SUFFICIENT SECOND-ORDER OPTIMALITY CONDITIONS FOR SEMILINEAR CONTROL PROBLEMS WITH POINTWISE STATE CONSTRAINTS*

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Abstract. Second-order sufficient optimality conditions are established for the optimal control of semilinear elliptic and parabolic equations with pointwise constraints on the control and the state. In contrast to former publications on this subject, the cone of critical directions is the smallest possible in the sense that the second-order sufficient conditions are the closest to the associated necessary ones. The theory is developed for elliptic distributed controls in domains up to dimension three. Moreover, problems of elliptic boundary control and parabolic distributed control are discussed in spatial domains of dimension two and one, respectively.

Key words. optimal control, elliptic equations, parabolic equations, pointwise state constraints, second-order necessary optimality conditions, second-order sufficient optimality conditions

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1. Introduction. In this paper, we essentially improve the theory of second-order sufficient optimality conditions for state-constrained optimal control problems of elliptic and parabolic type. We derive second-order sufficient conditions that are as close as possible to the associated necessary ones. In this way, we are able to complete the theory of second-order sufficient conditions for this class of problems, if the dimension of the spatial domain is sufficiently small.

For the theory of nonconvex differentiable mathematical programming in finite-dimensional spaces, second-order sufficient optimality conditions are indispensible both in the numerical analysis and for reliable numerical methods. If second-order information is not available, then local minima will not in general be stable and numerical methods will most likely not converge. For instance, the convergence analysis of SQP methods relies heavily on second-order conditions.

In the numerical analysis of nonlinear optimal control problems, second-order sufficient optimality conditions are even more important. If they are not satisfied, then the (strong) convergence of optimal controls or states and/or error estimates for numerical discretizations of the problems can hardly be shown. Also other types of perturbations are difficult to handle without second-order conditions.

As it is well known from the calculus of variations and the control theory for nonlinear ordinary differential equations, the theory of second-order conditions is more delicate and rich in function spaces. We mention, for instance, the work by Maurer [20] or Maurer and Zowe [21]. In particular, the well-known two-norm discrepancy occurs that essentially complicates the analysis, cf. the expositions in Ioffe [16] or Malanowski [18]. For the important but more difficult case of pointwise state constraints in the control of ordinary differential equations, we refer to Malanowski [19] and to the references therein.

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At present, the control of distributed parameter systems with pointwise state constraints is a very active field of research. Although the majority of papers is still devoted to convex problems with linear equations, the important case of nonlinear state equations is attracting more interest. Here, second-order conditions are needed. However, when pointwise state constraints are imposed, the situation is more complicated, since the Lagrange multipliers associated with them are measures. In contrast to the theory for ordinary differential equations, this causes severe restrictions on the dimension of the spatial domains of the equations and reduces the regularity of the adjoint state.

To our best knowledge, there exist only two contributions to the theory of second-order sufficient conditions for distributed problems with pointwise state constraints. The elliptic case was discussed in [12], while parabolic problems were investigated in [22]. The method of these papers was inspired by the splitting technique used in [11]. Applied to pointwise state constraints, the cones of critical directions established by this technique are too large so that the second-order sufficient conditions are based on slightly too strong assumptions. Moreover, the method was fairly complicated.

For other contributions to second-order optimality conditions for distributed parameter systems we mention, for instance, the work by Bonnans [3] and the exposition in the monography by Bonnans and Shapiro [4] on elliptic problems with control constraints. We also refer to [9], where second-order necessary optimality conditions were first treated for elliptic problems and to Casas and Mateos [7], who considered elliptic problems with control constraints and state constraints of integral type. Moreover, we refer to the references therein.

In our new paper, the sufficiency of second-order conditions is proven by a method that is close to the theory of nonlinear optimization in finite-dimensional spaces. We establish a cone of critical directions that is sharp, i.e. it is the one, closest to the cone for establishing second-order necessary conditions.

We present a detailed proof for the case of distributed elliptic problems in domains of spatial dimension $n \le 3$. Moreover, we briefly sketch the extension of this result to elliptic boundary control problems for $n \le 2$ and to the parabolic distributed case for $n = 1$.

2. Problem statement. Let Ω be an open and bounded domain in $\mathbb{R}^n$, $n \le 3$, with a Lipschitz boundary Γ. In this domain we consider the following state equation

$$\left\{ \begin{aligned} Ay + f(x,y) &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma, \end{aligned} \right. \tag{2.1}$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function and A denotes a second-order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y(x))$$

and the coefficients $a_{ij} \in L^\infty(\Omega)$ satisfy

$$\lambda_A \|\xi\|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \quad \text{for a.e. } x \in \Omega,$$

for some $\lambda_A > 0$. In (2.1), the function u denotes the control and y_u is the solution associated to the control u. We will state later the conditions leading to the existence and uniqueness of a solution of (2.1) in $C(\bar{\Omega}) \cap H^1(\Omega)$.

In this paper, we study the following optimal control problem

$$
(\mathrm{P}) \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))\, dx \\ \text{subject to } \ (y_u, u) \in (C(\bar\Omega) \cap H^1(\Omega)) \times L^\infty(\Omega), \\ \alpha(x) \le u(x) \le \beta(x) \quad \text{for a.e. } \ x \in \Omega, \\ g(x, y_u(x)) \le 0 \ \ \forall x \in K, \end{cases}
$$

where $\alpha(x) < \beta(x)$ for almost all $x \in \Omega$, $\alpha, \beta \in L^\infty(\Omega)$, and $K \subset \bar\Omega$ is a compact set. Let us state the assumptions on the functions L, f and g.
(A1) f is of class C^2 with respect to the second variable,

$$
f(\cdot, 0) \in L^2(\Omega), \qquad \frac{\partial f}{\partial y}(x, y) \ge 0 \quad \text{ for a.e. } x \in \Omega
$$

and for all $M > 0$ there exists a constant $C_{f,M} > 0$ such that

$$
\left|\frac{\partial f}{\partial y}(x, y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x, y)\right| \le C_{f,M} \text{ for a.e. } x \in \Omega \text{ and } |y| \le M.
$$

$$
|\frac{\partial^2 f}{\partial y^2}(x, y_2) - \frac{\partial^2 f}{\partial y^2}(x, y_1)| \le C_{f,M}|y_2 - y_1| \ \text{ for } |y_1|, |y_2| \le M \text{ and for a.e. } x \in \Omega.
$$

(A2) $L : \Omega \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second and third variables, $L(\cdot, 0, 0) \in L^1(\Omega)$, and for all $M > 0$ there is a constant $C_{L,M} > 0$ and a function $\psi_M \in L^2(\Omega)$ such that

$$
\left|\frac{\partial L}{\partial u}(x, y, u)\right| + \left|\frac{\partial L}{\partial y}(x, y, u)\right| \le \psi_M(x), \quad \|D^2_{(y,u)} L(x, y, u)\| \le C_{L,M},
$$

$$
\|D^2_{(y,u)} L(x, y_2, u_2) - D^2_{(y,u)} L(x, y_1, u_1)\| \le C_{L,M}(|y_2 - y_1| + |u_2 - u_1|),
$$

for a.e. $x \in \Omega$ and $|y|, |y_i|, |u|, |u_i| \le M$, $i = 1, 2$, where $D^2_{(y,u)} L$ denotes the second derivative of L with respect to (y, u).

(A3) The function $g : K \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, of class C^2 with respect to the second variable and $\partial_y g$, and $\partial^2_y g$ are also continuous functions in $K \times \mathbb{R}$. Moreover we will assume that $g(x, 0) < 0$ is satisfied for every $x \in K \cap \Gamma$.

The following result on the existence of a solution holds true for (2.1) as well as for problem (P):

THEOREM 2.1. *Suppose that* **(A1)** *holds. Then, for every* $u \in L^2(\Omega)$*, the state equation* (2.1) *has a unique solution* $y_u \in C(\bar\Omega) \cap H^1_0(\Omega)$*. Furthermore, if* $u_k \rightharpoonup u$ *weakly in* $L^2(\Omega)$*, then* $y_{u_k} \to y_u$ *strongly in* $C(\bar\Omega) \cap H^1_0(\Omega)$.

The existence of a unique solution of (2.1) in $H^1(\Omega) \cap L^\infty(\Omega)$ is classical. It is a consequence of the monotonicity of f with respect to the second component. The continuity of y_u is also a well known result; see for instance [15]. The continuity property is a consequence of the compactness of the inclusion $L^2(\Omega) \subset W^{-1,p}(\Omega)$ for any $p < 6$ and the fact that data $u \in W^{-1,p}(\Omega)$, with $6/5 < p < 6$, provide solutions in $C(\bar{\Omega}) \cap H_0^1(\Omega)$, the mapping $u \to y_u$ being continuous between these spaces.

THEOREM 2.2. *Let the function L be convex with respect to the third component and the set of feasible controls be nonempty. Then, under assumptions **(A1)-(A3)**, the control problem* (P) *has at least one solution.*

The proof of this theorem can be obtained by standard arguments. We refer, for instance, to [25], Thm. 5.8.

REMARK 2.3. *We should remark that the differentiability of the functions f, L and g is not necessary to prove the previous theorems. In fact, the only properties we need are the continuity of g and f with respect to the second variable, the continuity of L with respect to the second and third variables, the monotonicity of f with respect to y, the convexity of L with respect to u and, for every $M > 0$, the existence of two functions $\phi_{f,M} \in L^2(\Omega)$ and $\phi_{L,M} \in L^1(\Omega)$ such that*

$$|f(x,y)| \leq \phi_{f,M}(x) \quad \textit{and} \quad |L(x,y,u)| \leq \phi_{L,M}(x) \textit{ for a.e. } x \in \Omega \textit{ and } |y|, |u| \leq M.$$

*These properties are an immediate consequence of the assumptions **(A1)-(A3)**.*

We finish this section by recalling some results about the differentiability of the non linear mappings involved in the control problem. For the detailed proofs, the reader is referred to Casas and Mateos [7].

THEOREM 2.4. *If* **(A1)** *holds, then the mapping $G : L^2(\Omega) \longrightarrow C(\bar{\Omega}) \cap H_0^1(\Omega)$, defined by $G(u) = y_u$ is of class C^2. Moreover, for all $v, u \in L^2(\Omega)$, $z_v = G'(u)v$ is defined as the solution of*

$$\begin{cases} Az_v + \dfrac{\partial f}{\partial y}(x, y_u) z_v = v & \text{in } \Omega \\ z_v = 0 & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Finally, for every $v_1, v_2 \in L^2(\Omega)$, $z_{v_1 v_2} = G''(u) v_1 v_2$ is the solution of

$$\begin{cases} Az_{v_1v_2} + \dfrac{\partial f}{\partial y}(x, y_u) z_{v_1v_2} + \dfrac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} = 0 \text{ in } \Omega \\ \hfill z_{v_1v_2} = 0 \text{ on } \Gamma, \end{cases} \tag{2.3}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

REMARK 2.5. *This theorem shows why we assume $n \leq 3$: To prove Theorem 4.1 on second-order sufficient conditions, we need that the operator G is differentiable from $L^2(\Omega)$ to $C(\bar{\Omega})$. This result holds only true for $n \leq 3$.*

The proof can be obtained by the implicit function theorem; see, for instance, [7, Theorem 2.5] for the proof in the case of a Neumann problem, which can be translated straightforward to the Dirichlet case.

THEOREM 2.6. *Suppose that* **(A1)** *and* **(A2)** *hold. Then $J : L^\infty(\Omega) \to \mathbb{R}$ is a functional of class C^2. Moreover, for every $u, v, v_1, v_2 \in L^\infty(\Omega)$*

$$J'(u)v = \int_\Omega \left(\frac{\partial L}{\partial u}(x, y_u, u) + \varphi_{0u} \right) v \, dx \tag{2.4}$$

and

$$\begin{aligned} J''(u)v_1v_2 = \int_\Omega \Bigg[& \frac{\partial^2 L}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 L}{\partial y \partial u}(x, y_u, u)(z_{v_1} v_2 + z_{v_2} v_1) \\ & + \frac{\partial^2 L}{\partial u^2}(x, y_u, u) v_1 v_2 - \varphi_{0u} \frac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} \Bigg] dx, \end{aligned} \tag{2.5}$$

where $y_u = G(u)$ *and* $\varphi_{0u} \in W^{2,p}(\Omega)$ *is the unique solution of the problem*

$$\begin{cases} A^*\varphi + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u, u) & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma, \end{cases} \tag{2.6}$$

A^* *being the adjoint operator of* A *and* $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

Let us remark that the linear and quadratic functionals $J'(u)$ and $J''(u)$ can be extended from $L^\infty(\Omega)$ to $L^2(\Omega)$ by the formulas (2.4) and (2.5). To check this point it is enough to use the assumptions **(A1)** and **(A2)**. This extension will be used in the rest of the paper.

The previous theorem and the next one follow easily from Theorem 2.4 and the chain rule.

THEOREM 2.7. *Suppose that* **(A1)** *and* ***(A3)*** *hold. Then the mapping* $F : L^2(\Omega) \to C(K)$*, defined by* $F(u) = g(\cdot, y_u(\cdot))$*, is of class* C^2*. Moreover, for every* $u, v, v_1, v_2 \in L^2(\Omega)$

$$F'(u)v = \frac{\partial g}{\partial y}(\cdot, y_u(\cdot))z_v(\cdot) \tag{2.7}$$

and

$$F''(u)v_1v_2 = \frac{\partial^2 g}{\partial y^2}(\cdot, y_u(\cdot))z_{v_1}(\cdot)z_{v_2}(\cdot) + \frac{\partial g}{\partial y}(\cdot, y_u(\cdot))z_{v_1v_2}(\cdot) \tag{2.8}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, *and* $z_{v_1v_2} = G''(u)v_1v_2$.

REMARK 2.8. *A functional L that is very frequently appearing in the applications is given by*

$$L(x, y, u) = L_0(x, y) + \frac{N}{2}u^2.$$

In this case, the functional J *is twice differentiable not only in* $L^\infty(\Omega)$*, but also in* $L^2(\Omega)$*. Indeed,* $J : L^2(\Omega) \longrightarrow \mathbb{R}$ *is of class* C^2 *and the derivatives are given by the expressions*

$$J'(u)v = \int_\Omega (Nu(x) + \varphi_{0u})\, v\, dx \tag{2.9}$$

and

$$J''(u)v_1v_2 = \int_\Omega \left[\frac{\partial^2 L_0}{\partial y^2}(x, y_u)z_{v_1}z_{v_2} + Nv_1v_2 - \varphi_{0u}\frac{\partial^2 f}{\partial y^2}(x, y_u)z_{v_1}z_{v_2}\right] dx. \tag{2.10}$$

REMARK 2.9. *The adjoint state* φ_{0u} *allows us to get a simple expression of* $J'(u)$ *but it is not the complete adjoint state of the control problem because the adjoint state equation* (2.6) *does not include the Lagrange multiplier associated to the state constraint; see equation* (3.2) *below for the full definition.*

3. First order optimality conditions. We define the Hamiltonian associated with the problem (P), $H^\lambda : \Omega \times \mathbb{R}^3 \longrightarrow \mathbb{R}$, by

$$H^\lambda(x, y, u, \varphi) = \lambda \cdot L(x, y, u) + \varphi\,[u - f(x, y)].$$

We denote by $M(K)$ the Banach space of all real and regular Borel measures in K, which is identified with the dual space of $C(K)$.

In the rest of the paper, a local minimum of (P) is assumed to be a local solution in the sense of the topology of $L^\infty(\Omega)$. More precisely, we will say that $\bar u$ is a local minimum or a local solution of (P) in the sense of $L^q(\Omega)$, $1 \le q \le \infty$, if it is an admissible control of (P) and there exists $\varepsilon_{\bar u} > 0$ such that the minimum of J in the admissible set of (P) intersected with the ball $\bar B_{\varepsilon_{\bar u}}(\bar u) \subset L^q(\Omega)$ is achieved at $\bar u$.

The following result concerning the Pontryagin's principle for problem (P) is well known; look into [8] and [17] as well as in the references therein for the proof.

Theorem 3.1. *Let $\bar u$ be a local solution of* (P) *and suppose that the assumptions* **(A1)**–**(A3)** *hold. Then there exist a real number $\bar\lambda \ge 0$, a measure $\bar\mu \in M(K)$ and a function $\bar\varphi \in W_0^{1,s}(\Omega)$, for all $1 \le s < n/(n-1)$, such that*

$$\bar\lambda + \|\bar\mu\| > 0 \tag{3.1}$$

$$\begin{cases} A^*\bar\varphi + \dfrac{\partial f}{\partial y}(x,\bar y(x))\bar\varphi = \bar\lambda\,\dfrac{\partial L}{\partial y}(x,\bar y,\bar u) + \dfrac{\partial g}{\partial y}(x,\bar y(x))\bar\mu \ \text{ in } \Omega, \\ \bar\varphi = 0 \ \text{ on } \Gamma, \end{cases} \tag{3.2}$$

$$\int_K (z(x) - g(x,\bar y(x))d\bar\mu(x) \le 0 \quad \forall z \in C(K) \text{ such that } z(x) \le 0\ \forall x \in K, \tag{3.3}$$

$$H^{\bar\lambda}(x,\bar y(x),\bar u(x),\bar\varphi(x)) = \min_{t\in[\alpha_{\varepsilon_{\bar u}}(x),\beta_{\varepsilon_{\bar u}}(x)]} H^{\bar\lambda}(x,\bar y(x),t,\bar\varphi(x)) \quad \text{for a.e. } x \in \Omega, \tag{3.4}$$

where

$$\alpha_{\varepsilon_{\bar u}}(x) = \max\{\alpha(x), \bar u(x) - \varepsilon_{\bar u}\} \quad \text{and} \quad \beta_{\varepsilon_{\bar u}}(x) = \min\{\beta(x), \bar u(x) + \varepsilon_{\bar u}\},$$

assumed that $\bar u$ is a minimum of (P) *in the ball $\bar B_{\varepsilon_{\bar u}}(\bar u) \subset L^\infty(\Omega)$. Moreover, if the following linearized Slater condition holds*

$$\begin{aligned} &\exists u_0 \in L^\infty(\Omega), \text{with } \alpha(x) \le u_0(x) \le \beta(x) \ \text{ for a.e. } x \in \Omega, \text{such that} \\ &g(x,\bar y(x)) + \frac{\partial g}{\partial y}(x,\bar y(x))z_{u_0-\bar u}(x) < 0 \quad \forall x \in K, \end{aligned} \tag{3.5}$$

where $\bar y$ is the state associated to $\bar u$ and $z_{u_0-\bar u} = G'(\bar u)(u_0 - \bar u)$, then the choice $\bar\lambda = 1$ can be made.

¿From now on, we take $\bar\lambda = 1$ and denote the Hamilton function for short by $H := H^1$.

Remark 3.2. *Together with the inequality $g(x,\bar y(x)) \le 0$, relation* (3.3) *is equivalent to the well-known complementarity conditions*

$$g(x,\bar y(x)) \le 0\ \forall x \in K, \quad \bar\mu \ge 0 \text{ in } M(K), \quad \text{and} \int_K g(x,\bar y(x))\,d\bar\mu(x) = 0.$$

It is also well known that (3.3) *implies that $\bar\mu$ is a positive measure concentrated on the set of points*

$$K_0 = \{x \in K : g(x,\bar y(x)) = 0\};$$

*see, for instance, the references given before the statement of the previous theorem. From this property and Assumption **(A3)**, we deduce that* $\bar\mu(K \cap \Gamma) = 0$.

Remark 3.3. *By using elementary calculus, we obtain from* (3.4) *that*

$$\frac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x))(k - \bar u(x)) \geq 0 \quad \forall k \in [\alpha(x), \beta(x)], \tag{3.6}$$

and

$$\frac{\partial^2 H}{\partial u^2}(x, \bar y(x), \bar u(x), \bar\varphi(x)) \geq 0 \quad \text{if} \quad \frac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x)) = 0, \tag{3.7}$$

for a.e. $x \in \Omega$. *On the other hand, notice that*

$$\frac{\partial^2 L}{\partial u^2}(x, y, u) = \frac{\partial^2 H}{\partial u^2}(x, y, u, \varphi). \tag{3.8}$$

The inequality (3.6) *implies that*

$$\begin{cases} \dfrac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x)) \geq 0 & \text{if} \ \ \bar u(x) = \alpha(x), \\ \dfrac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x)) \leq 0 & \text{if} \ \ \bar u(x) = \beta(x), \\ \dfrac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x)) = 0 & \text{if} \ \ \alpha(x) < \bar u(x) < \beta(x). \end{cases} \tag{3.9}$$

Reciprocally we also deduce from (3.6)

$$\begin{cases} \bar u(x) = \alpha(x) & \text{if} \ \ \dfrac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x)) > 0, \\ \bar u(x) = \beta(x) & \text{if} \ \ \dfrac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x)) < 0. \end{cases} \tag{3.10}$$

The properties given by (3.8) *and* (3.9) *are satisfied almost everywhere in* Ω.

Remark 3.4. *If we consider* $\bar u$ *in Theorem 3.1 to be a local minimum of* (P) *in the sense of* $L^q(\Omega)$, $1 \leq q < +\infty$, *then* (3.4) *can be written in the form, see [8],*

$$H^{\bar\lambda}(x, \bar y(x), \bar u(x), \bar\varphi(x)) = \min_{t \in [\alpha(x), \beta(x)]} H^{\bar\lambda}(x, \bar y(x), t, \bar\varphi(x)) \quad \text{for a.e. } x \in \Omega.$$

Let us formulate the Lagrangian version of the optimality conditions (3.2)-(3.4). The Lagrange function $\mathcal{L} : L^\infty(\Omega) \times M(K) \longrightarrow \mathbb{R}$ associated to the problem (P) is defined by

$$\mathcal{L}(u, \mu) = J(u) + \int_K g(x, y_u(x))\, d\mu(x) = \int_\Omega L(x, y_u(x), u(x))\, dx + \int_K g(x, y_u(x))\, d\mu(x).$$

Using (2.4) we find that

$$\frac{\partial \mathcal{L}}{\partial u}(u, \mu)v = \int_\Omega \left(\frac{\partial L}{\partial u}(x, y_u(x), u(x)) + \varphi_u(x)\right) v(x)\, dx = \int_\Omega H_u(x) v(x)\ dx, \tag{3.11}$$

where

$$H_u(x) = \frac{\partial H}{\partial u}(x, y_u(x), u(x), \varphi(x)) \tag{3.12}$$

and $\varphi_u \in W_0^{1,s}(\Omega)$, for all $1 \le s < n/(n-1)$, is the solution of the Dirichlet problem

$$\left\{\begin{array}{l} A^*\varphi + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u, u) + \dfrac{\partial g}{\partial y}(x, y_u(x))\mu \ \text{ in } \Omega \\ \varphi = 0 \ \text{ on } \Gamma. \end{array}\right. \tag{3.13}$$

Notice that the subscript u in y_u and H_u has a different meaning. While y_u is used to indicate that y is the state associated with u, H_u denotes the partial derivative of H with respect to u. This short notation for partial derivatives is frequently used in the following and will not cause confusion. Later, we also write H_{uu}, H_{yu}, or H_{yu} for $\partial^2 H/\partial u^2$, $\partial^2 H/\partial y \partial u$ etc.

If we insert $(\bar y(x), \bar u(x), \bar\varphi(x))$ in the expression (3.12), then we denote $H_u(x)$ by $\bar H_u(x)$.

Now the inequality (3.6) along with (3.11) leads to

$$\frac{\partial \mathcal{L}}{\partial u}(\bar u, \bar\mu)(u - \bar u) \ge 0 \ \text{ if } \alpha(x) \le u(x) \le \beta(x) \text{ for a.e. } x \in \Omega, \tag{3.14}$$

for any local solution $\bar u$, where $\bar y$ is the associated state and $\bar\varphi$ is the adjoint state given by (3.2), provided that (3.5) holds.

Before finishing this section we provide the expression of the second derivative of the Lagrangian with respect to the control, which will be used in the next section. From (2.8) we get

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(u, \mu)v_1 v_2 = J''(u)v_1 v_2$$

$$+ \int_K \left[\frac{\partial^2 g}{\partial y^2}(x, y_u(x)) z_{v_1}(x) z_{v_2}(x) + \frac{\partial g}{\partial y}(x, y_u(x)) z_{v_1 v_2}(x)\right] d\mu(x).$$

By (2.5), this is equivalent to

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial u^2}(u, \mu)v_1 v_2 = &\int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 L}{\partial y \partial u}(x, y_u, u)(z_{v_1} v_2 + z_{v_2} v_1)\right. \\ &\left. + \frac{\partial^2 L}{\partial u^2}(x, y_u, u) v_1 v_2 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2}\right] dx \\ &+ \int_K \frac{\partial^2 g}{\partial y^2}(x, y_u(x)) z_{v_1}(x) z_{v_2}(x)\, d\mu(x), \end{aligned} \tag{3.15}$$

where φ_u is the solution of (3.13).

4. Second-order optimality conditions. Let $\bar u$ be a feasible control of problem (P) and $\bar y$ be the associated state. We assume that there exist $\bar\mu \in M(K)$ and $\bar\varphi \in W_0^{1,s}(\Omega)$, $1 \le s < n/(n-1)$, such that (3.2)-(3.4) are satisfied. As in the previous section, we use the notation

$$\bar H_u(x) := \frac{\partial H}{\partial u}(x, \bar y(x), \bar u(x), \bar\varphi(x)).$$

The partial derivative of H with respect to y at $(x, \bar y(x), \bar u(x), \bar\varphi(x)$ is denoted analogously by $\bar H_y(x)$.

Associated with $\bar{u}$, we define the cone of critical directions by

$$C_{\bar{u}} = \{h \in L^2(\Omega) : h \text{ satisfies (4.1), (4.2) and (4.3)}\},$$

$$h(x) = \begin{cases} \geq 0 \text{ if } \bar{u}(x) = \alpha(x), \\ \leq 0 \text{ if } \bar{u}(x) = \beta(x), \\ = 0 \text{ if } \bar{H}_u(x) \neq 0, \end{cases} \tag{4.1}$$

$$\frac{\partial g}{\partial y}(x, \bar{y}(x))z_h(x) \leq 0 \quad \text{if} \quad g(x, \bar{y}(x)) = 0, \tag{4.2}$$

$$\int_K \frac{\partial g}{\partial y}(x, \bar{y}(x))z_h(x)\, d\bar{\mu}(x) = 0. \tag{4.3}$$

If we think in terms of the finite dimensional-case, the inequality (4.2) says that the derivative of the state constraint in the direction h is non positive if the constraint is active and (4.3) states that this derivative is zero whenever the corresponding Lagrange multiplier is strictly positive. The relations (4.2)-(4.3) provide a convenient extension of the usual conditions in the finite-dimensional case.

We should mention that (4.3) is new in the context of infinite-dimensional optimization problems. In earlier papers on this subject, other extensions to the infinite-dimensional case were suggested. For instance, Maurer and Zowe [21] used first-order sufficient conditions to consider strict positivity of Lagrange multipliers. Inspired by their approach, in [12] an application to state-constrained elliptic boundary control was suggested. In terms of our problem, the equation (4.3) was relaxed by

$$\int_K \frac{\partial g}{\partial y}(x, \bar{y}(x))z_h(x)\, d\bar{\mu}(x) \geq -\varepsilon \int_{\Omega\setminus\Omega^\tau} |h(x)|\, dx$$

for some $\varepsilon > 0$, cf. [12], (5.15). Here, $\Omega^\tau \subset \Omega$ is the set of points, where $|\bar{H}_u(x)| \geq \tau$ holds true. We will prove that this relaxation is not necessary, which leads to a smaller cone of critical directions that seems to be optimal.

The sufficient second-order optimality conditions are given by the expressions (4.4) and (4.5) in the next theorem.

Theorem 4.1. *Let $\bar{u}$ a feasible control of problem* (P)*, $\bar{y}$ the associated state and $(\bar{\varphi}, \bar{\mu}) \in W_0^{1,s}(\Omega) \times M(K)$, for all $1 \leq s < n/(n-1)$, satisfying* (3.2)-(3.4)*. Assume further that there exist two constants $\omega > 0$ and $\tau > 0$ such that*

$$\frac{\partial^2 L}{\partial u^2}(x, \bar{y}(x), \bar{u}(x)) \geq \omega \quad \textit{if} \quad |\bar{H}_u(x)| \leq \tau, \quad \textit{for a.e. } x \in \Omega, \tag{4.4}$$

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})h^2 > 0 \quad \forall h \in C_{\bar{u}} \setminus \{0\}. \tag{4.5}$$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that for every admissible control u of problem (P) *the following inequality holds*

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \leq J(u) \quad \textit{if} \quad \|u - \bar{u}\|_{L^\infty(\Omega)} < \varepsilon. \tag{4.6}$$

Remark 4.2. *Thanks to* (3.8)*, we can compare the second-order necessary condition* (3.7) *with the sufficient one given by* (4.4)*. We do not only require the strict positivity on the second derivative of the Hamiltonian with respect to the control at* *the points where the first derivative vanishes, as in the finite dimensional case. We also impose the second derivative to be strictly positive whenever the first derivative is "small". This is the usual case when we pass from finite to infinite dimension. For an instructive example the reader is referred to [14].*

Inequality (4.4) *is satisfied if the second derivative of L with respect to u is strictly positive for any* $(y, u, \varphi) \in \mathbb{R}^3$ *and almost all* $x \in \Omega$. *This assumption implies that L is strictly convex with respect to u. We recall that the convexity of L with respect to u was necessary to prove the existence of an optimal control. Under this strict convexity assumption, the sufficient second-order optimality conditions are reduced to* (4.5). *This is the case when* $L(x, y, u) = L_0(x, y) + Nu^2/2$ *if* $N > 0$.

The condition (4.5) *seems to be natural. In fact, under some regularity assumption, we can expect the inequality*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})h^2 \geq 0 \quad \forall h \in C_{\bar{u}}$$

to be a necessary condition for local optimality. At least this is the case when the state constraints are of integral type, see [7], or when K is a finite set of points, see [6].

Proof of Theorem 4.1. We argue by contradiction. Suppose that $\bar{u}$ does not satisfy the quadratic growth condition (4.6). Then there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset L^2(\Omega)$ of feasible controls of (P) such that $u_k \to \bar{u}$ in $L^\infty(\Omega)$ and

$$J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} > J(u_k) \quad \forall k. \tag{4.7}$$

Let us take

$$\rho_k = \|u_k - \bar{u}\|_{L^2(\Omega)} \quad \text{and} \quad h_k = \frac{1}{\rho_k}(u_k - \bar{u}).$$

Since $\|h_k\|_{L^2(\Omega)} = 1$, we can extract a subsequence, denoted in the same way, such that $h_k \rightharpoonup h$ weakly in $L^2(\Omega)$. Now we split the proof in several steps.

Step 1: $\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})h = 0$. In the following, we write $y_k = y_{u_k}$.. Since u_k is feasible, it holds that $g(x, y_k(x)) \leq 0$ for every $x \in K$. By using (3.3) and (4.7) we obtain

$$J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} = \mathcal{L}(\bar{u}, \bar{\mu}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} > J(u_k) \geq \mathcal{L}(u_k, \bar{\mu}). \tag{4.8}$$

¿From the mean value theorem we know that

$$\mathcal{L}(u_k, \bar{\mu}) = \mathcal{L}(\bar{u}, \bar{\mu}) + \rho_k \frac{\partial \mathcal{L}}{\partial u}(v_k, \bar{\mu})h_k,$$

with v_k a point between $\bar{u}$ and u_k. This identity and (4.8) imply

$$\frac{\partial \mathcal{L}}{\partial u}(v_k, \bar{\mu})h_k < \frac{1}{k\rho_k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} = \frac{1}{k}\|u_k - \bar{u}\|_{L^2(\Omega)}.$$

Since $h_k \rightharpoonup h$ weakly in $L^2(\Omega)$, $v_k \to \bar{u}$ in $L^\infty(\Omega)$, $y_{v_k} \to \bar{y}$ in $C(\bar{\Omega}) \cap H^1_0(\Omega)$ and $\varphi_{v_k} \to \bar{\varphi}$ in $W^{1,s}_0(\Omega) \subset L^2(\Omega)$ for s close to $n/(n-1)$, we deduce from the above inequality and the expression of the derivative of the Lagrangian given by (3.11) that

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})h = \lim_{k\to\infty} \frac{\partial \mathcal{L}}{\partial u}(v_k, \bar{\mu})h_k \leq 0. \tag{4.9}$$

On the other hand, since $\alpha(x) \le u_k(x) \le \beta(x)$ holds for almost all $x \in \Omega$, we deduce from the variational inequality (3.14)

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})h_k = \rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})(u_k - \bar{u}) \ge 0,$$

which implies

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})h = \lim_{k\to\infty} \frac{\partial \mathcal{L}}{\partial u}(v_k, \bar{\mu})h_k \ge 0.$$

This inequality, along with (4.9), leads to

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})h = 0. \tag{4.10}$$

Step 2: $h \in C_{\bar{u}}$. We have to confirm (4.1)–(4.3). The set of functions of $L^2(\Omega)$ that are nonnegative if $\bar{u}(x) = \alpha(x)$ and nonpositive if $\bar{u}(x) = \beta(x)$, almost everywhere, is convex and closed. Therefore, it is weakly closed. Moreover $u_k - \bar{u}$ obviously belongs to this set, thus every h_k also does. Consequently, h belongs to the same set. Then (3.10), together with (3.12), implies

$$\int_\Omega |\bar{H}_u(x)h(x)|\,dx = \int_\Omega \bar{H}_u(x)h(x)\,dx = \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})h = 0,$$

hence $h(x) = 0$ if $\bar{H}_u(x) \ne 0$, which concludes the proof of (4.1).

Let us prove (4.2). From Theorem 2.4 we have

$$z_h = G'(\bar{u})h = \lim_{k\to\infty} \frac{(y_{\bar{u}+\rho_k h_k} - \bar{y})}{\rho_k} \quad \text{in } C(\bar{\Omega}) \cap H_0^1(\Omega),$$

which implies for every $x \in K$ such that $g(x, \bar{y}(x)) = 0$

$$\frac{\partial g}{\partial y}(x, \bar{y}(x))z_h(x) = \lim_{k\to\infty} \frac{[g(x, y_{\bar{u}+\rho_k h_k}(x)) - g(x, \bar{y}(x))]}{\rho_k} \le 0. \tag{4.11}$$

The last inequality follows from the fact that u_k is feasible, $\bar{u} + \rho_k h_k = u_k$, and consequently $g(x, y_{\bar{u}+\rho_k h_k}(x)) = g(x, y_{u_k}(x)) \le 0$ for every $x \in K$.

Finally, we prove (4.3). Taking $z = g(\cdot, y_{u_k}(\cdot))$ in (3.3), we get

$$\int_K \frac{\partial g}{\partial y}(x, \bar{y}(x))z_h(x)\,d\bar{\mu}(x) = \lim_{k\to\infty} \frac{1}{\rho_k} \int_K [g(x, y_{\bar{u}+\rho_k h_k}(x) - g(x, \bar{y}(x))]\,d\bar{\mu}(x) =$$

$$\lim_{k\to\infty} \frac{1}{\rho_k} \int_K [g(x, y_{u_k}(x)) - g(x, \bar{y}(x))]\,d\bar{\mu}(x) \le 0. \tag{4.12}$$

On the other hand, from (4.7) we find

$$J'(\bar{u})h = \lim_{k\to\infty} \frac{J(\bar{u} + \rho_k h_k) - J(\bar{u})}{\rho_k} = \lim_{k\to\infty} \frac{J(u_k) - J(\bar{u})}{\rho_k} \le \lim_{k\to\infty} \frac{\rho_k}{k} = 0. \tag{4.13}$$

Then (4.10), (4.12), (4.13) and the fact that

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu})h = J'(\bar{u})h + \int_K \frac{\partial g}{\partial y}(x, \bar{y}(x))z_h(x)\,d\bar{\mu}(x),$$

imply that

$$J'(\bar{u})h = \int_K \frac{\partial g}{\partial y}(x, \bar{y}(x)) z_h(x)\, d\bar{\mu}(x) = 0.$$

Thus (4.3) holds and we know $h \in C_{\bar{u}}$.

Step 3: $h = 0$. Taking into account (4.5), it is enough to prove that

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})h \leq 0. \tag{4.14}$$

For this purpose, we evaluate the Lagrangian. By a second-order Taylor expansion, we derive

$$\mathcal{L}(u_k, \bar{\mu}) = \mathcal{L}(\bar{u}, \bar{\mu}) + \rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu}) h_k + \frac{\rho_k^2}{2} \frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k, \bar{\mu}) h_k^2, \tag{4.15}$$

w_k being an intermediate point between $\bar{u}$ and u_k. From here we get

$$\rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu}) h_k + \frac{\rho_k^2}{2} \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) h_k^2$$

$$= \mathcal{L}(u_k, \bar{\mu}) - \mathcal{L}(\bar{u}, \bar{\mu}) + \frac{\rho_k^2}{2} \left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k, \bar{\mu}) \right] h_k^2. \tag{4.16}$$

Now (4.8) can be written

$$\mathcal{L}(u_k, \bar{\mu}) - \mathcal{L}(\bar{u}, \bar{\mu}) \leq \frac{\rho_k^2}{k}. \tag{4.17}$$

On the other hand, taking into account the expression (3.15) of the second derivative of the Lagrangian, the assumptions **(A1)-(A3)** and theorems 2.1 and 2.4, and the fact that $u_k \to \bar{u}$ in $L^\infty(\Omega)$ and $\|h_k\|_{L^2(\Omega)} = 1$, we obtain

$$\left| \left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k, \bar{\mu}) \right] h_k^2 \right| \leq \| \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k, \bar{\mu}) \|_{B(L^2(\Omega))} \|h_k\|^2_{L^2(\Omega)}$$

$$= \| \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k, \bar{\mu}) \|_{B(L^2(\Omega))} \to 0 \quad \text{when} \quad k \to \infty, \tag{4.18}$$

where $B(L^2(\Omega))$ is the space of quadratic forms in $L^2(\Omega)$.

Let us define

$$\Omega^\tau = \{x \in \Omega : |\bar{H}_u(x)| > \tau\}.$$

¿From (3.10) and the definition of h_k we know that $\bar{H}_u(x) h_k(x) \geq 0$ in Ω, therefore

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu}) h_k = \int_\Omega \bar{H}_u(x) h_k(x)\, dx \geq \int_{\Omega^\tau} |\bar{H}_u(x)||h_k(x)|\, dx \geq \tau \int_{\Omega^\tau} |h_k(x)|\, dx. \tag{4.19}$$

For any $\varepsilon > 0$ we can take k_ε such that

$$\|\rho_k h_k\|_{L^\infty(\Omega)} = \|\bar{u} - u_k\|_{L^\infty(\Omega)} < \varepsilon \quad \forall k \geq k_\varepsilon, \quad \text{for a.e. } x \in \Omega,$$

therefore

$$\frac{\rho_k^2 h_k^2(x)}{\varepsilon} \le \rho_k |h_k(x)| \quad \forall k \ge k_\varepsilon, \quad \text{for a.e. } x \in \Omega.$$

¿From this inequality and (4.19) it follows that

$$\rho_k \frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \bar{\mu}) h_k \ge \rho_k \tau \int_{\Omega^\tau} |h_k(x)|\, dx \ge \frac{\rho_k^2 \tau}{\varepsilon} \int_{\Omega^\tau} h_k^2(x)\, dx. \tag{4.20}$$

Collecting (4.16)-(4.18) and (4.20) and dividing by $\rho_k^2/2$ we obtain for any $k \ge k_\varepsilon$

$$\frac{2\tau}{\varepsilon} \int_{\Omega^\tau} h_k^2(x)\, dx + \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) h_k^2 \le \frac{2}{k} + \| \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k, \bar{\mu}) \|_{B(L^2(\Omega))}. \tag{4.21}$$

Next, we study the left hand side of this inequality. First of all let us notice that from (3.15) we obtain for any $v \in L^2(\Omega)$

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) v^2 = \int_\Omega \left[\bar{H}_{uu}(x) v^2(x) + 2\bar{H}_{uy}(x) z_v(x) v(x) + \bar{H}_{yy}(x) z_v^2(x) \right] dx$$

$$+ \int_K \frac{\partial^2 g}{\partial y^2}(x, \bar{y}(x)) z_v^2(x)\, d\bar{\mu}(x),$$

where

$$\bar{H}_{uu}(x) = \frac{\partial^2 H}{\partial u^2}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x))$$

and $\bar{H}_{uy}$ and $\bar{H}_{yy}$ are defined analogously. We also remind that

$$\bar{H}_{uu}(x) = \frac{\partial^2 L}{\partial u^2}(x, \bar{y}(x), \bar{u}(x)). \tag{4.22}$$

Then we have

$$\begin{aligned}
&\frac{2\tau}{\varepsilon} \int_{\Omega^\tau} h_k^2(x)\, dx + \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) h_k^2 \\
&\quad = \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x) \right) h_k^2(x)\, dx + \int_{\Omega \setminus \Omega^\tau} \bar{H}_{uu}(x) h_k^2(x)\, dx \\
&\quad + \int_\Omega \left[2\bar{H}_{uy}(x) z_{h_k}(x) h_k(x) + \bar{H}_{yy}(x) z_{h_k}^2(x) \right] dx \\
&\quad + \int_K \frac{\partial^2 g}{\partial y^2}(x, \bar{y}(x)) z_{h_k}^2(x)\, d\bar{\mu}(x).
\end{aligned} \tag{4.23}$$

From assumptions **(A1)-(A3)** we deduce the existence of $C > 0$ such that $|\bar{H}_{uu}(x)| \le C$ for a.e. $x \in \Omega$. Therefore we can take $\varepsilon > 0$ small enough so that the following inequality holds

$$\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x) \ge \frac{2\tau}{\varepsilon} - C > 0 \quad \text{for a.e. } x \in \Omega^\tau.$$

Thus

$$\liminf_{k \to \infty} \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x) \right) h_k^2(x)\, dx \ge \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x) \right) h^2(x)\, dx. \tag{4.24}$$

Moreover from (4.4) we have that $\bar{H}_{uu}(x) \geq \omega > 0$ in $\Omega \setminus \Omega^\tau$, therefore we also get

$$\liminf_{k\to\infty} \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h_k^2(x)\,dx \geq \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h^2(x)\,dx. \tag{4.25}$$

Finally, taking into account that $z_{h_k} \to z_h$ strongly in $C(\bar{\Omega}) \cap H_0^1(\Omega)$, we deduce from (4.21)-(4.24) and (4.18)

$$\int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x)\right) h^2(x)\,dx + \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h^2(x)\,dx$$

$$+ \int_\Omega [2\bar{H}_{uy}(x)h(x)z_h(x) + \bar{H}_{yy}(x)z_h^2(x)]dx + \int_K \frac{\partial^2 g}{\partial y^2}(x,\bar{y}(x))z_h^2(x)\,d\bar{\mu}(x) \leq 0. \tag{4.26}$$

This expression can be written as follows

$$\frac{2\tau}{\varepsilon} \int_{\Omega^\tau} h^2(x)\,dx + \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\mu})h^2 \leq 0,$$

which along with (4.5) and the fact that $h \in C_{\bar{u}}$ implies that $h = 0$.

Step 4: $h_k \to 0$ strongly in $L^2(\Omega)$. We have already proved that $h_k \rightharpoonup 0$ weakly in $L^2(\Omega)$, therefore $z_{h_k} \to 0$ strongly in $C(\bar{\Omega}) \cap H_0^1(\Omega)$. By using (4.21) and (4.23) and the fact that $\|h_k\|_{L^2(\Omega)} = 1$ we conclude

$$0 < \min\{\omega, \frac{2\tau}{\varepsilon} - C\} = \min\{\omega, \frac{2\tau}{\varepsilon} - C\} \limsup_{k\to\infty} \int_\Omega h_k^2(x)\,dx$$

$$\leq \limsup_{k\to\infty} \left\{ \int_{\Omega^\tau} \left(\frac{2\tau}{\varepsilon} + \bar{H}_{uu}(x)\right) h_k^2(x)\,dx + \int_{\Omega\setminus\Omega^\tau} \bar{H}_{uu}(x) h_k^2(x)\,dx \right\}$$

$$\leq \limsup_{k\to\infty} \left\{ \frac{2}{k} + \|\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\mu}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(w_k,\bar{\mu})\|_{B(L^2(\Omega))} \right.$$

$$\left. - \int_K \frac{\partial^2 g}{\partial y^2}(x,\bar{y}(x))z_{h_k}^2(x)\,d\bar{\mu}(x) - \int_\Omega \left[2\bar{H}_{uy}(x)z_{h_k}(x)h_k(x) + \bar{H}_{yy}(x)z_{h_k}^2(x)\right]dx \right\} = 0.$$

Thus we have got the contradiction. □

There is a very interesting particular case of (P) where Theorem 4.1 has a stronger formulation.

THEOREM 4.3. *Assume that $L(x,y,u) = L_0(x,y) + Nu^2/2$, with $N > 0$. If $\bar{u}$ is a feasible control of problem* (P)*, $\bar{y}$ is the associated state, $(\bar{\varphi},\bar{\mu}) \in W_0^{1,s}(\Omega) \times M(K)$, for all $1 \leq s < n/(n-1)$, and $(\bar{y},\bar{u},\bar{\varphi},\bar{\mu})$ satisfies* (3.2)-(3.4) *and* (4.5)*, then there exist $\varepsilon > 0$ and $\delta > 0$ such that for every admissible control u of problem* (P) *the following inequality holds*

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \leq J(u) \quad \text{if} \quad \|u - \bar{u}\|_{L^2(\Omega)} < \varepsilon. \tag{4.27}$$

We have already mentioned in Remark 4.2 that the first-order optimality conditions along with (4.5) are sufficient for optimality when $L(x,y,u) = L_0(x,y)+Nu^2/2$, with $N > 0$. But the above theorem includes another very important information. Relation (4.27) says that $\bar{u}$ is a strict local minimum of (P) in $L^2(\Omega)$. The fact that the control appears linearly in the state equation and quadratically in the cost functional allows us to get sufficient optimality conditions for a local minimum not only in $L^\infty(\Omega)$ but also in $L^2(\Omega)$. This fact is very important in the analysis of stability and convergence of numerical algorithms to solve (P). The proof of Theorem 4.3 follows the same arguments and steps than those given in the proof of Theorem 4.1. The essential fact is that the functional J is of class C^2 in $L^2(\Omega)$; see Remark 2.8.

5. Bilateral state constraints. In this section we will consider the extension of the control problem to the case of bilateral state constraints. More precisely we formulate the control problem as follows

$$
\text{(P)} \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))\,dx \\ \text{subject to } \ (y_u, u) \in (C(\bar{\Omega}) \cap H^1(\Omega)) \times L^\infty(\Omega), \\ \alpha(x) \le u(x) \le \beta(x) \quad \text{for a.e.} \quad x \in \Omega, \\ g_a(x) \le g(x, y_u(x)) \le g_b(x) \quad \forall x \in K, \end{cases}
$$

where $g_a, g_b : K \mapsto \mathbb{R}$ are continuous functions and $g_a(x) < g_b(x)$ for every $x \in K$. We assume the same hypotheses as in the previous sections. All the previous theorems remain valid with some obvious modifications that we are going to mention. The Slater assumption required in Theorem 3.5 is now formulated as follows

$$
\begin{aligned} &\exists u_0 \in L^\infty(\Omega), \text{with } \alpha(x) \le u_0(x) \le \beta(x) \text{ for a.e. } x \in \Omega, \text{such that} \\ &g_a(x) < g(x, \bar{y}(x)) + \frac{\partial g}{\partial y}(x, \bar{y}(x)) z_{u_0 - \bar{u}}(x) < g_b(x) \quad \forall x \in K. \end{aligned} \tag{5.1}
$$

Under this assumption, Theorem 3.1 remains valid except for (3.3) which is written now in the following way

$$
\int_K (z(x) - g(x, \bar{y}(x)) d\bar{\mu}(x) \le 0 \ \forall z \in C(K) \text{ with } g_a(x) \le z(x) \le g_b(x) \ \forall x \in K. \tag{5.2}
$$

From (5.2) we deduce that $\bar{\mu}$ is concentrated at the set of points K_0 where the state constraint is active

$$
K_0 = K_- \cup K_+ = \{x \in K : g(x, \bar{y}(x)) = g_a(x)\} \cup \{x \in K : g(x, \bar{y}(x)) = g_b(x)\}.
$$

Now the Lagrange multiplier $\bar{\mu}$ is not necessarily a positive measure. However, its Jordan decomposition into nonnegative measures $\bar{\mu}^+$, $\bar{\mu}^-$ is as follows

$$
\bar{\mu} = \bar{\mu}^+ - \bar{\mu}^-, \quad \text{with } \operatorname{supp}(\bar{\mu}^+) \subset K_+ \text{ and } \operatorname{supp}(\bar{\mu}^-) \subset K_-.
$$

The cone of critical directions $C_{\bar{u}}$ is formed by the functions $h \in L^2(\Omega)$ satisfying (4.1) and

$$\frac{\partial g}{\partial y}(x,\bar y(x))z_h(x) \le 0 \ \text{ if } \ g(x,\bar y(x)) = g_b(x), \tag{5.3}$$
$$\frac{\partial g}{\partial y}(x,\bar y(x))z_h(x) \ge 0 \ \text{ if } \ g(x,\bar y(x)) = g_a(x), \tag{5.4}$$
$$\int_K |\frac{\partial g}{\partial y}(x,\bar y(x))z_h(x)|\, d|\bar\mu|(x) = 0, \tag{5.5}$$

where $|\bar\mu| = \bar\mu^+ + \bar\mu^-$. Then Theorem 4.1 is still true and the only changes of the proof appear in the steps 1 and 2. In particular, (4.8) can be rewritten with the help of (3.3) in the following way

$$\mathcal{L}(\bar u,\bar\mu) + \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Omega)} - \int_K g(x,\bar y(x))\, d\bar\mu(x) = J(\bar u) + \frac{1}{k}\|u_k - \bar u\|^2_{L^2(\Omega)}$$

$$> J(u_k) \ge \mathcal{L}(u_k,\bar\mu) - \int_K g(x,\bar y(x))\, d\bar\mu(x) \ge \mathcal{L}(u_k,\bar\mu)$$

and the proof can continue as in Theorem 4.1.

On the other hand, relation (4.11) in step 2 must be replaced by

$$\frac{\partial g}{\partial y}(x,\bar y(x))z_h(x) = \begin{cases} \le 0 & \forall x \in K_+, \\ \ge 0 & \forall x \in K_-. \end{cases} \tag{5.6}$$

Relations (4.12) and (4.13) remain valid. Finally, using (4.10) and (5.6) we deduce the identity (5.5) as follows

$$\int_K |\frac{\partial g}{\partial y}(x,\bar y(x))z_h(x)|\, d|\bar\mu|(x) = -\int_K \frac{\partial g}{\partial y}(x,\bar y(x))z_h(x)\, d\bar\mu(x) = J'(\bar u)h = 0.$$

6. Elliptic boundary control.

6.1. Problem statement. The method of the preceding sections can be extended to other types of equations in a straightforward way. Here, we discuss the case of boundary control, while the next section is devoted to a one-dimensional distributed parabolic control problem. Instead of the equation (2.1), we consider now

$$\begin{cases} Ay + f(x,y) = 0 & \text{in } \Omega, \\ \partial_\nu y + \gamma\, y = u & \text{on } \Gamma, \end{cases} \tag{6.1}$$

where ∂_ν denotes the conormal-derivative associated with A and $\gamma \in L^\infty(\Gamma)$ is nonnegative with $\gamma \not\equiv 0$. In contrast to Section 1, we assume here that $n = 2$. We need this stronger assumption, since now the control-to-state mapping G must be twice continuously differentiable from $L^2(\Gamma)$ to $C(\bar\Omega)$, cf. Remark 2.5. The differential operator A is defined as in Section 1.

We consider the optimal boundary control problem

$$\text{(PB)} \begin{cases} \min J(u) = \displaystyle\int_\Omega L(x,y_u(x))\,dx + \int_\Gamma \ell(x,y_u(x),u(x))\,ds(x) \\ \text{subject to } \ (y_u,u) \in (C(\bar\Omega)\cap H^1(\Omega)) \times L^\infty(\Gamma), \\ \alpha(x) \le u(x) \le \beta(x) \quad \text{for a.e. } \ x \in \Gamma, \\ g(x,y_u(x)) \le 0 \ \ \forall x \in K. \end{cases} \tag{6.2}$$

Here, α, β are now functions from $L^\infty(\Gamma)$ with $\alpha(x) \le \beta(x)$ for a.a. $x \in \Gamma$, ds denotes the surface measure on Γ, y_u is the solution of (6.1) associated with $u \in L^2(\Gamma)$, and $K \subset \bar\Omega$ is again a compact set.

The following assumptions are imposed on the data: We assume (A1)–(A3) on f, L, and g (where, of course, the dependence of L on u in (A2) is redundant). Moreover, we require:

(A4) The function $\ell : \Gamma \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathbb{R}$ satisfies Assumption (A2) with ℓ substituted for L and Γ substituted for Ω.

REMARK 6.1. *We confine ourselves to a linear boundary condition. An extension to a nonlinear condition of the type $\partial_\nu y + b(x, y) = u$ is possible under associated assumptions on b. On the other hand, the assumption $\gamma \not\equiv 0$, that allows to deduce the existence of a unique solution of* (6.1)*, can be replaced by*

$$\frac{\partial f}{\partial y}(x, t) > 0 \quad \textit{for all } x \in E \textit{ and } t \in \mathbb{R},$$

where E is a measurable subset of Ω with a strictly positive measure.

The proof of the next theorems is completely analogous to that of Theorem 2.2 and Theorem 2.4; see Alibert and Raymond [1].

THEOREM 6.2. *Suppose that* **(A1)** *holds. Then, for every $u \in L^2(\Gamma)$, the state equation* (6.1) *has a unique solution $y_u \in C(\bar\Omega) \cap H^1(\Omega)$. Furthermore, if $u_k \rightharpoonup u$ weakly in $L^2(\Gamma)$, then $y_{u_k} \to y_u$ strongly in $C(\bar\Omega) \cap H^1(\Omega)$.*

Notice that controls of $L^2(\Gamma)$ are transformed continuously to states in the Hölder space $C^{0,\kappa}(\Omega)$ with some $0 < \kappa < 0$, cf. Stampacchia [23], Thm. 14.2. The second part of the statement is an immediate conclusion.

THEOREM 6.3. *Assume that **(A1)-(A4)** are fulfilled, the function ℓ is convex with respect to the third component and the set of feasible controls is nonempty. Then the control problem (PB) has at least one solution.*

The proof can be performed by standard methods.

6.2. Necessary optimality conditions. We first state results on the first- and second-order derivatives of the control-to-state mapping $G(u) = y_u$ and of the reduced objective functional J. The results are analogous to the theorems 2.6–2.7 so that we only collect them without proof, since the associated modifications are obvious. Under the assumptions **(A1)-(A4)**, all mappings listed below are of class C^2 from $L^\infty(\Gamma)$ to their respective image spaces. The associated derivatives can be obtained as follows:

We define, for $v \in L^2(\Gamma)$, the function z_v as the unique solution to

$$\begin{cases} Az_v + \dfrac{\partial f}{\partial y}(x, y_u) z_v = 0 \quad \text{in } \Omega \\ \partial_\nu z_v + \gamma z_v = v \quad \text{on } \Gamma. \end{cases} \tag{6.3}$$

Then G' is given by $G'(u)v = z_v$. Moreover, for v_1, $v_2 \in L^2(\Gamma)$, we introduce $z_{v_i} = G'(u)v_i$, $i = 1, 2$, and obtain $G''(u)v_1v_2 = z_{v_1v_2}$ where $z_{v_1v_2}$ is the solution to

$$\begin{cases} Az_{v_1v_2} + \dfrac{\partial f}{\partial y}(x, y_u) z_{v_1v_2} + \dfrac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} = 0 \text{ in } \Omega \\ \qquad\qquad\qquad\qquad \partial_\nu z_{v_1v_2} + \gamma\, z_{v_1v_2} = 0 \text{ on } \Gamma. \end{cases} \tag{6.4}$$

The adjoint state $\varphi_{0u} \in H_0^1(\Omega)$ associated with u and J is introduced as the unique solution to

$$\left\{\begin{array}{l} A^*\varphi + \dfrac{\partial f}{\partial y}(x,y_u)\varphi = \dfrac{\partial L}{\partial y}(x,y_u) \ \text{ in } \Omega \\ \partial_\nu \varphi + \gamma\,\varphi = \dfrac{\partial \ell}{\partial y}(x,y_u,u) \ \text{ on } \Gamma. \end{array}\right. \tag{6.5}$$

It holds that

$$J'(u)v = \int_\Gamma \left(\frac{\partial \ell}{\partial u}(x,y_u,u) + \varphi_{0u}\right) v\,ds, \tag{6.6}$$

$$\begin{aligned} J''(u)v_1v_2 \quad = \quad & \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,y_u,u)z_{v_1}z_{v_2} - \varphi_{0u}\frac{\partial^2 f}{\partial y^2}(x,y_u)z_{v_1}z_{v_2}\right] dx \\ & + \int_\Gamma \left[\frac{\partial^2 \ell}{\partial y^2}(x,y_u,u)z_{v_1}z_{v_2} + \frac{\partial^2 \ell}{\partial y \partial u}(x,y_u,u)(z_{v_1}v_2 + z_{v_2}v_1) \right. \\ & \left. + \frac{\partial^2 \ell}{\partial u^2}(x,y_u,u)v_1v_2\right] ds. \end{aligned} \tag{6.7}$$

Under **(A1)** and **(A3)**, the mapping $F : L^2(\Gamma) \to C(K)$, defined by $F(u) = g(\cdot, y_u(\cdot))$, is of class C^2. For every $u, v, v_1, v_2 \in L^2(\Gamma)$, its first- and second-order derivatives are given again by (2.7) and (2.8), respectively.

Now we introduce the Hamiltonian H by

$$H(x,y,u,\varphi) = \ell(x,y,u) + \varphi\,[u - \gamma\,y].$$

The first-order necessary conditions admit the following form:

THEOREM 6.4. *Let $\bar u$ be a local solution of* (PB). *Suppose that the assumptions* **(A1)**–**(A4)** *hold and assume the linearized Slater condition* (3.5) *with some $u_0 \in L^\infty(\Gamma)$, $\alpha(x) \le u_0(x) \le \beta(x)$ for a.e. $x \in \Gamma$. Then there exists a measure $\bar\mu \in M(K)$ and a function $\bar\varphi \in W^{1,s}(\Omega)$ for all $1 \le s < n/(n-1)$ such that*

$$\left\{\begin{array}{l} A^*\bar\varphi + \dfrac{\partial f}{\partial y}(x,\bar y(x))\bar\varphi = \dfrac{\partial L}{\partial y}(x,\bar y,\bar u) + \dfrac{\partial g}{\partial y}(x,\bar y(x))\bar\mu_{|\Omega} \ \textit{ in } \Omega, \\ \partial_\nu \bar\varphi + \gamma\,\bar\varphi = \dfrac{\partial g}{\partial y}(x,\bar y(x))\bar\mu_{|\Gamma} \ \textit{ on } \Gamma, \end{array}\right. \tag{6.8}$$

$$\int_K (z(x) - g(x,\bar y(x))d\bar\mu(x) \le 0 \quad \forall z \in C(K) \textit{ such that } z(x) \le 0\ \forall x \in K, \tag{6.9}$$

$$H(x,\bar y(x),\bar u(x),\bar\varphi(x)) = \min_{t\in[\alpha_{\varepsilon_{\bar u}}(x),\beta_{\varepsilon_{\bar u}}(x)]} H(x,\bar y(x),t,\bar\varphi(x)) \ \textit{ for a.e. } x \in \Gamma, \tag{6.10}$$

where $\alpha_{\varepsilon_{\bar u}}$ and $\beta_{\varepsilon_{\bar u}}$ are defined similarly as in Theorem 3.1 and $\bar\mu_{|\Omega}$ and $\bar\mu_{|\Gamma}$ denote the restrictions of μ to Ω and Γ, respectively,

At the optimal point, the derivatives of H fulfil the relations (3.6) – (3.10) with obvious modification: We have to substitute $x \in \Gamma$ for $x \in \Omega$. Moreover, we have to replace (3.8) by

$$\frac{\partial^2 \ell}{\partial u^2}(x,y,u) = \frac{\partial^2 H}{\partial u^2}(x,y,u,\varphi). \tag{6.11}$$

The Lagrangian function $\mathcal{L} : L^\infty(\Gamma) \times M(K) \longrightarrow \mathbb{R}$ associated to the problem (PB) is defined by

$$\mathcal{L}(u,\mu) = \int_\Omega L(x, y_u(x))\,dx + \int_\Gamma \ell(x, y_u(x), u(x))\,ds + \int_K g(x, y_u(x))\,d\mu(x).$$

Using (6.6) we deduce that

$$\frac{\partial \mathcal{L}}{\partial u}(u,\mu)v = \int_\Gamma H_u(x)v(x)\ ds, \tag{6.12}$$

where

$$H_u(x) = \frac{\partial H}{\partial u}(x, y(x), u(x), \varphi_u(x)) \tag{6.13}$$

and φ_u is obtained from the adjoint equation (6.8), where y_u is substituted for $\bar{y}$, u for $\bar{u}$, and μ for $\bar{\mu}$, respectively. We finally indicate the expression for the second-order derivative of $\mathcal{L}$,

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial u^2}(u,\mu)v_1v_2 = &\int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_u)z_{v_1}z_{v_2} - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)z_{v_1}z_{v_2}\right] dx\\
&+ \int_\Gamma \left[\frac{\partial^2 \ell}{\partial y^2}(x, y_u, u)z_{v_1}z_{v_2} + \frac{\partial^2 \ell}{\partial y \partial u}(x, y_u, u)(z_{v_1}v_2 + z_{v_2}v_1)\right.\\
&\left. + \frac{\partial^2 \ell}{\partial u^2}(x, y_u, u)v_1v_2\right] ds\\
&+ \int_K \frac{\partial^2 g}{\partial y^2}(x, y_u(x))z_{v_1}(x)z_{v_2}(x)\,d\mu(x),\end{aligned} \tag{6.14}$$

where φ_u is defined as after (6.13).

6.3. Second-order sufficient optimality conditions. Let $\bar{u}$ be a feasible control of problem (PB) and $\bar{y}$ be the associated state. We assume that there exist $\bar{\mu} \in M(K)$ and $\bar{\varphi} \in W_0^{1,s}(\Omega)$, $1 \le s < n/(n-1)$, such that the first-order necessary conditions (6.8)-(6.10) are satisfied. Associated with $\bar{u}$, we introduce the function

$$\bar{H}_u(x) = \frac{\partial H}{\partial u}(x, \bar{y}(x), \bar{u}(x), \bar{\varphi}(x))$$

and define the cone of critical directions by

$$C_{\bar{u}} = \{h \in L^2(\Gamma) : h \text{ satisfies (4.1)– (4.3) with } x \in \Gamma\}. \tag{6.15}$$

Notice that this cone is only formally the same as in (4.1)– (4.3), since x varies here through Γ. The second-order sufficient condition admits now the following form:

THEOREM 6.5. *Assume that $n = 2$ and let $\bar{u}$ be a feasible control of problem* (PB)*, $\bar{y}$ the associated state and $(\bar{\varphi}, \bar{\mu}) \in W^{1,s}(\Omega) \times M(K)$, for all $1 \le s < n/(n-1)$, satisfying* (6.8)-(6.10)*. Let there exist two constants $\omega > 0$ and $\tau > 0$ such that*

$$\frac{\partial^2 \ell}{\partial u^2}(x, \bar{y}(x), \bar{u}(x)) \ge \omega \quad \textit{if} \quad |\bar{H}_u(x)| \le \tau, \quad \textit{for a.e. } x \in \Gamma, \tag{6.16}$$

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu})h^2 > 0 \quad \forall h \in C_{\bar{u}} \setminus \{0\}, \tag{6.17}$$

where $C_{\bar u}$ *is defined in* (6.15) *and* $\partial^2\mathcal{L}/\partial u^2$ *is taken from* (6.14) *with* $u := \bar u$ *and* $\mu := \bar\mu$.

Then there exist $\varepsilon > 0$ *and* $\delta > 0$ *such that, for every admissible control* u *of problem* (PB)*, the following inequality holds:*

$$J(\bar u) + \frac{\delta}{2}\|u-\bar u\|^2_{L^2(\Gamma)} \le J(u) \quad \textit{if} \quad \|u-\bar u\|_{L^\infty(\Gamma)} < \varepsilon. \tag{6.18}$$

Proof. The proof is almost identical with the one of Theorem 4.1. Therefore, we only mention, where essential changes occur.

Throughout the proof, we have to perform the obvious modification that $L^2(\Gamma)$, $L^\infty(\Gamma)$ and $H^1(\Omega)$ must be substituted for $L^2(\Omega)$, $L^\infty(\Omega)$, and $H^1_0(\Omega)$, respectively. Moreover, in some integrals, Ω must obviously be replaced by Γ. Then the steps 1 and 2 can be adopted without further changes.

Step 3: The arguments up to (4.18) do not need changes. Next, we modify Ω^τ by

$$\Gamma^\tau = \{x \in \Gamma \,:\, |\bar H_u(x)| > \tau\}.$$

Hereafter, Ω and Ω^τ are replaced by Γ and Γ^τ, respectively. In (4.22), ℓ must be substituted for L, and in equation (4.23) we add the integral over $\partial^2 L/\partial u^2$ to arrive at

$$\begin{aligned}
&\frac{2\tau}{\varepsilon}\int_{\Gamma^\tau} h_k^2(x)\,ds + \frac{\partial^2\mathcal{L}}{\partial u^2}(\bar u,\bar\mu)h_k^2\\
&\quad = \int_{\Gamma^\tau}\left(\frac{2\tau}{\varepsilon} + \bar H_{uu}(x)\right)h_k^2(x)\,ds + \int_{\Gamma\setminus\Gamma^\tau}\bar H_{uu}(x)h_k^2(x)\,ds\\
&\quad + \int_\Gamma\left[2\bar H_{uy}(x)z_{h_k}(x)h_k(x) + \bar H_{yy}(x)z_{h_k}^2(x)\right]ds\\
&\quad + \int_K \frac{\partial^2 g}{\partial y^2}(x,\bar y(x))z_{h_k}^2(x)\,d\bar\mu(x) + \int_\Omega \frac{\partial^2 L}{\partial y^2}(x,\bar y(x))z_{h_k}^2(x)\,dx.
\end{aligned}\tag{6.19}$$

Analogously, this term must be added to the left-hand side of (4.26)..

Step 4: First, we conclude from $h_k \rightharpoonup 0$ in $L^2(\Gamma)$ that $z_{h_k} \to 0$ strongly in $C(\bar\Omega)$. Proceeding as in the former Step 4, we finally conclude with

$$\begin{aligned}
0 \quad < \quad \limsup_{k\to\infty}\Bigg\{&\frac{2}{k} + \left\|\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar u,\bar\mu) - \frac{\partial^2\mathcal{L}}{\partial u^2}(w_k,\bar\mu)\right\|_{B(L^2(\Gamma))}\\
&- \int_K \frac{\partial^2 g}{\partial y^2}(x,\bar y(x))z_{h_k}^2(x)\,d\bar\mu(x) - \int_\Omega \frac{\partial^2 L}{\partial y^2}(x,\bar y(x))z_{h_k}^2(x)\,dx\\
&- \int_\Omega\left[2\bar H_{uy}(x)z_{h_k}(x)h_k(x) + \bar H_{yy}(x)z_{h_k}^2(x)\right]dx\Bigg\} = 0.
\end{aligned}$$

∎

7. The parabolic case.

7.1. Problem statement. Finally we prove that our method can also be extended to one-dimensional parabolic problems with distributed control. This extension is addressed here. To define the parabolic problem, we consider the one-dimensional domain $\Omega = (a,b)$ and the time interval [0,T] for given $T > 0$. We fix an initial value $y_0 \in C[a,b]$ and introduce the set $Q = (a,b)\times(0,T)$. Moreover, we introduce the space $W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) \,:\, \frac{dy}{dt} \in L^2(0,T;H^1(\Omega)')\}$.

REMARK 7.1. *Again, the restriction on the dimension of* Ω *comes from the requirement that the control-to-state mapping is of class* C^2 *from* $L^2(Q)$ *to* $C(\bar{Q})$. *This holds only true for* $n = 1$. *We should mention here that boundary controls cannot be handled by our approach. Neumann boundary data from* $L^2(0,T)$ *are not in general transformed into continuous states.*

The parabolic equation is defined by

$$\begin{cases} \dfrac{dy}{dt} + Ay + f(x,t,y) &=& u & \text{in } (a,b)\times(0,T), \\ -\partial_x y(a,t) &=& 0 & \text{in } (0,T), \\ \partial_x y(b,t) &=& 0 & \text{in } (0,T), \\ y(\cdot,0) &=& y_0 & \text{in } (a,b), \end{cases} \tag{7.1}$$

where ∂_x denotes the partial derivative with respect to x. The associated optimal control problem is

$$\text{(PP)} \begin{cases} \min J(u) = \displaystyle\int_0^T \int_a^b L(x,t,y_u(x,t),u(x,t))\,dxdt + \int_a^b r(x,y(x,T))\,dx \\ \qquad\qquad + \displaystyle\int_0^T \ell_a(t,y_u(a,t))\,dt + \int_0^T \ell_b(t,y_u(b,t))\,dt \\ \text{subject to } \ (y_u,u) \in (C(\bar{Q}) \cap W(0,T)) \times L^\infty(Q), \\ \alpha(x,t) \le u(x,t) \le \beta(x,t) \quad \text{for a.e. } \ (x,t) \in Q, \\ g(x,t,y_u(x,t)) \le 0 \ \ \forall (x,t) \in K. \end{cases} \tag{7.2}$$

Here, α, β are functions from $L^\infty(Q)$ with $\alpha(x,t) \le \beta(x,t)$ for a.a. $(x,t) \in Q$, y_u is the solution of (7.1) associated with $u \in L^2(Q)$, and $K \subset \bar{Q}$ is a compact set.

The following assumptions are required:

(A5) The function $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the modification of Assumption (A1) that is obtained by substituting Q for Ω, and (x,t) for x, respectively.

(A6) The function $L : Q \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathbb{R}$ satisfies the modified assumption (A2) obtained by substituting Q for Ω, and (x,t) for x, respectively.

(A7) The function $g : K \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, is of class C^2 with respect to the second variable and $\partial_y g$, and $\partial_y^2 g$ are also continuous functions in $K \times \mathbb{R}$. Moreover, the strict inequality

$$g(x,0,y_0(x)) < 0 \tag{7.3}$$

holds for every $x \in K \cap \bar{\Omega}$.

(A8) The functions $\ell_k : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $k \in \{a,b\}$, are Carathéodory functions of class C^2 with respect to the second variable with $\ell_k(\cdot,0) \in L^1(0,T)$. For all $M > 0$, there exist a constant $C_M > 0$ and a function $\psi_M \in L^2(0,T)$ such that

$$\left|\frac{\partial \ell_k}{\partial u}(t,y)\right| \le \psi_M(x), \quad \left|\frac{\partial^2 \ell_k}{\partial y^2}(t,y)\right| \le C_M,$$
$$\left|\frac{\partial^2 \ell_k}{\partial y^2}(t,y_2) - \frac{\partial^2 \ell_k}{\partial y^2}(t,y_1)\right| \le C_M\,|y_2 - y_1|,$$

holds for $k \in \{a, b\}$, for a.e. $t \in [0,T]$ and $|y|, |y_i| \le M$, $i = 1, 2$.

Analogously, $r : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable with $r(\cdot, 0) \in L^1(a,b)$. It satisfies the assumptions on ℓ_k above with ℓ_k replaced by r, (a,b) substituted for $(0,T)$ and x substituted for t.

For the parabolic equation, the following result on existence and regularity holds true:

THEOREM 7.2. *Suppose that* **(A5)** *is satisfied. Then, for every $u \in L^2(Q)$, the state equation* (7.1) *has a unique solution $y_u \in C(\bar{Q}) \cap W(0,T)$. If $u_k \rightharpoonup u$ weakly in $L^2(Q)$, then $y_{u_k} \to y_u$ strongly in $C(\bar{Q})$.*

The proof of the theorem is postponed to Section 7.4.

THEOREM 7.3. *Assume that* ***(A5)-(A8)*** *are fulfilled, the function L is convex with respect to the third component and the set of feasible controls is nonempty. Then the control problem* (PP) *has at least one solution.*

This theorem is a standard consequence of Theorem 7.2.

7.2. Necessary optimality conditions. Also here, the control-to-state mapping $G(u) = y_u$, $G : L^2(Q) \to C(\bar{Q}) \cap W(0,T)$, and the reduced objective functional J are of class C^2 from $L^\infty(Q)$ to their image spaces, provided that the assumptions **(A5)-(A8)** are satisfied. Since this is known, see [5], we state the associated derivatives for convenience below.

We define, for $v \in L^2(Q)$, the function z_v as the unique solution to

$$\left\{\begin{array}{rcll} \dfrac{dz_v}{dt} + Az_v + \dfrac{\partial f}{\partial y}(x,t,y_u)z_v &=& v & \text{in } Q, \\ -\partial_x z_v(a,t) &=& 0 & \text{in } (0,T), \\ \partial_x z_v(b,t) &=& 0 & \text{in } (0,T), \\ y(x,0) &=& 0 & \text{in } (a,b). \end{array}\right. \tag{7.4}$$

Then $G'(u)$, $G : L^2(Q) \to C(\bar{Q}) \cap W(0,T)$ is given by $G'(u)v = z_v$. Moreover, for v_1, $v_2 \in L^2(Q)$, we introduce $z_{v_i} = G'(u)v_i$, $i = 1, 2$, and obtain $G''(u)v_1v_2 = z_{v_1v_2}$, where $z_{v_1v_2}$ is the solution to

$$\left\{\begin{array}{rcll} \dfrac{dz_{v_1v_2}}{dt} + Az_{v_1v_2} + \dfrac{\partial f}{\partial y}(x,t,y_u)z_{v_1v_2} + \dfrac{\partial^2 f}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2} &=& 0 & \text{in } Q, \\ -\partial_x z_{v_1v_2}(a,t) &=& 0 & \text{in } (0,T), \\ \partial_x z_{v_1v_2}(b,t) &=& 0 & \text{in } (0,T), \\ z_{v_1v_2}(x,0) &=& 0 & \text{in } (a,b). \end{array}\right. \tag{7.5}$$

The adjoint state $\varphi_{0u} \in W(0,T)$ associated with u and J is introduced as the unique solution to

$$\left\{\begin{array}{rcll} -\dfrac{d\varphi}{dt} + A^*\varphi + \dfrac{\partial f}{\partial y}(x,t,y_u)\varphi &=& \dfrac{\partial L}{\partial y}(x,t,y_u,u) & \text{in Q}, \\ -\partial_x\varphi(a,t) &=& \dfrac{\partial \ell_a}{\partial y}(t, y_u(a,t)) & \text{in } (0,T), \\ \partial_x\varphi(b,t) &=& \dfrac{\partial \ell_b}{\partial y}(t, y_u(b,t)) & \text{in } (0,T), \\ \varphi(x,T) &=& \dfrac{\partial r}{\partial y}(x, y_u(x,T)) & \text{in } (a,b). \end{array}\right. \tag{7.6}$$

We have that

$$J'(u)v = \int_Q \left(\frac{\partial L}{\partial u}(x,t,y_u,u) + \varphi_{0u}\right) v\,dxdt, \tag{7.7}$$

$$\begin{aligned} J''(u)v_1v_2 \quad = \quad & \int_Q \left[\frac{\partial^2 L}{\partial y^2}(x,t,y_u,u) z_{v_1} z_{v_2} + \frac{\partial^2 L}{\partial y \partial u}(x,t,y_u,u)(z_{v_1}v_2 + z_{v_2}v_1)\right. \\ & \quad \left. + \frac{\partial^2 L}{\partial u^2}(x,t,y_u,u)v_1v_2 - \varphi_{0u}\frac{\partial^2 f}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2}\right] dxdt, \\ & + \int_0^T \frac{\partial^2 \ell_a}{\partial y^2}(t,y_u(a,t))z_{v_1}(a,t)z_{v_2}(a,t)\,dt \\ & + \int_0^T \frac{\partial^2 \ell_b}{\partial y^2}(t,y_u(b,t))z_{v_1}(b,t)z_{v_2}(b,t)\,dt \\ & + \int_\Omega \frac{\partial^2 r}{\partial y^2}(x,y_u(x,T))z_{v_1}(x,T)z_{v_2}(x,T)\,dx. \end{aligned} \tag{7.8}$$

We require the following linearized Slater condition: There exists $u_0 \in L^\infty(Q)$ with $\alpha(x,t) \le u_0(x,t) \le \beta(x,t)$ for a.e. $(x,t) \in Q$ such that

$$g(x,t,\bar y(x,t)) + \frac{\partial g}{\partial y}(x,t,\bar y(x,t)) z_{u_0-\bar u}(x,t) < 0 \quad \forall (x,t) \in K. \tag{7.9}$$

Notice that we have assumed (7.3), since this is needed to satisfy (7.9). The Hamiltonian H is defined by

$$H(x,t,y,u,\varphi) = L(x,t,y,u) + \varphi\,[u - f(x,t,y)],$$

and the first-order necessary conditions admit the following form (see Casas [5]).

THEOREM 7.4. *Let $\bar u$ be a local solution of* (PP). *Suppose that the assumptions* **(A5)**–**(A8)** *hold and assume the Slater condition* (7.9) *with some $u_0 \in L^\infty(Q)$, $\alpha(x,t) \le u_0(x,t) \le \beta(x,t)$ for a.e. $(x,t) \in Q$. Then there exists a measure $\bar\mu \in M(K)$ and a function $\bar\varphi \in L^\tau(0,T;W^{1,\sigma}(\Omega))$, for all τ, $\sigma \in [1,2)$ with $\frac{1}{\tau} + \frac{1}{\sigma} > \frac{3}{2}$ such that*

$$\left\{ \begin{aligned} -\frac{d\bar\varphi}{dt} + A^*\varphi + \frac{\partial f}{\partial y}(x,t,\bar y)\,\bar\varphi \quad &= \quad \frac{\partial L}{\partial y}(x,t,\bar y,\bar u) + \frac{\partial g}{\partial y}(x,t,\bar y)\bar\mu_{|_Q}, \\ -\partial_x\bar\varphi(a,t) \quad &= \quad \frac{\partial \ell_a}{\partial y}(t,y_u(a,t)) + \frac{\partial g}{\partial y}(a,t,\bar y(a,t))\bar\mu_{|_{\{a\}\times(0,T)}}, \\ \partial_x\bar\varphi(b,t) \quad &= \quad \frac{\partial \ell_b}{\partial y}(t,\bar y(b,t)) + \frac{\partial g}{\partial y}(b,t,\bar y(b,t))\bar\mu_{|_{\{b\}\times(0,T)}}, \\ \bar\varphi(x,T) \quad &= \quad \frac{\partial r}{\partial y}(x,\bar y(x,T)) + \frac{\partial g}{\partial y}(x,T,\bar y(x,T))\bar\mu_{|_{\Omega\times\{T\}}} \end{aligned} \right. \tag{7.10}$$

for a.a. $x \in (a,b)$, $t \in (0,T)$, where $\bar\mu_{|_Q}$, $\bar\mu_{|_{\{a\}\times(0,T)}}$, $\bar\mu_{|_{\{b\}\times(0,T)}}$, and $\bar\mu_{|_{\Omega\times\{T\}}}$ denote the restrictions of μ to Q, $\{a\}\times(0,T)$, $\{b\}\times(0,T)$, and $\Omega\times\{T\}$, respectively,

$$\int_K (z(x,t) - g(x,t,\bar y(x,t))d\bar\mu(x,t) \le 0 \;\forall z \in C(K) \text{ with } z(x,t) \le 0 \;\forall (x,t) \in K, \tag{7.11}$$

and, for almost all $(x,t) \in Q$,

$$H(x,t,\bar y(x,t),\bar u(x,t),\bar\varphi(x,t)) = \min_{s\in[\alpha_{\varepsilon_{\bar u}}(x,t),\beta_{\varepsilon_{\bar u}}(x,t)]} H(x,t,\bar y(x,t),s,\bar\varphi(x,t)), \tag{7.12}$$

where $\alpha_{\varepsilon_{\bar u}}$ and $\beta_{\varepsilon_{\bar u}}$ are defined along these lines of Theorem 3.1.

The Lagrange function is defined in a standard way by

$$\begin{aligned}\mathcal{L}(u,\mu) \quad = \quad & \int_Q L(x,t,y_u(x,t),u(x,t))\,dxdt + \int_0^T \ell_a(t,y_u(a,t))\,dt \\ & + \int_0^T \ell_b(t,y_u(b,t))\,dt + \int_K g(x,t,y_u(x,t))\,d\mu(x,t).\end{aligned}$$

For convenience, we only establish the second-order derivative of $\mathcal{L}$:

$$\begin{aligned}&\frac{\partial^2\mathcal{L}}{\partial u^2}(u,\mu)v_1v_2 = \\ &\int_Q \left[\frac{\partial^2 L}{\partial y^2}(x,t,y_u,u)z_{v_1}z_{v_2} + \frac{\partial^2 L}{\partial y\partial u}(x,t,y_u,u)(z_{v_1}v_2 + z_{v_2}v_1) + \frac{\partial^2 L}{\partial u^2}(x,t,y_u,u)v_1v_2\right. \\ &\left.-\varphi_u\frac{\partial^2 f}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2}\right]dxdt + \int_\Omega \frac{\partial^2 r}{\partial y^2}(x,y_u(x,T))z_{v_1}(x,T)z_{v_2}(x,T)\,dx \\ &+\int_0^T\left[\frac{\partial^2\ell_a}{\partial y^2}(t,y_u(a,t))z_{v_1}(a,t)z_{v_2}(a,t) + \frac{\partial^2\ell_b}{\partial y^2}(t,y_u(b,t))z_{v_1}(b,t)z_{v_2}(b,t)\right]dt \\ &+\int_K \frac{\partial^2 g}{\partial y^2}(x,t,y_u(x,t))z_{v_1}(x)z_{v_2}(x)\,d\bar\mu(x,t),\end{aligned} \tag{7.13}$$

where φ_u is the solution of (7.10), where u is taken for $\bar u$, y_u instead of $\bar y$, and μ for $\bar\mu$.

7.3. Second-order sufficient optimality conditions. The prerequisites of the preceding section at hand, the extension of the second-order sufficient optimality conditions to the parabolic case is straightforward. We define the cone of critical directions associated with $\bar u$ by

$$C_{\bar u} = \{h \in L^2(Q) : h \text{ satisfies (7.14), (7.15) and (7.16) below}\},$$

$$h(x,t) = \begin{cases} \geq 0 \text{ if } \bar u(x,t) = \alpha(x,t), \\ \leq 0 \text{ if } \bar u(x,t) = \beta(x,t), \\ = 0 \text{ if } \bar H_u(x,t) \neq 0, \end{cases} \tag{7.14}$$

$$\frac{\partial g}{\partial y}(x,t,\bar y(x,t))z_h(x,t) \leq 0 \quad \text{if} \quad g(x,t,\bar y(x,t)) = 0, \tag{7.15}$$

$$\int_K \frac{\partial g}{\partial y}(x,t,\bar y(x,t))z_h(x,t)\,d\bar\mu(x,t) = 0. \tag{7.16}$$

The sufficient second-order optimality conditions for $\bar u$ are stated in the following result:

THEOREM 7.5. *Let $\bar u$ be a feasible control of problem* (PP) *that satisfies, together with the associated state $\bar y$ and $(\bar\varphi,\bar\mu) \in L^\tau(0,T;W^{1,\sigma}(\Omega)) \times M(K)$ for all τ, $\sigma \in [1,2)$ with $\frac{1}{\tau}+\frac{1}{\sigma} > \frac{3}{2}$, the first-order conditions* (7.10)-(7.12)*. Assume in addition that there exist two constants $\omega > 0$ and $\tau > 0$ such that*

$$\frac{\partial^2 L}{\partial u^2}(x,t,\bar y(x,t),\bar u(x,t)) \geq \omega \quad \textit{if} \quad |\bar H_u(x,t)| \leq \tau, \quad \textit{for a.e. } (x,t) \in Q, \tag{7.17}$$

$$\frac{\partial^2\mathcal{L}}{\partial u^2}(\bar u,\bar\mu)h^2 > 0 \quad \forall h \in C_{\bar u} \setminus \{0\}. \tag{7.18}$$

Then there exist $\varepsilon > 0$ *and* $\delta > 0$ *such that, for every admissible control* u *of problem* (PP)*, the following inequality holds*

$$J(\bar{u}) + \frac{\delta}{2}\|u-\bar{u}\|^2_{L^2(Q)} \le J(u) \quad \textit{if} \quad \|u-\bar{u}\|_{L^\infty(Q)} < \varepsilon. \tag{7.19}$$

The proof is analogous to the one of Theorem 4.1. We have to perform obvious modifications that are along the ones explained in the proof of Theorem 6.5. Therefore, we skip these details.

7.4. Proof of Theorem 7.2. To prepare the proof of Theorem 7.2, we first state some results on maximal parabolic regularity of the elliptic differential operator A. In the one-dimensional case we study here, A admits the form

$$A = \frac{\partial}{\partial x}[a_{11}(\cdot)\frac{\partial}{\partial x}].$$

Let us consider A on its natural domain

$$D := D(A) = \left\{ w \in H^2(\Omega) \,:\, \frac{\partial w}{\partial x}(a) = \frac{\partial w}{\partial x}(b) = 0 \right\} \tag{7.20}$$

that is dense in $L^2(\Omega)$. It is known that, for all $\tau \in (0,1)$,

$$D(A^\tau) = \begin{cases} H^{2\tau}(\Omega) \cap \left\{ w \,:\, \frac{\partial w}{\partial x}(a) = \frac{\partial w}{\partial x}(b) = 0 \right\}, & \text{if } \tau > \frac{3}{4} \\ H^{2\tau}(\Omega), & \text{if } \tau < \frac{3}{4}, \end{cases}$$

cf. [24]. In particular, we have $D(A^{\frac{1}{2}}) = H^1(\Omega)$. To shorten the notation, we write below $S := (0,T)$ with closure $\bar{S}$. Moreover, for a Banach space $X \subset L^1(\Omega)$ and $1 < p < \infty$, we introduce the space

$$W^{1,p}(S,X) = \left\{ y \in L^p(S,X) \,:\, \frac{\partial y}{\partial t} \in L^p(S,X) \right\}.$$

It is known that, for all $1 < p < \infty$, A exhibits maximal parabolic $L^p(S, L^p(\Omega))$-regularity. This means that, for all $f \in L^p(S, L^p(\Omega))$, there is a unique solution $y \in W^{1,p}(S, L^p(\Omega)) \cap L^p(S, D(A))$ of

$$\frac{\partial y}{\partial t} + A\, y = f \text{ in } S, \quad y(0) = 0, \tag{7.21}$$

where the differential equation is to be understood in the distributional sense, cf. [13]. Here, the definition of $D(A)$ must be adapted by replacing $W^{2,p}(\Omega)$ for $H^2(\Omega)$ in (7.20). In all what follows, we apply this result with $p = 2$ for $X = H := L^2(\Omega)$. Therefore, for all $f \in L^2(S,H) \cong L^2(Q)$, there is a unique solution $y \in W^{1,2}(S,H) \cap L^2(S,D)$ of (7.21). The mapping $f \mapsto y$ is surjective, hence continuous.

Our proof relies on the following result:

LEMMA 7.6. *For all* $0 < \tau < \eta < 1$ *and* $\kappa = \frac{\eta-\tau}{2\eta}$*, there holds the continuous injection* $W^{1,2}(S,H) \cap L^2(S,D) \hookrightarrow C^\kappa(S, H^\tau(\Omega))$.

Proof. We show first that $W^{1,2}(S,H) \hookrightarrow C^{\frac{1}{2}}(S,H)$. To this aim, let $y \in W^{1,2}(S,H)$ and $t,s \in \bar{S}$ be given. Then

$$\begin{aligned}
\|y(t)-y(s)\|_H &= \Big\| \int_s^t y'(\rho)\, d\rho \Big\|_H \le \int_s^t \|y'(\rho)\|_H \, d\rho \\
&\le \Big(\int_s^t \|y'(\rho)\|_H^2 \, d\rho \Big)^{\frac{1}{2}} \Big(\int_s^t d\rho \Big)^{\frac{1}{2}} \le \|y\|_{W^{1,2}(S,H)} \, |t-s|^{\frac{1}{2}}
\end{aligned}$$

verifies the injection claimed above. Next, we prove the statement of the Lemma. We denote by $[\cdot,\cdot]_\theta$ the complex interpolation functor, see Triebel [24]. It follows from [2], Chpt. III, Thm. 4.10.2 and [24], Chpt. 1.8 that the continuous injection

$$W^{1,2}(S,H) \cap L^2(S,D) \hookrightarrow C(\bar{S},[H,D]_{1/2}) = C(\bar{S},H^1(\Omega)) \tag{7.22}$$

takes place. The interpolation identity $[H,D]_{1/2} = H^1(\Omega)$ is well known and can be found, for instance, in [24].

We fix now τ and η by $0 < \tau < \eta < 1/2$ and put $\lambda = \tau/\eta$. Then we obtain with a generic constant c that

$$\frac{\|y(t)-y(s)\|_{[H,D]_\tau}}{|t-s|^{\frac{1}{2}(1-\lambda)}} \le c\, \frac{\|y(t)-y(s)\|_{[H,[H,D]_\eta]_\lambda}}{|t-s|^{\frac{1}{2}(1-\lambda)}} \tag{7.23}$$

$$\begin{aligned}
&\le c\, \frac{\|y(t)-y(s)\|_H^{1-\lambda}}{|t-s|^{\frac{1}{2}(1-\lambda)}} \, \|y(t)-y(s)\|_{[H,D]_\eta}^\lambda \\
&\le c \left(\frac{\|y(t)-y(s)\|_H}{|t-s|^{\frac{1}{2}}} \right)^{1-\lambda} \|y(t)-y(s)\|_{[H,D]_\eta}^\lambda ,
\end{aligned} \tag{7.24}$$

where we have applied the complex re-iteration theorem, [24], Chpt. 1.9.3. In the last estimate, the first factor is bounded, since $W^{1,2}(S,H) \hookrightarrow C^{\frac{1}{2}}(S,H)$. In view of the injection (7.22) and $[H,D]_\eta = H^{2\eta}(\Omega)$ with $0 < 2\eta < 1$, the second factor can be estimated by

$$\begin{aligned}
\|y(t)-y(s)\|_{[H,D]_\eta}^\lambda &\le c\, \|y(t)-y(s)\|_{H^{2\eta}(\Omega)}^\lambda \le \Big(2c\, \|y\|_{C(\bar{S},H^{2\eta}(\Omega))} \Big)^\lambda \\
&\le \Big(c\, \|y\|_{C(\bar{S},H^1(\Omega))} \Big)^\lambda \le c\, \|y\|_{W^{1,2}(S,H)\cap L^2(S,D)}^\lambda .
\end{aligned}$$

In the last estimate, we have used the embedding (7.22). Moreover, we took advantage of the equivalence of the norms of $[H,D]_\eta$ and $H^{2\eta}(\Omega)$. Therefore, the second factor in (7.24) is bounded too. The statement of the Lemma follows now from (7.24) after inserting $\tau := 2\tau$, $\eta := 2\eta$, $[H,D]_\tau = H^{2\tau}(\Omega)$ and $\kappa = \frac{1}{2}(1-\lambda)$ in (7.23). □

Proof of Theorem 7.2: The existence result of Theorem 7.2 is well known, we refer to Casas [5]. Therefore, we only show that weakly converging sequences of controls are transformed to strongly converging sequences of states.

Let a sequence (u_k) be given that converges weakly in $L^2(Q)$ to u. Consider the equation for y_k and u_k

$$\begin{aligned}
\frac{\partial y_k}{\partial t} + Ay_k + f(x, y_k) = u_k, &\quad \text{in } Q\\
\frac{\partial y_k}{\partial x}(a,t) = 0, &\quad \text{in } S\\
\frac{\partial y_k}{\partial x}(b,t) = 0 &\quad \text{in } S\\
y_k(0) = y_0, &\quad \text{in } \Omega.
\end{aligned}$$

Standard arguments show $y_k \rightharpoonup y$ in $W(0,T) \cap C(\bar{Q})$, where $y = y_u$. The functions y_k are uniformly bounded in $C(\bar{Q})$, hence the sequence $\big(d(\cdot, y_k)\big)$ is bounded in $L^2(Q)$ and we can select a weakly converging subsequence indexed by k_l. We write $f_k = u_k - d(\cdot, y_k)$ and split $y_k = v + w_k$, where w_k solves

$$\frac{\partial w_k}{\partial t} + Aw_k = f_k$$

with homogeneous inital and boundary conditions, while v solves

$$\frac{\partial v}{\partial t} + Av = 0$$

with inhomogeneous initial condition $v(0) = y_0$ and homogeneous boundary conditions. Thanks to Lemma 7.6, the sequence (w_{k_l}) converges weakly in $C^\kappa(S, H^\tau(\Omega))$, where $\kappa > 0$ and $\tau > 1/2$ can be chosen. Therefore, the functions w_{k_l} belong to a space $C^\sigma(\bar{Q})$ with some positive σ so that, by compact embedding into $C(\bar{Q})$, the sequence converges strongly in $C(\bar{Q})$. Consequently, $y_{k_l} = v + w_{k_l}$ converges strongly in $C(\bar{Q})$ towards y. Moreover, it follows by standard arguments that $y = y_u$. Since this holds for all subsequences with the same limit y, the whole sequence (y_k) converges uniformly to y_u. $\square$

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REFERENCES

[1] J.J. Alibert and J.P. Raymond, *Boundary Control of Semilinear Elliptic Equations with Discontinuous Leading Coefficients and Unbounded Controls*, Numer. Funct. Anal. and Optimiz., 18 (1997), pp. 235–250.

[2] H. Amann, *Linear and quasilinear parabolic problems.*, Birkhäuser, Basel-Boston-Berlin, 1995.

[3] F. Bonnans. Second-order analysis for control constrained optimal control problems of semilinear elliptic systems. *Appl. Math. and Optimization*, 38:303–325, 1998.

[4] F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems.* Springer-Verlag, New York, 2000.

[5] E. Casas, *Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations*, SIAM J. Control and Optimization, 35 (1997), pp. 1297–1327.

[6] ———, *Necessary and sufficient optimality conditions for elliptic control problems with finitely many pointwise state constraints*, ESAIM: COCV, to appear.

[7] E. Casas and M. Mateos, *Second order optimality conditions for semilinear elliptic control problems with finitely many state constraints*, SIAM J. Control Optim., 40 (2002), pp. 1431–1454.

[8] E. Casas, J. Raymond, and H. Zidani, *Pontryagin's principle for local solutions of control problems with mixed control-state constraints*, SIAM J. Control Optim., 39 (2000), pp. 1182–1203.

[9] E. Casas and F. Tröltzsch, *Second-order necessary optimality conditions for some state-constrained control problems of semilinear elliptic equations*, Applied Math. and Optimization, 39 (1999), pp. 211–227.

[10] ———, *Second order necessary and sufficient optimality conditions for optimization problems and applications to control theory*, SIAM J. Optim., 13 (2002), pp. 406–431.

[11] E. Casas, F. Tröltzsch, and A. Unger, *Second order sufficient optimality conditions for a nonlinear elliptic control problem*, J. for Analysis and its Appl., 15 (1996), pp. 687–707.

[12] ———, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations*, SIAM J. Control Optim., 38 (2000), pp. 1369–1391.

[13] G. Dore, *l^p-regularity for abstract differential equations.*, in Komatsu, H. (Ed.), Functional analysis and related topics, Proceed. Int. Conf. in memory of Prof. K. Yosida held at RIMS, Kyoto University, Japan, July 29 – Aug. 2, 1991, vol. 1540 of Lect. Notes Math., Berlin, 1993, Springer, pp. 25–38.

[14] J. Dunn, *l^2 sufficient optimality conditions for end-constrained optimal control problems with inputs in a polyhedron*, SIAM J. Control Optim., 36 (1998), pp. 1833–1851.

[15] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[16] A. D. Ioffe. Necessary and sufficient conditions for a local minimum 3: Second order conditions and augmented duality. *SIAM J. Control and Optimization*, 17:266–288, 1979.

[17] X. Li and J. Yong, *Necessary conditions of optimal control for distributed parameter systems*, SIAM J. Control Optim., 29 (1991), pp. 895–908.

[18] K. Malanowski. Two-norm approach in stability and sensitivity analysis of optimization and optimal control problems. *Adv. Math. Sci. Appl.*, 2:397–443, 1993.

[19] K. Malanowski. Sufficient optimality conditions for optimal control subject to state constraints. *SIAM J. Control Optimization*, 35:205–227, 1997.

[20] H. Maurer. First and second order sufficient optimality conditions in mathematical programming and optimal control. *Math. Programming Study*, 14:163–177, 1981.

[21] H. Maurer and J. Zowe, *First- and second-order conditions in infinite-dimensional programming problems*, Math.Programming, 16 (1979), pp. 98–110.

[22] J.-P. Raymond and F. Tröltzsch, *Second order sufficient optimality conditions for nonlinear parabolic control problems with state constraints*, Discrete and Continuous Dynamical Systems, 6 (2000), pp. 431–450.

[23] G. Stampacchia, *Problemi al contorno ellittici, con dati discontinui, dotati di soluzioni hölderiane*, Annali di Matematica, 51 (1960), pp. 1–37.

[24] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North Holland Publishing Company, 1978.

[25] F. Tröltzsch, *Optimale Steuerung partieller Differentialgleichungen – Theorie, Verfahren und Anwendungen*, Vieweg, Wiesbaden, 2005.