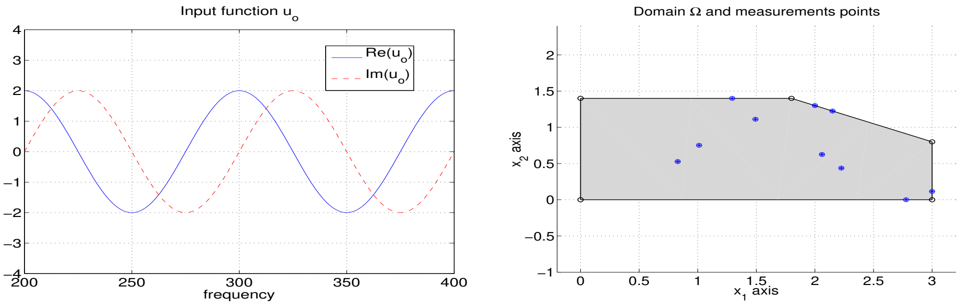

FIGURE 3. Run 2: Input function $u_\circ$ for the frequencies from 200 to $400Hz$ (left plot); observation points in $\Omega \cup \Gamma_N$ (right plot).

$1 \le i \le 10$; see Figure 3. Introducing the quadratic cost functional

$$J(p,u) = \frac{1}{200}\sum_{i=1}^{10} \big|p(\mathbf{x}_i) - p_i^{\mathrm{m}}\big|^2 + \frac{1}{2}\big|u - u_\circ\big|^2, \tag{5.2}$$

where $u_n = u_\circ$, and $|u|$ is the absolute modulus of the convex number u, we consider the optimal control problem

$$\min J(p,u) \quad \text{subject to} \quad (p,u) \text{ solves (5.1)} \tag{5.3}$$

over the frequency range from 200 to 400 $[Hz]$. Notice that $(p_\circ, u_\circ)$ must be the optimal solution to (5.3). The optimality conditions to (5.3) consist of the state equation, the adjoint system

$$\begin{aligned}
\Delta\lambda_\circ(\mathbf{x}) + k^2\lambda_\circ(\mathbf{x}) &= \frac{1}{100}\sum_{i=1}^{10}(p_i^{\mathrm{m}} - p_\circ(\mathbf{x}_i))\delta_{\mathbf{x}_i}(\mathbf{x}), && \text{for all } \mathbf{x} \in \Omega,\\
\frac{\jmath}{\varrho_\circ\omega}\frac{\partial\lambda_\circ(\mathbf{x})}{\partial n} &= -\frac{\lambda_\circ(\mathbf{x})}{\overline{Z}} && \text{for all } \mathbf{x} \in \Gamma_R,\\
\frac{\jmath}{\varrho_\circ\omega}\frac{\partial\lambda_\circ(\mathbf{x})}{\partial n} &= 0 && \text{for all } \mathbf{x} \in \Gamma_N
\end{aligned} \tag{5.4}$$

and the equation

$$\big(u_\circ - u_n\big) - \int_\Omega b(\mathbf{x})\lambda_\circ(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \quad \text{in } \mathbb{C}, \tag{5.5}$$

where $\delta_{\mathbf{x}_i}$ denotes the Dirac delta distribution satisfying

$$\langle \delta_{\mathbf{x}_i}, \varphi\rangle = \varphi(\mathbf{x}_i) \quad \text{for } \varphi \in C(\Omega \cup \Gamma_N) \text{ and } i = 1, \dots, 10.$$

Remark 5.14. The functional J contains point observations, hence the problem – besides the fact that the state equation is of different type than in the sections before – does formally not fit in our theory. Nevertheless, the perturbation analysis can be extended, and the numerical results show the efficiency of our approach. $\Diamond$

The domain Ω is discretized utilizing a standard piecewise linear FE discretization with $m = 2108$ degrees of freedom. To generate the snapshot ensemble we compute the FE solution p_h^j to (5.1) for the frequencies $f = 200, 201, \dots, 400$ and for $u = 1, \jmath$. Thus, we have $n = 402$ snapshots. Recall that also ω, k, and Z depend on f. In the context of Section 4.2 we choose the real part $y_j = \Re e(p_h^j) \in H^1(\Omega)$

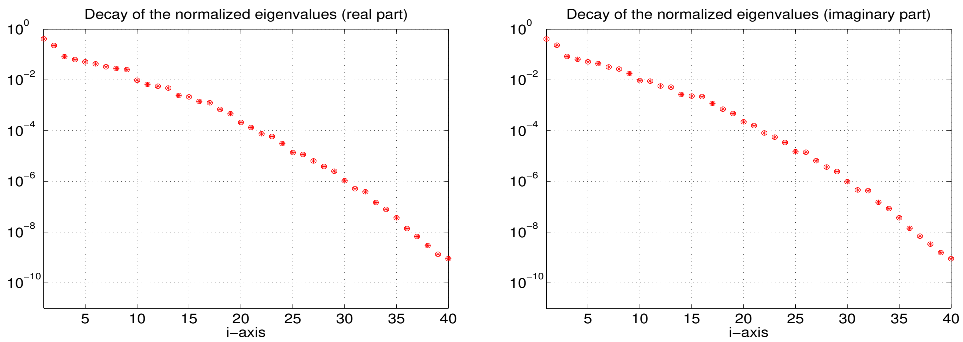

FIGURE 4. Run 2: Decay of the largest 40 normalized eigenvalues $\lambda_i / \sum_{i=1}^{d} \lambda_i$ for the real and the imaginary part of the snapshots.

of p_h^j, $1 \le j \le n$, and compute the solution to the eigenvalue problem 4.8. For the decay of the largest 40 eigenvalues we refer to Figure 4. Setting

$$\mathcal{E}_{\Re e}(\ell) = \frac{\sum_{i=1}^{\ell} \lambda_i}{\sum_{i=1}^{d} \lambda_i} \text{ (real part)} \quad \text{and} \quad \mathcal{E}_{\Im m}(\ell) = \frac{\sum_{i=1}^{\ell} \lambda_i}{\sum_{i=1}^{d} \lambda_i} \text{ (imaginary part)}$$

we found that $\mathcal{E}_{\Re e}(\ell)$ and $\mathcal{E}_{\Im m}(\ell)$ are approximately $1 - 4 \cdot 10^{-10}$, i.e., very close to one. Hence, we determine a POD basis of rank 40. By $\{\psi_i\}_{i=1}^{\ell}$ we denote the POD basis of rank ℓ for the real part. For the imaginary part $\Im m(p_h^j)$ of p_h^j we proceed analogously. The obtained POD basis is denoted by $\{\phi_i\}_{i=1}^{\ell}$ and the largest $\ell_{\max} = 40$ eigenvalues are shown in the right plot of Figure 4. For simplicity of the representation, we choose the same number of POD ansatz functions for the real and the imaginary parts which is not necessary. Now we make the POD Galerkin ansatz

$$p^\ell(\mathbf{x}; f) = \sum_{i=1}^{\ell} a_i \psi_i + \jmath b_i \phi_i, \quad a_i,\, b_i \in \mathbb{R},\ \mathbf{x} \in \Omega,\ 200\,[Hz] \le f \le 400\,[Hz]$$

with $3 \le \ell \le \ell_{\max}$ and derive the reduced-order model for (5.3). Then, we apply Algorithm 2. In Figure 5 the change of the number ℓ of POD basis functions depending on the frequencies is plotted.

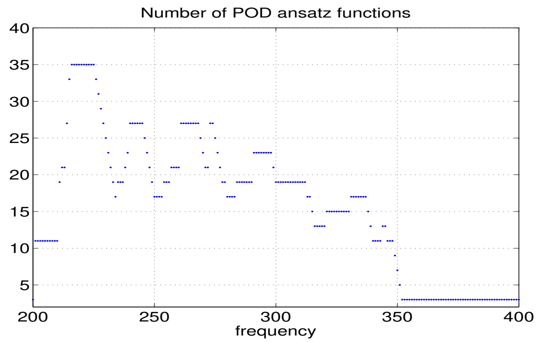

FIGURE 5: Change of the number ℓ of POD basis functions.

ℓ	$\lvert \bar{u}^{h,\tau} - \bar{u}^\ell \rvert_{\mathbb{C}}$	$\frac{1}{\sigma} \lvert \zeta^\ell \rvert_{\mathbb{C}}$
15	1.765e-1	1.770e-1
20	1.709e-2	1.714e-2
25	2.054e-3	2.060e-3
30	8.634e-5	8.660e-5
35	1.751e-6	1.756e-6
40	8.555e-8	8.581e-8

TABLE 3: $\lvert \bar{u} - \bar{u}^\ell \rvert_{\mathbb{C}}$ and $\frac{1}{\sigma} \lvert \zeta^\ell \rvert_{\mathbb{C}}$ for different ℓ and fixed frequency $f = 400\,[Hz]$.

Algorithm 2 Solver for (5.3) with POD a-posteriori estimator.

Choose $\ell = 3$, $\ell^{\max} = 40$, and $\varepsilon = 10^{-3}$. Set flag $fl = 0$.
for $i = 0$ **to** 200 **do**
  repeat
    Set the frequency $f = 200 + i$ and the number $\ell = \max(3, \ell - 2)$.
    Calculate the solution $\bar{u}^\ell \in \mathbb{C}$ to the reduced-order model for (5.3).
    Evaluate the solution $\hat{p} = \hat{p}(\bar{u}^\ell)$ to (5.1) for $u = \bar{u}^\ell$ and compute the solution $\hat{\lambda} = \hat{\lambda}(\bar{u}^\ell)$ to (5.4), where we replace $p_\circ$ by $\hat{p}$.
    Due to (5.5) set $\zeta^\ell = (\bar{u}^\ell - u_n) - \int_\Omega b(\mathbf{x})\hat{\lambda}(\mathbf{x})\,\mathrm{d}\mathbf{x}$.
    if $\|\zeta^\ell\|_{\mathbb{C}} < \varepsilon$ **or** $\ell = \ell^{\max}$ **then**
      Set $fl = 1$.
      Return fl, ℓ and suboptimal control $\bar{u}^\ell$ and STOP.
    else
      Set $\ell = \ell + 2$.
    end if
  until $\ell > \ell^{\max}$
  if $fl = 0$ **then**
    Increase $\ell^{\max}$ and restart the algorithm.
    BREAK.
  end if
end for

The decay of the error $|u_\circ - \bar{u}^\ell|_{\mathbb{C}}$ and the estimator $|\zeta^\ell|_{\mathbb{C}}$ are presented in Table 3 for fixed frequency $f = 400\,[Hz]$. It turns out that in this example the estimate is very close to the actual error in the optimal control. For other frequencies f in the frequency range from 200 to 400 $[Hz]$ the convergence behavior is similar. $\Diamond$

Appendix

Proof of Lemma 2.4. Let $v \in L^2(\mathfrak{I})$ be chosen arbitrarily. The claim is proven if we show

$$\big\langle \mathcal{S}^\star(\Xi(z_1, z_2) - \Theta(\bar{y})), v\big\rangle_{L^2(\mathfrak{I})} = \langle \mathcal{B}^\star \bar{p}, v\rangle_{L^2(\mathfrak{I})}. \tag{A.1}$$

Setting $w = \mathcal{S}v \in W(0,T)$ we infer $w(0) = 0$. From (2.2), (2.6), (2.8)-(2.10) and $w(0) = 0$ it follows that

$$\begin{aligned}
&\big\langle \mathcal{S}^\star(\Xi(z_1, z_2) - \Theta(\bar{y})), v\big\rangle_{L^2(\mathfrak{I})} \\
&\quad = \big\langle \Xi(z_1, z_2) - \Theta(\bar{y}), \mathcal{S}v\big\rangle_{W(0,T)', W(0,T)} = \langle \Xi(z_1, z_2) - \Theta(\bar{y}), w\rangle_{W(0,T)', W(0,T)} \\
&\quad = \alpha_1 \langle z_1 - \mathcal{C}\bar{y}, \mathcal{C}w\rangle_{W_1} + \alpha_2 \langle z_2 - \mathcal{D}\bar{y}(T), \mathcal{D}w(T)\rangle_{W_2} \\
&\quad = \int_0^T -\langle \bar{p}_t(t), w(t)\rangle_{V',V} + a(\bar{p}(t), w(t))\,\mathrm{d}t + \langle \bar{p}(T), w(T)\rangle_H \\
&\quad = \int_0^T \langle w_t(t), \bar{p}(t)\rangle_{V',V} + a(w(t), \bar{p}(t))\,\mathrm{d}t + \langle w(0), \bar{p}(0)\rangle_H \\
&\quad = \int_0^T \langle (\mathcal{B}v)(t), \bar{p}(t)\rangle_{V',V}\,\mathrm{d}t = \langle \mathcal{B}v, \bar{p}\rangle_{L^2(0,T;V'), L^2(0,T;V)} = \langle \mathcal{B}^\star \bar{p}, v\rangle_{L^2(\mathfrak{I})}
\end{aligned}$$

so that (A.1) holds. $\Box$

Proof of Proposition 4.3. For almost all $t \in [0,T]$ let

$$y^\ell - y(t) = y^\ell(t) - \mathcal{P}^\ell y(t) + \mathcal{P}^\ell y(t) - y(t) = \vartheta^\ell(t) + \varrho^\ell(t)$$

where $\vartheta^\ell(t) = y^\ell(t) - \mathcal{P}^\ell y(t) \in V^\ell$ and $\varrho^\ell(t) = \mathcal{P}^\ell y(t) - y(t)$. From (4.6) we have

$$\begin{aligned} \big\|\varrho^\ell\big\|^2_{W(0,T)} &= \int_0^T \big\|y(t) - \mathcal{P}^\ell y(t)\big\|^2_V + \big\|y_t(t) - \mathcal{P}^\ell y_t(t)\big\|^2_{V'}\,\mathrm{d}t \\ &= \sum_{i=\ell+1}^\infty \lambda_i + \big\|y_t - \mathcal{P}^\ell y_t\big\|^2_{L^2(0,T;V')}. \end{aligned} \tag{A.2}$$

Using (2.2) and (4.11) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \vartheta^\ell(t), \psi\rangle_H + a(\vartheta^\ell(t),\psi) = \langle y_t(t) - \mathcal{P}^\ell y_t(t), \psi\rangle_{V',V} \tag{A.3}$$

for all $\psi \in V^\ell$ and almost all $t \in [0,T]$. From choosing $\psi = \vartheta^\ell(t)$, (2.1) and Young's inequality we find

$$\begin{aligned} \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\big\|\vartheta^\ell(t)\big\|^2_H + \big\|\vartheta^\ell(t)\big\|^2_V &\le \big\|y_t(t) - \mathcal{P}^\ell y_t(t)\big\|_{V'}\big\|\vartheta^\ell(t)\big\|_V \\ &\le \frac{1}{2}\,\big\|y_t(t) - \mathcal{P}^\ell y_t(t)\big\|^2_{V'} + \frac{1}{2}\,\big\|\vartheta^\ell(t)\big\|^2_V \end{aligned}$$

which easily gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\big\|\vartheta^\ell(t)\big\|^2_H + \big\|\vartheta^\ell(t)\big\|^2_V \le \big\|y_t(t) - \mathcal{P}^\ell y_t(t)\big\|^2_{V'} \tag{A.4}$$

for almost all $t \in [0,T]$. Integrating (A.4) over the interval $(0,T)$, $t \in [0,T]$, and using (4.6) we arrive at

$$\big\|\vartheta^\ell(t)\big\|^2_H + \int_0^t \big\|\vartheta^\ell(s)\big\|^2_V\,\mathrm{d}s \le \big\|\vartheta^\ell(0)\big\|^2_H + \big\|y_t - \mathcal{P}^\ell y_t\big\|^2_{L^2(0,T;V')} \tag{A.5}$$

for almost all $t \in [0,T]$. From $\vartheta^\ell(0) = y^\ell(0) - \mathcal{P}^\ell y(0) = y^\ell(0) - \mathcal{P}^\ell y_0$ and (A.5) we have

$$\big\|\vartheta^\ell\big\|^2_{L^2(0,T;V)} \le \big\|y^\ell(0) - \mathcal{P}^\ell y_0\big\|^2_H + \big\|y_t - \mathcal{P}^\ell y_t\big\|^2_{L^2(0,T;V')}. \tag{A.6}$$

Utilizing (A.3) we find

$$\langle \vartheta^\ell_t(t), \psi\rangle_{V',V} = \langle y_t(t) - \mathcal{P}^\ell y_t(t), \psi\rangle_{V',V} - a(\vartheta^\ell(t),\psi)$$

for all $\psi \in V^\ell$ and almost all $t \in [0,T]$. Hence, from (2.1) and Cauchy-Schwarz inequality we derive

$$\begin{aligned} \big\|\vartheta^\ell\big\|_{L^2(0,T;V')} &= \int_0^T \sup_{\|\varphi\|_V=1} \langle \vartheta^\ell_t(t), \varphi\rangle_{V',V}\,\mathrm{d}t \\ &\le \int_0^T \Big(\big\|y_t(t) - \mathcal{P}^\ell y_t(t)\big\|_{V'} + \big\|\vartheta^\ell(t)\big\|_V\Big)\,\mathrm{d}t \\ &\le \sqrt{T}\,\Big(\big\|y_t - \mathcal{P}^\ell y_t\big\|_{L^2(0,T;V')} + \big\|\vartheta^\ell\big\|_{L^2(0,T;V)}\Big). \end{aligned}$$

Using (A.6) we find

$$\begin{aligned}\|\vartheta^\ell\|^2_{L^2(0,T;V')} &\le T\,\Big(\|y_t-\mathcal{P}^\ell y_t\|_{L^2(0,T;V')}+\|\vartheta^\ell\|_{L^2(0,T;V)}\Big)^2\\ &\le 2T\,\Big(\|y_t-\mathcal{P}^\ell y_t\|^2_{L^2(0,T;V')}+\|\vartheta^\ell\|^2_{L^2(0,T;V)}\Big)\\ &\le 2T\,\Big(2\,\|y_t-\mathcal{P}^\ell y_t\|^2_{L^2(0,T;V')}+\|y^\ell(0)-\mathcal{P}^\ell y_0\|^2_H\Big).\end{aligned}$$

Hence

$$\|\vartheta^\ell\|^2_{L^2(0,T;V')} \le 4T\,\Big(\|y_t-\mathcal{P}^\ell y_t\|^2_{L^2(0,T;V')}+\|y^\ell(0)-\mathcal{P}^\ell y_0\|^2_H\Big). \tag{A.7}$$

Consequently, (A.2), (A.6) and (A.7) imply that

$$\begin{aligned}\|y-y^\ell\|^2_{W(0,T)} &\le 2\,\Big(\|\vartheta^\ell\|^2_{L^2(0,T;V)}+\|\vartheta^\ell\|^2_{L^2(0,T;V')}+\|\varrho^\ell\|^2_{W(0,T)}\Big)\\ &= C\bigg(\|y^\ell(0)-\mathcal{P}^\ell y_0\|^2_H+\|y_t-\mathcal{P}^\ell y_t\|^2_{L^2(0,T;V')}+\sum_{i=\ell+1}^\infty\lambda_i\bigg)\end{aligned}$$

with $C=4(2T+1)$, so that the claim follows. □

Proof of Proposition 4.6. As in the proof of Proposition 4.3 we write

$$p^\ell(t)-p(t)=p^\ell(t)-\mathcal{P}^\ell p(t)+\mathcal{P}^\ell p(t)-p(t)=\theta^\ell(t)+\rho^\ell(t)$$

for almost all $t\in[0,T]$, where $\theta^\ell(t)=p^\ell(t)-\mathcal{P}^\ell p(t)\in V^\ell$ and $\rho^\ell(t)=\mathcal{P}^\ell p(t)-p(t)$ hold. From (2.6) and (4.19) we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\theta^\ell(t),\psi\rangle_H+a(\theta^\ell(t),\psi)=\langle\mathcal{C}(y-y^\ell)(t),\mathcal{C}\psi\rangle_{W_1}+\langle p_t(t)-\mathcal{P}^\ell p_t(t),\psi\rangle_{V',V}$$

for all $\psi\in V^\ell$ and almost all $t\in[0,T]$. Applying similar arguments as in the proof of Proposition 4.3 we arrive at

$$\|\theta^\ell\|^2_{L^2(0,T;V)}\le\|\theta^\ell(T)\|^2_H+c_1^2\,\|y-y^\ell\|^2_{L^2(0,T;H)}+\|p_t-\mathcal{P}^\ell p_t\|^2_{L^2(0,T;V')},$$

where $c_1=\sup_{\varphi\neq0}\|\mathcal{C}\varphi\|_{W_1}/\|\varphi\|_{L^2(0,T;H)}$. Utilizing $p(T)=\alpha_2\mathcal{D}^\star(z_2-\mathcal{D}y(T))\in H$ we derive

$$\begin{aligned}0&=\langle p^\ell(T)-\alpha_2\mathcal{D}^\star(z_2-\mathcal{D}y^\ell(T)),p^\ell(T)-\mathcal{P}^\ell p(T)\rangle_H\\ &=\langle p^\ell(T)-\mathcal{P}^\ell p(T)+\mathcal{P}^\ell p(T)-\alpha_2\mathcal{D}^\star(z_2-\mathcal{D}y^\ell(T)),p^\ell(T)-\mathcal{P}^\ell p(T)\rangle_H\\ &=\|p^\ell(T)-\mathcal{P}^\ell p(T)\|^2_H+\langle\mathcal{P}^\ell p(T)-p(T),p^\ell(T)-\mathcal{P}^\ell p(T)\rangle_H\\ &\quad+\langle\alpha_2\mathcal{D}^\star\mathcal{D}(y^\ell(T)-y(T)),p^\ell(T)-\mathcal{P}^\ell p(T)\rangle_H\end{aligned}$$

whivh gives

$$\begin{aligned}&\|p^\ell(T)-\mathcal{P}^\ell p(T)\|^2_H\\ &\quad=\langle p(T)-\mathcal{P}^\ell p(T)+\alpha_2\mathcal{D}^\star\mathcal{D}(y(T)-y^\ell(T)),p^\ell(T)-\mathcal{P}^\ell p(T)\rangle_H\\ &\quad\le\big(\|p(T)-\mathcal{P}^\ell p(T)\|_H+\alpha_2c_2^2\,\|y(T)-y^\ell(T)\|_H\big)\,\|p^\ell(T)-\mathcal{P}^\ell p(T)\|_H\end{aligned}$$

with $c_2=\sup_{\chi\neq0}\|\mathcal{D}\chi\|_{W_2}/\|\chi\|_H$. Hence,

$$\|\theta^\ell(T)\|_H=\|p^\ell(T)-\mathcal{P}^\ell p(T)\|_H\le\|p(T)-\mathcal{P}^\ell p(T)\|_H+\alpha_2c_2^2\,\|y(T)-y^\ell(T)\|_H$$

Consequently, there exists a constant $C > 0$ such that

$$\begin{aligned}\|\theta^\ell\|^2_{L^2(0,T;V)} &\leq C\Big(\big\|p(T) - \mathcal{P}^\ell p(T)\big\|_H^2 + \|p_t - \mathcal{P}^\ell p_t\|^2_{L^2(0,T;V)}\Big)\\ &\quad + C\Big(\big\|y(T) - y^\ell(T)\big\|_H^2 + \big\|y - y^\ell\big\|^2_{L^2(0,T;H)}\Big)\end{aligned}$$

so that (4.21) follows. By assumption, $p \in H^1(0,T;V)$ holds. Analogously to (4.14) and (4.15) we find

$$\begin{aligned}\lim_{\ell\to\infty}\big\|p - \mathcal{P}^\ell p\big\|^2_{W(0,T)} &= \big\|p - \mathcal{P}^\ell p\big\|^2_{L^2(0,T;V)} + \big\|p_t - \mathcal{P}^\ell p_t\big\|^2_{L^2(0,T;V')}\\ &\leq C\lim_{\ell\to\infty}\int_0^T\sum_{i=\ell+1}^\infty\Big(\big|\langle p(t),\psi_i\rangle_V\big|^2 + \big|\langle p_t(t),\psi_i\rangle_V\big|^2\Big)\,\mathrm{d}t = 0.\end{aligned}$$

From Proposition 4.4, Remark 4.5 and (4.21) we have $\lim_{\ell\to\infty}\|p-p^\ell\|_{L^2(0,T;V)} = 0.$ □

Proof of Proposition 4.8. Let $\bar u$ and $\bar u^\ell$ be the optimal solutions to $(\hat{\mathbf{P}})$ and $(\hat{\mathbf{P}}^\ell)$, respectively. From (2.7) and (4.22) we find

$$\langle \sigma\bar u - \mathcal{B}^\star\bar p, \bar u^\ell - \bar u\rangle_{L^2(\mathfrak{I})} \geq 0 \quad\text{and}\quad \langle \sigma\bar u^\ell - \mathcal{B}^\star\bar p^\ell, \bar u - \bar u^\ell\rangle_{L^2(\mathfrak{I})} \geq 0.$$

Adding both inequalities we deduce

$$\big\langle \sigma\big(\bar u - \bar u^\ell\big) + \mathcal{B}^\star\big(\bar p^\ell - \bar p\big), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})} \geq 0.$$

Applying Lemma 2.4 and (4.20) it follows that

$$\begin{aligned}\sigma\,\|\bar u - \bar u^\ell\|^2_{L^2(\mathfrak{I})} &\leq \big\langle \mathcal{B}^\star\big(\bar p^\ell - \bar p\big), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})}\\ &= \big\langle (S^\ell)^\star\big(\Xi(z_1,z_2) - \Theta(\bar y^\ell)\big) - \mathcal{S}^\star\big(\Xi(z_1,z_2) - \Theta(\bar y)\big), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})}\\ &= \big\langle \big((S^\ell)^\star - \mathcal{S}^\star\big)\Xi(z_1,z_2) + \mathcal{S}^\star\Theta(\bar y) - (S^\ell)^\star\Theta(\bar y^\ell), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})}.\end{aligned}$$

Recall that $\bar y = \hat y_0 + \mathcal{S}\bar u$ and $\bar y^\ell = \hat y_0^\ell + \mathcal{S}^\ell\bar u^\ell$ holds. Since Θ is a linear operator, we obtain

$$\begin{aligned}\big\langle \mathcal{S}^\star\Theta(\bar y) - (S^\ell)^\star\Theta(\bar y^\ell), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})} &= \big\langle \mathcal{S}^\star\Theta(\hat y_0) - (S^\ell)^\star\Theta(\hat y_0^\ell), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})}\\ &\quad + \big\langle \mathcal{S}^\star\Theta(\mathcal{S}\bar u) - (S^\ell)^\star\Theta(\mathcal{S}^\ell\bar u^\ell), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})}.\end{aligned}$$

From

$$\begin{aligned}&\big\langle \mathcal{S}^\star\Theta(\mathcal{S}\bar u) - (S^\ell)^\star\Theta(\mathcal{S}^\ell\bar u^\ell), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})}\\ &\quad = \big\langle \mathcal{S}^\star\Theta(\mathcal{S}\bar u) - (S^\ell)^\star\Theta(\mathcal{S}^\ell\bar u), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})} + \big\langle \mathcal{S}_\ell^\star\Theta(\mathcal{S}^\ell\bar u) - (S^\ell)^\star\Theta(\mathcal{S}^\ell\bar u^\ell), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})}\end{aligned}$$

and

$$\begin{aligned}\big\langle \mathcal{S}_\ell^\star\Theta(\mathcal{S}^\ell\bar u) - (S^\ell)^\star\Theta(\mathcal{S}^\ell\bar u^\ell), \bar u^\ell - \bar u\big\rangle_{L^2(\mathfrak{I})} &= -\big\langle \Theta\big(\mathcal{S}^\ell(\bar u - \bar u^\ell)\big), (S^\ell)^\star\big(\bar u - \bar u^\ell\big)\big\rangle_{L^2(\mathfrak{I})}\\ &= -\alpha_1\big\|\mathcal{C}(S^\ell)^\star(\bar u - \bar u^\ell)\big\|^2_{W_1} - \alpha_2\big\|\mathcal{D}(S^\ell)^\star(\bar u - \bar u^\ell)(T)\big\|^2_{W_2} \leq 0\end{aligned}$$

we conclude that

$$\begin{aligned}
\sigma \left\|\bar u - \bar u^\ell\right\|^2_{L^2(\mathfrak{I})} &\le \left\langle \mathcal{S}^\star\big(\Theta(\bar y) - \Xi(z_1,z_2)\big), \bar u^\ell - \bar u\right\rangle_{L^2(\mathfrak{I})}\\
&\quad + \left\langle (S^\ell)^\star\big(\Xi(z_1,z_2) - \Theta(\hat y_0^\ell + \mathcal{S}^\ell \bar u)\big), \bar u^\ell - \bar u\right\rangle_{L^2(\mathfrak{I})}\\
&= \left\langle \mathcal{B}^\star(\bar p - \hat p^\ell), \bar u^\ell - \bar u\right\rangle_{L^2(\mathfrak{I})}\\
&\le \|\mathcal{B}\|_{L(L^2(\mathfrak{I}),L^2(0,T;V))}\left\|\bar p - \hat p^\ell\right\|_{L^2(0,T;V)}\left\|\bar u - \bar u^\ell\right\|_{L^2(\mathfrak{I})},
\end{aligned} \tag{A.8}$$

where $\hat p^\ell$ solves (4.24) and $\hat y^\ell$ is the solution to (4.25). Thus, (A.8) implies that

$$\|\bar u - \bar u^\ell\|_{L^2(\mathfrak{I})} \le c\,\|\bar p - \hat p^\ell\|_{L^2(0,T;V)}.$$

with the constant $c = \|\mathcal{B}\|_{L(L^2(\mathfrak{I}),L^2(0,T;V))}/\sigma$. □

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