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## Optimal regularity for elliptic transmission problems including $C^1$ interfaces

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## ABSTRACT

We prove an optimal regularity result for elliptic operators  $-\nabla \cdot \mu \nabla : W_0^{1,q} \rightarrow W^{-1,q}$  for a  $q > 3$  in the case when the coefficient function  $\mu$  has a jump across a  $C^1$  interface and is continuous elsewhere. A counterexample shows that the  $C^1$  condition cannot be relaxed in general. Finally, we draw some conclusions for corresponding parabolic operators.

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# 1 Introduction

The item of this work is situated on the intersection of two mathematical questions: the first is on the regularity for the solutions of elliptic transmission problems (see, e.g. [41, 48, 52, 55, 24, 2, 3, 43, 18, 53, 40, 21, 22], and references therein). The other is on the isomorphism property for elliptic operators  $-\nabla \cdot \mu \nabla : X \rightarrow Y$  between suitable Banach spaces  $X, Y$  in case of nonsmooth domains and/or discontinuous coefficient functions  $\mu$ , see [7, 20, 29, 35, 53, 64, 12]. In particular, the latter question in view of transmission problems for spaces  $X := W^{1,q}, Y := W^{-1,q}$  (boundary conditions incorporated) has been treated in [29, 12, 46, 7], see also [34] and references therein. All of these have in common that they transfer geometrical properties of the underlying domain or/and geometrical properties of the smoothness regions for the coefficient function to the functional analytic quality of the occurring spaces  $W^{1,q}$  and  $W^{-1,q}$ , respectively. Exactly this is also the case in this paper; our aim is to prove a sharpened (and optimal) version of the results from [12, Ch. 4], namely:

**1.1 Theorem.** *Assume that  $\Omega \subset \mathbb{R}^d$  is a bounded domain with Lipschitz boundary. Further, let  $\Omega_\circ \subset \Omega$  be another domain which is supposed to satisfy one of the following conditions:*

- i)  $\Omega_\circ$  is  $C^1$  domain which does not touch the boundary of  $\Omega$ .
- ii) The dimension  $d$  equals 3,  $\Omega_\circ$  is a Lipschitz domain, and  $\partial\Omega_\circ \cap \Omega$  is a  $C^1$  surface. Moreover,  $\partial\Omega$  and  $\partial\Omega_\circ$  meet suitably (see the definition below).

Let  $\mu$  be a function on  $\Omega$  with values in the set of real, symmetric  $d \times d$  matrices which is uniformly continuous on both of the sets  $\Omega_\circ$  and  $\Omega \setminus \bar{\Omega}_\circ$ . Additionally,  $\mu$  is supposed to satisfy the usual ellipticity condition

$$\operatorname{ess\,inf}_{\mathbf{x} \in \Omega} \inf_{\xi \in \mathbb{C}^d, \|\xi\|_{\mathbb{C}^d} = 1} \mu(\mathbf{x})\xi \cdot \bar{\xi} > 0. \quad (1.1)$$

Then there is a  $q_1 > 3$  such that for every  $\lambda$  from the (closed) right complex half plane

$$-\nabla \cdot \mu \nabla + \lambda : W_0^{1,q}(\Omega) \rightarrow W^{-1,q}(\Omega) \quad (1.2)$$

provides a topological isomorphism for all  $q \in ]q_1', q_1[$ . If  $\Omega$  itself is also a  $C^1$  domain and  $\Omega_\circ$  fulfills i), then  $q_1$  may be taken as  $\infty$ .

**1.2 Definition.** We say that  $\partial\Omega$  and  $\partial\Omega_\circ$  meet suitably if for any point  $\mathbf{x}$  from the boundary of  $\partial\Omega \cap \partial\Omega_\circ$  within  $\partial\Omega$  there is an open neighbourhood  $\mathcal{U}_\mathbf{x}$  of  $\mathbf{x}$  in  $\mathbb{R}^3$  and a  $C^1$  diffeomorphism  $\Phi_x$  from  $\mathcal{U}_\mathbf{x}$  onto an open subset of  $\mathbb{R}^3$  such that

- $\Phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega)$  equals a convex polyhedron  $\mathcal{K}_\mathbf{x}$
- $\Phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega \cap \partial\Omega_\circ) = \mathcal{K}_\mathbf{x} \cap \mathcal{H}_\mathbf{x}$ , where  $\mathcal{H}_\mathbf{x}$  is a plane which contains  $\Phi_\mathbf{x}(\mathbf{x})$  and an inner point of  $\mathcal{K}_\mathbf{x}$ .

The proof rests heavily on nontrivial regularity results for adequate model problems within the same scale of spaces: concerning i), an isomorphy result for the Dirichlet Laplacian on a domain with Lipschitz boundary [35] is required and, secondly, a result for  $\nabla \cdot \sigma \nabla$  on  $\mathbb{R}^d$ , where  $\sigma$  equals a (real, symmetric, positive definite)  $d \times d$  matrix on a half space and another  $d \times d$  matrix on the complementing half space, see Theorem 3.11 below. In case ii) an isomorphy result for interface problems on polyhedra is additionally needed, see [23]. Note that our result is a certain complement to [20], where for 3D-problems with mixed boundary conditions, but without heterogeneities isomorphism theorems within the  $W^{1,q} \leftrightarrow W^{-1,q}$  scales are obtained. Furthermore, it is somewhat similar to the results of [43], where piecewise Hölder continuity of the first order derivatives is proved under slightly stronger assumptions on the data. Last but not least Theorem 1.1 is related to the results of [15], where  $W_{loc}^{1,\infty}$  regularity is proved for the solution if the right hand side is sufficiently regular.

Operators of type (1.2) – which may be seen as the principal part of the homogenized version of an elliptic operator with inhomogeneous Dirichlet data – are of fundamental significance in many application areas. This is the case not only in the mechanics (see [42, Ch. IV.3]), thermodynamics [57, 54, 13], and electrodynamics [56] of heterogeneous media, but also in mining, multiphase flow and mathematical biology. Especially in biological models it often seems unavoidable to take into account heterogeneties, see [25] or [11] and references therein. Moreover, such operators are also of interest for the description of submicron devices by means of a Schrödinger operator in effective mass approximation (see for example [10, 62, 60, 44]). Here heterostructures are the determining features of many fundamental

effects (see for instance [9, 37]). With ongoing miniaturisation of electronic devices the resolution of material interfaces becomes ever more important, so that one definitely has to deal with discontinuous coefficient functions here. Besides, a large amount of papers exist on the numerics of such problems (see e.g. [1, 33, 14, 61] and references therein).

The  $W_0^{1,q} \leftrightarrow W^{-1,q}$  setting is attractive for many problems for the following reasons: if the gradient of the solution belongs to a summability class  $q$ , larger than the space dimension  $d$ , then the solution is automatically Hölder continuous - what often is of use for auxiliary problems. By the way, this cannot be achieved within the  $W^{s,2}$  scale because  $W^{3/2,2}$  is a principal threshold in case of jumping coefficients, see [53] for further results. Secondly, the result has far reaching consequences for the treatment of quasilinear parabolic equations in  $L^p$  spaces - as is carried out in [46, 51, 36]. Moreover, our elliptic regularity theorem, combined with a result from [8], yields maximal parabolic regularity on  $W^{-1,q}$ , too.

Another important application of the information  $q > d$  is the possibility to obtain uniqueness results for associated nonlinear equations and systems, see for example [26, 27]. Of course, these things are most relevant in the 'physical' space dimension 3. Last, but not least,  $W^{-1,q}$  is large enough to contain (suitable, say bounded) surface densities and even (not too singular) measures, see [65, Ch. 4]. In particular, this enables to include prescribed jump conditions for the conormal derivative of the solution across the interface, see [14].

The outline of the paper is as follows: First we introduce some notation. In the next chapter we derive some technical prerequisites and afterwards prove Theorem 1.1. Chapter 4 contains some perturbation results concerning first order operators. In Chapter 5 it is shown by a counterexample that if the  $C^1$  condition on the subdomain is violated in only one point, then one completely loses the result. Chapter 6 is devoted to conclusions for corresponding parabolic operators, such as maximal parabolic regularity on  $W^{-1,q}$ . Finally, in the Appendix we prove a technical lemma on domains with Lipschitz boundary.

## 2 Notations, general assumptions

The real scalar product  $\sum_{j=1}^d x_j y_j$  of two vectors  $\mathbf{x} = (x_1, \dots, x_d)$ ,  $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{C}^d$  is denoted by  $\mathbf{x} \cdot \mathbf{y}$ . Throughout this paper,  $\Omega$  and  $\Lambda$  are always domains in  $\mathbb{R}^d$ . Concerning the definition of a Lipschitz domain and a domain with Lipschitz boundary we refer the reader primarily to [28, Ch. 1.2], see also [63, Ch. 1.2]. If  $X$  is a complex Banach space, then we denote the space of  $X$ -valued, Bochner measurable,  $p$ -integrable functions on  $\Lambda$ , ( $p \in [1, \infty]$ ), by  $L^p(\Lambda; X)$ , whereas  $L^\infty(\Lambda; X)$  denotes the space of Lebesgue measurable, essentially bounded functions on  $\Lambda$  with values in  $X$ . If  $X = \mathbb{C}$ , then we write simply  $L^p(\Lambda)$ .  $W^{1,q}(\Lambda)$  stands for the usual (complex) Sobolev space on the set  $\Lambda$  (see [28] or [59]). Further, we use the symbol  $W_0^{1,q}(\Lambda)$  for the closure of  $\{v|_\Lambda : v \in C_0^\infty(\mathbb{R}^d), \text{supp } v \subset \Lambda\}$  in  $W^{1,q}(\Lambda)$ .  $W^{-1,q'}(\Lambda)$  denotes the dual to  $W_0^{1,q}(\Lambda)$ , where  $q'$  here and in the sequel always denotes the adjoint exponent  $q' := \frac{q}{q-1}$ . If  $\rho$  is a Lebesgue measurable, essentially bounded function on the domain  $\Lambda$ , taking its values in the set of real, symmetric  $d \times d$  matrices, then we define

$$-\nabla \cdot \rho \nabla : W_0^{1,2}(\Lambda) \longmapsto W^{-1,2}(\Lambda) \quad (2.1)$$

by

$$\langle -\nabla \cdot \rho \nabla v, w \rangle := \int_\Lambda \rho \nabla v \cdot \nabla w \, d\mathbf{x}; \quad v, w \in W_0^{1,2}(\Lambda). \quad (2.2)$$

Here and in the following  $\langle \cdot, \cdot \rangle$  always denotes the dual pairing between  $W_0^{1,2}$  and  $W^{-1,2}$ . The maximal restriction of  $-\nabla \cdot \rho \nabla$  to any of the spaces  $W^{-1,q}(\Lambda)$  ( $q > 2$ ) we will denote by the same symbol. If we are given a function, defined on a subset of  $\mathbb{R}^d$  and uniformly continuous there, then we identify it canonically with its (uniquely determined) extension to the closure of this set. The norm in a Banach space  $X$  will be always indicated by  $\|\cdot\|_X$ . For two Banach spaces  $X$  and  $Y$  we denote the space of linear, bounded operators from  $X$  into  $Y$  by  $\mathcal{B}(X; Y)$ . If  $X = Y$ , then we abbreviate  $\mathcal{B}(X)$ . Finally, we introduce the following model sets which will be used later: by  $\mathcal{E}$  we denote the open unit cube in  $\mathbb{R}^d$ , that means the set

$$\left\{ \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : -\frac{1}{2} < x_1, \dots, x_d < \frac{1}{2} \right\}.$$

$\mathcal{E}_-, \mathcal{E}_+$  are used as symbols for the lower and upper open half cubes

$$\mathcal{E}_- := \mathcal{E} \cap \left\{ \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : -\frac{1}{2} < x_d < 0 \right\}$$

and

$$\mathcal{E}_+ := \mathcal{E} \cap \left\{ \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : 0 < x_d < \frac{1}{2} \right\}.$$

Finally, we denote by  $\mathcal{E}_0$  the upper plate of  $\mathcal{E}_-$ ,  $\mathcal{E}_0 := \mathcal{E} \cap \{\mathbf{x} : x_d = 0\}$ .

### 3 Proof of Theorem 1.1

#### 3.a Known results and preliminaries

In this chapter we will prove Theorem 1.1. In order to do so, we first quote a classical perturbation theorem on the bounded invertibility for operators (see [38, Ch. IV.1.4 Thm. 1.16] which we will use repeatedly in the sequel:

**3.1 Proposition.** *Let  $X, Y$  be Banach spaces. Assume that  $A, B : X \rightarrow Y$  are linear, continuous operators, such that  $\|A^{-1}\|_{\mathcal{B}(Y;X)} \|B\|_{\mathcal{B}(X;Y)} < 1$ . Then  $A + B$  is a topological isomorphism between  $X$  and  $Y$  and*

$$\|A^{-1} - (A + B)^{-1}\|_{\mathcal{B}(Y;X)} \leq \frac{\|B\|_{\mathcal{B}(X;Y)} \|A^{-1}\|_{\mathcal{B}(Y;X)}}{1 - \|B\|_{\mathcal{B}(X;Y)} \|A^{-1}\|_{\mathcal{B}(Y;X)}} \|A^{-1}\|_{\mathcal{B}(Y;X)}$$

Next, we quote a result of Jerison/Kenig (see [35, Thm. 1.1]), which is a cornerstone for all what follows:

**3.2 Proposition.** *If  $\Lambda \subset \mathbb{R}^d$  is a bounded domain with Lipschitz boundary, then there is a number  $q_1 > 3$ , depending only on the Lipschitz constant of  $\Lambda$ , such that the Dirichlet Laplacian provides a topological isomorphism between  $W_0^{1,q}(\Lambda)$  and  $W^{-1,q}(\Lambda)$  for all  $q \in ]q'_1, q_1[$ . If  $\Lambda$  is a  $C^1$  domain,  $q_1$  may be chosen  $\infty$ .*

**3.3 Remark.** The second assertion may also be directly concluded from [58] Thm. 4.6.



In order to generalize Proposition 3.2 to operators  $\nabla \cdot \rho \nabla$  we need the following lemma, which is proved in the Appendix:

**3.4 Lemma.** *Let  $\Lambda$  be a bounded domain with Lipschitz boundary and Lipschitz constant  $\gamma$ . If  $K$  is a linear bijection of  $\mathbb{R}^n$  onto itself, then  $K\Lambda$  is again a domain with Lipschitz boundary and the Lipschitz constant of  $K\Lambda$  does not exceed  $\|K\| \|K^{-1}\| (\gamma + 1)$ .*

This at hand, we can draw the following conclusion from Proposition 3.2:

**3.5 Corollary.** *Let  $\Lambda \subset \mathbb{R}^d$  be a bounded domain with Lipschitz boundary. If the coefficient function  $\rho$  is a constant real, symmetric, positive definite  $d \times d$  matrix on  $\Lambda$ , then there is a number  $q_1 > 3$  such that the operator  $-\nabla \cdot \rho \nabla + 1$  provides a topological isomorphism between  $W_0^{1,q}(\Lambda)$  and  $W^{-1,q}(\Lambda)$  for all  $q \in ]q_1', q_1[$ . The number  $q_1$  may be taken uniformly with respect to any set of (symmetric)  $\rho$ 's which is, together with the set of inverses, bounded in  $\mathcal{B}(\mathbb{C}^d)$ . If  $\Lambda$  is a  $C^1$  domain, then  $q_1$  may be chosen as  $\infty$ .*

*Proof.* The assertion may be deduced immediately from Proposition 3.2: namely one transforms  $-\nabla \cdot \rho \nabla$  with respect to the coordinate transform  $\rho^{1/2}$  and ends up with a multiple of the Dirichlet Laplacian. Under the supposition on the  $\rho$ 's the Lipschitz constants of the transformed domains  $\rho^{1/2}\Lambda$  are uniformly bounded by Lemma 3.4. Thus, every  $-\nabla \cdot \rho \nabla$  provides a topological isomorphism between  $W_0^{1,q}(\Lambda)$  and  $W^{-1,q}(\Lambda)$  for the asserted range of  $q$ 's. The same is true for the operators  $-\nabla \cdot \rho \nabla + 1$  because the corresponding resolvents are compact and  $-1$  is not an eigenvalue for any of these operators.  $\square$

Having in mind operators with non-constant coefficients, we need the following interpolation result:

**3.6 Theorem.** *Assume that  $\Lambda \subseteq \mathbb{R}^d$  is an open set. Let the linear mapping  $F : W^{-1,q}(\Lambda) \rightarrow W_0^{1,q}(\Lambda)$  be continuous for  $q = q_1 \in ]1, \infty[$  and  $q = q_2 \in ]1, \infty[$ . Then it is continuous for any  $q = \left(\frac{\theta}{q_1} + \frac{1-\theta}{q_2}\right)^{-1} \in ]q_1, q_2[$  and*

$$\|F\|_{\mathcal{B}(W^{-1,q}(\Lambda); W_0^{1,q}(\Lambda))} \leq \|F\|_{\mathcal{B}(W^{-1,q_1}(\Lambda); W_0^{1,q_1}(\Lambda))}^\theta \|F\|_{\mathcal{B}(W^{-1,q_2}(\Lambda); W_0^{1,q_2}(\Lambda))}^{(1-\theta)}. \quad (3.1)$$

The proof is carried out with help of the following representation theorem:

**3.7 Proposition.** *Let  $\Lambda \subseteq \mathbb{R}^d$  be open and  $q \in ]1, \infty[$ .*

*i) Any element  $T \in (W^{1,q'}(\Lambda))^*$  may be represented as*

$$\langle T, \psi \rangle = \int_{\Lambda} f_0 \psi + \sum_{j=1}^d \frac{\partial \psi}{\partial x_j} f_j \, d\mathbf{x}, \quad \psi \in W^{1,q'}(\Lambda) \quad (3.2)$$

*with  $f = (f_0, f_1, \dots, f_d) \in L^q(\Lambda; \mathbb{C}^{d+1})$  and the additional property*

$$\|f\|_{L^q(\Lambda; \mathbb{C}^{d+1})} = \|T\|_{(W^{1,q'}(\Lambda))^*}. \quad (3.3)$$

*ii) The same representation (3.2) holds true for any continuous linear form  $T$  which is defined on a closed subspace of  $W^{1,q'}(\Lambda)$ , in particular for  $T \in W^{-1,q}(\Lambda)$ . In this case  $f$  can be chosen such that  $\|T\| = \|f\|_{L^q(\Lambda; \mathbb{C}^{d+1})}$ .*

A proof of the representation formula (3.2) is given in [65, Ch. 4.3]. The norm equality (3.3) is obtained by an inspection of the proof given there. ii) is obtained from i) by extending the linear form  $T$  (norm preserving) to whole  $W^{1,q'}(\Lambda)$ .

We give now the proof of Theorem 3.6: Assume  $q \in [q_1, q_2]$ . Then for any  $f = (f_0, f_1, \dots, f_d) \in L^q(\Lambda; \mathbb{C}^{d+1})$  we define an element  $(1 + \text{div})f \in W^{-1,q}(\Lambda)$  by

$$\langle (1 + \text{div})f, \psi \rangle := \int_{\Lambda} f_0 \psi + \sum_{j=1}^d \frac{\partial \psi}{\partial x_j} f_j \, d\mathbf{x}, \quad \psi \in W_0^{1,q'}(\Lambda).$$

Further, for any  $q \in [q_1, q_2]$  we define a mapping  $G : L^q(\Lambda; \mathbb{C}^{d+1}) \rightarrow L^q(\Lambda; \mathbb{C}^{d+1})$  by putting  $G = (1 \oplus \text{grad})F(1 + \text{div})$ . The crucial point is the equality

$$\|G\|_{\mathcal{B}(L^q(\Lambda; \mathbb{C}^{d+1}))} = \|F\|_{\mathcal{B}(W^{-1,q}(\Lambda), W_0^{1,q}(\Lambda))},$$

which results from the following facts:

- $1 \oplus \text{grad}$  is an isometry from  $W_0^{1,q}(\Lambda)$  into  $L^q(\Lambda; \mathbb{C}^{d+1})$ .
- $(1 + \text{div})$  is non-expansive by Hölder's inequality, but, additionally, Proposition 3.7 holds.

Thus, an application of the Riesz-Thorin interpolation theorem to the mappings  $G : L^q(\Lambda; \mathbb{C}^{d+1}) \rightarrow L^q(\Lambda; \mathbb{C}^{d+1})$  gives the assertion. Next we present a localization principle similar to Lemma 2 of [29]. In essence, this will permit us to conclude the isomorphism property (1.2) from the same isomorphism property for adequate local model constellations.

**3.8 Lemma.** *Let  $\Lambda \subset \mathbb{R}^d$  be a bounded Lipschitz domain and  $\mathcal{O} \subset \mathbb{R}^d$  be open such that  $\Lambda_\bullet := \Lambda \cap \mathcal{O}$  is again a Lipschitz domain. We fix an arbitrary function  $\eta \in C_0^\infty(\mathbb{R}^d)$  with  $\text{supp } \eta \subset \mathcal{O}$ . Let  $\rho_\bullet$  denote the restriction of the coefficient function  $\rho$  to  $\Lambda_\bullet$ . Assume  $u \in W_0^{1,2}(\Lambda)$  to be the solution of*

$$-\nabla \cdot \rho \nabla u + u = f \in W^{-1,2}(\Lambda); \quad (3.4)$$

then the following holds true:

i) *The linear form*

$$f_\bullet : w \rightarrow \langle f, \widetilde{\eta w} \rangle$$

*(where  $\widetilde{\eta w}$  means the extension by zero to whole  $\Lambda$ ) is well defined and continuous on  $W_0^{1,r'}(\Lambda_\bullet)$  whenever  $f \in W^{-1,r}(\Lambda)$ .*

ii) *Let  $T_u$  denote the linear form*

$$w \longrightarrow \int_{\Lambda_\bullet} u \rho_\bullet \nabla \eta \cdot \nabla w \, d\mathbf{x}$$

*on  $W_0^{1,2}(\Lambda_\bullet)$ . If  $u \in W^{1,r}(\Lambda)$ , then  $-\rho_\bullet \nabla u|_{\Lambda_\bullet} \cdot \nabla \eta|_{\Lambda_\bullet} + T_u \in W^{-1,s}(\Lambda_\bullet)$ , where  $s = s(r)$  is given by*

$$s = \begin{cases} \frac{rd}{d-r} & \text{if } r \in [2, d[ \\ \text{any (large) positive number} & \text{if } r \geq d. \end{cases} \quad (3.5)$$

iii) *Let the operator  $-\nabla \cdot \rho_\bullet \nabla : W_0^{1,2}(\Lambda_\bullet) \rightarrow W^{-1,2}(\Lambda_\bullet)$  be defined analogously to (2.2). Then  $v := \eta u|_{\Lambda_\bullet}$  belongs to  $W_0^{1,2}(\Lambda_\bullet)$  and satisfies*

$$-\nabla \cdot \rho_\bullet \nabla v + v = -\rho_\bullet \nabla u|_{\Omega_\bullet} \cdot \nabla \eta|_{\Omega_\bullet} + T_u + f_\bullet. \quad (3.6)$$

*Proof.* i) The mapping  $f \mapsto f_\bullet$  is the adjoint to  $w \mapsto \widetilde{\eta w}$  which maps  $W_0^{1,r'}(\Lambda_\bullet)$  continuously into  $W_0^{1,r'}(\Lambda)$ .

ii) The case  $r \geq d$  may be reduced by the embedding  $W^{1,r}(\Lambda) \hookrightarrow W^{1,d-\epsilon}(\Lambda)$  to the case  $r < d$ ; we treat this latter one: clearly, one has  $\rho_\bullet \nabla u|_{\Lambda_\bullet} \cdot \nabla \eta \in L^r(\Lambda_\bullet)$ , what gives by Sobolev embedding and duality  $\rho_\bullet \nabla u|_{\Lambda_\bullet} \cdot \nabla \eta|_{\Lambda_\bullet} \in W^{-1, \frac{rd}{d-r}}(\Lambda_\bullet)$  for  $r \in [2, d]$ . Concerning  $T_u$ , we will show that it is a continuous linear form on  $W_0^{1, (\frac{rd}{d-r})'}(\Lambda_\bullet)$ : one can estimate

$$|\langle T_u, w \rangle| \leq \|u\|_{L^{\frac{rd}{d-r}}(\Lambda_\bullet)} \|\rho\|_{L^\infty(\Lambda; \mathcal{B}(\mathbb{C}^d))} \|\nabla \eta\|_{L^\infty(\Lambda_\bullet)} \|\nabla w\|_{L^{(\frac{rd}{d-r})'}(\Lambda_\bullet)}. \quad (3.7)$$

Using again Sobolev embedding, the right hand side of (3.7) may be estimated by

$$\gamma \|u\|_{W^{1,r}(\Lambda_\bullet)} \|\rho\|_{L^\infty(\Lambda; \mathcal{B}(\mathbb{C}^d))} \|\nabla \eta\|_{L^\infty(\Lambda_\bullet)} \|w\|_{W^{1, (\frac{rd}{d-r})'}(\Lambda_\bullet)}.$$

iii) For every  $u \in W_0^{1,2}(\Lambda)$  there is a sequence  $\{u_l\}_l$  consisting of  $C_0^\infty(\mathbb{R}^d)$  functions with support within  $\Lambda$  such that  $\lim_{l \rightarrow \infty} u_l|_\Lambda = u$  in  $W^{1,2}(\Lambda)$ . Obviously, then any function  $\eta u_l$  has its support within  $\Lambda_\bullet$  and  $\lim_{l \rightarrow \infty} \eta u_l|_{\Lambda_\bullet} = \eta u|_{\Lambda_\bullet}$  in  $W^{1,2}(\Lambda_\bullet)$ . Secondly, for every  $w \in W_0^{1,2}(\Lambda_\bullet)$  we have

$$\begin{aligned} \langle -\nabla \cdot \rho_\bullet \nabla v, w \rangle + \langle v, w \rangle &= \int_{\Lambda_\bullet} \rho_\bullet \nabla(\eta u) \cdot \nabla w \, d\mathbf{x} + \int_{\Lambda_\bullet} \eta u w \, d\mathbf{x} = \\ &= - \int_{\Lambda_\bullet} w \rho_\bullet \nabla u \cdot \nabla \eta \, d\mathbf{x} + \int_{\Lambda_\bullet} u \rho_\bullet \nabla \eta \cdot \nabla w \, d\mathbf{x} + \int_{\Lambda} \rho \nabla u \cdot \nabla(\widetilde{\eta w}) \, d\mathbf{x} + \int_{\Lambda} u \widetilde{\eta w} \, d\mathbf{x}. \end{aligned} \quad (3.8)$$

Applying the definitions of  $T_u$  and  $f_\bullet$ , this gives the assertion.  $\square$

Next we want to show the assertion of Theorem 1.1 under the additional assumption that the coefficient function is uniformly continuous on whole  $\Omega$ .

**3.9 Theorem.** *Let  $\Lambda \subset \mathbb{R}^d$  be a bounded domain with Lipschitz boundary and  $\rho$  a real, symmetric-valued, uniformly continuous coefficient function on  $\Lambda$ , elliptic in the sense of (1.1).*

i) Then there is a  $q_1 > 3$  such that for all  $q \in ]q'_1, q_1[$  it holds true:

$$\sup_{\mathbf{x} \in \bar{\Lambda}} \|(-\nabla \cdot \rho(\mathbf{x})\nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,q}(\Lambda); W_0^{1,q}(\Lambda))} < \infty.$$

ii) The operator

$$-\nabla \cdot \rho \nabla + 1 : W_0^{1,q}(\Lambda) \longrightarrow W^{-1,q}(\Lambda) \quad (3.9)$$

is a topological isomorphism for the same range of  $q$ 's.

iii) If  $\Lambda$  is a  $C^1$  domain, then  $q_1$  may be chosen  $\infty$ .

*Proof.* The proof will be concluded from Corollary 3.5, for this reason the corresponding  $q$ 's are identical with those from Corollary 3.5. i) The set  $\{\rho(\mathbf{x}) : \mathbf{x} \in \bar{\Lambda}\}$  is bounded in  $\mathcal{B}(\mathbb{R}^d)$  while  $\{\rho(\mathbf{x})^{-1} : \mathbf{x} \in \bar{\Lambda}\}$  is also bounded by the ellipticity condition and the (uniform) continuity of  $\rho$ . Thus, by Corollary 3.5, there is a  $q_1 > 3$  such that for any  $q \in ]q'_1, q_1[$  and for any  $\mathbf{x} \in \bar{\Omega}$  the operator  $-\nabla \cdot \rho(\mathbf{x})\nabla + 1$  provides a topological isomorphism between  $W_0^{1,q}(\Omega)$  and  $W^{-1,q}(\Omega)$ . If  $\Omega$  is a  $C^1$  domain, then  $q_1 = \infty$ . Hence, the function

$$\bar{\Lambda} \ni \mathbf{x} \mapsto (-\nabla \cdot \rho(\mathbf{x})\nabla + 1)^{-1} \in \mathcal{B}(W^{-1,q}(\Lambda); W_0^{1,q}(\Lambda)) \quad (3.10)$$

is well defined and, additionally, the mapping

$$\bar{\Lambda} \ni \mathbf{x} \mapsto \rho(\mathbf{x}) \mapsto -\nabla \cdot \rho(\mathbf{x})\nabla + 1 \in \mathcal{B}(W_0^{1,q}(\Lambda); W^{-1,q}(\Lambda))$$

is continuous. By Proposition 3.1 the function (3.10) is also continuous and, hence, bounded.

ii) First we consider the case  $q \in ]2, q_1[$ ; then (3.9) is injective by Lax-Milgram. Choose for every point  $\mathbf{x} \in \bar{\Lambda}$  a ball  $B_{\mathbf{x}}$  around  $\mathbf{x}$  with radius  $R_{\mathbf{x}}$  such that for  $\mathbf{y} \in B_{\mathbf{x}} \cap \bar{\Lambda}$

$$\begin{aligned} \|\rho(\mathbf{y}) - \rho(\mathbf{x})\|_{\mathcal{B}(\mathbb{R}^d)} &< \\ &\frac{1}{\sup_{t \in [2, q]} \sup_{\mathbf{z} \in \bar{\Lambda}} \|(-\nabla \cdot \rho(\mathbf{z})\nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\Lambda); W_0^{1,t}(\Lambda))}} \end{aligned} \quad (3.11)$$

holds true. This radius  $R_{\mathbf{x}}$  is indeed nonzero, namely: the Lax-Milgram lemma yields

$$\sup_{\mathbf{z} \in \bar{\Lambda}} \|(-\nabla \cdot \rho(\mathbf{z})\nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,2}(\Lambda); W_0^{1,2}(\Lambda))} < \infty.$$

This, together with i) and interpolation (Theorem 3.6) implies

$$\sup_{t \in [2, q]} \sup_{\mathbf{z} \in \bar{\Lambda}} \|(-\nabla \cdot \rho(\mathbf{z})\nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\Lambda); W_0^{1,t}(\Lambda))} < \infty.$$

We choose a finite subcovering  $B_{\mathbf{x}_1} \dots B_{\mathbf{x}_m}$  for  $\bar{\Lambda}$ . Let  $\eta_1 \dots \eta_m$  be a partition of unity on  $\bar{\Lambda}$  which is subordinated to this subcovering. Assume now  $f \in W^{-1,q}(\Lambda)$  and let  $u$  be a solution of

$$-\nabla \cdot \rho \nabla u + u = f. \quad (3.12)$$

By the Lax-Milgram lemma  $u$  must be from  $W_0^{1,2}(\Lambda)$ . Putting  $\mathcal{O} := \cup_{l=1}^m B_{\mathbf{x}_l}$  we get from Lemma 3.8

$$-\nabla \cdot \rho \nabla(\eta_l u) + \eta_l u = g_l, \quad (3.13)$$

where  $g_l$  is from  $W^{-1, \min(s(2), q)}(\Lambda)$  (see Lemma 3.8). We now set  $t := \min(s(2), q)$  and define for every  $l \in \{1, \dots, m\}$  a modified coefficient function  $\rho_l$  on  $\Lambda$  as follows:

$$\rho_l(\mathbf{y}) = \begin{cases} \rho(\mathbf{y}) & \text{if } \mathbf{y} \in B_{\mathbf{x}_l} \cap \Lambda \\ \rho(\mathbf{x}_l) & \text{elsewhere on } \Lambda. \end{cases} \quad (3.14)$$

Because  $\eta_l u$  has its support in  $B_{\mathbf{x}_l}$ , it satisfies besides (3.13) also the equation

$$-\nabla \cdot \rho_l \nabla(\eta_l u) + \eta_l u = g_l. \quad (3.15)$$

We will now show that  $g_l \in W^{-1,t}(\Lambda)$  implies  $\eta_l u \in W_0^{1,t}(\Lambda)$ . Rewriting (3.15) as

$$-\nabla \cdot \rho(\mathbf{x}_l) \nabla(\eta_l u) + \eta_l u + \nabla \cdot [\rho(\mathbf{x}_l) - \rho_l] \nabla(\eta_l u) = g_l,$$

one estimates

$$\|\nabla \cdot [\rho_l - \rho(\mathbf{x}_l)] \nabla\|_{\mathcal{B}(W_0^{1,t}(\Lambda); W^{-1,t}(\Lambda))} \leq \|\rho(\mathbf{x}_l) - \rho_l\|_{L^\infty(\Lambda; \mathcal{B}(\mathbb{C}^d))} =$$

$$= \|\rho(\mathbf{x}_l) - \rho\|_{L^\infty(B_r(\mathbf{x}_l) \cap \Lambda; \mathcal{B}(\mathbb{C}^d))}.$$

Taking into account (3.11), we obtain for all  $l \in \{1 \dots m\}$

$$\|\nabla \cdot [\rho_l - \rho(\mathbf{x}_l)] \nabla\|_{\mathcal{B}(W_0^{1,t}(\Lambda); W^{-1,t}(\Lambda))} \|(-\nabla \cdot \rho(\mathbf{x}_l) \nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\Lambda); W_0^{1,t}(\Lambda))} < 1.$$

Now one can apply again the perturbation result (Proposition 3.1), which says that  $-\nabla \cdot \rho_l \nabla + 1 : W_0^{1,t}(\Lambda) \mapsto W^{-1,t}(\Lambda)$  is boundedly invertible. Thus, each  $\eta_l u$  must be from  $W_0^{1,t}(\Lambda)$ , what gives  $u \in W_0^{1,t}(\Lambda)$ . Repeating these considerations with the improved information on the integrability exponent of  $\nabla u$  — each time using Lemma 3.8 — one, after finitely many steps, ends up with  $u \in W_0^{1,q}(\Lambda)$ . Hence, (3.9) is surjective and thus, by the Open mapping theorem, a topological isomorphism. The case  $q < 2$  is obtained by duality.  $\square$

Further, we need the following technical lemma, the proof of which can be found in [39, Remark 2.1.3]:

**3.10 Lemma.** *Let  $\Lambda$  be a domain with Lipschitz boundary. Then for any  $\mathbf{x} \in \partial\Lambda$  and any neighbourhood of  $\mathbf{x}$  there is a (possibly) smaller open neighbourhood  $\mathcal{V}_{\mathbf{x}}$  of  $\mathbf{x}$  such that  $\Lambda \cap \mathcal{V}_{\mathbf{x}}$  is a (even starlike) domain with Lipschitz boundary.*

### 3.b Core of the proof

Before we prove Theorem 1.1 we have to show a result on our first model constellation for operators  $\nabla \cdot \sigma \nabla$ , when  $\sigma$  is discontinuous:

**3.11 Theorem.** *Let  $\sigma$  be a coefficient function on  $\mathbb{R}^d$  which equals a real, symmetric, positive definite  $d \times d$  matrix  $\sigma^-$  on  $\mathbb{R}_-^d = \{\mathbf{x} \in \mathbb{R}^d : x_d < 0\}$  and another real, symmetric, positive definite  $d \times d$  matrix  $\sigma^+$  on  $\mathbb{R}_+^d = \{\mathbf{x} \in \mathbb{R}^d : x_d > 0\}$ . Then  $-\nabla \cdot \sigma \nabla + 1$  provides a topological isomorphism between  $W^{1,q}(\mathbb{R}^d)$  and  $W^{-1,q}(\mathbb{R}^d)$  for all  $q \in ]1, \infty[$ .*

*Proof.* Let  $\mathbf{x} = (x', x_d) \in \mathbb{R}^d$ ,  $x' \in \mathbb{R}^{d-1}$ , and  $\partial_i = \partial_{x_i}$ ,  $1 \leq i \leq d$ . Moreover, we identify  $\{\mathbf{x} \in \mathbb{R}^d : x_d = 0\}$  with  $\mathbb{R}^{d-1}$ . It is sufficient to prove that the unique solution  $u \in W^{1,2}(\mathbb{R}^d)$  for each of the equations

$$-\nabla \cdot \sigma \nabla u + u = f, \quad f \in L^q(\mathbb{R}^d), \quad 2 < q < \infty \quad (3.16)$$

$$-\nabla \cdot \sigma \nabla u + u = \partial_i f, \quad f \in L^q(\mathbb{R}^d), \quad 2 < q < \infty \quad (3.17)$$

$i \in \{1, \dots, d\}$ , belongs to  $W^{1,q}(\mathbb{R}^d)$ . To do this, it is enough to show the estimate

$$\|u\|_{W^{1,q}(\mathbb{R}^d)} \leq c \|f\|_{L^q(\mathbb{R}^d)}, \quad f \in \tilde{C}^\infty, \quad (3.18)$$

where  $c$  denotes a generic positive constant and  $\tilde{C}^\infty$  stands for the dense subset of  $L^q(\mathbb{R}^d)$  defined by

$$\tilde{C}^\infty = \{\psi \in C_0^\infty(\mathbb{R}^d) : \psi = 0 \text{ in some neighbourhood of } \mathbb{R}^{d-1}\}.$$

Applying classical elliptic theory of transmission problems (e.g., [52]) to the equation

$$-\nabla \cdot \sigma \nabla v + v = f, \quad f \in \tilde{C}^\infty, \quad (3.19)$$

we obtain the inequality

$$\|v\|_{W^{2,q}(\mathbb{R}_-^d \cup \mathbb{R}_+^d)} \leq c \|f\|_{L^q(\mathbb{R}^d)}. \quad (3.20)$$

This assures (3.18) in case of (3.16). We establish (3.18) also in case of (3.17): looking for the solution of (3.17) in the form  $u = \partial_i v + w$ , we observe that  $w$  has to satisfy the following transmission problem:

$$\begin{aligned} -\nabla \cdot \sigma^\pm \nabla w^\pm + w^\pm &= 0 \quad \text{in } \mathbb{R}_\pm^d, \quad [w] = -[\partial_i v] =: g, \\ [\partial_{\nu, \sigma} w] &= -[\partial_{\nu, \sigma} \partial_i v] =: h, \end{aligned} \quad (3.21)$$

where  $w^\pm = w|_{\mathbb{R}_\pm^d}$ ,  $[w] = (w^- - w^+)|_{\mathbb{R}^{d-1}}$  and

$$[\partial_{\nu, \sigma} w] = (\sigma^- \nu \cdot \nabla w^- - \sigma^+ \nu \cdot \nabla w^+)|_{\mathbb{R}^{d-1}}, \quad \nu = (0, \dots, 0, 1).$$

Since  $v^\pm$  satisfy the homogeneous differential equations near  $\mathbb{R}^{d-1}$ , the term  $[\partial_{\nu, \sigma} \partial_d v]$  is a linear combination of  $\partial_j \partial_d v^\pm|_{\mathbb{R}^{d-1}}$  for  $j = 1, \dots, d-1$ . Thus, by the trace theorem and the continuity of differentiation in tangential direction, we obtain from (3.20) that the estimate

$$\|[\partial_i v]\|_{W^{1-1/q, q}(\mathbb{R}^{d-1})} + \|[\partial_{\nu, \sigma} \partial_i v]\|_{W^{-1/q, q}(\mathbb{R}^{d-1})} \leq c \|f\|_{L^q(\mathbb{R}^d)} \quad (3.22)$$

holds for  $i = 1, \dots, d$ . We refer to [59, Ch. 2] for the required properties of Sobolev spaces.



To prove (3.18), in view of (3.20) and (3.22), it now suffices to show that the solution of (3.21) satisfies

$$\|w\|_{W^{1,q}(\mathbb{R}_-^d \cup \mathbb{R}_+^d)} \leq c \left( \|h\|_{W^{-1/q,q}(\mathbb{R}^{d-1})} + \|g\|_{W^{1-1/q,q}(\mathbb{R}^{d-1})} \right). \quad (3.23)$$

We will reduce (3.23) to well known continuity properties of Poisson operators (see [30]), the symbols of which can be calculated explicitly. In order to do so, we solve (3.21) by taking partial Fourier transform with respect to  $x'$  denoted by  $\mathcal{F}u = \mathcal{F}u(\xi', x_d)$  for a function  $u(\mathbf{x})$  on  $\mathbb{R}^d$ , with  $\mathcal{F}^{-1}$  being the inverse transform. We set

$$B^\pm = (\sigma_{ij}^\pm)_{i,j=1}^{d-1}, \quad a^\pm = (\sigma_{1d}^\pm, \dots, \sigma_{d-1d}^\pm), \quad b^\pm = \sigma_{dd}^\pm,$$

where  $\sigma_{ij}^\pm$  are the entries of the matrices  $\sigma^\pm$ . Applying the partial Fourier transform to (3.21), we obtain

$$\begin{aligned} (-b^\pm \partial_d^2 + 2ia^\pm \cdot \xi' \partial_d + B^\pm \xi' \cdot \xi' + 1) \mathcal{F}w^\pm(\xi', x_d) &= 0 \quad \text{in } \mathbb{R}_\pm^d, \\ \mathcal{F}w^-(\xi', 0) - \mathcal{F}w^+(\xi', 0) &= \mathcal{F}g(\xi'), \\ (b^- \partial_d - ia^- \cdot \xi') \mathcal{F}w^-(\xi', 0) - (b^+ \partial_d - ia^+ \cdot \xi') \mathcal{F}w^+(\xi', 0) &= \mathcal{F}h(\xi'). \end{aligned} \quad (3.24)$$

Ignoring the exponentially increasing solutions of the homogeneous differential equations in (3.24), we have

$$\mathcal{F}w^\pm(\xi', x_d) = C^\pm(\xi') \exp\{\mp x_d (\mathcal{A}^\pm(\xi') + ia^\pm \cdot \xi')/b^\pm\} \quad (3.25)$$

with  $\mathcal{A}^\pm(\xi') = (b^\pm(1 + B^\pm \xi' \cdot \xi') - (a^\pm \cdot \xi')^2)^{1/2}$ . Then we determine  $C^\pm(\xi')$  from the transmission conditions in (3.24),

$$\begin{aligned} C^-(\xi') - C^+(\xi') &= \mathcal{F}g(\xi'), \\ \mathcal{A}^-(\xi') C^-(\xi') + \mathcal{A}^+(\xi') C^+(\xi') &= \mathcal{F}h(\xi'), \end{aligned}$$

which gives

$$C^\pm = (\mathcal{A}^- + \mathcal{A}^+)^{-1} \mathcal{F}h \mp \mathcal{A}^\mp (\mathcal{A}^- + \mathcal{A}^+)^{-1} \mathcal{F}g. \quad (3.26)$$

Note that the ellipticity of  $\nabla \cdot \sigma \nabla$  implies the lower bound

$$\mathcal{A}^\pm(\xi') \geq c \langle \xi' \rangle, \quad \langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}.$$

We will only prove the corresponding estimate (3.23) for the upper half-space since the proof for  $\mathbb{R}_-^d$  is completely analogous. From (3.25) and (3.26) we obtain the representation

$$w(x', x_d) = \mathcal{F}^{-1} k_1(\xi', x_d) \mathcal{F}h(\xi') + \mathcal{F}^{-1} k_2(\xi', x_d) \mathcal{F}g(\xi') =: \mathcal{K}_1 h + \mathcal{K}_2 g \quad (3.27)$$

for  $x_d > 0$ . Here  $\mathcal{K}_1, \mathcal{K}_2$  are Poisson operators with the symbols

$$\begin{aligned} k_1(\xi', x_d) &= (\mathcal{A}^-(\xi') + \mathcal{A}^+(\xi'))^{-1} \exp\{-x_d (\mathcal{A}^+(\xi') + ia^+ \cdot \xi')\}, \\ k_2(\xi', x_d) &= -\mathcal{A}^-(\xi') k_1(\xi', x_d). \end{aligned} \quad (3.28)$$

Using (3.28) and the expressions for  $\mathcal{A}^\pm$ , it is not difficult to check that  $k_2$  is a symbol of order  $-1$ , i.e., it satisfies the estimates

$$\|x_d^m \partial_d^n \partial_{\xi'}^\alpha k_1(\xi', \cdot)\|_{L^2(\mathbb{R}^+)} \leq c_{mn\alpha} \langle \xi' \rangle^{-3/2 - |\alpha| - m + n} \quad (3.29)$$

for all  $\xi' \in \mathbb{R}^{d-1}$ ,  $x_d \in \mathbb{R}^+$ ,  $m, n \in \mathbb{N}$  and all multi-indices  $\alpha$ . Analogously,  $k_2$  is a symbol of order 0, i.e., the  $-3/2$  in the exponent of  $\langle \xi' \rangle$  in (3.29) has to be replaced by  $-1/2$ . Therefore, from [30, Thm. 3.1] we obtain the continuity of the operators

$$\mathcal{K}_1 : W^{s-1/q, q}(\mathbb{R}^{d-1}) \rightarrow W^{s+1, q}(\mathbb{R}_+^d), \quad \mathcal{K}_2 : W^{s-1/q, q}(\mathbb{R}^{d-1}) \rightarrow W^{s, q}(\mathbb{R}_+^d)$$

for all  $s \in \mathbb{Z}$ . In particular, together with (3.27) this implies that the  $W^{1, q}$  norm of  $w$  on  $\mathbb{R}_+^d$  can be estimated by the right hand side of (3.23).  $\square$

We now come to the proof of Theorem 1.1, starting with the setting defined under *i*): first, one easily notices that the operator in (1.2) is well defined and continuous for any  $q \in ]1, \infty[$ . Concerning the continuity of the inverse, we restrict the considerations first to the case  $q > 2$ . For these  $q$ , (1.2) is injective by the Lax-Milgram lemma. Hence, by the Open mapping theorem it suffices to show that (1.2) is surjective for suitable  $q$ 's, what we will do in the sequel. Let for any  $\mathbf{x} \in \partial\Omega$  an open neighbourhood  $\mathcal{O}_{\mathbf{x}}$  be given which satisfies the following two conditions:

*i*)  $\mathcal{O}_{\mathbf{x}} \cap \bar{\Omega}_o = \emptyset$ .

*ii*) If  $\Omega$  is  $C^1$ , then  $\mathcal{A}_{\mathbf{x}} := \mathcal{O}_{\mathbf{x}} \cap \Omega$  is  $C^1$ ; and if  $\Omega$  has a Lipschitz boundary, then  $\mathcal{A}_{\mathbf{x}} := \mathcal{O}_{\mathbf{x}} \cap \Omega$  has a Lipschitz boundary.

The existence of such a neighbourhood is almost obvious in the  $C^1$  case and follows from Lemma 3.10 in the other case.

We choose a finite subcovering  $\mathcal{O}_{\mathbf{x}_1}, \dots, \mathcal{O}_{\mathbf{x}_k}$  of  $\partial\Omega$  and fix from now on a number  $q \in ]3, \infty[$  such that

$$-\nabla \cdot \mu|_{\mathcal{A}_{\mathbf{x}_l}} \nabla : W_0^{1,q}(\mathcal{A}_{\mathbf{x}_l}) \rightarrow W^{-1,q}(\mathcal{A}_{\mathbf{x}_l}) \quad (3.30)$$

is a topological isomorphism for every  $l \in \{1, \dots, k\}$ . This is possible by Theorem 3.9; in particular,  $q$  may be chosen as an arbitrarily large number, if  $\Omega$  is  $C^1$ . Additionally, observe that (3.30) is then, by interpolation, a topological isomorphism for any other number from the interval  $[2, q[$ .

Assume now  $f \in W^{-1,q}(\Omega) \hookrightarrow W^{-1,2}(\Omega)$  and  $u$  to be a solution of

$$-\nabla \cdot \mu \nabla u + u = f \quad (3.31)$$

(which belongs to  $W_0^{1,2}(\Omega)$  by the Lax-Milgram lemma). We will show that then  $u \in W_0^{1,q}(\Omega)$ .

The  $C^1$  property of  $\partial\Omega_\circ$  assures for every  $\mathbf{x} \in \partial\Omega_\circ$  the existence of a positive number  $\alpha_{\mathbf{x}}$ , an open neighborhood  $\mathcal{V}_{\mathbf{x}} \subset \Omega$  of  $\mathbf{x}$  and a  $C^1$  diffeomorphism  $\Phi_{\mathbf{x}} : \mathcal{V}_{\mathbf{x}} \mapsto \alpha_{\mathbf{x}}\mathcal{E}$  such that  $\Phi_{\mathbf{x}}(\partial\Omega_\circ \cap \mathcal{V}_{\mathbf{x}}) = \alpha_{\mathbf{x}}\mathcal{E}_0$ ,  $\Phi_{\mathbf{x}}(\mathbf{x}) = 0$  and the corresponding Jacobian is identical 1, see [63, Ch.I Satz 2.5]. Without loss of generality we may assume that the closure of  $\mathcal{V}_{\mathbf{x}}$  is also contained in  $\Omega$ . The transformed of  $(-\nabla \cdot \mu \nabla + 1)|_{\mathcal{V}_{\mathbf{x}}}$  under  $\Phi_{\mathbf{x}}$  (see [7, Ch. 0.8]) is then of the form  $-\nabla \cdot \hat{\mu}_{\mathbf{x}} \nabla + 1$ , where  $\hat{\mu}_{\mathbf{x}}$  is uniformly continuous on  $\alpha_{\mathbf{x}}\mathcal{E}_-$  and on  $\alpha_{\mathbf{x}}\mathcal{E}_+$ , respectively. We denote  $\lim_{\mathbf{y} \in \mathcal{E}_-, \mathbf{y} \rightarrow 0} \hat{\mu}_{\mathbf{x}}(\mathbf{y})$  by  $\sigma_{\mathbf{x}}^-$  and  $\lim_{\mathbf{y} \in \mathcal{E}_+, \mathbf{y} \rightarrow 0} \hat{\mu}_{\mathbf{x}}(\mathbf{y})$  by  $\sigma_{\mathbf{x}}^+$ . Now let  $\sigma_{\mathbf{x}}$  be the coefficient function on  $\mathbb{R}^d$  defined by

$$\sigma_{\mathbf{x}} = \sigma_{\mathbf{x}}^\pm \quad \text{on} \quad \mathbb{R}_\pm^d.$$

By Theorem 3.11,  $-\nabla \cdot \sigma_{\mathbf{x}} \nabla + 1$  is a topological isomorphism between  $W^{1,t}(\mathbb{R}^d)$  and  $W^{-1,t}(\mathbb{R}^d)$  for all  $\mathbf{x} \in \partial\Omega_\circ$  and all  $t \in ]1, \infty[$ . Let  $\beta_{\mathbf{x}} \in ]0, \alpha_{\mathbf{x}}[$  be a number such that

$$\|\sigma_{\mathbf{x}} - \hat{\mu}_{\mathbf{x}}\|_{L^\infty(\beta_{\mathbf{x}}\mathcal{E}; \mathcal{B}(\mathbb{C}^d))} \|(\nabla \cdot \sigma_{\mathbf{x}} \nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\mathbb{R}^d); W^{1,t}(\mathbb{R}^d))} < 1 \quad (3.32)$$

holds for  $t = 2$  and  $t = q$ . Such  $\beta_{\mathbf{x}}$  exists because the second factor is finite by Theorem 3.11 and the first factor can be made arbitrarily small by the properties of  $\hat{\mu}_{\mathbf{x}}$  and  $\sigma_{\mathbf{x}}$  for  $\beta_{\mathbf{x}} \mapsto 0$ . Please notice that, by our

interpolation result Theorem 3.6, (3.32) remains true for any other  $t \in ]2, q[$ . Define  $\mathcal{U}_{\mathbf{x}}$  as the inverse image of  $\beta_{\mathbf{x}}\mathcal{E}$  under  $\Phi_{\mathbf{x}}$ . Finally, for any  $\mathbf{x} \in (\Omega \setminus (\cup_{l=1}^n \mathcal{O}_{\mathbf{x}_l} \cup \bar{\Omega}_o))$  let  $B_{\mathbf{x}}$  be an open ball around  $\mathbf{x}$  which does not intersect  $\partial\Omega \cup \partial\Omega_o$ . Obviously, the systems  $\{\mathcal{U}_{\mathbf{x}}\}_{\mathbf{x} \in \partial\Omega_o}, \{B_{\mathbf{x}}\}_{\mathbf{x} \in (\Omega \setminus (\cup_{l=1}^k \mathcal{O}_{\mathbf{x}_l} \cup \bar{\Omega}_o))}$  form an open covering of the (compact) set  $\Omega \setminus (\cup_{l=1}^k \mathcal{O}_{\mathbf{x}_l} \cup \Omega_o)$ . Let the system  $\mathcal{U}_{\mathbf{x}_{k+1}} \dots \mathcal{U}_{\mathbf{x}_m}, B_{\mathbf{x}_{m+1}}, \dots, B_{\mathbf{x}_n}$ , be a finite subcovering. Clearly, then the sets  $\mathcal{O}_{\mathbf{x}_1}, \dots, \mathcal{O}_{\mathbf{x}_k}, \mathcal{U}_{\mathbf{x}_{k+1}} \dots \mathcal{U}_{\mathbf{x}_m}, B_{\mathbf{x}_{m+1}}, \dots, B_{\mathbf{x}_n}, \Omega_o$  form an open covering of  $\bar{\Omega}$ . Let  $\eta_1, \dots, \eta_k, \eta_{k+1}, \dots, \eta_m, \eta_{m+1}, \dots, \eta_n, \eta_o$  be a partition of unity over  $\bar{\Omega}$  subordinated to this subcovering. Recalling (3.5), from now on we set  $t := \min(s(2), q)$ . Assume  $l \in \{1, \dots, k\}$ . Then  $v_l := \eta_l u|_{\mathcal{A}_{\mathbf{x}_l}}$ , due to the property  $u \in W_0^{1,2}(\Omega)$  and Lemma 3.8, satisfies an equation

$$-\nabla \cdot \mu_l \nabla v_l + v_l = f_l \quad (3.33)$$

where  $\mu_l := \mu|_{\mathcal{A}_{\mathbf{x}_l}}$  and  $f_l \in W^{-1,t}(\mathcal{A}_{\mathbf{x}_l})$ . Because (3.30) also is a topological isomorphism if  $q$  is replaced by  $t$  there, we get  $v_l \in W_0^{1,t}(\mathcal{A}_{\mathbf{x}_l})$  what gives  $\eta_l v_l \in W_0^{1,t}(\Omega)$ . Let next  $l$  be from  $\{k+1, \dots, m\}$ . Then the property  $u \in W_0^{1,2}(\Omega)$  and Lemma 3.8 imply that  $v_l := \eta_l u|_{\mathcal{U}_{\mathbf{x}_l}}$  satisfies an equation (3.33), where this time  $\mu_l := \mu|_{\mathcal{U}_{\mathbf{x}_l}}$  and  $f_l \in W^{-1,t}(\mathcal{U}_{\mathbf{x}_l})$ . Moreover, it is clear that both,  $v_l$  and  $f_l$ , have their supports within  $\mathcal{U}_{\mathbf{x}_l}$ . We transform (3.33) via the  $C^1$ -mapping  $\Phi_{\mathbf{x}_l}$ . This leads to the following equation for the transformed objects

$$-\nabla \cdot \hat{\mu}_{\mathbf{x}_l} \nabla \hat{v}_l + \hat{v}_l = \hat{f}_l \quad (3.34)$$

on  $\beta_{\mathbf{x}_l}\mathcal{E}$ , where  $\hat{f}_l \in W^{-1,t}(\beta_{\mathbf{x}_l}\mathcal{E})$ . Additionally,  $\hat{f}_l$  has its support in  $\beta_{\mathbf{x}_l}\mathcal{E}$ , what is also true for  $\hat{v}_l$ . Let  $\check{\sigma}_l$  be the following coefficient function, defined on  $\mathbb{R}^d$ :

$$\check{\sigma}_l = \begin{cases} \hat{\mu}_{\mathbf{x}_l} & \text{on } \beta_{\mathbf{x}_l}\mathcal{E} \\ \sigma_{\mathbf{x}_l} & \text{on } \mathbb{R}^d \setminus \beta_{\mathbf{x}_l}\mathcal{E}. \end{cases}$$

Because  $\hat{f}_l$  and  $\hat{v}_l$  have their supports in  $\beta_{\mathbf{x}_l}\mathcal{E}$ , (3.34) can be extended to an equation on whole  $\mathbb{R}^d$ ; namely: let  $\zeta_l$  be a  $C^\infty$  function on  $\mathbb{R}^d$  which is identical 1 on  $\text{supp}(\hat{v}_l) \cup \text{supp}(\hat{f}_l)$  and which has its support within  $\beta_{\mathbf{x}_l}\mathcal{E}$ . If we define  $F_l$  by  $\langle F_l, w \rangle = \langle \hat{f}_l, \zeta_l w \rangle$  for  $w \in W^{1,t'}(\mathbb{R}^d)$  and  $V_l$  as the extension of  $\hat{v}_l$  by zero to whole  $\mathbb{R}^d$ , then  $F_l \in W^{-1,t}(\mathbb{R}^d)$  and the following equation is fulfilled:

$$-\nabla \cdot \check{\sigma}_l \nabla V_l + V_l = -\nabla \cdot \sigma_{\mathbf{x}_l} \nabla V_l + V_l + \nabla \cdot (\sigma_{\mathbf{x}_l} - \check{\sigma}_l) \nabla V_l = F_l. \quad (3.35)$$

Because (3.32) is in particular true for our specified  $t$ , this implies

$$\begin{aligned} & \|\nabla \cdot (\sigma_{\mathbf{x}_l} - \check{\sigma}_l) \nabla\|_{\mathcal{B}(W^{1,t}(\mathbb{R}^d); W^{-1,t}(\mathbb{R}^d))} \|(-\nabla \cdot \sigma_{\mathbf{x}_l} \nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\mathbb{R}^d); W^{1,t}(\mathbb{R}^d))} \leq \\ & \leq \|\sigma_{\mathbf{x}_l} - \check{\sigma}_l\|_{L^\infty(\mathbb{R}^d; \mathcal{B}(\mathbb{C}^d))} \|(\nabla \cdot \sigma_{\mathbf{x}_l} \nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\mathbb{R}^d); W^{1,t}(\mathbb{R}^d))} = \\ & = \|\sigma_{\mathbf{x}_l} - \hat{\mu}_{\mathbf{x}_l}\|_{L^\infty(\beta_{\mathbf{x}_l} \mathcal{E}; \mathcal{B}(\mathbb{C}^d))} \|(\nabla \cdot \sigma_{\mathbf{x}_l} \nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\mathbb{R}^d); W^{1,t}(\mathbb{R}^d))} < 1. \end{aligned}$$

This, together with Proposition 3.1, then implies that  $-\nabla \cdot \check{\sigma}_l \nabla + 1 : W^{1,t}(\mathbb{R}^d) \mapsto W^{-1,t}(\mathbb{R}^d)$  is also a topological isomorphism. Consequently,  $V_l \in W^{1,t}(\mathbb{R}^d)$ , what gives  $\hat{v}_l \in W_0^{1,t}(\beta_{\mathbf{x}_l} \mathcal{E})$  and, hence,  $v_l = \eta_l u|_{\mathcal{U}_{\mathbf{x}_l}} \in W_0^{1,t}(\mathcal{U}_{\mathbf{x}_l})$ . Because the support of  $\eta_l u$  is within  $\mathcal{U}_{\mathbf{x}_l}$  we obtain  $\eta_l u \in W_0^{1,t}(\Omega)$  for all  $l = k+1, \dots, m$ . Lastly, if  $l \in \{m+1, \dots, n\}$ , then one also ends up for  $v_l := \eta_l u|_{B_{\mathbf{x}_l}}$  with an equation of type (3.33) and this same is true for  $v_0 := \eta_\circ u|_{\Omega_\circ}$ . The corresponding right hand sides are from  $W^{-1,t}(B_{\mathbf{x}_l})$  and  $W^{-1,t}(\Omega_\circ)$ , respectively (see Lemma 3.8). By Theorem 3.9  $\eta_l u|_{B_{\mathbf{x}_l}}$  and  $\eta_\circ u|_{\Omega_\circ}$  are then from  $W_0^{1,t}(B_{\mathbf{x}_l})$  and  $W_0^{1,t}(\Omega_\circ)$ , respectively. Clearly, then  $\eta_l u$  and  $\eta_\circ u$  must be from  $W_0^{1,t}(\Omega)$  what altogether gives  $u \in W_0^{1,t}(\Omega)$ . Exploiting this and iterating the above considerations one improves the summability of  $\nabla u$  in the light of Lemma 3.8 step by step and finally ends up with  $u \in W_0^{1,q}(\Omega)$ . This proves the assertion for  $\lambda = 1$ . For all other  $\lambda$ 's we obtain the proof by the compactness of the resolvent and the fact that no  $\lambda$  with  $\Re \lambda \leq 0$  can be an eigenvalue. The case  $q < 2$  is obtained by duality. We will now point out how to prove Theorem 1.1 if Condition *ii*) is fulfilled. The only difference in the proofs of *i*) and *ii*) in Theorem 1.1 is that the boundary points must be treated in different ways; for this we prove the following

**3.12 Lemma.** *For any  $\mathbf{x} \in \partial\Omega$  there is a neighbourhood  $\mathcal{O}_{\mathbf{x}}$  and a  $q = q_{\mathbf{x}} > 3$  such that  $\mathcal{O}_{\mathbf{x}} \cap \Omega$  is a Lipschitz domain and*

$$\nabla \cdot \mu \nabla + 1 : W_0^{1,q}(\mathcal{O}_{\mathbf{x}} \cap \Omega) \rightarrow W^{-1,q}(\mathcal{O}_{\mathbf{x}} \cap \Omega)$$

*is a topological isomorphism.*

In contrast to case *i*) one cannot treat the points from  $\partial\Omega$  in common, but has to divide  $\partial\Omega$  into three subsets which have to be treated separately:  
a)  $\partial\Omega \setminus \partial\Omega_\circ$

b) the inner points of  $\partial\Omega \cap \partial\Omega_o$  within  $\partial\Omega$   
c) the boundary points of  $\partial\Omega \cap \partial\Omega_o$  within  $\partial\Omega$   
a) If  $\mathbf{x} \in \partial\Omega \setminus \partial\Omega_o$ , then there is an open neighbourhood  $\mathcal{W}_x$  of  $\mathbf{x}$  such that  $\mathcal{W}_x \cap \Omega$  does not intersect  $\bar{\Omega}_o$ . Namely, if this were not the case, then  $\mathbf{x}$  would be an accumulation point of  $\bar{\Omega}_o$ , and, hence, belongs to  $\bar{\Omega}_o$ . Because  $\mathbf{x}$  is not from  $\Omega_o$  this would mean  $\mathbf{x} \in \partial\Omega_o$ , what is wrong. By Lemma 3.10 we can pass to a (possibly) smaller open neighbourhood  $\mathcal{O}_x$  such that  $\mathcal{O}_x \cap \Omega$  is again a domain with Lipschitz boundary. Thus, the coefficient function is uniformly continuous on  $\mathcal{O}_x \cap \Omega$  and the assertion follows from Theorem 3.9. Let us now consider case b). What we want to show is the following: if  $\mathbf{x}$  is an inner point of  $\partial\Omega \cap \partial\Omega_o$  within  $\partial\Omega$ , then one can find a neighbourhood  $\mathcal{O}_x$  of  $\mathbf{x}$  such that

i)  $\mathcal{O}_x \cap \Omega = \mathcal{O}_x \cap \Omega_o$

and

ii)  $\mathcal{O}_x \cap \Omega$  is a domain with Lipschitz boundary.

First we construct an open neighbourhood  $\mathcal{M}_x$  of  $\mathbf{x}$  which fulfills  $\mathcal{M}_x \cap \Omega = \mathcal{M}_x \cap \Omega_o$ . Namely, because  $\Omega$  is a Lipschitz domain (see [28, Ch. 1.2] or [63, Ch. I.2.3]) there is an open neighbourhood  $\mathcal{W}_x$  of  $\mathbf{x}$  and a bi-Lipschitz map  $\Psi_x : \mathcal{W}_x \rightarrow \mathcal{E}$  such that  $\Psi_x(\mathbf{x}) = 0$ ,  $\Psi_x(\Omega \cap \mathcal{W}_x) = \mathcal{E}_+$  and  $\Psi_x(\partial\Omega \cap \mathcal{W}_x) = \mathcal{E}_o$ . Because  $\mathbf{x}$  was an inner point of  $\partial\Omega \cap \partial\Omega_o$ , there is a positive number  $r_x$  such that  $r_x \mathcal{E}_o \subset \Psi(\partial\Omega \cap \partial\Omega_o) \subset \Psi(\partial\Omega_o)$ . But, by supposition,  $\Omega_o$  itself was a Lipschitz domain, too; thus there is a number  $s_x \in ]0, r_x]$  such that

$$\Psi_x(\partial\Omega_o) \cap s_x \mathcal{E} = s_x \mathcal{E}_o. \quad (3.36)$$

Now we define  $\mathcal{M}_x := \Psi_x^{-1}(s_x \mathcal{E})$  and write

$$\mathcal{M}_x \cap \Omega = (\mathcal{M}_x \cap \Omega_o) \cup (\mathcal{M}_x \cap \Omega \cap \partial\Omega_o) \cup (\mathcal{M}_x \cap (\Omega \setminus \bar{\Omega}_o)). \quad (3.37)$$

From the definition of  $\mathcal{M}_x$  and (3.36) it is clear that  $\mathcal{M}_x \cap \Omega \cap \partial\Omega_o$  is empty. Thus, (3.37) reduces to

$$\mathcal{M}_x \cap \Omega = (\mathcal{M}_x \cap \Omega_o) \cup (\mathcal{M}_x \cap (\Omega \setminus \bar{\Omega}_o)). \quad (3.38)$$

But  $\mathcal{M}_x \cap \Omega$  is -as a continuous image of a connected set- itself connected. Thus, one of the (open) sets on the right hand side of (3.38) must be empty, what is definitely not true for  $\mathcal{M}_x \cap \Omega_o$ . This gives  $\mathcal{M}_x \cap \Omega = \mathcal{M}_x \cap \Omega_o$ . Due to Lemma 3.10 we may pass to a neighbourhood  $\mathcal{O}_x \subset \mathcal{M}_x$  which then

(obviously) also satisfies *i*) and, additionally, *ii*). Hence, the coefficient function is also uniformly continuous on  $\mathcal{O}_{\mathbf{x}} \cap \Omega$  and one can again argue by Theorem 3.9.

It remains case *c*), which we will consider now. For doing so, we first establish some preliminaries:

**3.13 Proposition.** [23] *Assume that  $\mathcal{K} \subset \mathbb{R}^3$  is a convex polyhedron and that  $\mathcal{H} \subset \mathbb{R}^3$  is a plane which contains an inner point of  $\mathcal{K}$ . Let  $\mathcal{K}_+$  and  $\mathcal{K}_-$  be the two components of  $\mathcal{K} \setminus \mathcal{H}$ , and let  $\rho$  be a function on  $\mathcal{K}$ , constant on  $\mathcal{K}_+$  and  $\mathcal{K}_-$ , and whose values are two real, symmetric, positive definite  $3 \times 3$  matrices there. Then there is a  $q > 3$  such that*

$$-\nabla \cdot \rho \nabla : W_0^{1,q}(\Omega) \rightarrow W^{-1,q}(\Omega)$$

*is a topological isomorphism.*

**3.14 Lemma.** *Let  $\mathcal{K} \subset \mathbb{R}^3$  be a convex set whose closure contains 0. Assume that  $\rho$  is a bounded, measurable, elliptic coefficient function on  $\mathcal{K}$ , taking its values in the set of real, symmetric  $3 \times 3$  matrices and which additionally satisfies*

$$\rho(\alpha \mathbf{x}) = \rho(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{K}, \alpha \in ]0, 1[. \quad (3.39)$$

*Let for any  $\alpha \in ]0, 1[$  the space  $W_0^{1,q}(\alpha \mathcal{K})$  be equipped with the norm  $\psi \mapsto \left( \int_{\alpha \mathcal{K}} |\nabla \psi|^q dx \right)^{\frac{1}{q}}$ . Then*

$$\|(-\nabla \cdot \rho|_{\alpha \mathcal{K}} \nabla)^{-1}\|_{\mathcal{B}(W_0^{-1,q}(\alpha \mathcal{K}); W_0^{1,q}(\alpha \mathcal{K}))} = \|(-\nabla \cdot \rho \nabla)^{-1}\|_{\mathcal{B}(W_0^{-1,q}(\mathcal{K}); W_0^{1,q}(\mathcal{K}))}. \quad (3.40)$$

*Proof.* One easily checks that for any  $q \in [1, \infty[$  and any  $\alpha \in ]0, 1[$  the mapping

$$T_{q,\alpha} : W_0^{1,q}(\mathcal{K}) \ni \psi \longmapsto \alpha^{1-\frac{3}{q}} \psi(\alpha^{-1}(\cdot))$$

provides an isometric isomorphism from  $W_0^{1,q}(\mathcal{K})$  onto  $W_0^{1,q}(\alpha \mathcal{K})$ . Afterwards one verifies the identity

$$T_{q',\alpha}^* (-\nabla \cdot \rho|_{\alpha \mathcal{K}} \nabla) T_{q,\alpha} = -\nabla \cdot \rho \nabla.$$

□

Assume now that  $\mathbf{x}$  is a boundary point of  $\partial\Omega \cap \partial\Omega_\circ$  within  $\partial\Omega$ . Then, by supposition, there is an open neighbourhood  $\mathcal{W}_\mathbf{x}$ , a  $C^1$  mapping  $\Phi_\mathbf{x}$ , a convex polyhedron  $\mathcal{K}_\mathbf{x}$  and a plane  $\mathcal{H}_\mathbf{x}$  which together satisfy the conditions of Definition 1.2. Modulo a translation we may additionally assume  $\Phi_\mathbf{x}(\mathbf{x}) = 0$ . Let  $\rho_\mathbf{x}$  be the coefficient function on  $\mathcal{K}_\mathbf{x}$  which is induced by  $\mu|_{\mathcal{W}_\mathbf{x}}$  under the mapping  $\Phi_\mathbf{x}$ . If  $\mathcal{K}_\mathbf{x}^+$  and  $\mathcal{K}_\mathbf{x}^-$  are the two components of  $\mathcal{K}_\mathbf{x} \setminus \mathcal{H}_\mathbf{x}$ , then  $\rho_\mathbf{x}$  is uniformly continuous on both of them. Define the matrices

$$\rho_\mathbf{x}^+ := \lim_{\mathbf{y} \rightarrow 0, \mathbf{y} \in \mathcal{K}_+} \rho_\mathbf{x}(\mathbf{y}) \quad \text{and} \quad \rho_\mathbf{x}^- := \lim_{\mathbf{y} \rightarrow 0, \mathbf{y} \in \mathcal{K}_-} \rho_\mathbf{x}(\mathbf{y}) \quad (3.41)$$

and the coefficient function  $\tilde{\rho}_\mathbf{x}$  on  $\mathcal{K}_\mathbf{x}$  by

$$\tilde{\rho}_\mathbf{x} := \begin{cases} \rho_\mathbf{x}^+ & \text{on } \mathcal{K}_\mathbf{x}^+ \\ \rho_\mathbf{x}^- & \text{on } \mathcal{K}_\mathbf{x}^- \end{cases} \quad (3.42)$$

Let  $\alpha_\mathbf{x} \in ]0, 1[$  be a number for which the following is true:

$$\alpha_\mathbf{x} \overline{\Phi_\mathbf{x}(\mathcal{W}_\mathbf{x})} \subset \Phi_\mathbf{x}(\mathcal{W}_\mathbf{x}) \quad (3.43)$$

and

$$\operatorname{ess\,sup}_{\mathbf{y} \in \alpha_\mathbf{x} \mathcal{K}_\mathbf{x}} \|\rho_\mathbf{x}(\mathbf{y}) - \tilde{\rho}_\mathbf{x}(\mathbf{y})\|_{\mathcal{B}(\mathbb{C}^3)} \|(-\nabla \cdot \tilde{\rho}_\mathbf{x} \nabla)^{-1}\|_{\mathcal{B}(W_0^{-1,q}(\mathcal{K}); W_0^{1,q}(\mathcal{K}))} < 1.$$

This is possible due to (3.41) and (3.42). In view of Lemma 3.14 then also

$$\operatorname{ess\,sup}_{\mathbf{y} \in \alpha_\mathbf{x} \mathcal{K}_\mathbf{x}} \|\rho_\mathbf{x}(\mathbf{y}) - \tilde{\rho}_\mathbf{x}(\mathbf{y})\|_{\mathcal{B}(\mathbb{C}^3)} \|(-\nabla \cdot \tilde{\rho}_\mathbf{x}|_{\alpha_\mathbf{x} \mathcal{K}} \nabla)^{-1}\|_{\mathcal{B}(W_0^{-1,q}(\alpha_\mathbf{x} \mathcal{K}); W_0^{1,q}(\alpha_\mathbf{x} \mathcal{K}))} < 1$$

is true. Completely analogous to the above considerations one obtains by the perturbation theorem that

$$-\nabla \cdot \rho_\mathbf{x}|_{\alpha_\mathbf{x} \mathcal{K}} \nabla : W_0^{1,q}(\alpha_\mathbf{x} \mathcal{K}) \rightarrow W^{-1,q}(\alpha_\mathbf{x} \mathcal{K})$$

is also a topological isomorphism. If one defines  $\mathcal{O}_\mathbf{x} := \Phi_\mathbf{x}^{-1}(\alpha_\mathbf{x} \Phi_\mathbf{x}(\mathcal{W}_\mathbf{x}))$  (what makes sense in view of (3.43)) then  $\Phi_\mathbf{x}(\mathcal{O}_\mathbf{x} \cap \Omega) = \alpha_\mathbf{x} \mathcal{K}_\mathbf{x}$ . The latter is a domain with Lipschitz boundary and, hence, a Lipschitz domain. Because  $\Phi_\mathbf{x}^{-1}$  is in particular bi-Lipschitz in a neighbourhood of  $\alpha_\mathbf{x} \mathcal{K}_\mathbf{x}$ ,  $\mathcal{O}_\mathbf{x} \cap \Omega$  itself is a Lipschitz domain (see [28, Ch. 1.2 Lem. 1.2.1.3]). Moreover,

$$-\nabla \cdot \mu \nabla : W_0^{1,q}(\mathcal{O}_\mathbf{x} \cap \Omega) \rightarrow W^{-1,q}(\mathcal{O}_\mathbf{x} \cap \Omega)$$



is a topological isomorphism. But the resolvent is compact and  $-1$  obviously not an eigenvalue, hence  $\mathcal{O}_x$  also fulfills the assertion of Lemma 3.12. With the help of Lemma 3.12 the proof for case *ii*) of Theorem 1.1 can be carried out as in case *i*).

**3.15 Remark.** The reader may possibly ask why in case *ii*) we restrict ourself to  $d = 3$ . The answer is: the essential aim of this paper is to prove the isomorphism property for a  $q$  which is larger than the space dimension  $d$ . In this spirit, the two dimensional case (even under more general assumptions) is covered by [29]. If  $d > 3$  we do not have results for the corresponding model sets. Nevertheless,  $d = 3$  as the 'physical' dimension seems to us the most important case.

In fact, in [23] more general geometric (nonconvex) settings are treated. However, the technicalities here would get much more involved.

**3.16 Remark.** If  $\Omega_\circ$  does not touch the boundary of  $\Omega$ , then one can prove the analogous result for the Neumann operator, namely:  $-\nabla \cdot \mu \nabla + \lambda$  provides a topological isomorphism between  $W^{1,q}(\Omega)$  and  $\left(W^{-1,q'}(\Omega)\right)^*$  for a  $q > 3$  and all  $\lambda$  from the open right half plane. In this case one uses Zanger's result [64] instead that of Jerison/Kenig.

**3.17 Remark.** The reader should notice that the result generalizes to the case where finitely many  $C^1$  domains are included in  $\Omega$  having positive distance to each other and the coefficient function being uniformly continuous one each of them and, of course, on the complement of their union. The proof runs along the same lines and has not been carried out here only for notational simplicity.

**3.18 Remark.** The isomorphy property claimed in Theorem 1.1 remains true in case of real spaces  $W_0^{1,q}(\Omega)$ ,  $W^{-1,q}(\Omega)$  and real  $\lambda$ 's, because  $-\nabla \cdot \mu \nabla + 1$  commutes with complex conjugation.

## 4 Perturbation by lower order terms

In this chapter we will present a class of first order terms under the perturbation of which our regularity result is (essentially) maintained:

**4.1 Theorem.** *Let  $q \geq 2$  be a number such that*

$$-\nabla \cdot \rho \nabla : W_0^{1,q}(\Lambda) \mapsto W^{-1,q}(\Lambda)$$

*is a topological isomorphism. Assume  $r > d, \epsilon > 0, \delta \in ]0, 2[$ ,*

$$s := \begin{cases} q & \text{if } q > d \\ d + \epsilon & \text{if } q \leq d \end{cases}, \quad t := \begin{cases} \left(\frac{1}{q} + \frac{1}{d}\right)^{-1} & \text{if } q > d \\ \frac{d}{\delta} & \text{if } q \leq d. \end{cases}$$

*and  $a_1, \dots, a_d \in L^r(\Lambda), b_1, \dots, b_d \in L^s(\Lambda), c \in L^t(\Lambda)$ .*

*i) The first order operator*

$$u \mapsto \sum_{l=1}^d a_l \frac{\partial u}{\partial x_l} + \frac{\partial(b_l u)}{\partial x_l} + c u$$

*is relatively compact with respect to  $-\nabla \cdot \rho \nabla$ .*

*ii) The operator*

$$u \mapsto -\nabla \cdot \rho \nabla u + \sum_{l=1}^d a_l \frac{\partial u}{\partial x_l} + \frac{\partial(b_l u)}{\partial x_l} + c u \quad (4.1)$$

*also has  $W_0^{1,q}(\Lambda)$  as its domain of definition.*

*iii) The spectrum of the operator from (4.1) consists of countably many isolated eigenvalues with finite (algebraic) multiplicities.*

*Proof.* i) It suffices to show the assertion for the terms separately: each operator  $\frac{\partial}{\partial x_l}$  maps  $W_0^{1,q}(\Lambda)$  continuously into  $L^q(\Lambda)$ . The multiplication operators  $a_1, \dots, a_d$  then continuously map  $L^q(\Lambda)$  into  $L^{(\frac{1}{q} + \frac{1}{r})^{-1}}(\Lambda) \hookrightarrow W^{-\frac{d}{r},q}(\Lambda)$ . Thus, for the terms  $a_l \frac{\partial}{\partial x_l}$  the assertion results from the compactness of the embedding  $W^{-\frac{d}{r},q}(\Lambda) \hookrightarrow W^{-1,q}(\Lambda)$ . Concerning the terms  $\frac{\partial(b_l u)}{\partial x_l}$  we first consider the case  $q > d$ : then one has the compact embedding  $W_0^{1,q}(\Lambda) \hookrightarrow L^\infty(\Lambda)$  which implies the compactness of the mappings

$$W_0^{1,q}(\Lambda) \ni u \mapsto b_l u \in L^q(\Lambda) \quad (4.2)$$

and

$$W_0^{1,q}(\Lambda) \ni u \rightarrow \frac{\partial(b_l u)}{\partial x_l} \in W^{-1,q}(\Lambda). \quad (4.3)$$

If  $q = d$ , then the mapping  $W_0^{1,q}(\Lambda) \hookrightarrow L^{\frac{d(d+\epsilon)}{\epsilon}}(\Lambda)$  is compact and so are the mappings (4.2) and (4.3) in this case. It remains the case  $q < d$ : putting  $\tau := \frac{d}{d+\epsilon}$ , one verifies the compactness of  $W_0^{1,q}(\Lambda) \hookrightarrow W_0^{\tau,q}(\Lambda) \hookrightarrow L^{(\frac{1}{q}-\frac{\tau}{d})^{-1}}(\Lambda)$  and, hence, again the compactness of the mappings (4.2) and (4.3). We inspect the  $c$ -term, first considering the case  $q > d$ : then the mapping  $W_0^{1,q}(\Lambda) \hookrightarrow L^\infty(\Lambda)$  is compact. Consequently, the mapping

$$W_0^{1,q}(\Lambda) \ni u \rightarrow cu \in L^{(\frac{1}{q}+\frac{1}{d})^{-1}}(\Lambda) \hookrightarrow W^{-1,q}(\Lambda)$$

is compact, too. If  $q = d$ , then the embedding  $W_0^{1,q}(\Lambda) \hookrightarrow L^{\frac{d}{2-\delta}}(\Lambda)$  is compact. This implies the compactness of the mapping

$$W_0^{1,d}(\Lambda) \hookrightarrow L^{\frac{d}{2-\delta}}(\Lambda) \ni u \rightarrow cu \in L^{\frac{d}{2}}(\Lambda) \hookrightarrow W^{-1,d}(\Lambda).$$

What concerns the case  $q < d$ , it suffices to consider  $\delta$ 's from ]1, 2[. Then we have the embedding  $W_0^{1,q}(\Lambda) \hookrightarrow L^{(\frac{1}{q}-\frac{1}{d})^{-1}}(\Lambda)$ . Consequently, the mapping

$$W_0^{1,q}(\Lambda) \ni u \rightarrow cu \in L^{(\frac{1}{q}+\frac{\delta-1}{d})^{-1}}(\Lambda) \hookrightarrow W^{1-\delta,q}(\Lambda) \hookrightarrow W^{-1,q}(\Lambda)$$

is also compact. ii) follows from a well known theorem on relatively compact perturbations (see [38, Ch. IV.1.3 Thm. 1.11]).

iii) Obviously, the resolvent of  $-\nabla \cdot \rho \nabla$  is compact. Hence, the essential spectrum of this operator is empty (see [38, Ch. III.6.8 Thm. 6.29]). Because the perturbation is relatively compact, the essential spectrum of the perturbed operator (c.f. (4.1)) is also empty (see [38, Ch. IV.5.6 Thm. 5.35]). Thus, the assertion follows from another well known theorem (see [38, Ch. IV.5.6 Thm. 5.33]).  $\square$

## 5 Nonsmooth interfaces: a counterexample

The reader may have possibly asked himself whether the  $C^1$  property is necessary or may be weakened without changing the result. The following

counterexample (see [23]) shows that the situation changes dramatically if the interface has only one corner point. In particular, this shows that piecewise  $C^1$  is (by far) not sufficient for our result. Namely, quite parallelly to the classical example of Meyers (see [47]) the integrability exponent for the gradient of the solution of the (planar) homogeneous elliptic equation tends to 2 in dependence of a suitable parameter. The difference to Meyer's example is that there the ellipticity constant tends to zero, while here a nonsmooth interface occurs and the norms of the coefficient matrices tend to infinity.

The background for the considerations in this chapter is the well known connection between singularities for the solution of an elliptic equation and the eigenvalues of an associated operator pencil of Sturm-Liouville operators, see [46] or [23].

We consider the following coefficient function on  $\mathbb{R}^2$ :

$$\mu(x, y) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} & \text{if } x, y > 0 \\ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} & \text{elsewhere on } \mathbb{R}^2, t > 0, \end{cases}$$

and, correspondingly, the following elliptic problem

$$\nabla \cdot \mu \nabla u = 0. \quad (5.1)$$

Proceeding as in [46] we are looking for solutions  $\tilde{u} \in W^{1,2}([0, 2\pi])$  of the (generalized) Sturm-Liouville equation

$$-(b_2 \tilde{u}')' - \lambda(b_1 \tilde{u})' - \lambda b_1 \tilde{u}' - \lambda^2 b_0 \tilde{u} = 0, \quad (5.2)$$

combined with the compatibility conditions

$$\begin{aligned} w(\pi/2) &= v(\pi/2), \quad w(0) = v(2\pi), \\ (b_2 \partial_\theta w + \lambda b_1 w)|_0 &= (b_2 \partial_\theta v + \lambda b_1 v)|_{2\pi}, \\ (b_2 \partial_\theta w + \lambda b_1 w)|_{\pi/2} &= (b_2 \partial_\theta v + \lambda b_1 v)|_{\pi/2}, \end{aligned} \quad (5.3)$$

if  $w = \tilde{u}|_{[0, \pi/2]}$  and  $v = \tilde{u}|_{[\pi/2, 2\pi]}$ .

The coefficient functions  $b_0, b_1, b_2$  are defined as follows:

$$\begin{aligned} b_0(\theta) &\stackrel{\text{def}}{=} \begin{cases} \cos^2 \theta + t^2 \sin^2 \theta, & \text{if } \theta \in [0, \pi/2[ \\ t, & \text{if } \theta \in [\pi/2, 2\pi[ \end{cases} \\ b_2(\theta) &\stackrel{\text{def}}{=} \begin{cases} \sin^2 \theta + t^2 \cos^2 \theta, & \text{if } \theta \in [0, \pi/2[ \\ t, & \text{if } \theta \in [\pi/2, 2\pi[ \end{cases} \\ b_1(\theta) &\stackrel{\text{def}}{=} \begin{cases} (t^2 - 1) \sin \theta \cos \theta, & \text{if } \theta \in [0, \pi/2[ \\ 0, & \text{if } \theta \in [\pi/2, 2\pi[ \end{cases} \end{aligned} \quad (5.4)$$

In order to determine the  $\lambda$  with the smallest possible (positive) real part, we use the ansatz functions (see [17])

$$w(\theta) := c_+(t \cos \theta + i \sin \theta)^\lambda + c_-(t \cos \theta - i \sin \theta)^\lambda$$

and

$$v(\theta) := d_+ \cos \lambda \theta + d_- \sin \lambda \theta$$

with unknown coefficients  $c_\pm$  and  $d_\pm$ . Using (5.3) and (5.4), we can eliminate  $c_\pm$  and then get the equations

$$\begin{aligned} d_+(t^\lambda - \cos 2\pi\lambda) - d_- \sin 2\pi\lambda &= 0, \\ d_+ \sin 2\pi\lambda + d_-(t^\lambda - \cos 2\pi\lambda) &= 0. \end{aligned} \quad (5.5)$$

Obviously, the system (5.5) is nontrivially solvable in  $d_+, d_-$  iff

$$(t^\lambda - \cos 2\pi\lambda)^2 + \sin^2 2\pi\lambda = 0,$$

or, what is the same,

$$\cos 2\pi\lambda = \frac{t^\lambda + t^{-\lambda}}{2} = \cosh(\lambda \ln t). \quad (5.6)$$

Writing  $\cosh(\lambda \ln t) = \cos(i\lambda \ln t)$  and taking into account the identity

$$\cos \theta - \cos \rho = -2 \sin \frac{\theta + \rho}{2} \sin \frac{\theta - \rho}{2},$$

(5.6) is equivalent to

$$\sin\left(\frac{\lambda}{2}(2\pi + i \ln t)\right) \sin\left(\frac{\lambda}{2}(2\pi - i \ln t)\right) = 0.$$

This is the case iff

$$\frac{\lambda}{2}(2\pi \pm i \ln t) = 2k\pi, \quad k \in \mathbb{Z}.$$

Thus, the  $\lambda$  with the smallest (positive) real part is

$$\lambda = \frac{8\pi^2}{4\pi^2 + \ln^2 t} \pm i \frac{4\pi \ln t}{4\pi^2 + \ln^2 t}.$$

One easily notices: If  $t \rightarrow \infty$ , then the real parts of these  $\lambda$ 's converge to zero. Assume that  $\lambda$  with  $\Re\lambda \in (0, 1)$  is a complex number and  $\tilde{u}_\lambda \in W^{1,2}(0, 2\pi)$  a corresponding function which satisfies (5.2) together with the compatibility conditions (5.3). Then the function

$$u(\mathbf{x}) := (x_1^2 + x_2^2)^{\lambda/2} \tilde{u}_\lambda(\arg(\mathbf{x})) \in W_{loc}^{1,2}(\mathbb{R}^2)$$

is a solution of equation (5.1) in the distributional sense. Moreover,  $\tilde{u}_\lambda$  does not vanish identically and, hence, its absolute value has a strictly positive lower bound at least on a (nontrivial) subinterval of  $(0, 2\pi)$ . Thus,  $u \in W_{loc}^{1,q}(\mathbb{R}^2)$  for  $q \in [2, (\frac{1-\Re\lambda}{2})^{-1})$ , but not for  $q = (\frac{1-\Re\lambda}{2})^{-1}$ . Tending with  $t$  to  $\infty$ , these solutions lack any common (local) integrability exponent larger than 2 for their first order derivatives.

**5.1 Remark.** The example is not restricted to two dimensions. One can add arbitrarily many dimensions by extending the solution constantly in these directions – at least in a neighbourhood of zero.

## 6 Parabolic operators

Very often elliptic operators in divergence form occur as the elliptic part of parabolic operators (see [5] or [31]). In this chapter we will deduce functional analytic properties for the corresponding parabolic operators from our

elliptic regularity result. If  $X$  is a complex Banach space, then we denote by  $W^{1,r}(\]0, T[; X)$  the set of elements from  $L^r(\]0, T[; X)$  whose distributional derivatives also belong to  $L^r(\]0, T[; X)$  (see [4, Ch. III 1.1] for details). The main result reads as follows:

**6.1 Theorem.** *Let  $\Lambda$  be a bounded domain with Lipschitz boundary and  $\rho$  a measurable, essentially bounded, elliptic coefficient function which takes its values in the set of real, symmetric  $d \times d$  matrices. Assume that  $q \in ]1, \infty[$  is a number such that*

$$-\nabla \cdot \rho \nabla : W_0^{1,q}(\Lambda) \rightarrow W^{-1,q}(\Lambda)$$

*is a topological isomorphism.*

*Then  $\frac{\partial}{\partial t} - \nabla \cdot \rho \nabla$  satisfies maximal parabolic regularity on  $W^{-1,q}(\Lambda)$ , precisely: If  $r \in ]1, \infty[$  is fixed, then for any  $f \in L^r(\]0, T[, W^{-1,q}(\Lambda))$  there is exactly one function  $w \in L^r(\]0, T[; W^{1,q}(\Lambda)) \cap W^{1,r}(\]0, T[; W^{-1,q}(\Lambda))$  such that*

$$\frac{\partial w}{\partial t} - \nabla \cdot \rho \nabla w = f \quad \text{and} \quad w(0) = 0. \quad (6.1)$$

**6.2 Corollary.** *Under the above suppositions  $-\nabla \cdot \rho \nabla$  generates an analytic semigroup on  $W^{-1,q}(\Lambda)$ .*

In order to prove this theorem we first establish some auxiliary results:

**6.3 Theorem.** *Let  $\Lambda$  be a Lipschitz domain and  $\rho$  as in the previous theorem. Assume  $q \in ]1, \infty[$  and let  $A_q$  be the  $L^q(\Lambda)$  realization of  $\nabla \cdot \rho \nabla$ , further  $D_q$  the domain of this realization. Then  $\frac{\partial}{\partial t} - A_q$  satisfies maximal regularity over  $L^q(\Lambda)$ , in other words: If  $r \in ]1, \infty[$  is fixed, then for any  $f \in L^r(\]0, T[; L^q(\Lambda))$  there is exactly one function  $w \in L^r(\]0, T[; D_q) \cap W^{1,r}(\]0, T[; L^q(\Lambda))$  such that (6.1) is satisfied.*

*Proof.* The semigroup generated by  $A_2$  on  $L^2(\Lambda)$  admits upper Gaussian estimates, see [8] or [6]. But upper Gaussian estimates imply maximal parabolic regularity on  $L^p$  spaces [32], see also [19].  $\square$

**6.4 Theorem.** *Under the suppositions of Theorem 6.1  $(-\nabla \cdot \rho \nabla)^{1/2}$  provides a topological isomorphism between  $W_0^{1,s}(\Lambda)$  and  $L^s(\Lambda)$  and between  $L^s(\Lambda)$  and  $W^{-1,s}(\Lambda)$  for all  $s \in [q', q]$ .*

*Proof.* First, interpolation (see Theorem 3.6) and duality show that  $-\nabla \cdot \rho \nabla$  is a topological isomorphism between  $W_0^{1,s}(\Lambda)$  and  $W^{-1,s}(\Lambda)$  for all  $s \in [q', q]$ . A deep result of [8, Thm. 4] yields the continuity of the map

$$(-\nabla \cdot \rho \nabla)^{1/2} : W_0^{1,s}(\Lambda) \rightarrow L^s(\Lambda) \quad (6.2)$$

for all  $s \in ]1, \infty[$ . By duality one obtains the continuity of

$$(-\nabla \cdot \rho \nabla)^{1/2} : L^s(\Lambda) \mapsto W^{-1,s}(\Lambda) \quad (6.3)$$

for all  $s \in ]1, \infty[$ . Hence, for  $s \in [q', q]$  we can estimate

$$\begin{aligned} & \|(-\nabla \cdot \rho \nabla)^{-1/2}\|_{\mathcal{B}(L^s(\Lambda); W_0^{1,s}(\Lambda))} \leq \\ & \leq \|(-\nabla \cdot \rho \nabla)^{1/2}\|_{\mathcal{B}(L^s(\Lambda); W^{-1,s}(\Lambda))} \|(-\nabla \cdot \rho \nabla)^{-1}\|_{\mathcal{B}(W^{-1,s}(\Lambda); W_0^{1,s}(\Lambda))}. \end{aligned}$$

This proves that (6.2) in fact is a topological isomorphism, if  $s \in [q', q]$ . The isomorphism property between  $L^s(\Lambda)$  and  $W^{-1,s}(\Lambda)$  follows from this by duality.  $\square$

**6.5 Corollary.** *Let  $D_q$  denote the domain of the  $L^q(\Lambda)$  realization of  $-\nabla \cdot \rho \nabla$ . Then  $(-\nabla \cdot \rho \nabla)^{1/2}$  provides a topological isomorphism between  $D_q$  and  $W_0^{1,q}(\Lambda)$ .*

*Proof.*  $-\nabla \cdot \rho \nabla$  is a topological isomorphism between  $D_q$  and  $L^q(\Lambda)$  while  $(-\nabla \cdot \rho \nabla)^{1/2}$  is a topological isomorphism between  $W_0^{1,q}(\Lambda)$  and  $L^q(\Lambda)$ .  $\square$

We will now give the proof of Theorem 6.1: it is clear that the established isomorphisms for  $(-\nabla \cdot \rho \nabla)^{1/2}$  induce the following isomorphisms:

$$(-\nabla \cdot \rho \nabla)^{-1/2} : L^r(]0, T[; W^{-1,q}(\Lambda)) \rightarrow L^r(]0, T[; L^q(\Lambda)) \quad (6.4)$$

$$(-\nabla \cdot \rho \nabla)^{1/2} : L^r(]0, T[; D_q) \rightarrow L^r(]0, T[; W_0^{1,q}(\Lambda)) \quad (6.5)$$

$$(-\nabla \cdot \rho \nabla)^{1/2} : W^{1,r}(]0, T[; L^q(\Lambda)) \rightarrow W^{1,r}(]0, T[; W^{-1,q}(\Lambda)) \quad (6.6)$$

Further, it is well known that the solution  $w$  of (6.1) is obtained as  $w(t) = \int_0^t e^{(t-s)\nabla \cdot \rho \nabla} f(s) ds$ . Hence, the parabolic solution operator commutes with  $(-\nabla \cdot \rho \nabla)^{1/2}$ . Consequently, the maximal regularity property on  $L^q(\Lambda)$  transports via the isomorphisms (6.4), (6.5), (6.6) to the space  $W^{-1,q}(\Lambda)$ . Corollary 6.2 is implied by the well known fact that maximal parabolic regularity implies the generation property of an analytic semigroup.



**6.6 Remark.** The authors are convinced that the results on the parabolic operators are adequate instruments for the treatment of (even non-autonomous) semilinear (see [49, Ch. 5.6]) and quasilinear parabolic problems [45, 50, 16]. The key point concerning quasilinear equations of, say, the type

$$\frac{\partial w}{\partial t} - \nabla \cdot G(w)\mu\nabla w = H(t, w, \nabla w)$$

is the fact that in case of three-dimensional domains and  $q > 3$  suitable interpolation spaces between  $W_0^{1,q}$  and  $W^{-1,q}$  embed continuously into Hölder spaces. Thus, if  $G$  is a strictly positive  $C^1$  function, then the coefficient functions  $G(w)\mu$  are of the same quality as  $\mu$  (in the spirit of Theorem 1.1). Hence, the domains of the operators  $\nabla \cdot G(w)\mu\nabla$  do not depend on  $u$  if this runs through a suitable interpolation space (see [51]) - what often is required in quasilinear parabolic theory. We will accomplish these things elsewhere in detail.

## 7 Appendix

In the appendix we give the announced proof of Lemma 3.4: let  $\mathbf{x}_0$  be any point from  $\partial\Lambda$ . Then for every  $\epsilon > 0$  there is an orthonormal basis  $e_1, \dots, e_d$  of  $\mathbb{R}^d$  such that  $\partial\Lambda$  can be parametrized in a neighbourhood of  $\mathbf{x}_0$  via a Lipschitz function  $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  by

$$\mathbf{x} \cdot e_d = \varphi(\mathbf{x} \cdot e_1, \dots, \mathbf{x} \cdot e_{d-1})$$

and the Lipschitz constant  $\text{lip}(\varphi)$  of  $\varphi$  does not exceed  $\gamma + \epsilon$ . If  $K : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a linear bijection, then  $\partial(K\Lambda)$  may be parametrized in a neighbourhood of  $K\mathbf{x}_0$  by

$$K\mathbf{x} \cdot (K^{-1})^*e_d = \varphi(K\mathbf{x} \cdot (K^{-1})^*e_1, \dots, K\mathbf{x} \cdot (K^{-1})^*e_{d-1}). \quad (7.1)$$

Clearly,  $\{(K^{-1})^*e_1, (K^{-1})^*e_2, \dots, (K^{-1})^*e_d\}$  is not necessarily an orthonormal system. In the sequel we will modify the representation (7.1) in such a way that the required orthogonality of the representing coordinates is re-established. Let  $\{f_1, \dots, f_{d-1}\}$  be any orthonormal basis in the subspace

which is generated by  $\{(K^{-1})^*e_1, \dots, (K^{-1})^*e_{d-1}\}$ . Then, if  $k \in \{1, \dots, d-1\}$ , any  $(K^{-1})^*e_k$  may be written as

$$(K^{-1})^*e_k = \sum_{j=1}^{d-1} \alpha_{kj} f_j. \quad (7.2)$$

In this notation, (7.1) reads as

$$K\mathbf{x} \cdot (K^{-1})^*e_d = \varphi\left(\sum_{j=1}^{d-1} \alpha_{1j} K\mathbf{x} \cdot f_j, \dots, \sum_{j=1}^{d-1} \alpha_{dj} K\mathbf{x} \cdot f_j\right). \quad (7.3)$$

Let  $f_d$  be a unit vector, orthogonal to  $\{f_1, \dots, f_{d-1}\}$ . Then, according to  $(K^{-1})^*e_d = \sum_{j=1}^d (K^{-1})^*e_d \cdot f_j f_j$ , (7.3) can be expressed as

$$\begin{aligned} K\mathbf{x} \cdot f_d &= \frac{1}{(K^{-1})^*e_d \cdot f_d} \left( \varphi\left(\sum_{j=1}^{d-1} \alpha_{1j} K\mathbf{x} \cdot f_j, \dots, \sum_{j=1}^{d-1} \alpha_{d-1j} K\mathbf{x} \cdot f_j\right) - \right. \\ &\quad \left. - \sum_{j=1}^{d-1} K\mathbf{x} \cdot f_j (K^{-1})^*e_d \cdot f_j \right). \end{aligned} \quad (7.4)$$

We denote the mapping which assigns to the vector  $(K\mathbf{x} \cdot f_1, \dots, K\mathbf{x} \cdot f_{d-1})$  the right hand side of (7.4) by  $\psi$ .

Finally we have to estimate the Lipschitz constant  $lip(\psi)$  of  $\psi$ : obviously, one has

$$lip(\psi) \leq \frac{1}{|f_d \cdot (K^{-1})^*e_d|} \left( lip(\varphi) \|\alpha_{kj}\|_{B(\mathbb{R}^{d-1})} + \sqrt{\sum_{j=1}^{d-1} ((K^{-1})^*e_d \cdot f_j)^2} \right). \quad (7.5)$$

Next we will derive a bound for  $\frac{1}{|(K^{-1})^*e_d \cdot f_d|} = \frac{1}{|e_d \cdot K^{-1}f_d|}$ : one has

$$K^{-1}f_d \cdot e_j = f_d \cdot (K^{-1})^*e_j = 0 \quad \text{for } j = 1, \dots, d-1.$$

Hence,  $K^{-1}f_d = \lambda e_d$ , or, equivalently,  $Ke_d = \frac{1}{\lambda} f_d$ . This implies

$$\frac{1}{|\lambda|} = \left\| \frac{1}{\lambda} f_d \right\| = \|Ke_d\| \leq \|K\|.$$

Altogether, we obtain

$$\frac{1}{|(K^{-1})^* e_d \cdot f_d|} = \frac{1}{|e_d \cdot K^{-1} f_d|} = \frac{1}{|\lambda|} \leq \|K\|. \quad (7.6)$$

By definition,  $\{\alpha_{kj}\}_{kj}$  is the matrix representation of  $(K^{-1})^*|_{\text{span}\{e_1, \dots, e_{d-1}\}}$  with respect to two orthonormal bases  $\{e_1, \dots, e_{d-1}\}$  and  $\{f_1, \dots, f_{d-1}\}$ . Consequently, one has

$$\|\alpha_{kj}\|_{\mathcal{B}(\mathbb{R}^{d-1})} = \|(K^{-1})^*|_{\text{span}\{e_1, \dots, e_{d-1}\}}\| \leq \|(K^{-1})^*\| = \|K^{-1}\|. \quad (7.7)$$

Lastly, one estimates

$$\sqrt{\sum_{j=1}^{d-1} ((K^{-1})^* e_d \cdot f_j)^2} \leq \sqrt{\sum_{j=1}^d (e_d \cdot K^{-1} f_j)^2},$$

and recognizes that the expression of the right hand side equals  $\|PK^{-1}\| \leq \|K^{-1}\|$ , where  $P$  is the orthoprojector  $\mathbf{x} \rightarrow \mathbf{x} \cdot e_d e_d$ . This, together with (7.5), (7.6) and (7.7) gives

$$\text{lip}(\psi) \leq \|K\| \|K^{-1}\| (\text{lip}(\varphi) + 1) \leq \|K\| \|K^{-1}\| (\gamma + \epsilon + 1).$$

## References

- [1] Adams, L. Li, Z. *The immersed interface/multigrid methods for interface problems*, SIAM J. Sci. Comput. **24** No.2 (2002) 463-479
- [2] Agranovich, M.S. *Spectral properties of potential type operators for a class of strongly elliptic systems on smooth and Lipschitz surfaces*, (Russian, English) Trans. Mosc. Math. Soc. 2001 (2001), 1-47; translation from Tr. Mosk. Mat. O.-va 62, (2001) 3-53
- [3] Agranovich, M.S. *Spectral problems for second-order strongly elliptic systems in domains with smooth and nonsmooth boundaries*, Russ. Math. Surv. **57**, No.5, (2002) 847-920; translation from Usp. Mat. Nauk **57**, No.5, (2002) 3-79

- [4] Amann, H. *Linear and quasilinear parabolic problems, Volume I: Abstract linear theory*, Birkhäuser (1995)
- [5] Amann, H. *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in: Function spaces, differential operators and nonlinear analysis, H.-J. Schmeisser (ed.) et al., 9–126, Teubner-Texte Math., Vol. 133, Stuttgart, 1993
- [6] Arendt, W. & terElst, A.F.M. *Gaussian estimates for second order elliptic operators with boundary conditions*, J. Oper. Th. **38** (1997) 87-130
- [7] Auscher, P. & Tchamitchian, P. *Square root problem for divergence operators and related topics*, Asterisque **249**, Paris, (1998)
- [8] Auscher, P. & Tchamitchian, P. *Square roots of elliptic second order divergence operators on strongly Lipschitz domains:  $L^p$  theory*, Math. Ann. **320** No. 3, (2001) 577-623
- [9] Bandelow, U., Hünlich, R. & Koprucki, T. *Simulation of static and dynamic properties of edge-emitting multiple-quantum-well lasers*, IEEE JSTQE **9** (2003), pp. 798-806
- [10] BenDaniel, D. & Duke, C. *Space-charge effects on electron tunneling*, Phys. Rev. **152** No. 2 (1966) 683-692
- [11] Berestycki, H., Hamel, F. & Roques, L. *Analysis of the periodically fragmented environment model: I- species persistence*, J. Math. Biol. **51**, No.1, (2005) 75-113
- [12] Caffarelli, L.A. & Peral, I. *On  $W^{1,p}$  estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51**, No.1, (1998) 1-21
- [13] Carslaw, H. & Jaeger, J. *Conduction of heat in solids*, University Press Oxford London (1959)
- [14] Chen, Z. & Zou, J. *Finite element methods and their convergence for elliptic and parabolic interface problems*, Numer. Math. **79**, No.2, (1998) 175-202

- 
- [15] Chipot, M., Kinderlehrer, D. & Caffarelli, G.V. *Smoothness of linear laminates*, Arch. Ration. Mech. Anal. 96, (1986) 81-96
- [16] Clement, P. & Li, S. *Abstract parabolic quasilinear equations and application to a groundwater flow problem*, Adv. Math. Sci. Appl. 3, (1994) 17-32
- [17] Costabel, M., Dauge, M. & Lafranche, Y. *Fast semi-analytic computation of elastic edge singularities*, Comput. Meth. Appl. Mech. Engrg. **190** (2001), 2111–2134
- [18] Costabel, M. & Dauge, M. *General edge asymptotics of solutions of second-order elliptic boundary value problems I*, Proc. R. Soc. Edinb., Sect. A **123**, No.1, (1993) 109-155
- [19] Coulhon, T. & Duong, X. *Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss*, Adv. Differential Equations **5** (2000), No. 1-3, 343–368
- [20] Dauge, M. *Neumann and mixed problems on curvilinear polyhedra*, Integral Equations Oper. Theory **15**, No.2, 227-261 (1992)
- [21] Duduchava, R., Sändig, A.M. & Wendland, W. *Interface cracks in anisotropic composites*, Math. Methods Appl. Sci. **22**, No.16, (1999) 1413-1446
- [22] Costabel, M., Dauge, M. & Duduchava, R. *Asymptotics without logarithmic terms for crack problems*, Commun. Partial Differ. Equations **28**, No.5-6, (2003) 869-926
- [23] Elschner, J., Kaiser, H.-C. & Rehberg, J., Schmidt, G.  *$W^{1,q}$  regularity results for elliptic transmission problems on heterogeneous polyhedra*, submitted
- [24] Escauriaza, L., Fabes, E.B. & Verchota, G. *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proc. AMS **115** No. 4 (1992) 1069-1076

- [25] Franzone, P.C., Guerri, L. & Rovida, S. *Wavefront propagation in an activation model of the anisotropic cardiac tissue: asymptotic analysis and numerical simulation*, J. Math. Biol. **28** (1990) 121-176
- [26] Gajewski, H. & Gröger, K. *Semiconductor equations for variable mobilities based on Boltzmann statistics or Fermi-Dirac statistics*, Math. Nachr. **140** (1987), pp. 7-36
- [27] Gajewski, H. & Gröger, K. *Reaction-diffusion processes of electrical charged species*, Math. Nachr. **177** (1996), pp. 109-130
- [28] Grisvard, P. *Elliptic problems in nonsmooth domains*, Pitman, Boston, 1985.
- [29] Gröger, K. *A  $W^{1,p}$ -estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann. **283** (1989), 679–687.
- [30] Grubb, G. *Pseudo-differential boundary problems in  $L^p$  spaces*, Comm. Part. Diff. Eq. **15** (1990) 289-340
- [31] Henry, D. *Geometric theory of semilinear parabolic equations*, Lect. Notes Math. **840** Springer (1981)
- [32] Hieber, M. & Prüss, J. *Heat kernels and maximal  $L^p - L^q$  estimates for parabolic evolution equations*, Comm. Part. Diff. Eq. **22** No. 9/10 (1997) 1647-1669
- [33] Hwang, T., Lin, W., Liu, J. & Wang, W. *Jacobi-Davidson methods for cubic eigenvalue problems*, Numer. Linear. Algebra Appl. **12** (2005) 605-624
- [34] Jentsch, L., Natroshvili, D. & Wendland, W. *General transmission problems in the theory of elastic oscillations of anisotropic bodies (Mixed interface problems)*, J. Math. Anal. Appl. **235**, No.2, (1999) 418-434
- [35] Jerison, D. & Kenig, C. *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130**, No.1, (1995) 161-219

- [36] Kaiser, H.-C., Neidhardt, H. & Rehberg, J. *Classical solutions of quasi-linear parabolic systems on two dimensional domains*, to appear in NoDEA.
- [37] Kapon, E. *Semiconductor Lasers I*, Academic Press, Boston, 1999
- [38] Kato, T. *Perturbation theory for linear operators*, Grundlehren der mathematischen Wissenschaften, vol. **132**, Springer Verlag, Berlin, 1984.
- [39] Kenig, C. *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, Amer. Math. Soc., Providence, 1994
- [40] Knees, D. *On the regularity of weak solutions of quasi-linear elliptic transmission problems on polyhedral domains*, Z. Anal. Anwend. **23** No.3 (2004) 509-546
- [41] Ladyshenskaja, O.A., Rivkind, V.Y. & Uralzeva, N.N. *The classical solvability of diffraction problems*, (Russian) Tr. Mat. Inst. Steklova **92**, (1966) 116-146
- [42] Leguillon, D. & Sanchez-Palenzia, E. *Computation of Singular Solutions in Elliptic Problems and Elasticity*, Wiley, Chichester, 1987
- [43] Li, Y.Y. & Vogelius, M. *Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients*, Arch. Rat. Mech. Anal. **153** (2000) 91-151
- [44] Li, Y., Liu, j., Voskoboynikov, O., Lee, C. & Sze, S. *Electron energy level calculations for cylindrical narrow gap semiconductor quantum dot*, Comp. Phys. Comm. **140** (2001) 399-404
- [45] Lunardi, A. *Abstract quasilinear parabolic equations*, Math. Ann. **267** (1984) 395-415
- [46] Maz'ya, V., Elschner, J., Rehberg, J. & Schmidt, G. *Solutions for quasi-linear evolution systems in  $L^p$* , Arch. Rat. Mech. Anal. **171** (2004) 219-262

- [47] Meyers, N.G. *An  $L^p$ -estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa **17**, (1963), 189-206
- [48] Olejnik, O.A. *Boundary value problems for linear equations of elliptic and parabolic type with discontinuous coefficients*, Izv. Akad. Nauk SSSR, Ser. Mat. **25** (1961), 3-20
- [49] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, Springer, (1983)
- [50] Prüss, J. *Maximal regularity for evolution equations in  $L^p$ -spaces*, Conferenze del Seminario di Matematica dell'Università di Bari, **285** 1-39
- [51] Rehberg, J. *Quasilinear parabolic equations in  $L^p$* , in: Progress in Non-linear Differential Equations and Their Applications, Vol **64**, M. Chipot, J. Escher, (Eds.) Birkhäuser, (2005) 413-419
- [52] Roitberg, J.A. & Sheftel, Z.G. *On equations of elliptic type with discontinuous coefficients*, Sov. Math. Dokl. **3** (1962), 1491–1494
- [53] Savare, G. *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal. **152** (1998) 176-201
- [54] Schaefer, C. *Einführung in die Theoretische Physik II: Theorie der Wärme, Molekular-kinetische Theorie*, de Gruyter, Berlin, 1958
- [55] Schechter, M. *A generalization of the problem of transmission*, Ann. Scuola Norm. Sup. Pisa **14** (1960), 207–236
- [56] Sommerfeld, A. *Electrodynamics*, Lectures on theoretical physics, Vol. III, Academic Press, New York, 1952
- [57] Sommerfeld, A. *Thermodynamics and Statistical Mechanics*, Lectures on theoretical physics, Vol. V, Academic Press, New York, 1956.
- [58] Simader, C.G. *On Dirichlet's boundary value problem*, Lect. Notes Math. Vol. 268, Berlin, Heidelberg, New York, 1972



- 
- [59] Triebel, H. *Interpolation theory, function spaces, differential operators*, North Holland Publishing Company, 1978.
- [60] Wang, W., Hwang, T., Lin, W. & Liu, J. *Numerical methods for semiconductor heterostructures with band nonparabolicity*, J. Comp. Phys. **190** (2003) 141-158
- [61] Wang, W. *A jump condition capturing finite difference scheme for elliptic interface problems*, SIAM J. Sci. Comp. **25** No.5 (2004) 1479-1496
- [62] Weisbuch, C. & Vinter, B. *Quantum Semiconductor Structures: Fundamentals and Applications*, Academic Press, Boston, 1991
- [63] Wloka, J. *Partial differential equations*, Cambridge University Press, Cambridge
- [64] Zanger, D. *The inhomogeneous Neumann problem in Lipschitz domains*, Commun. Partial Differ. Equations **25**, No.9-10, (2000) 1771-1808.
- [65] Ziemer, W.P. *Weakly Differentiable Functions*, Springer, Berlin, 1989