

Finite utility on financial markets with asymmetric information and the theorem of Bichteler-Dellacherie-Mokobodsky *

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Abstract

We consider financial markets with two kinds of small traders: regular traders who perceive the asset price process S through its natural filtration, and insiders who possess some information advantage which makes the filtrations through which they perceive the evolution of the market richer. The basic question we discuss is the link between (NFLVR), the semimartingale property of S viewed from the agent's perspective, and bounded expected utility. We show that whenever an agent's expected utility is finite, S is a semimartingale with a Doob-Meyer decomposition featuring a martingale part and an information drift. The expected utility gain of an insider with respect to a regular trader is calculated in a completely general setting. In particular, for the logarithmic utility function, utility gain is a function of the relative information drift alone, regardless of the completeness of the market.

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Introduction

Asymmetry of information on financial markets has been a subject of increasing interest in recent years. Several mathematical models have been designed to deal with financial markets on which traders with different information levels are active. See Wu [29] for an overview. The model to capture basic facts of insider action on markets which motivated this paper is very simple. Two kinds of traders are considered: regular agents who usually know not more than the natural evolution of the assets of the market, and insiders whose knowledge at any given time in the trading interval is larger than the σ -field generated by the asset price process up to that time. The insider may, for example, possess some additional information on the price of an asset at maturity, or at some later time. He might anticipate the time when an asset price reaches a favorable level, or be able to stop the time at which some final level crossing of the price process occurs. Situations of this type have been modelled for example by Karatzas, Pikovsky [21], Amendinger [1], Amendinger, Becherer and Schweizer [2], Grorud, Pontier [15], and [3], [16], [17], [18]. In most of these papers, questions of utility gain of the insider relative to the regular trader were discussed. It turned out that for many types of additional information the expected increment of utility gained by the insider may become infinite quite easily, and might provide opportunities for free lunch or even arbitrage in an equally easy way. Baudoin [6] and Baudoin, Nguyen-Ngoc [7] develop a model in which additional information on some random variable unknown to the regular trader is only weakly available, i.e. in form of some knowledge of its law instead of the precise anticipation of its value. In this framework the insider's utility is more likely to be finite and can be computed for example by means of the fundamental results by Kramkov, Schachermayer [22]. In [10], the precise observation of some random element by the insider which is inaccessible to the natural trader is blurred dynamically by some exterior independent noise to produce a weaker information advantage in the same spirit, and keep the additional utility from getting out of control.

A natural mathematical toolbox to use in the context of the models described contains the techniques of *grossissement de filtrations* developed in some deep work mostly by French authors [9], [19], [20], [23], [30], [31], [32], [33], [28]. This is just one of numerous examples in which the direct impact of Meyer's Strasbourg school on contemporary financial mathematics becomes evident. Another example is initiated in a recent paper by Biagini, Oksendal [8]. In this paper a question is raised which appears of purely mathematical interest at the first glance: knowing that the expected utility of an insider is finite, what can be said about the regularity of the asset price process from the insider's point of view? The authors show that given finite utility and the existence of an optimal investment strategy for the insider, the asset price process must be a semimartingale in the insider's enlarged filtration. This way, they address one of the basic questions of the theory of grossissements de filtrations, and at the same time raise a problem which goes to the heart of stochastic analysis: the relationship between semimartingales and the stochastic integrator property. To describe the utility of the insider in his enlarged filtration, they use extended notions of stochastic integrals investigated in anticipative stochastic calculus, such as Skorokhod's integral (see Nualart [24]) and the forward Itô integral introduced by Russo and Vallois [26].

The deep and central theorem of Bichteler-Dellacherie-Mokobodsky characterizes semimartingales as *stochastic integrators*. A process S is a semimartingale if and only if the stochastic integrals of uniformly bounded simple processes, i.e. predictable step processes, with respect to S form a bounded set in the topological vector space of random variables with the (L^0-) topology of convergence in probability. This key theorem allows to deal with the problem posed by Biagini and Oksendal [8] from a different perspective. Suppose an agent invests on a financial market with asset price process S and perceives the utility of his final wealth through a utility function U which is unbounded. Then the hypothesis that the expected utilities the agent is able to attain be a bounded function of the simple investment strategies he is allowed to use due to his information horizon should be closely related to the L^0 -boundedness of the set of stochastic integrals of simple admissible strategies. Hence the theorem should provide a direct link between finite utility of agents on financial markets and the semimartingale property of the asset price processes with respect to the evolution of their information. This basic observation is the starting point for the analysis presented in this paper.

A related link is exploited in the fundamental paper by Delbaen and Schachermayer [11]. It is shown that if an asset price process S fulfills the (NFLVR) condition, i.e. allow no admissible simple strategies which lead with positive probability to a final gain with controllable risk, then the agent views S as a semimartingale. In addition, M being the martingale part of S , its Doob-Meyer decomposition is given by the special formula

$$S = M + \alpha \cdot \langle M, M \rangle.$$

We start by proving that if an agent has bounded expected utility with respect to his information horizon, then he cannot have (FLVR). This allows us to hook up to the result by Delbaen and Schachermayer, show that bounded utility implies the semimartingale property of S , and investigate more thoroughly the relationship between the properties (*NFLVR*), the *semimartingale property* of S in the agent's filtration, and *bounded expected utility*. The drift density α may be considered a function of the agent's information horizon, i.e. its filtration. Passing from one filtration to a bigger one while keeping utility finite will change α to β , and we may well call $\beta - \alpha$ the corresponding *information drift*. We will keep an attentive eye on logarithmic utility. In this particular case we will show that a better informed agent's additional utility is a function of the information drift alone, regardless of whether we face a complete or an incomplete market. This result is derived in an entirely abstract framework. We do not have to specify the type of information advantage the insider possesses. Based on the fundamental result by Kramkov and Schachermayer [22], we will describe the additional expected utility of an insider in a complete market setting for all reasonable utility functions and express it as a function of relative information drifts.

Here is a brief outline of the paper. In section 1, we shall investigate the relationship between (NFLVR), the semimartingale property, and finite utility. Section 2 is devoted to investigate conditions under which optimal utility can be obtained by maximizing over simple instead of general predictable admissible strategies. In section 3 we investigate the link between singularities of the integral process of the squared information

drift α and infinite utility. In the following two sections 4 and 5 we restrict our attention to logarithmic utility. We calculate it in general incomplete market settings, and derive the expected utility increment of an insider. In the final section 6 we give some formulas expressing the additional utility of an insider in complete markets as a function of information drift for general ways of information evolution and show that the logarithm gives essentially the only way of measuring utility which allows portfolios that are optimal at any time in the trading interval.

1 Finite Utility and semimartingales

Delbaen and Schachermayer [11] establish a link between the (NFLVR) condition and the semimartingale property of an asset price process on a financial market. In this section we shall compare these two properties with a third one: the boundedness of expected utility with respect to wealth processes based on simple admissible integrands, and non-bounded utility functions. Our main result will roughly show that boundedness of utility implies the semimartingale property of the price process with respect to the filtration of a fixed agent on the market.

Let (Ω, F, P) be a probability space and $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ an arbitrary filtration satisfying the usual conditions, T being the time horizon. Suppose that $S : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a stochastic process. S will take the role of the asset price process on our financial market. The wealth of the agent on our market with information horizon \mathcal{F} will be determined in this section by simple investment strategies (integrands) of the following form. A simple integrand is a linear combination of processes of the form $f1_{]T_1, T_2]}$ where f is a bounded and \mathcal{F}_{T_1} -measurable random variable and T_1 and T_2 are finite stopping times with respect to the filtration \mathcal{F} . The collection of simple integrands will be denoted by \mathcal{S} and the stochastic integral process of simple or more general predictable integrands with respect to a cadlag process X by $\theta \cdot X$. We now recall some terminology introduced in [11]. If a is a positive real number, then a strategy θ is called *a-admissible*, if for all $t \in [0, T]$ we have $(\theta \cdot S)_t \geq -a$ almost surely. It will be called *admissible* if it is *a-admissible* for some $a \geq 0$. We put

$$\mathcal{K}^s = \{(\theta \cdot S)_T | \theta \in \mathcal{S} \text{ admissible}\}$$

and $C = \mathcal{K}^s \cap L_+^\infty$. The process S is said to satisfy the *no free lunch with vanishing risk* (NFLVR) property for simple integrands, if

$$\bar{C} \cap L_+^\infty = \{0\},$$

where \bar{C} denotes the closure of C in L^∞ . For the general (NFLVR) condition, we refer to \mathcal{K} defined as \mathcal{K}^s just with general \mathcal{F} -predictable θ with well defined stochastic integral. If the intersection contains more than the trivial element 0, we will say that S satisfies (FLVR).

The following is a useful reformulation of the (FLVR) property.

Lemma 1.1 *S satisfies the (FLVR) property for simple integrands if and only if there is a sequence $(\theta^n)_{n \geq 0}$ of admissible simple integrands such that the following two conditions are satisfied*

i) $f_n = (\theta^n \cdot S)_T, n \in \mathbb{N}$, converges a.s. to a nonnegative function f satisfying $P(f > 0) > 0$ and

ii) $\|f_n^-\|_\infty \rightarrow 0$.

Proof. It is an immediate consequence of the definition of (FLVR), that this condition is implied by i) and ii) .

For the reverse direction, suppose that the (FLVR) property holds. Then there is a sequence $(\theta^n)_{n \in \mathbb{N}}$ of simple integrands such that the integrals $g_n = (\theta^n \cdot S)_T, n \in \mathbb{N}$, satisfy

i') $\|g_n^-\|_\infty \rightarrow 0$ and

ii') $g_n^+ \rightarrow 0$ in probability.

One can find an $\alpha > 0$ such that for any $n \geq 0$ there exists a $k \geq n$ with $P(g_k > \alpha) > \alpha$. By taking a subsequence, still denoted by $(g_n)_{n \in \mathbb{N}}$, we assume that $P(g_n > \alpha) > \alpha$ holds for all $n \geq 0$. From lemma A.1.1 in [11] we know that there are $f_n \in \text{conv}(g_k : k \geq n)$ converging almost surely to some f with $P(f > 0) > 0$. Observe that every f_n is still an integral of some simple process with respect to S . i) and ii) follow and the claim is proven. \square

If not stated otherwise in the sequel, we mean by a *utility function* a function $U : \mathbb{R} \rightarrow [-\infty, \infty)$ which is strictly concave and strictly increasing on $\text{dom}(U) = \{y : U(y) > -\infty\}$. We will interpret the integral $EU(x + (\theta \cdot S)_T)$ as the expected utility of a trader possessing an initial wealth x and choosing his investments following the strategy θ . Note that the integral might not exist. For ease of notation, we use the convention $EU(x + (\theta \cdot S)_T) = -\infty$, if both the positive and the negative part of $U(x + (\theta \cdot S)_T)$ have infinite expectation.

The following proposition provides the link between the boundedness of the agent's utility for simple strategies and the (NFLVR) condition.

Proposition 1.2 *Let $U : \mathbb{R} \rightarrow [-\infty, \infty)$ be a utility function with $\lim_{x \rightarrow +\infty} U(x) = +\infty$. Then for all $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$ (recall $\sup \emptyset = -\infty$) the following implication holds.*

If $\sup_{S \ni \theta \text{ admissible}} E[U(x + (\theta \cdot S)_T)] < \infty$, then (NFLVR) for simple integrands.

Proof. Let $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$. Then there is a $\delta > 0$ for which $x - \delta > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$. We put $D = U(x - \delta) \wedge 0 > -\infty$.

Suppose that the (NFLVR) property for simple integrands is violated. By the preceding lemma we can find a sequence $(\theta^n)_{n \in \mathbb{N}}$ of admissible simple integrands such that the final payoffs $f_n = (\theta^n \cdot S)_T, n \in \mathbb{N}$, satisfy

i) $f_n = (\theta^n \cdot S)_T \rightarrow f$ a.s. , where f is nonnegative with $P(f > 0) > 0$ and

ii) $\|f_n^-\|_\infty \rightarrow 0$.

For $n \in \mathbb{N}$ we set $\varepsilon_n = \|f_n^-\|_\infty$. For all but finitely many $n \in \mathbb{N}$ we have $\varepsilon_n < \delta$. To simplify notation we assume that this holds for all $n \in \mathbb{N}$. We now define new simple integrands

$$\pi^n = \frac{\delta}{\varepsilon_n} \theta^n$$

for all $n \in \mathbb{N}$. It is clear that all the integrals $(\pi^n \cdot S)_T$ exceed the bound $-\delta$. Furthermore the random variables $U(x + (\pi^n \cdot S)_T)$ are bounded from below by the constant D . More formally,

$$\begin{aligned} U(x + (\pi^n \cdot S)_T) &= U\left(x + \frac{\delta}{\varepsilon_n} (\theta^n \cdot S)_T\right) \\ &\geq U\left(x + \frac{\delta}{\varepsilon_n} (-\varepsilon_n)\right) \\ &= U(x - \delta) \\ &= D > -\infty. \end{aligned}$$

Since f_n converges to the nontrivial nonnegative function f , one can find an integer n_0 and real numbers $\alpha > 0$ and $\beta > 0$ such that

$$P((\theta^n \cdot S)_T > \alpha) > \beta$$

for all $n \geq n_0$. This implies

$$\begin{aligned} \liminf_{n \rightarrow \infty} E[U(x + (\pi^n \cdot S)_T)] &= \liminf_{n \rightarrow \infty} E\left[U\left(x + \frac{\delta}{\varepsilon_n} (\theta^n \cdot S)_T\right)\right] \\ &\geq \liminf_{n \rightarrow \infty} E\left[D 1_{\{(\theta^n \cdot S)_T \leq \alpha\}} + U\left(x + \frac{\delta}{\varepsilon_n} \alpha\right) 1_{\{(\theta^n \cdot S)_T > \alpha\}}\right] \\ &\geq \liminf_{n \rightarrow \infty} \left[D(1 - \beta) + U\left(x + \frac{\delta}{\varepsilon_n} \alpha\right) \beta\right] \\ &= \infty. \end{aligned}$$

Hence

$$\sup_{S \ni \theta \text{ admissible}} E[U(x + (\theta \cdot S)_T)] = \infty.$$

This proves the proposition. \square

Remark: Proposition (1.2) holds in particular for all increasing functions U with $\lim_{x \rightarrow \infty} U(x) = \infty$.

Combining Proposition (1.2) with the results of the fundamental paper by Delbaen and Schachermayer [11] we obtain the intuitively plausible relationship between boundedness of the expected utility and the semimartingale property for the continuous asset price process with respect to the agent's filtration.

Corollary 1.3 *Let S be a cadlag and locally bounded adapted process, $U : \mathbb{R} \rightarrow [-\infty, \infty)$ a utility function with $\lim_{x \rightarrow +\infty} U(x) = +\infty$ and $x > \sup\{y \in \mathbb{R} : U(y) = -\infty\}$. If $\sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_T)] < \infty$, then S is a semimartingale with respect to \mathcal{F} . If S is, moreover, continuous, it satisfies the general (NFLVR) property.*

Proof. By proposition (1.2), the process S satisfies the (NFLVR) property for simple integrands. Theorem 7.2 in Delbaen and Schachermayer [11] states that in this case S is already a semimartingale. Since S is continuous, the general (NFLVR) property follows from the (NFLVR) property for simple integrands (see Theorem 7.6 in Delbaen and Schachermayer [11]). \square

Note that Delbaen and Schachermayer [11] use a slight different definition of simple integrands. They admit unbounded processes. But one can show that (NFLVR) for bounded simple integrands is equivalent to (NFLVR) for all (possibly unbounded) simple processes.

We can sharpen the result of the preceding Corollary. In fact, we can show in the following theorem that boundedness of expected utility over all uniformly bounded simple strategies is sufficient for the semimartingale property of S to follow.

Theorem 1.4 *Let S be a cadlag and locally bounded adapted process indexed by $[0, T]$ and U a utility function satisfying $\lim_{x \rightarrow \infty} U(x) = \infty$. If $\sup_{\theta \in \mathcal{S}, |\theta| \leq 1} E[U(x + (\theta \cdot S)_T)] < \infty$, then S is a semimartingale.*

Proof. Our proof is similar to the one of theorem 7.2 in Delbaen and Schachermayer [11]. Since S is locally bounded we can find a sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ such that the stopped processes S^{T_n} are bounded. It is sufficient to prove that each S^{T_n} is a semimartingale. To put it simply we assume that S is already bounded by some constant C . Our proof shall proceed in two lemmas for which we will suppose that the assumptions of theorem 1.4 hold.

Lemma 1.5 *Let Θ be a set of simple integrands θ satisfying $|\theta| \leq 1$. If $\{\sup_{0 \leq t \leq T} (\theta \cdot S)_t^- : \theta \in \Theta\}$ is bounded in L^0 , then the set $\{\sup_{0 \leq t \leq T} (\theta \cdot S)_t^+ : \theta \in \Theta\}$ is also bounded in L^0 .*

Proof. Suppose that $\{\sup_{0 \leq t \leq T} (\theta \cdot S)_t^+ : \theta \in \Theta\}$ is not bounded in L^0 . Then one can find a sequence $(c_n)_{n \in \mathbb{N}}$ of real numbers and $(\theta_n)_{n \in \mathbb{N}}$ in Θ satisfying $c_n \rightarrow \infty$ and $P(\sup_{0 \leq t \leq T} (\theta_n \cdot S)_t > c_n + 2C) \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. Since $\{\sup_{0 \leq t \leq T} (\theta \cdot S)_t^- : \theta \in \Theta\}$ is bounded in L^0 , there is a constant K for which

$$\sup_{\theta \in \Theta} P\left(\sup_{0 \leq t \leq T} (\theta \cdot S)_t^- \geq K\right) < \frac{\varepsilon}{2}.$$

Consider the stopping times

$$T_n = \inf\{t > 0 : (\theta^n \cdot S)_t^- \geq K \text{ or } (\theta^n \cdot S)_t \geq c_n + 2C\} \wedge T, \quad n \in \mathbb{N}.$$

We then have for $n \in \mathbb{N}$

- i) $(\theta^n 1_{[0, T_n]} \cdot S)_T \geq -K - 2C$
- ii) $P((\theta^n 1_{[0, T_n]} \cdot S)_T \geq c_n) \geq \frac{\varepsilon}{2}$.

We choose $\delta > 0$ such that $U(x - \delta) > -\infty$ still holds and a sequence of real numbers $(\gamma_n)_{n \in \mathbb{N}}$ with $\gamma_n \in (0, 1)$ converging to 0 slowly enough to guarantee $\gamma_n c_n \rightarrow \infty$. Now define new simple integrands by

$$\pi^n = \frac{\delta \gamma_n}{K + 2C} 1_{[0, T^n]} \theta^n, \quad n \in \mathbb{N}.$$

For all but finitely many $n \in \mathbb{N}$ we remark $|\frac{\delta \gamma_n}{K + 2C}| \leq 1$, because $\gamma_n \rightarrow 0$. Without loss of generality we suppose $|\pi^n| \leq 1, n \in \mathbb{N}$. The integrands π^n satisfy the properties

$$\begin{aligned} (\pi^n \cdot S)_T &= \frac{\delta \gamma_n}{K + 2C} (1_{[0, T^n]} \theta^n \cdot S)_T & (1) \\ &\geq \frac{\delta \gamma_n}{K + 2C} (-K - 2C) \geq -\delta \gamma_n \\ &\geq -\delta, \end{aligned}$$

$$P((\pi^n \cdot S)_T \geq \frac{\delta \gamma_n c_n}{K + 2C}) = P((1_{[0, T^n]} \theta^n \cdot S)_T \geq c_n) > \frac{\varepsilon}{2}. \quad (2)$$

Due to (1), the random variables $U(x + (\pi^n \cdot S)_T)$ are bounded from below by the constant $D = U(x - \delta) > -\infty$. Set $a_n = \frac{\delta \gamma_n c_n}{K + 2C}, n \in \mathbb{N}$, and observe that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Hence

$$\begin{aligned} E[U(x + (\pi^n \cdot S)_T)] &= E[U(x + (\pi^n \cdot S)_T) 1_{\{(\pi^n \cdot S)_T < a_n\}}] \\ &\quad + E[U(x + (\pi^n \cdot S)_T) 1_{\{(\pi^n \cdot S)_T \geq a_n\}}] \\ &\geq DP[(\pi^n \cdot S)_T < a_n] + U(x + a_n)P[(\pi^n \cdot S)_T \geq a_n] \\ &\geq D(1 - \frac{\varepsilon}{2}) + U(x + a_n)\frac{\varepsilon}{2} \longrightarrow \infty. \end{aligned}$$

But this is in contradiction with the hypothesis $\sup_{\theta \in \mathcal{S}, |\theta| \leq 1} E[U(x + (\theta \cdot S)_T)] < \infty$. \square

As in [11] one can show that the preceding lemma implies

Lemma 1.6 (Lemma 7.4. in [11]) *The set*

$$\left\{ \sum_{k=0}^n (S_{T_{k+1}} - S_{T_k})^2 \mid n \in \mathbb{N}, 0 \leq T_0 \leq \dots \leq T_{n+1} \leq T \right\}$$

is bounded in L^0 .

We can now complete the proof of Theorem (1.4) as in [11]. The two preceding lemmas imply that S is a semimartingale (proof of theorem 7.2 in [11]). \square

2 Simple versus general strategies

In the preceding section we have seen that if the expected utility maximized over the set of simple strategies is finite, the price process S is a semimartingale. As a consequence, S is a stochastic integrator, and its stochastic integral is defined not only for simple

integrands, but for a much wider class of \mathcal{F} -predictable strategies. A natural question arising in this context is the following: can a trader increase his optimal utility by using general S -integrable strategies? While this may be the case for discontinuous S , as is shown by an example at the end of this section, its main result will prove that for continuous asset price processes S the answer is no.

The utility functions $U : \mathbb{R} \rightarrow [-\infty, \infty)$ considered in this section have to fulfill the following further requirements. We suppose that U is strictly increasing, strictly concave and continuously differentiable on $\text{dom}(U) = \{y : U(y) > -\infty\}$. Furthermore we assume that

$$U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0. \quad (3)$$

We remark at this point that all results of this section could equally well be stated for infinite time horizon. For homogeneity reasons (some results we refer to in a later section are formulated for finite T) we refrain from doing so. Fix an initial wealth $x > \sup\{y : U(y) = -\infty\}$. We denote by \mathcal{A} the set of all S -integrable processes θ recalling the convention made in the previous section. From the economic point of view it is no restriction to admit only processes from \mathcal{A} .

We next define two quantities to be compared to the maximal expected utility taken over simple strategies. Let

$$u_a(x) = \sup_{\mathcal{A} \ni \theta \text{ } a\text{-adm.}} E[U(x + (\theta \cdot S)_T)],$$

and

$$u(x) = \sup_{\mathcal{A} \ni \theta \text{ } \text{adm.}} E[U(x + (\theta \cdot S)_T)].$$

Before stating the main result of this section, some preliminary steps are in order. The following auxiliary results deal with some aspects of a -admissible strategies.

Lemma 2.1 *Let S be a continuous semimartingale satisfying (NFLVR). If θ is a -admissible then almost surely*

$$(\theta \cdot S)_T > -a \implies (\theta \cdot S)_t > -a \text{ for all } 0 \leq t \leq T.$$

Proof. Let $A = \{\text{there exists a } t \in [0, T] \text{ for which } (\theta \cdot S)_t = -a\}$. A is measurable due to continuity of S . We have to show that $A \cap \{(\theta \cdot S)_T > -a\}$ has probability 0. Define the entrance time $T' = \inf\{t > 0 : (\theta \cdot S)_t = -a\} \wedge T$. Observe that the strategy $\pi = 1_A 1_{]T', T]} \theta$ satisfies

- i) $(\pi \cdot S)_T = 1_A [(\theta \cdot S)_T - (\theta \cdot S)_{T'}] \geq -a + a = 0$,
- ii) $P((\pi \cdot S)_T > 0) = P(T' < T, (\theta \cdot S)_T > -a) = P(A \cap \{(\theta \cdot S)_T > -a\})$.

If $P(A \cap \{(\theta \cdot S)_T > -a\}) > 0$, then i) and ii) would qualify π as an arbitrage opportunity. But this violates (NFLVR). \square

In a similar way we obtain

Proposition 2.2 *Let S be a continuous semimartingale satisfying (NFLVR). If $(\theta \cdot S)_T \geq -a$ a.s., then the process θ is a -admissible.*

Proof. For every $\varepsilon > 0$ define a stopping time by

$$T_\varepsilon = \inf\{t > 0 : (\theta \cdot S)_t = -a - \varepsilon\} \wedge T.$$

Suppose θ is not a -admissible. Then for some $\varepsilon > 0$ we must have $P(T_\varepsilon < T) > 0$. The strategy $\pi = 1_{]T_\varepsilon, T]} \theta$ satisfies

$$\begin{aligned} (\pi \cdot S)_T &= 1_{\{T_\varepsilon < T\}} [(\theta \cdot S)_T - (\theta \cdot S)_{T_\varepsilon}] \geq 0, \\ P((\pi \cdot S)_T > 0) &= P(T_\varepsilon < T) > 0. \end{aligned}$$

Hence π is an arbitrage opportunity. But this is a contradiction to (NFLVR). \square

In the following proposition we approximate admissible general strategies by simple ones.

Proposition 2.3 *Let S be a continuous semimartingale satisfying (NFLVR). For every a -admissible integrand θ there is a sequence of a -admissible simple processes $(\theta^n)_{n \geq 0}$ for which*

$$(\theta^n \cdot S)_T \longrightarrow (\theta \cdot S)_T \text{ a.s.}$$

Proof. Let $(\pi^n)_{n \in \mathbb{N}}$ be an arbitrary sequence of simple integrands such that a.s. the trajectories of $\pi^n \cdot S$ converge uniformly to those of $\theta \cdot S$. For $n \in \mathbb{N}$, we put

$$T_n = \inf\{t > 0 : (\pi^n \cdot S)_t \leq -a\} \wedge T.$$

We first show that T_n converges to T a.s. on the set $\{(\theta \cdot S)_T > -a\}$. According to Lemma 2.1 almost all $\omega \in \{(\theta \cdot S)_T > -a\}$ satisfy:

$$(\theta \cdot S)_t > -a \text{ for all } 0 \leq t \leq T.$$

Since $\theta \cdot S$ is continuous, for almost all $\omega \in \{(\theta \cdot S)_T > -a\}$ there exists a $\delta = \delta(\omega) > 0$ such that

$$(\theta \cdot S)_t(\omega) > -a + \delta.$$

Since $(\pi^n \cdot S)$ converges uniformly to $(\theta \cdot S)$, we find for almost every $\omega \in \{(\theta \cdot S)_T > -a\}$ some n_0 such that

$$(\pi^n \cdot S)_t > -a \text{ for all } 0 \leq t \leq T \text{ and } n \geq n_0.$$

It follows that $T_n \longrightarrow T$ a.s. on the set $\{(\theta \cdot S)_T > -a\}$.

Furthermore, the simple processes $\theta^n = 1_{[0, T_n]} \pi^n$, $n \in \mathbb{N}$, are obviously a -admissible and satisfy

$$\begin{aligned} & |(\theta^n \cdot S)_T - (\theta \cdot S)_T| \\ &= |(\theta^n \cdot S)_T - (\theta \cdot S)_T| 1_{\{T_n < T, (\theta \cdot S)_T > -a\}} \\ &\quad + |(\theta^n \cdot S)_T - (\theta \cdot S)_T| 1_{\{T_n = T\}} \\ &\leq |(\theta^n \cdot S)_T - (\theta \cdot S)_T| 1_{\{T_n < T, (\theta \cdot S)_T > -a\}} + |(\pi^n \cdot S)_T - (\theta \cdot S)_T| \end{aligned}$$

The first summand converges to 0 a.s., because T_n converges to T on the set $\{(\theta \cdot S)_T > -a\}$. Since the second summand also converges to 0, we obtain that $(\theta^n \cdot S)_T$ converges to $(\theta \cdot S)_T$ a.s. \square

The preceding proposition now allows to prove the result we aim at if for a fixed a we concentrate on a -admissible strategies.

Proposition 2.4 *If $a > 0$ is such that $U(x - a) > -\infty$, then*

$$u_a(x) = \sup_{\mathcal{S} \ni \theta \text{ } a\text{-adm.}} E[U(x + (\theta \cdot S)_T)].$$

Proof. We have to prove that the right hand side is not smaller than the left hand side. Let therefore θ be an a -admissible integrand. Proposition (2.3) states that we can find a sequence of a -admissible simple processes $(\theta^n)_{n \in \mathbb{N}}$ such that $(\theta^n \cdot S)_T \rightarrow (\theta \cdot S)_T$ a.s. Since the random variables $U(\theta^n \cdot S)_T, n \in \mathbb{N}$, are bounded from below by $U(x - a) > -\infty$, we conclude by using Fatou's lemma and the fact that U is continuous on $\{y : U(y) > -\infty\}$:

$$\begin{aligned} E[U(x + (\theta \cdot S)_T)] &= E[\lim_{n \rightarrow \infty} U(x + (\theta^n \cdot S)_T)] \\ &\leq \liminf_{n \rightarrow \infty} E[U(x + (\theta^n \cdot S)_T)] \\ &\leq \sup_{\mathcal{S} \ni \theta' \text{ } a\text{-adm.}} E[U(x + (\theta' \cdot S)_T)] = u_a(x). \end{aligned}$$

□

Remark. The proposition remains valid if $u_a(x) = \infty$.

We are now ready to state and prove the main result of this section.

Theorem 2.5 *Let S be a continuous semimartingale satisfying (NFLVR). If $x > \sup\{y : U(y) = -\infty\}$, then*

$$u(x) = \sup_{\mathcal{S} \ni \theta \text{ } adm.} E[U(x + (\theta \cdot S)_T)]. \quad (4)$$

In particular, the maximal expected utility $u(x)$ is infinite if and only if $\sup_{\mathcal{S} \ni \theta \text{ } adm.} E[U(x + (\theta \cdot S)_T)] = \infty$.

Proof. The proof will be executed in several steps. The utility functions admitted by the hypotheses above will be subdivided into several classes. This leads to distinguishing the following cases.

We start with

case 1: $\{y : U(y) > -\infty\} = \mathbb{R}$.

Observe that the exponential utility function $U(x) = -e^{-\alpha x}, x \in \mathbb{R}$, with $\alpha > 0$, is covered by case 1.

If the domain of U is \mathbb{R} , any admissible strategy leads to a utility bounded from below. This is the main observation needed to prove the assertion in this case. Let ζ be any admissible integrand. According to proposition 2.4 the expected utility $EU(x + (\zeta \cdot S)_T)$ is not greater than $\sup_{\mathcal{S} \ni \theta \text{ } adm.} E[U(x + (\theta \cdot S)_T)]$. Hence we have

$$u(x) \leq \sup_{\mathcal{S} \ni \theta \text{ } adm.} E[U(x + (\theta \cdot S)_T)].$$

Since the left hand side is obviously not smaller than the right hand side, equality holds.

case 2: $c = \sup\{y : U(y) = -\infty\} \in \mathbb{R}$ and $U(c) > -\infty$.

Think of the power utility function $U(x) = \frac{x^\alpha}{\alpha}, x > 0$, extended to be $-\infty$ for $x \leq 0$, where $\alpha \in (0, 1)$, as a typical example. If ζ is $(x - c)$ -admissible, then by proposition 2.4 the expected utility $EU(x + (\zeta \cdot S)_T)$ is dominated by $\sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_T)]$. Suppose now that ζ is not $(x - c)$ -admissible. By proposition 2.2 we have $(\theta \cdot S)_T < -x + c$ on a set of positive probability. Since $U(z) = -\infty$ for all $z < c$, the expected utility $EU(x + (\zeta \cdot S)_T)$ must equal $-\infty$. This provides the asserted equation in this case.

case 3: $c = \sup\{y : U(y) = -\infty\} \in \mathbb{R}$ and $U(c) = -\infty$

For example the logarithmic utility function is covered by this case.

To simplify notation we assume that $c = 0$. The fact that $\lim_{x \rightarrow 0^+} U(x) = -\infty$ implies

$$U'(0) = \lim_{x \rightarrow 0^+} U'(x) = \infty.$$

Hence the Inada conditions are satisfied and we are in the setting of [22]. We make use of Theorem 2.1 in [22], according to which the following statement holds true. If $u(x_0) < \infty$ for some $x_0 > 0$, then $u(x) < \infty$ for all $x > 0$ and the function u is continuously differentiable on $(0, \infty)$. With the help of this result we are able to prove the assertion in the given case.

Let $x > 0$. Assume first that $u(x) < \infty$. Due to the quoted result u is continuous on $(0, \infty)$. Hence for any $\varepsilon > 0$ there exists a $0 < y < x$ such that $u(x) - u(y) < \frac{\varepsilon}{2}$. Let ζ be an admissible strategy satisfying

$$u(y) - EU(y + (\zeta \cdot S)_T) \leq \frac{\varepsilon}{2}.$$

Proposition 2.2 guarantees that ζ is y -admissible. Starting with the initial wealth x , the utility process $U(x + (\zeta \cdot S)_t)$ will be bounded from below by the constant $D = U(x - y) > -\infty$. Again with proposition 2.4 we obtain

$$\begin{aligned} u(x) - \sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_T)] &\leq u(x) - EU(x + (\zeta \cdot S)_T) \\ &= [u(x) - u(y)] + [u(y) - EU(x + (\zeta \cdot S)_T)] \\ &\leq [u(x) - u(y)] + [u(y) - EU(y + (\zeta \cdot S)_T)] \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, the assertion follows.

Next suppose that $u(x) = \infty$. Then by the preceding theorem for $0 < y < x$ the maximal utility $u(y)$ is also infinite. Choose y -admissible integrands $\theta^n, n \in \mathbb{N}$, such that

$$EU(y + (\theta^n \cdot S)_T) \geq n \quad \text{for } n \in \mathbb{N}.$$

Given the initial wealth x we have

$$U(x + (\theta^n \cdot S)_t) \geq U(x - y) > -\infty$$

for all $t \geq 0, n \in \mathbb{N}$. Hence we can apply proposition 2.4 to obtain for $n \in \mathbb{N}$

$$\begin{aligned} \sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_T)] &\geq EU(x + (\theta^n \cdot S)_T) \\ &\geq EU(y + (\theta^n \cdot S)_T) \\ &\geq n. \end{aligned}$$

This shows that $\sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_T)] = \infty$.

This completes the proof in the final case. \square

Combining theorem 2.5 with the results of section 1 we get

Corollary 2.6 *Let S be an arbitrary adapted continuous process indexed by $[0, T]$, U a utility function with $\lim_{x \rightarrow \infty} U(x) = \infty$ and $x > \sup\{y : U(y) = -\infty\}$. If $\sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_T)] < \infty$, then S is a semimartingale satisfying (NFLVR) and expected utility maximized over general admissible integrands is also finite and given by $\sup_{S \ni \theta \text{ adm.}} E[u(x + (\theta \cdot S)_T)]$.*

Proof. This follows by combining corollary 1.3 and theorem 2.5. \square

We close this section with an example inspired by example 7.5 in [12] and showing that in theorem 2.5 the requirement that S is continuous cannot be dropped.

Example 2.7 *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of Gaussian unit variables and $(\phi_n)_{n \in \mathbb{N}}$ a sequence of random variables satisfying $P(\phi_n = 1) = 2^{-n}$ and $P(\phi_n = 0) = 1 - 2^{-n}$. Furthermore suppose that Z is a random variable with distribution $P(Z = a) = P(Z = b) = \frac{1}{2}$, where $0 < a < 1$ and $b > 1$. We assume that all these random variables are independent. Choose an enumeration $(q_n)_{n \in \mathbb{N}}$ of the rationals in $[0, 1]$. The process defined by*

$$S = 1_{[0,1[}(t) + Z 1_{\{1\}}(t) + \sum_{\{n: q_n \leq t\}} \phi_n X_n, \quad 0 \leq t \leq 1,$$

is cadlag. We start by showing that S is a semimartingale satisfying the (NFLVR) property. For this purpose denote by \tilde{P} the restriction of P to $\sigma(Z)$. It is obvious, that there is a probability measure $\tilde{Q} \sim \tilde{P}$ on $\sigma(Z)$ such that the expectation of Z with respect to \tilde{Q} is equal to 1. Note that the extension $dQ = \frac{d\tilde{Q}}{d\tilde{P}} dP$ is a probability measure such that

- i) $Q = \tilde{Q}$ on $\sigma(Z)$,
- ii) $Q = P$ on $\sigma(\phi_n X_n, n \in \mathbb{N})$ and
- iii) $Q \sim P$.

Hence the process S is a Q -martingale with respect to its natural filtration. By the fundamental theorem of asset pricing (see corollary 1.2 in [12]) this implies that S is a semimartingale satisfying the (NFLVR) property.

As in example 7.5 in [11] one can show that $\theta = 0$ is the only simple integrand which is admissible for S . Hence we have

$$\sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_1)] = U(x).$$

However, the non-simple strategy $\theta = 1_{\{1\}}$ has as final payoff

$$x + (\theta \cdot S)_1 = x + (S_1 - S_{-1}) = x + (Z - 1) \quad a.s.$$

If $\lim_{x \rightarrow \infty} U(x) = \infty$, choose x , a and b such that

$$EU(x + (\theta \cdot S)_1) = \frac{1}{2}U(x + a - 1) + \frac{1}{2}U(x + b - 1) > U(x).$$

For example if $U = \log$, $x = 1$, a and b are such that $ab = e^2$, then

$$EU(x + (\theta \cdot S)_1) = \frac{1}{2} \log(a) + \frac{1}{2} \log(b) = \frac{1}{2} \log(ab) = 1 > 0 = U(x).$$

Thus we have

$$u(x) \neq \sup_{S \ni \theta \text{ adm.}} E[U(x + (\theta \cdot S)_1)].$$

3 Instantaneous infinite utility

According to the previous sections bounded utility for an agent with an information horizon \mathcal{F} implies (NFLVR). Under this condition, Delbaen and Schachermayer [12] show that the process of bounded variation in the Doob-Meyer decomposition of S must be controlled by the martingale (*uncertainty*) part M of S , i.e. there is an \mathcal{F} -predictable process α such that

$$S = M + \alpha \cdot \langle M, M \rangle.$$

In this section, we shall start in this framework, to establish a relationship between the intensity of the intrinsic drift $\alpha \cdot \langle M, M \rangle$ of S and the boundedness of expected utility. We shall prove that if this drift has an instantaneously infinite increase at some stopping time T' , then at this same time there is an equally infinite increase of expected utility with respect to unbounded utility functions. Due to close connections between (NFLVR) and finite utility, explained in section 1, our treatment will in some parts heavily rely on similar arguments in Delbaen and Schachermayer [12].

This is the case in the following lemma in which a link between infinite intrinsic drift and the existence of admissible strategies inducing large wealths is established.

Lemma 3.1 *Suppose $P(\int_0^T \alpha^2 d\langle M, M \rangle = \infty) = \eta > 0$. Then for all $a, \xi > 0$ we can find an a -admissible integrand θ such that $P((\theta \cdot S)_T \geq 1) \geq \eta - \xi$.*

Proof. The proof is essentially the same as the one of Lemma 3.8 in [12], and is therefore omitted. \square

As an immediate consequence of the preceding, infinite drift with positive probability entails that free lunches are possible.

Corollary 3.2 *If $\int_0^T \alpha^2 d\langle M, M \rangle = \infty$ on a set with positive probability, then S satisfies (FLVR).*

For later use we are mainly interested in another consequence of the lemma. It says that infinite drift with positive probability also implies that the expected utility becomes infinite.

Theorem 3.3 *Suppose U is a utility function satisfying $\lim_{x \rightarrow \infty} U(x) = \infty$. If*

$$\int_0^T \alpha^2 d\langle M, M \rangle = \infty$$

on a set with positive probability, then for all $a > 0$ and $x > \sup\{y : U(y) = -\infty\}$ we have

$$u_a(x) = \infty.$$

Proof. Choose $a > 0$ and x so that $U(x) > -\infty$. By eventually reducing a we may assume that $D = U(x - a) > -\infty$. By lemma 3.1 there is an $\alpha > 0$ and a sequence $(\theta^n)_{n \in \mathbb{N}}$ of a -admissible integrands satisfying

$$P((\theta^n \cdot S)_T \geq n) > \alpha.$$

Since $U(x) \rightarrow \infty$, we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} EU(x + (\theta^n \cdot S)_T) &\geq \liminf_{n \rightarrow \infty} U(x + n)\alpha + D(1 - \alpha) \\ &= \infty, \end{aligned}$$

which proves the theorem. □

Remark. The theorem does neither follow from the preceding corollary nor from the ‘Immediate Arbitrage Theorem’ of Delbaen and Schachermayer in [12]. This is because there are situations where (NA) is violated, but $u_a(x)$ is finite for some a .

In the preceding findings about infinite utility the agent may need an arbitrarily long time to obtain unbounded utility. For completeness, we shall now generalize this to arbitrarily short time intervals after a stopping time. The following notion is related to the notion of *immediate arbitrage* (Definition 3.2 in [12]).

Definition 3.4 *Let U be a utility function with $U(x) \rightarrow \infty$ as $x \rightarrow \infty$ and x an initial wealth such that $U(x) > -\infty$. The semimartingale S admits instantaneous infinite utility at the stopping time \tilde{T} , where we suppose $P(\tilde{T} < T) > 0$, if for all $\varepsilon > 0$*

$$\sup_{\mathcal{A} \ni \theta \text{ adm.}} EU(x + (\theta 1_{] \tilde{T}, \tilde{T} + \varepsilon]} \cdot S)_T) = \infty.$$

Theorem 3.5 *(instantaneous infinite utility theorem)*
Suppose \tilde{T} is a stopping time with $P(\tilde{T} < T) > 0$. If

$$\int_{\tilde{T}}^{(\tilde{T} + \varepsilon) \wedge T} \alpha^2 d\langle M, M \rangle = \infty \quad \text{for all } \varepsilon > 0,$$

then S admits immediate infinite utility at time T .

Proof. Let $\varepsilon > 0$. We define $S'_t = S_{t \wedge \tilde{T} + \varepsilon} - S_{t \wedge \tilde{T}}$, $0 \leq t \leq T$. The semimartingale S' satisfies the conditions of theorem 3.3. Hence it admits infinite utility. Let $(\theta^n)_{n \in \mathbb{N}}$ be a sequence with $\lim_{n \rightarrow \infty} EU(x + (\theta^n \cdot S')_T) = \infty$. On the interval $]\tilde{T} \wedge T, (\tilde{T} + \varepsilon) \wedge T]$ the process S' coincides with S . Hence $\theta^n \cdot S' = \theta^n 1_{] \tilde{T} \wedge T, (\tilde{T} + \varepsilon) \wedge T]} \cdot S$. The claim is now obvious. \square

4 Logarithmic utility of an agent

In this section we uniquely consider the case of logarithmic utility. So let $U = \log$ throughout the section. We also assume throughout that the asset price process of the market be continuous. Assume that an agent with information horizon \mathcal{F} possess bounded logarithmic utility. According to section 1 we therefore know that S enjoys the (NFLVR) property, and thus it is a semimartingale with Doob-Meyer decomposition

$$S = M + \alpha \cdot \langle M, M \rangle. \quad (5)$$

The aim of this section consists in computing explicitly the expected logarithmic utility of the agent, and to prove implicitly that it only depends on the drift density α . In fact, we shall prove that

$$u(x) = \log x + \frac{1}{2} E \int_0^T \alpha_s^2 d\langle M, M \rangle_s, \quad x > 0. \quad (6)$$

We shall even show that (6) is valid irrespective of whether (NFLVR) holds, provided (5) is guaranteed. So we do not assume completeness for the underlying financial market. Our analysis in fact does not use equivalent martingale measures for S . The method of proof of (6) we employ consists in using the linear stochastic equation link allowing to describe the optimal portfolio θ^* as a function of the drift process α in a completely general framework. In subsection 4.1., we shall consider the case of positive wealth, in subsection 4.2. we extend the results to the case in which wealth may become negative.

4.1 Maximal utility if wealth stays positive

If wealth always stays positive, we may consider the following class of admissible strategies.

Definition 4.1 *An S -integrable and predictable process θ is called a -superadmissible if almost surely $(\theta \cdot S)_t > -a$ for all $t \in [0, T]$.*

Our goal is to find

$$u^+(x) = \sup_{\mathcal{A} \ni \theta x\text{-superadm.}} E \log(x + (\theta \cdot S)_T).$$

It will be helpful to express the right hand side by the so-called *optimal portfolio*, i.e. the process $\theta^* \in \mathcal{A}$ which satisfies $u^+(x) = E \log(x + (\theta^* \cdot S)_T)$. Before we can show

that the optimal portfolio exists and may be expressed as a function of α , we have to prove some auxiliary results which will ultimately turn out to present the optimal portfolio as the unique solution of a linear stochastic equation. Recall that we do not assume (NFLVR) here. We start by proving

Proposition 4.2 *Suppose $E \int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$. If π is a progressively measurable and S -integrable process, the product $\mathcal{E}(\pi \cdot S)\mathcal{E}(-\alpha \cdot M)$ is a local martingale.*

Proof: Let $Z = \mathcal{E}(-\alpha \cdot M)$. Note that Z is a local martingale. By applying the integration by parts formula we obtain for $t \in [0, T]$

$$\begin{aligned}
Z_t \mathcal{E}(\pi \cdot S)_t &= 1 + \int_0^t \mathcal{E}(\pi \cdot S) dZ + \int_0^t Z d\mathcal{E}(\pi \cdot S) + \langle \mathcal{E}(\pi \cdot S), Z \rangle_t \\
&= 1 + \int_0^t \mathcal{E}(\pi \cdot S) dZ + \int_0^t \pi \mathcal{E}(\pi \cdot S) Z dS + \int_0^t \pi \mathcal{E}(\pi \cdot S) d\langle S, Z \rangle \\
&= 1 + \int_0^t \mathcal{E}(\pi \cdot S) dZ + \int_0^t \pi \mathcal{E}(\pi \cdot S) Z dM \\
&\quad + \int_0^t \pi \mathcal{E}(\pi \cdot S) \alpha Z d\langle M, M \rangle - \int_0^t \pi \mathcal{E}(\pi \cdot S) \alpha Z d\langle M, M \rangle \\
&= 1 + \int_0^t \mathcal{E}(\pi \cdot S) dZ + \int_0^t \pi \mathcal{E}(\pi \cdot S) Z dM.
\end{aligned}$$

This shows that $Z\mathcal{E}(\pi \cdot S)$ is a local martingale. \square

Remark: Proposition (4.2) states that $\mathcal{E}(-\alpha \cdot S)$ is a strict martingale density for $\mathcal{E}(\pi \cdot S)$ in the sense of Schweizer (1995).

Lemma 4.3 *Suppose that $x > 0$ and $E \int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$. The process $H = x\alpha\mathcal{E}(\alpha \cdot S)$ is x -superadmissible, belongs to \mathcal{A} and solves the integral equation*

$$H_t = \alpha_t \left(x + \int_0^t H_r dS_r \right), \quad 0 \leq t \leq T. \quad (7)$$

Proof. We observe that the process $H = x\alpha\mathcal{E}(\alpha \cdot S)$ is progressively measurable and satisfies for all $t \in [0, T]$

$$\begin{aligned}
x + (H \cdot S)_t &= x + x \int_0^t \alpha_r \mathcal{E}(\alpha \cdot S)_r dS_r \\
&= x \left(1 + \int_0^t \alpha_r \mathcal{E}(\alpha \cdot S)_r dS_r \right) \\
&= x \mathcal{E}(\alpha \cdot S)_t > 0.
\end{aligned}$$

This yields that H is x -superadmissible.

The expression

$$E \log(x + (H \cdot S)_T) = \log x + E(\alpha \cdot S)_T - \frac{1}{2} E \int_0^T \alpha_s^2 d\langle M, M \rangle_s$$

makes sense due to the integrability condition $E \int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$. Hence H belongs to \mathcal{A} .

We now prove the last claim. The process $Y = x\mathcal{E}(\alpha \cdot S)$ is the unique solution of the SDE

$$\begin{aligned} Y_0 &= x \\ dY_t &= Y_t \alpha_t dS_t = Y_t d(\alpha \cdot S)_t. \end{aligned}$$

Obviously Y solves also

$$\alpha_t Y_t = \alpha_t \left(x + \int_0^t \alpha_r Y_r dS_r \right).$$

Hence $H = x\alpha\mathcal{E}(\alpha \cdot S)$ is a solution of equation (7). \square

We now state the main result of this subsection. It generalizes Theorem 3.5. of [3], where it was proved in the special case of a semimartingale given by an SDE.

Theorem 4.4 *For any $x > 0$ the following equation holds*

$$u^+(x) = \log x + \frac{1}{2} E \int_0^T \alpha_s^2 d\langle M, M \rangle_s. \quad (8)$$

If $E \int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$, then the process

$$\theta^* = x\alpha\mathcal{E}(\alpha \cdot S)$$

is the unique optimal portfolio.

Proof. We first assume that $E \int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$.

Let $\theta \in \mathcal{A}$ be x -superadmissible. Then $x + (\theta \cdot S)_t > 0$ a.s. for all $t \in [0, T]$ and hence we can define a new process by

$$\pi_t = \frac{\theta_t}{x + (\theta \cdot S)_t}, \quad 0 \leq t \leq T.$$

Since π is progressively measurable, the integral $\pi \cdot S$ is defined.

The SDE

$$\begin{aligned} Y_0 &= x, \\ dY_t &= \pi_t Y_t dS_t = Y_t d(\pi \cdot S)_t \end{aligned}$$

is uniquely solved by the process $Y = x\mathcal{E}(\pi \cdot S)$. On the other hand the process $x + (\theta \cdot S)_t$ is also easily seen to be a solution. By uniqueness this implies

$$x + (\theta \cdot S) = x\mathcal{E}(\pi \cdot S). \quad (9)$$

In the next step we will show that the expected logarithmic utility of $x + (\theta \cdot S)_T$ is not greater than $\log x + \frac{1}{2} E \int_0^T \alpha_s^2 d\langle M, M \rangle_s$. Applying the inequality $\log z \leq z - 1$, valid for positive z , to the product of two positive numbers a, b we get the inequality

$$\log a \leq ab - \log b - 1.$$

If we take $a = x\mathcal{E}(\pi \cdot S)$ and $b = \frac{1}{x}\mathcal{E}(-\alpha \cdot M)$ we obtain

$$\log x\mathcal{E}(\pi \cdot S) \leq \mathcal{E}(\pi \cdot S)\mathcal{E}(-\alpha \cdot M) - \log \frac{1}{x}\mathcal{E}(-\alpha \cdot M) - 1.$$

By proposition (4.2) the product $\mathcal{E}(\pi \cdot S)\mathcal{E}(-\alpha \cdot M)$ is a local martingale. Since it is nonnegative, it is also a supermartingale and therefore by (9)

$$\begin{aligned} E[\log(x + (\theta \cdot S)_T)] &= E[\log x\mathcal{E}(\pi \cdot S)_T] \\ &\leq E[\mathcal{E}(\pi \cdot S)_T\mathcal{E}(-\alpha \cdot M)_T - \log \frac{1}{x}\mathcal{E}(-\alpha \cdot M)_T - 1] \\ &\leq -E[\log \frac{1}{x}\mathcal{E}(-\alpha \cdot M)_T] \\ &= \log x - E\left[-\int_0^T \alpha_t dM_t - \frac{1}{2}\int_0^T \alpha^2 d\langle M, M \rangle\right] \\ &= \log x + \frac{1}{2}E\int_0^T \alpha^2 d\langle M, M \rangle. \end{aligned}$$

This implies

$$u^+(x) \leq \log x + \frac{1}{2}E\int_0^T \alpha^2 d\langle M, M \rangle.$$

Before we prove that in fact equality holds, we note

$$E\log(x\mathcal{E}(\alpha \cdot S)_T) = \log x + \frac{1}{2}E\int_0^T \alpha^2 d\langle M, M \rangle.$$

Therefore it is enough to show that there is a process θ such that $E\log(x + (\theta \cdot S)_T) = E\log(x\mathcal{E}(\alpha \cdot S)_T)$.

According to lemma 4.3 the process $\theta^* = x\alpha\mathcal{E}(\alpha \cdot S)$ belongs to \mathcal{A} , is x -superadmissible and satisfies

$$\alpha = \frac{\theta^*}{x + (\theta^* \cdot S)}.$$

As in the first part of the proof we deduce

$$x + (\theta^* \cdot S)_t = x\mathcal{E}(\alpha \cdot S)_t.$$

This proves the theorem in the case where $E\int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$.

We now claim that equation (8) is still true if $E\int_0^T \alpha_s^2 d\langle M, M \rangle_s = \infty$. Suppose $\int_0^T \alpha_s^2 d\langle M, M \rangle_s = \infty$ on a set with positive probability. Then theorem 3.3 yields $u^+(x) = \infty$.

If $\int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$ almost surely, we can find stopping times $(T_n)_{n \in \mathbb{N}}$ such that $T_n \rightarrow T$ and

$$E\int_0^{T_n} \alpha_s^2 d\langle M, M \rangle_s < \infty.$$

With the first part of the proof we deduce

$$u^+(x) \geq \log x + \frac{1}{2}E\int_0^{T_n} \alpha_s^2 d\langle M, M \rangle_s$$

for all $n \in \mathbb{N}$. By dominated convergence the right hand side goes to infinity as $n \rightarrow \infty$. Hence $u^+(x) = \infty$, which completes the proof. \square

4.2 Maximal utility if wealth may become negative

In this section we allow the wealth process to take negative values and again deduce the desired formula for $u(x)$.

Let S be a continuous semimartingale satisfying (NFLVR). So, as usual, we can write

$$dS = dM + \alpha d\langle M, M \rangle,$$

due to [12]. If $\theta \in \mathcal{A}$ is not x -superadmissible, then by lemma 2.1

$$(\theta \cdot S)_T \leq -x$$

on a set with positive probability. But this implies $E \log(x + (\theta \cdot S)_T) = -\infty$ and therefore $u(x) = u^+(x)$. Hence we have shown

Theorem 4.5 *Let S be a continuous semimartingale satisfying (NFLVR). The maximal expected logarithmic utility is given by*

$$u(x) = \log x + \frac{1}{2} E \int_0^T \alpha_s^2 d\langle M, M \rangle_s.$$

Remark. Kramkov and Schachermayer [22] show that under the assumption of (NFLVR) a more general result can be obtained. They give explicit formulas for the maximal expected utility not only for the logarithm but for a large class of utility functions.

We mention that $E \int_0^T \alpha_s^2 d\langle M, M \rangle_s < \infty$ does not imply the (NFLVR) property. In the following example the integral of the drift is finite, but arbitrage is possible and hence $u(x)$ is infinite (see proposition 1.2). Hence the assumption of (NFLVR) in theorem 4.5 cannot be dropped.

Example 4.6 *Let W be a Brownian motion on some probability space (Ω, F, P) . We denote by $(\mathcal{F}_t)_{t \geq 0}$ the completed filtration generated by W . We will study the price process*

$$S_t = \mathcal{E}(W)_t, \quad t \geq 0,$$

not under $(\mathcal{F}_t)_{t \geq 0}$ but with respect to a larger filtration. Choose for example $T = 1$, let $a, b \in \mathbb{R}$ such that $a < b$, let $G = 1_{[a,b]}(W_1)$, and take the right continuous and completed version of $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G), t \in [0, 1]$. It has been shown in [3] that an agent in this filtration possesses finite logarithmic utility, if wealth has to be positive. $u^+(x)$ is given by the entropy of G , or, alternatively, by $\frac{1}{2} E \int_0^1 \alpha_s^2 ds$ with the corresponding information drift α .

We will see now that there are arbitrage strategies. Define a stopping time by

$$T = \inf\{t \geq 0 : W_t \leq a - 1\} \wedge 1.$$

The strategy $\theta = 1_{\{W_1 \in [a,b]\}} 1_{[T,1]}$ is admissible, because

$$(\theta \cdot S)_t \geq -e^{a-1}, \quad 0 \leq t \leq 1.$$

Furthermore θ satisfies

- i) $(\theta \cdot S)_1 = 1_{\{W_1 \in [a,b]\}}(S_1 - S_T) \geq 0$ and
- ii) $P((\theta \cdot S)_1 > 0) = P(T < 1, W_1 \in [a,b]) > 0$,

which shows that θ is an arbitrage strategy. In particular S doesn't have the (NFLVR) property.

As a summarizing consequence of the results obtained so far we note the following Corollary which computes the (bounded) expected logarithmic utility of an agent with information horizon \mathcal{F} from first principles in a very general setting, and therefore generalizes Theorem 3.7 in [8].

Corollary 4.7 *Let S be an \mathcal{F} -adapted continuous stochastic process and $x > 0$. If $\sup_{S \ni \theta \text{ adm.}} E \log(x + (\theta \cdot S)_T) < \infty$, then S is a semimartingale satisfying (NFLVR) possessing a Doob-Meyer decomposition*

$$S = M + \alpha \cdot \langle M, M \rangle.$$

with a local martingale M , and an \mathcal{F} -predictable process α . Furthermore there is a process θ^* solving the optimal portfolio problem and satisfying

$$\theta^* = x\alpha\mathcal{E}(\alpha \cdot S).$$

The maximal expected logarithmic utility is

$$u(x) = \log x + \frac{1}{2}E \int_0^T \alpha_s^2 d\langle S, S \rangle_s.$$

Proof. By corollary 1.3 S is a semimartingale satisfying (NFLVR). Theorem 2.5 states that $u(x) = \sup_{S \ni \theta \text{ adm.}} E[\log(x + (\theta \cdot S)_T)]$, hence $u(x)$ is finite. By theorem 4.4 we know that $\theta^* = x\alpha\mathcal{E}(\alpha \cdot S)$ is the solution of the optimal portfolio problem. The last assertion follows from theorem 4.5. \square

5 Additional logarithmic utility of an insider

We now return to the setting of a financial market with small agents possessing asymmetric information. But as before we keep our analysis very general. So we assume that each of the agents (regular trader and insider) takes his portfolio decisions on the basis of his individual information horizon, given by different filtrations \mathcal{F} and \mathcal{G} . We just suppose that the insider's filtration is bigger, but do not specify at all what the sources for the additional information in \mathcal{G} are. The asset price process S will again be continuous, and as in the preceding section we shall consider logarithmic utility $U = \log$. Our main result will show that in this general setting the finite utility advantage an insider has compared to the regular trader is given by

$$\frac{1}{2}E \int_0^T \gamma_s^2 d\langle M, M \rangle_s,$$

if γ is the density of the relative information drift obtained by passing from \mathcal{F} to \mathcal{G} .

Let us first specify those agents who possess finite utility on the basis of their knowledge.

Definition 5.1 A filtration \mathcal{F} satisfying the usual conditions is a finite utility filtration for S , if S is a \mathcal{F} -semimartingale with decomposition $dS = dM + \alpha \cdot d\langle M, M \rangle$, where α is predictable and belongs to $L^2(P_M)$, i.e. $E \int_0^T \alpha^2 d\langle M, M \rangle < \infty$.

Given an initial wealth $x > 0$, we denote by $u^+(\mathcal{F}, x)$ the corresponding maximal expected utility if wealth has always to be positive. Note that if \mathcal{F} is a finite utility filtration, $u^+(\mathcal{F}, x)$ is finite for all $x > 0$ (see theorem 4.4).

Let \mathcal{F} and \mathcal{G} be two finite utility filtrations. We denote by

$$S = M + \alpha \cdot \langle M, M \rangle \quad (10)$$

the semimartingale decomposition with respect to \mathcal{F} and by

$$S = N + \beta \cdot \langle N, N \rangle \quad (11)$$

the decomposition with respect to \mathcal{G} . Obviously,

$$\langle M, M \rangle = \langle S, S \rangle = \langle N, N \rangle$$

and therefore equations (10) and (11) imply

$$M = N - (\alpha - \beta) \cdot \langle M, M \rangle \quad a.s. \quad (12)$$

If $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$, equation (12) can be interpreted as the semimartingale decomposition of M with respect to \mathcal{G} .

A bigger filtration must clearly lead to a bigger maximal utility. The following Theorem will quantify this increase.

Theorem 5.2 If $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$, then for any $x > 0$

$$u^+(\mathcal{G}, x) - u^+(\mathcal{F}, x) = \frac{1}{2} E \int_0^T (\beta - \alpha)^2 d\langle M, M \rangle. \quad (13)$$

Proof. Since α and β are in $L^2(P_M)$, we can write

$$\frac{1}{2} E \int_0^T (\beta - \alpha)^2 d\langle M, M \rangle = \frac{1}{2} E \int_0^T (\beta^2 - \alpha^2) d\langle M, M \rangle + E \int_0^T (\alpha^2 - \alpha\beta) d\langle M, M \rangle.$$

Since α is both \mathcal{F} - and \mathcal{G} - adapted we have

$$\begin{aligned} E \int_0^T (\alpha^2 - \alpha\beta) d\langle M, M \rangle &= E \int_0^T \alpha dN - E \int_0^T \alpha dM \\ &= 0. \end{aligned}$$

Hence

$$\frac{1}{2} E \int_0^T (\beta - \alpha)^2 d\langle M, M \rangle = \frac{1}{2} E \int_0^T (\beta^2 - \alpha^2) d\langle M, M \rangle = u^+(\mathcal{G}, x) - u^+(\mathcal{F}, x). \quad \square$$

We see that the additional utility depends only on the relative drift density process $(\beta - \alpha)$. On the other hand equation (12) shows that $(\beta - \alpha)$ is the density of the process of bounded variation which has to be subtracted from M in order to obtain a \mathcal{G} -local martingale. This relationship motivates the following notion.

Definition 5.3 Let \mathcal{F} be a finite utility filtration and $S = M + \alpha \cdot \langle M, M \rangle$ the Doob-Meyer decomposition of S with respect to \mathcal{F} . Suppose that \mathcal{G} is a filtration such that $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$. The \mathcal{G} -predictable process μ satisfying

$$M - \int_0^\cdot \mu_t d\langle M, M \rangle_t \quad \text{is a } \mathcal{G}\text{-local martingale}$$

is called information drift (see [18]) of \mathcal{G} with respect to \mathcal{F} .

Notice that any finite utility filtration containing \mathcal{F} has an information drift with respect to \mathcal{F} . Using this terminology we can paraphrase Theorem 5.2 as follows.

Theorem 5.4 Let \mathcal{F} and \mathcal{G} be two finite utility filtrations such that $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$. If μ is the information drift of \mathcal{G} w.r.t. \mathcal{F} , we have

$$u^+(\mathcal{G}, x) - u^+(\mathcal{F}, x) = \frac{1}{2} E \int_0^T \mu^2 d\langle M, M \rangle.$$

Relative information drifts are additive with respect to successive refinements of filtrations. Indeed, let \mathcal{F} , \mathcal{G} and \mathcal{H} be three finite utility filtrations such that $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$ for all $t \in [0, T]$. Suppose that μ is the information drift of \mathcal{G} with respect to \mathcal{F} . Then by the definition we know that $\tilde{M} = M - \mu \cdot \langle M, M \rangle$ is a \mathcal{G} -local martingale. If λ is the information drift of \mathcal{H} with respect to \mathcal{G} , then $\tilde{M} - \lambda \cdot \langle M, M \rangle$ is a \mathcal{H} -local martingale. As a consequence $\kappa = \mu + \lambda$ is the information drift of \mathcal{H} with respect to \mathcal{F} . We obtain

Proposition 5.5 The information drift of \mathcal{H} w.r.t. \mathcal{F} is the sum of the information drift of \mathcal{G} w.r.t. \mathcal{F} and the information drift of \mathcal{H} w.r.t. \mathcal{G} .

In the same situation, we will now show that the information drift of \mathcal{G} with respect to \mathcal{F} can be expressed as the optional projection of the information drift of \mathcal{H} with respect to \mathcal{F} . To this end, we have to recall some basic facts about this notion of projection.

Proposition 5.6 (Theorem 43, chapter VI in [14]) Let X be a measurable process which is either nonnegative or bounded. If \mathcal{K} is a filtration satisfying the usual conditions, then there is a unique (up to indistinguishability) \mathcal{K} -predictable process Y such that

$$Y_T 1_{\{T < \infty\}} = E[X_T 1_{\{T < \infty\}} | \mathcal{K}_{T-}]$$

a.s. for every predictable stopping time T . Y is called the predictable projection of X on \mathcal{K} .

The predictable projection has the following useful property.

Proposition 5.7 Let X be a measurable process which is either nonnegative or bounded and let A be an increasing process adapted to \mathcal{K} . The \mathcal{K} -predictable projection Y of X satisfies

$$E \left[\int_T^\infty X_s dA_s | \mathcal{K}_{T-} \right] = E \left[\int_T^\infty Y_s dA_s | \mathcal{K}_{T-} \right]$$

for every predictable stopping time T .

Proof. This is shown in remark b after theorem 57, chapter VI, in [14]. \square

We now generalize proposition 5.6 and 5.7 to measurable processes in $L^2(P_M)$.

Proposition 5.8 *Let X be a measurable process in $L^2(P_M)$. If $\mathcal{K} \supset \mathcal{F}$ satisfies the usual conditions, there is a unique (up to indistinguishability) \mathcal{K} -predictable process $Y \in L^2(P_M)$ such that*

$$Y_T 1_{\{T < \infty\}} = E[X_T 1_{\{T < \infty\}} | \mathcal{K}_{T-}]$$

a.s. for every predictable stopping time T .

Proof. Let Y^+ be the predictable projection of X^+ and let Z^+ be the predictable projection of $(X^+)^2$ on \mathcal{K} . For all $t \geq 0$ we have

$$(Y_t^+)^2 = E[X_t^+ | \mathcal{K}_{t-}]^2 \leq E[(X_t^+)^2 | \mathcal{K}_{t-}] = Z_t^+$$

a.s. by Jensen's inequality. Since Y^+ and Z^+ are cadlag, almost everywhere

$$(Y_t^+)^2 \leq Z_t^+ \quad \text{for all } t \geq 0.$$

Hence by Proposition 5.7

$$\begin{aligned} E \left[\int_0^\infty (Y_t^+)^2 d\langle M, M \rangle_t \right] &\leq E \left[\int_0^\infty Z_t^+ d\langle M, M \rangle_t \right] \\ &= E \left[\int_0^\infty (X_t^+)^2 d\langle M, M \rangle_t \right] \\ &< \infty. \end{aligned}$$

Thus we have shown that Y^+ lies in $L^2(P_M)$. In a similar way we obtain $Y^- \in L^2(P_M)$ and therefore $Y \in L^2(P_M)$. Obviously Y satisfies $Y_T 1_{\{T < \infty\}} = E[X_T 1_{\{T < \infty\}} | \mathcal{K}_{T-}]$. \square

The generalization of Proposition 5.9 is now immediate.

Proposition 5.9 *Let $X \in L^2(P_M)$ be a measurable process and $\mathcal{K} \supset \mathcal{F}$. The \mathcal{K} -predictable projection Y of X satisfies*

$$E \left[\int_T^\infty X_s d\langle M, M \rangle_s | \mathcal{K}_{T-} \right] = E \left[\int_T^\infty Y_s d\langle M, M \rangle_s | \mathcal{K}_{T-} \right]$$

for every predictable stopping time T .

Equipped with these prerequisites we can state our theorem about the predictable projection property of information drifts. Recall that $\mathcal{F} \subset \mathcal{G} \subset \mathcal{H}$ are finite utility filtrations, $S = M + \alpha \cdot \langle M, M \rangle$ is the decomposition w.r.t. \mathcal{F} , μ is the information drift of \mathcal{G} with respect to \mathcal{F} and κ the information drift of \mathcal{H} with respect to \mathcal{F} . Also note that we return to finite horizon T now.

Theorem 5.10 *Let μ be the information drift of \mathcal{G} with respect to \mathcal{F} and ${}^p\kappa$ the predictable projection of κ on \mathcal{G} . Then P_M -a.s.*

$$\mu = {}^p\kappa.$$

Proof: Assume that $S = M + \alpha \cdot \langle M, M \rangle$ is the semimartingale decomposition with respect to \mathcal{F} . We may assume, by localizing the processes with some stopping time, that both M and $\kappa \cdot \langle M, M \rangle$ are bounded. We have to show that $M - {}^p\kappa \cdot \langle M, M \rangle$ is a \mathcal{G} -local martingale. Choose $0 \leq s < t$, a set $A \in \mathcal{G}_s$, and $\epsilon > 0$ such that $s + \epsilon < t$. Since \mathcal{G} is a sub-filtration of \mathcal{H} we obtain, using Proposition 5.7,

$$\begin{aligned} E(1_A(M_t - M_{s+\epsilon})) &= E(1_A \int_{s+\epsilon}^t \kappa_r d\langle M, M \rangle_r) \\ &= E(1_A E \left[\int_{s+\epsilon}^t \kappa_r d\langle M, M \rangle_r \middle| \mathcal{G}_{(s+\epsilon)-} \right]) \\ &= E(1_A E \left[\int_{s+\epsilon}^t {}^p\kappa_r d\langle M, M \rangle_r \middle| \mathcal{G}_{(s+\epsilon)-} \right]) \\ &= E(1_A \int_{s+\epsilon}^t {}^p\kappa_r d\langle M, M \rangle). \end{aligned}$$

We next let $\epsilon \rightarrow 0$ in the extremes of the chain of equations just obtained, and use dominated convergence to get

$$E(1_A(M_t - M_s)) = E(1_A \int_s^t {}^p\kappa_r d\langle M, M \rangle).$$

This proves the claim. □

In particular we have

Corollary 5.11 *If μ is the information drift of \mathcal{G} with respect to \mathcal{F} , then the predictable projection of μ on \mathcal{F} vanishes.*

6 Additional utility of an insider on a complete market

The main aim of this section is to describe the additional utility of an insider with respect to a regular trader for fairly arbitrary utility functions. Besides, we shall briefly discuss *always optimal strategies*, i.e. strategies that optimize expected utility from terminal wealth if any time t in the trading interval may be chosen as terminal. Again, the setting is very general: we specify the information advantage between the insider with filtration \mathcal{G} and the regular trader with filtration \mathcal{F} just by the relative information drift. As opposed to the previous section, we however assume the market to be complete here, so that we may invoke the general results by Kramkov and Schachermayer [22] about maximal utility. We assume time horizon $T > 0$ to be finite, and as usual the asset price process S indexed by $[0, T]$ to be continuous. Completeness entails that there is a unique equivalent local martingale measure, which we will denote by Q . By the fundamental theorem of asset pricing the NFLVR property holds and hence we may decompose S into

$$S = M + \alpha \cdot \langle M, M \rangle,$$

where M is a P -local martingale and α an \mathcal{F} -predictable process. The Radon-Nikodym density of the martingale measure given P is known to be described by the exponential of $\alpha \cdot M$:

$$\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = \mathcal{E}(-\alpha \cdot M)_t, \quad t \in [0, T]. \quad (14)$$

(see [13].)

In the following we shall abbreviate

$$Z = \mathcal{E}(-\alpha \cdot M).$$

Let us next describe the class of utility functions for which the maximal expected utility can be explicitly calculated by means of Z . Let U be strictly increasing, strictly concave and continuously differentiable on $(0, \infty)$. Furthermore we assume that U satisfies

$$\lim_{x \rightarrow 0^+} U'(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(x) = 0 \quad (15)$$

and that

$$u(x_0) < \infty \quad \text{for some } x_0 > 0. \quad (16)$$

On $(0, \infty)$ the derivative of U has an inverse function, which we will denote by I . Observe that I is a function with domain $(0, \infty)$ and with range $(0, \infty)$. The following formula for the maximal expected utility is obtained by Kramkov, Schachermayer [22].

Theorem 6.1 (theorem 2.0 in [22]) *Assume that the conditions (15) and (16) are satisfied. For all $x > 0$ we have*

$$u(x) = EU(I(yZ_T)),$$

where y is the real number satisfying $E[Z_T I(yZ_T)] = x$. Furthermore the process $I(yZ_t)$ is a uniformly integrable martingale under Q .

6.1 Always optimal strategies

The maximal expected utility $u(x)$ depends of course on the time interval in which the traders are allowed to act. We will denote by $u_t(x)$ the maximal expected utility of a trader of initial wealth x who is not allowed to hold any shares of the stock after time $t \leq T$, i.e.

$$u_t(x) = \sup_{\theta \in \mathcal{A}} EU(x + (\theta 1_{[0,t]} \cdot S)_T) = \sup_{\theta \in \mathcal{A}} EU(x + (\theta \cdot S)_t).$$

Definition 6.2 *A strategy $\theta^* \in \mathcal{A}$ is called always optimal, if for all $t \in [0, T]$ and $x > 0$*

$$EU(x + (\theta^* \cdot S)_t) = u_t(x).$$

We will now analyze to which extent always optimal strategies exist.

Consider at first the case where the drift α is equal to 0. In this case the price process S is a P -local martingale and intuitively one would expect that a risk averse trader will not trade at all. Theorem 6.1 confirms that the maximal expected utility is the utility of the initial capital $U(x)$. Hence in this case the trivial strategy $\theta = 0$ is always optimal, whatever the utility function U looks like.

If the drift α is not trivial, however, the situation is different. It turns out that in general always optimal strategies exist only for logarithmic utility functions. Before proving this we define

$$\bar{Z}_T = \sup_{0 \leq t \leq T} Z_t$$

and

$$\underline{Z}_T = \inf_{0 \leq t \leq T} Z_t.$$

We will only consider the case where

$$\text{ess inf } \underline{Z}_T = 0 \quad \text{and} \quad \text{ess sup } \bar{Z}_T = \infty. \quad (17)$$

Theorem 6.3 *Assume that $I = (U')^{-1}$ is twice continuously differentiable on $(0, \infty)$ and that the conditions (15), (16) and (17) are satisfied. Then an always optimal strategy exists if and only if U is the logarithm up to affine transformations, i.e.*

$$U(x) = a \log(x) + b$$

for some constants $a > 0$ and $b \in \mathbb{R}$.

Proof. Suppose at first that $U(x) = \log(x) + c$. By theorem 6.1 we have for any $t \in [0, T]$

$$\begin{aligned} u_t(x) &= EU(I(yZ_t)) = EU\left(\frac{1}{yZ_t}\right) \\ &= E \log(xZ_t^{-1}) + c = E \log[x\mathcal{E}(\alpha \cdot S)_t] + c = E \log[x + (\alpha\mathcal{E}(\alpha \cdot S) \cdot S)_t] + c. \end{aligned}$$

This shows that $\theta^* = \alpha\mathcal{E}(\alpha \cdot S)$ is always optimal.

We now prove the converse statement. Let θ^* be an always optimal strategy. By theorem 6.1 the process

$$x + (\theta^* \cdot S) = I(yZ)$$

is a Q -martingale. Hence

$$ZI(yZ)$$

is a P -martingale. Since the function $\phi : (0, \infty) \rightarrow \mathbb{R}$, $\phi(x) = xI(yx)$ is twice continuously differentiable, we may apply Itô's formula and obtain for $t \in [0, T]$

$$Z_t I(yZ_t) = \phi(Z_t) = \phi(1) + \int_0^t \phi'(Z_s) dZ_s + \frac{1}{2} \int_0^t \phi''(Z_s) d\langle Z, Z \rangle_s.$$

From this equation we can deduce that the continuous process of bounded variation

$$\int_0^\cdot \phi''(Z_s) d\langle Z, Z \rangle_s = \int_0^\cdot \phi''(Z_s) \alpha_s^2 Z_s^2 d\langle M, M \rangle_s$$

is a P -martingale and hence vanishes. We will now show that $\phi''(z) = 0$ for all $z > 0$. Suppose that this is not true. Then $\phi'' \neq 0$ on an interval (p, q) , where $0 < p < q$. Equation (17) implies that on the set

$$A = \{(t, \omega) : Z_t(\omega) \in (p, q)\}$$

we have

$$\alpha = 0 \quad P_M - a.s.$$

This means that the process $\int_0^\cdot \alpha^2 d\langle M, M \rangle$ is constant on A . Hence also the processes $\int_0^\cdot \alpha dM$ and $Z = \mathcal{E}(\alpha \cdot M)$ are constant on A (see [25]), i.e.

$$1_A(t, \omega) Z_t(\omega) = \text{is constant} \quad a.s.$$

In other words, the paths $t \mapsto Z_t(\omega)$ are a.s. constant on (p, q) .

Suppose first that $q < 1$ or $p > 1$. Since $Z_0 = 1$, it follows that the entire trajectories of Z are above q or below p , respectively. This contradicts (17).

Suppose next that $p < 1 < q$. Since Z is constant on (p, q) , we must have $Z = 1$, which also contradicts property (17).

Thus we have shown $\phi'' = 0$.

On the other hand we know that

$$\phi'(x) = I(yx) + yxI'(yx)$$

and

$$\phi''(x) = 2yI'(yx) + xy^2I''(yx).$$

Hence I' solves the differential equation

$$2I'(z) = -zI''(z), \quad z > 0.$$

By assumption (15) the function $I' : (0, \infty) \rightarrow (-\infty, 0)$ satisfies

$$\lim_{z \rightarrow 0^+} I'(z) = -\infty.$$

Hence

$$I'(z) = -\frac{a}{z^2}$$

and

$$I(z) = \frac{a}{z} + c_1$$

for some constants $a > 0$ and $c_1 \in \mathbb{R}$. It follows

$$U'(x) = \frac{a}{x - c_1}$$

and

$$U(x) = a \log(x - c_1) + c_2$$

for some $c_2 \in \mathbb{R}$. Note that $c_1 = 0$, because $\lim_{x \rightarrow 0^+} U(x) = -\infty$. This completes the proof. \square

6.2 The additional expected utility of an insider

Let \mathcal{F} and \mathcal{G} be filtrations such that S is complete with respect to both filtrations. We write

$$S = M + \alpha \cdot \langle M, M \rangle$$

for the semimartingale decomposition with respect to \mathcal{F} and

$$S = N + \beta \cdot \langle N, N \rangle$$

for the decomposition with respect to \mathcal{G} . Furthermore we denote by Q the ELMM with respect to \mathcal{F} and by Q' the ELMM with respect to \mathcal{G} . Notice that

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_T} = \mathcal{E}(-\alpha \cdot M)_T$$

and

$$\frac{dQ'}{dP} \Big|_{\mathcal{G}_T} = \mathcal{E}(-\beta \cdot N)_T.$$

Consider now the case where \mathcal{F} is contained in \mathcal{G} , i.e. $\mathcal{F}_t \subset \mathcal{G}_t$ for all $0 \leq t \leq T$. The following Lemma observes that the two ELMMs agree on the smaller world.

Lemma 6.4 *On \mathcal{F}_T the measures Q and Q' are equal, i.e.*

$$Q' \Big|_{\mathcal{F}_T} = Q \Big|_{\mathcal{F}_T}.$$

Proof. On the one hand, S is a (Q', \mathcal{G}) -local martingale. Since on the other hand S is adapted to \mathcal{F} , it is also a (Q', \mathcal{F}) -local martingale. Completeness of the market implies that the ELMM on \mathcal{F} is unique. Hence Q' coincides with Q on \mathcal{F}_T . \square

By applying theorem 6.1 we obtain the following expression for the utility increment

$$\begin{aligned} u(\mathcal{G}, x) - u(\mathcal{F}, x) &= EU(I(y \frac{dQ'}{dP})) - EU(I(y \frac{dQ}{dP})) \\ &= EU(I(y \mathcal{E}(-\beta \cdot N)_T)) - EU(I(y \mathcal{E}(-\alpha \cdot M)_T)). \end{aligned}$$

Again we want to express the additional expected utility by means of the information drift μ .

Recall the representation

$$M = N - \int_0^\cdot (\alpha - \beta) d\langle M, M \rangle \quad a.s.$$

with $\mu = \alpha - \beta$ as information drift. Note that for $t \in [0, T]$

$$\begin{aligned} \mathcal{E}(-\beta \cdot N)_t &= \exp \left[- \int_0^t \beta dN - \frac{1}{2} \int_0^t \beta^2 d\langle M, M \rangle \right] \\ &= \exp \left[\int_0^t \mu dN - \int_0^t \alpha dM + \int_0^t \alpha(\beta - \alpha) d\langle M, M \rangle - \frac{1}{2} \int_0^t \beta^2 d\langle M, M \rangle \right] \\ &= \mathcal{E}(-\alpha \cdot M)_t \mathcal{E}(-\mu \cdot N)_t, \end{aligned}$$

which implies

$$\frac{dQ'}{dP} \Big|_{\mathcal{G}_T} = \frac{dQ}{dP} \Big|_{\mathcal{F}_T} \mathcal{E}(-\mu \cdot N)_T.$$

Conditioning on \mathcal{F}_T leads to

$$\frac{dQ'}{dP} \Big|_{\mathcal{F}_T} = \frac{dQ}{dP} \Big|_{\mathcal{F}_T} E[\mathcal{E}(-\mu \cdot N)_T | \mathcal{F}_T],$$

and by lemma 6.4 we get

$$E[\mathcal{E}(-\mu \cdot N)_T | \mathcal{F}_T] = \frac{dQ'}{dQ} \Big|_{\mathcal{F}_T} = 1. \quad (18)$$

We may summarize our findings on the expected additional utility in the following Proposition.

Proposition 6.5 *The additional expected utility of the insider is equal to*

$$u(\mathcal{G}, x) - u(\mathcal{F}, x) = E[U(I(y\mathcal{E}(-\alpha \cdot M)_T \mathcal{E}(-\mu \cdot N)_T)) - U(I(y\mathcal{E}(-\alpha \cdot M)_T))].$$

By definition, the insider's expected utility must exceed the regular trader's. In case $U \circ I$ is convex, which is the case for the exponential, power, and logarithmic utility functions for example, but in general does not hold true, the projection result of Lemma 6.4 gives us a direct argument to show this starting with the representation obtained in the preceding Proposition. Since $U \circ I$ is convex, Jensen's inequality and equation (18) yield

$$\begin{aligned} u(\mathcal{G}, x) - u(\mathcal{F}, x) &= E \left[U \left(I \left(y \frac{dQ'}{dP} \right) \right) \right] - E \left[U \left(I \left(y \frac{dQ}{dP} \right) \right) \right] \\ &\geq E \left[U \left(I \left(y E \left[\mathcal{E}(-\alpha \cdot M)_T \mathcal{E}(-\mu \cdot N)_T \mid \mathcal{F}_T \right] \right) \right) \right] \\ &\quad - E [U(I(y\mathcal{E}(-\alpha \cdot M)_T))] \\ &= E \left[U \left(I \left(y \mathcal{E}(-\alpha \cdot M)_T E \left[\mathcal{E}(-\mu \cdot N)_T \mid \mathcal{F}_T \right] \right) \right) \right] \\ &\quad - E [U(I(y\mathcal{E}(-\alpha \cdot M)_T))] \\ &= E [U(I(y\mathcal{E}(-\alpha \cdot M)_T))] - E [U(I(y\mathcal{E}(-\alpha \cdot M)_T))] \\ &= 0. \end{aligned}$$

Remark. 1. We conclude that in general, the utility increment depends - besides the information drift - on the initial wealth and on the intrinsic drift α . This is not the case for logarithmic utility functions, where it only depends on the information drift.
2. The assumption that in the bigger filtration's information evolution the market is still complete is not as restrictive as it might seem. Some examples can be found in [3] and [2].

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