A wave-type limit for the 3D Landau-Lifshitz-Gilbert equation

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1 Introduction

We consider the dynamics of unit magnetization fields $\mathbf{m} = \mathbf{m}(t) : \Omega(h) \to \mathbb{S}^2$ on a magnetic film $\Omega(h) = \Omega \times (0, h)$ of small relative height h, governed by the Landau-Lifshitz-Gilbert equation (LLG)

$$\mathbf{m}_t + \mathbf{m} \times (\gamma\, \mathbf{H} - \alpha\, \mathbf{m}_t) = 0,$$

see [4, 7]. The parameters γ and α are the gyromagnetic ratio and the Gilbert damping factor, respectively. $\mathbf{H} = \mathbf{H}_{\text{eff}}$ is the effective field that is given by the negative functional gradient of the interaction energy $E = E(\mathbf{m})$. The equation prescribes a damped precession of the magnetization vector about the effective field. The parameter γ has the dimension of frequency, and the number $1/(\gamma\alpha)$ is the typical relaxation time. Upon nondimensionalization, we assume them to be positive and finite, so that the associate dynamical system is dissipative. In a basic micromagnetic model for an isotropic material, see [7, 1], the interaction energy $E(\mathbf{m}) = E(\mathbf{m}, \Omega(h))$ consist in exchange energy (with exchange constant A) and magnetostatic energy

$$E(\mathbf{m}, \Omega(h)) = \frac{A}{2} \int_{\Omega(h)} |\nabla \mathbf{m}|^2 \, d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla U|^2 \, d\mathbf{x}.$$

The gradient field $-\nabla U$ is the associated stray-field that is related to $\mathbf{m}$ via the magnetostatic Maxwell equation

$$\Delta U = \text{div}\left(\mathbf{m}\chi_{\Omega(h)}\right) \quad \text{distributionally on } \mathbb{R}^3.$$

It is well-known that for thin film geometries ($h \ll 1$) the leading order energy contribution is a quadratic shape anisotropy effect that favors in-plane magnetization, see [1, 6]. The equilibrium condition is $\mathbf{m} \times \mathbf{H} = 0$ where $\mathbf{H} = -\nabla E(\mathbf{m})$. Here the gradient is taken with respect to the standard L^2 inner product on $\Omega(h)$, that is $\mathbf{H} = A\Delta\mathbf{m} - \nabla U$. If $h \ll 1$, the stray-field $-\nabla U$ includes a singular forcing term that pushes the magnetization vector towards the film plane. In a dynamic context, however, the shape anisotropy effect competes with gyrotropic forces. We investigate this competition and possible singular limits as $h \to 0$ when shape anisotropy becomes a hard constraint, geometrically incompatible with LLG dynamics.

2 Thin-film limits for the rescaled LLG

We are mainly interested in situations where the renormalized energy

$$E_h(\mathbf{m}) = h^{-2} E(\mathbf{m}, \Omega(h))$$

is bounded or at most logarithmically divergent when $h \ll 1$. A particular feature of this energy regime is that the stray-field induced anisotropy asymptotically leads to a hard in-plane constraint for the magnetization field. The renormalized Landau-Lifshitz-Gilbert equation reads:

$$\mathbf{m}_t + \mathbf{m} \times \big(\gamma\, h\, \mathbf{H}^h - \alpha\, \mathbf{m}_t\big) = 0 \quad \text{with renormalized} \quad \mathbf{H}^h = \mathbf{H}/h.$$

The renormalized effective field can also be identified with a functional gradient

$$-\nabla_h E_h(\mathbf{m}) = \mathbf{H}^h,$$

which is the negative gradient of the renormalized energy with respect to the averaged L^2 inner product. Due to possible energy divergence as $h \to 0$, the limiting object might only formally by an energy gradient. A crucial analytical assumption is that the energy is exchange dominated that is $\lim_{h\to 0} A(h)/h = \varepsilon$ where $\varepsilon > 0$ is small but finite. This assumption features a rather microscopic view point like in case of domain wall models or particles of rather small lateral size. Interesting thin film features are usually revealed in the limit $\varepsilon \to 0$ when stray-field interaction dominates. A dynamic theory in such a limit is mathematically more subtle and a challenge for future work. Here we always assume $h \ll \varepsilon \ll 1$.
The competition of strong shape anisotropy due to stray-field interaction and gyroscopic forces might lead to asymptotic limiting dynamics. There are two dynamic regimes that have been considered in literature: If $\gamma \sim 1/h$ and $\alpha \sim 1$ (slow time scale and fixed damping), the gyromagnetic term simply drops out. The corresponding formal limit for LLG is a gradient flow which has first been derived formally by Garcia-Cervera & E in [3], see also [5]. A rigorous result for a full 3D model has been obtained by Moser in [8]. The case when $\gamma \sim 1/\sqrt{h}$ and $\alpha \sim \sqrt{h}$ (intermediate regime with small damping) has been considered in [2] in case of a reduced model for infinitely extended Néel walls with a small parameter $\varepsilon = \sqrt{h}$. It is shown that the asymptotic evolution equation for the transition angle θ is a damped wave equation $\partial_t^2\theta + \nu\, \partial_t\theta + \nabla\mathcal{E}(\theta) = 0$ where $\nu > 0$ is an effective damping constant and $\mathcal{E} = \mathcal{E}(\theta)$ is the reduced Néel wall energy. The main simplification in [2] is that the magnetization field is assumed to have no variations in the vertical direction, which amounts to a significant mathematical simplification of stray-field energies. The common feature of both dynamic regimes is uniform averaged energy dissipation along smooth trajectories. As it turns out, among all such dynamic regimes, the one discussed in [2] is special since stray-field anisotropy and gyrotropic forcing are in balance. Here we present a more general convergence result for the full 3D model. Let $\mathbf{m}^h : \Omega(h) \times (0,\infty) \to \mathbb{S}^2 \in L^\infty\big(H^1(\Omega(h))\big) \cap \dot{H}^1\big(L^2(\Omega(h))\big)$ be a family of weak solutions of the Landau-Lifshitz-Gilbert equation subject to homogeneous Neumann boundary conditions.

Theorem 1. *Suppose that there are positive constants ν and ε such that for some sequence $h = h_k \to 0$:*

$$\alpha(h)/\sqrt{h} \to \nu, \quad \gamma(h)\sqrt{h} \to 1, \quad A(h)/h \to \varepsilon$$

$$\fint_0^h m^h \overset{*}{\rightharpoonup} m \quad \textit{weakly* in} \quad L^\infty_{\mathrm{loc}}\big(H^1_{\mathrm{loc}}(\Omega)\big) \cap \dot{H}^1\big(L^2(\Omega)\big)$$

$$\sup_{h,t} \fint_0^h \int_\Omega \left(\left| \frac{\partial \mathbf{m}^h}{\partial x_3} \right|^2 + \left|\nabla m_3^h\right|^2 \right) d\mathbf{x} < \infty$$

$$U^h/h \overset{*}{\rightharpoonup} u \quad \textit{weakly* in} \quad L^\infty(\dot{H}^1(\mathbb{R}^3)).$$

Then $\mathbf{m} = (m, 0) : \Omega \times (0, T) \to \mathbb{S}^1$ *is a distributional solutions of*

$$\Big[\partial_t^2 m + \nu\, \partial_t m - \varepsilon\, \Delta' m + \nabla' u\Big] \perp T_m\mathbb{S}^1.$$

The proof will be given in a forthcoming paper. A crucial step is the identification

$$m \wedge m_t = \lim_{h\to 0} \frac{1}{\sqrt{h}} \fint_0^h m_3^h \, dx_3 = \lim_{h\to 0} \frac{1}{\sqrt{h}} \fint_0^h \frac{\partial U^h}{\partial x_3}\, dx_3$$

weakly in $H^{-1}(\Omega \times (0,\infty))$ that reflects the balance of dynamic and energetic forces. The function $u = u(t)$ is, for almost every $t > 0$, a solutions of the reduced stray-field equation,

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \phi = \int_{\mathbb{R}^2} m \cdot \nabla' \phi$$

for any $\phi \in C_0^\infty(\mathbb{R}^3)$ where $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$. In the limiting equation, $\nabla' u = \nabla' u|_{\mathbb{R}^2}$ is understood in the sense of traces. The limit m exhibits improved regularity properties. Indeed, $m \wedge m_t \in L^\infty(L^2(\Omega))$ can be shown to be weakly continuous in time, and Cauchy data can be identified. Moreover, in the finite energy case, an inequality for the total energy holds true.

Applications. In case of microscopic domain wall models, see [1, 2, 6], uniform energy bounds imply the requisite compactness conditions. Expressed in collective coordinates for the wall center, the result strongly supports mechanical particle models for domain wall motion and motivates the notion of wall mass, see [2]. From a mesoscopic view point, however, topological defects and energy divergence have to be taken into account. In case of a bounded simply connected base domain Ω, the so-called boundary vortex regime as investigated in [8] by R. Moser is characterized by the energy scaling $E_h(\mathbf{m}) \sim \varepsilon\,\pi\, \log\log(1/h) + L$ as $h \to 0$ (i.e. finite energy perturbations of minimizing configurations that indeed develop vortices near the boundary, see [8]). In the dynamic regime of E and Cervera discussed earlier in this note, Moser derived under various technical assumptions limiting gradient flow dynamics off the (static) boundary vortices, see [8]. Alternatively, in our intermediate dynamic regime with small damping, the local compactness conditions requisite for Theorem 1 are met by suitable weak solutions of LLG, effectively leading to wave-type dynamics in the interior. The derivation of effective dynamic laws for magnetic point singularities near the boundary on a suitable time scale remain a challenging open problem.

References

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