Equilibrium trading of climate and weather risk and numerical simulation in a Markovian framework *

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Abstract

We consider financial markets with agents exposed to external sources of risk caused for example by short term climate events such as the South Pacific sea surface temperature anomalies widely known under the name El Nino. Since such risks cannot be hedged through investments on the capital market alone, we face a typical example of an incomplete financial market. In order to make this risk tradable, we use a financial market model in which an additional insurance asset provides another possibility of investment besides the usual capital market. Given one of many possible market prices of risk each agent can maximize his individual exponential utility from his income obtained from trading in the capital market, the additional security, and his risk exposure function. Under the equilibrium market clearing condition for the insurance security the market price of risk is uniquely determined by a backward stochastic differential equation. We translate these stochastic equations via the Feynman-Kac formalism into semi-linear parabolic partial differential equations. Numerical schemes are available by which these semilinear pde can be simulated. We choose two simple qualitatively interesting models to describe sea surface temperature, and with an ENSO risk exposed fisher and farmer and a climate risk neutral bank three model agents with simple risk exposure functions. By simulating the expected appreciation price of risk trading, the optimal utility of the agents as a function of temperature, and their optimal investment into the risk trading security we obtain first insight into the dynamics of such a market in simple situations.

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Introduction

In this paper, we are concerned with the control and dynamical hedging of risks exterior to usual financial markets, residing in climate and weather influences onto parts of economies. Let us first give some details about one of the typical examples of risk sources we think of.

The most well known randomly periodic climate event is known under the name El Nino or more precisely El Nino Southern Oscillation (ENSO). It has been known to Peruvian fishermen before the arrival of the Spaniards through its spectacular economical effects. The normal large scale of atmospheric pressure distribution over the Southern Pacific shows a zone of high pressure over the eastern part near the South American coast, while a zone of lower pressure prevails over the western part of the ocean. This pressure difference on sea level is expressed in the so-called *Southern Oscillation Index (SOI)* which is usually positive. Positive SOI forces trade winds to blow east to west. At randomly periodic times - every 3-8 years (the *El Nino years*) - however, the SOI becomes negative forcing the trade winds to relax or even blow in the reverse direction. Ocean currents are largely influenced by trade winds at sea level. In particular, the Humboldt Current along the South American coast may be affected. It normally transports cold water northward. During an El Nino event, the relaxation of trade winds allows warm water to appear on the surface of the Southern Pacific near the South American coast.

The effects of this change of the sea surface temperature on marine life are tremendous. The trade wind shift disrupts the upwelling of oxygen and nutrient rich cold water, one of the basic conditions for dense concentrations of marine life. Let us give some numbers first for the local effect on the Peruvian fishing industry. As in most developing countries in the tropics with economies depending largely on few branches for example in food production, the sensitivity to climatic fluctuations is very high. According to a study of the World Resources Institute (1994), El Nino contributed to the collapse of the Peruvian fishing industry. From the early 1950s through 1971 the harvest increased, peaking at more than 12 million tons per year. With the arrival of the 1972/1973 El Nino, a disastrous drop of the harvest to 2.5 million tons was recorded.

The consequences of this distortion of ocean currents due to changes in the SOI are much more global than one may conjecture at first glance. The change of the tropical Pacific sea surface temperatures induced by the fluctuation of trade winds affects the atmosphere in turn directly by causing convection. Dense tropical rain clouds are created which, besides increasing the amount of precipitation the western hemisphere receives, distort the atmospheric air flow in altitudes of 5-10 km above sea level. This

effect may be compared to rocks which disturb the flow of a water stream. However, it has to be imagined on a much larger horizontal length scale of several thousands of miles. The waves in the air flow at these altitudes determine the belts of jet streams along which systems of high and low pressure travel, and thus also the positions of storms and monsoons on a global scale. One *global* pattern becomes clearly visible: in El Nino periods, rain areas usually centered above Indonesia and the far western Pacific move eastward into the central Pacific, which affects waves in the tropospheric air flow causing unusual weather over many regions of the globe.

We shall underline the *global impacts* of El Nino by some examples. Changes in the health and population sizes of northern fur seals and California sea lions appear to be related to changes in marine mammal prey availability caused by missing upwelling of nutrient rich water. According to a study conducted by the Oregon State University 1996, a similar effect is observed with the population of coho salmon during the El Nino of 1982/1983 believed to have been the worst of the 20th century. Fishery experts had predicted, based on numbers in the previous years, that about 1.6 million wild coho salmon would return to spawn in Pacific North West streams in 1982. In fact, only about 42 percent, an estimated 667.000 showed up. Statistical correlations (*teleconnections*) between El Nino and atypical weather events globally have been found in droughts in Central America, Southern India, Indonesia, the Philippines, Africa, and Australia, culminating in large scale brush and bush fires for example in Australia and Kalimantan (Borneo). They also are found in floods in the US, Cuba, Peru, other states in South America, and even Western Europe. As consequences of droughts or floods, ecosystem dynamics may be disrupted. For example, the natural balance between predators like owls and snakes, and rodent prey animals in the US and in Southern Africa has been found to be destroyed following El Nino years. It is clear that these teleconnections have extensive social and economical impacts, eventually leaving whole national economies disturbed.

The model in which we face and treat risks of this type was discussed in [16]. We consider an economy with a finite number of agents. They may for example represent individual farmers or fishers, bigger farming cooperatives, or companies like insurance or even reinsurance companies. Their common feature is an eventually big exposure to the risks caused by extreme weather or climate events. We emphasize here the global or at least transnational composition of this market, which is due to the global features of the risk causing climate event, as explained above. One component of the economy is a stock market on which all participants of the economy are able to trade. In fact they are considered as small traders unable to influence the stock prices - a hypothesis made for simplicity, which needs some further development in view of the role big agents like re-insurers may play. The risks caused by the external factors cannot be hedged by the stocks. For this reason we of course face a typical incomplete market.

As opposed to other approaches, in view of the global interests meeting in facing the risk source, our goal consists in *completing the market*. This is done by constructing a special security which is physically added to the market. Through this security climate risk becomes tradable. Agents participating in the market may buy or sell individual amounts of risk trading money according to their random exposures to climate risk.

If a particular market price of risk is given, every agent is able to price his share of risk to be traded. He will then choose an investment strategy which optimizes the individual utility from his total income composed of his investment both into the usual capital market and into the risk trading security, and of his random risky income subject to climate hazards. There will be a unique market price of risk for which a market equilibrium is achieved, i.e. for which there is equality between the total offer and demand of risk trading money on the market. This pricing rule is determined by the intervention of one of the main tools of stochastic control theory in incomplete markets, backward stochastic differential equations. Utility maximization techniques for complete markets using martingale methods are independently treated for example in [18], [8] and [27]. The construction of a unique equilibrium in a Brownian filtration is given in [19], where the expected utility of consumption in a trading interval is maximized .

So the introduction of the special risk trading security leads to a process of relocating and shuffling risky claims between the different market participants to reach an equilibrium, which will be particularly efficient in the presence of groups of agents with complementary interests. Let us illustrate this effect by giving some examples. As explained above, American fishing industries from Peru to Canada are strongly affected during El Nino years by seriously dropping catch numbers. Quite opposite changes are observed on the other "pole" of the Southern Oscillation. According to Gaol and Manurung [12] and Mizuno [25], catch numbers for big eye tuna in the South Java sea waters, one of the most important tuna fishing regions of the world, during El Nino periods increase by about 30 percent, due to an opposite effect on the sea surface temperatures in the Western South Pacific. Another pair of groups of economic agents with complementary interests is given by farmers and fishers even in the same national economy. Warm El Nino years are unfavorable for fishers for the reasons given, but may be favorable to farmers in parts of the country normally dry due to increased amounts of precipitation. Cold years usually following in the heels of El Nino years are welcomed by fishermen, but not by farmers, because of droughts and crop failures. For example, rice and cotton, two of the primary crops grown in Northern Peru, are highly sensitive to the quantities and timing of rainfall. Hence there are various scenarios of natural risk sharing possibilities for fishermen and farmers, eventually in combination with predictions of the event. Finally, also farmers or fishers exposed to climate risk as explained on the one hand, and banks with no such exposure, but the desire to diversify their portfolios on the other hand might be considered as pairs of agents with complementary interests.

The model

We next recall the model of [16] in more formal details. The stock market is represented by a stock price process X_1 indexed by the trading interval $[0, T]$. The external (climate) risk component is represented by a stochastic process K, indexed by the trading interval as well. Both processes live on a Wiener space and are adapted to a Brownian filtration. Agents on the market are symbolized by the elements a of a finite set I. The number of agents is small compared to the number of traders at the stock market. Every agent

$a \in I$ is supposed to be endowed with an initial capital $v_0^a \geq 0$. At the end T of the trading interval he receives a stochastic income H^a which describes the profits that he or the company he represents obtains from his usual business. The income H^a is supposed to be a bounded function of the processes X_1 and K, i.e.

$$H^a = g^a(X_1, K).$$

Should one of the companies represented by agents be traded at the stock exchange, then its stock price is supposed to be only a small fraction of the index X_1. The sum of the random incomes $\sum_{a \in I} H^a$ is assumed to be small compared to the value of the securities traded at the stock market.

One of the main aims of this paper is to model and simulate different scenarios for climate processes qualitatively correct. The main example will be given if we interpret K as the sea surface temperature process in the South Pacific. As explained above, this process exhibits randomly periodic fluctuations with periods between 3-8 years. There is a number of simple mathematical models available to describe such a behavior. The most common used for the purpose of predictions of the event is based on an Ornstein-Uhlenbeck process in dimension 15, where the dimensionality comes from statistical data fitting (see Penland [26]). Periodicity in this simple model is generated by a non-trivial rotational part in the matrix determining the drift in the stochastic differential equation giving the Ornstein-Uhlenbeck process. Another 2-dimensional conceptual model with intrinsic random periodicity is obtained from a deterministic nonlinear equation coupling thermocline depth and sea surface temperature perturbed by random noise representing trade wind coupling at sea level (see Fang, Barcilon, Wang [1]). Another way to obtain the random periodicity in a simple conceptual model is given by Battisti [4]. Here the delay coupling to the state the randomly perturbed system experienced before it sent Kelvin waves from the South American Pacific coast across the ocean, which were reflected at the Japanese coast and travelled back to their origin, is responsible for an intrinsic periodicity. In our simulations below we use two simple models to describe this climate phenomenon. In the simplest one, K is a one-dimensional Ornstein-Uhlenbeck process. The second model, a more realistic one, is given by a conceptual bi-stable diffusion model driven by a Brownian motion with a time-periodic potential function which has two minima the depths of which fluctuate periodically. The noise is given intensities at which the solution trajectories show some random periodicity which can be measured by means of quality of periodic tuning notions in the theory of stochastic resonance (see [17], [15], [14]).

The typical toy agents we have in mind in our simulations will be just a pair composed on the one side of a fisher or a (rice) farmer (f or r) subject to the hazard of ENSO, for example, and whose random income $H^f(H^r)$ depends uniquely on the climate process K. The exposures of fishermen or farmers f resp. r will be given by some cumulative functional of the form

$$H = \int_0^T \phi(K_s)ds \quad \text{or} \quad H = \int_0^T \phi(s, K_s, X_1(s))ds,$$

where ϕ is an individual bounded revenue function taking its maximum for example at some low temperature k_1 close to the normal sea surface temperature in the case

of the fisher, or at some higher temperature k_2 in the case of the rice farmer. The functions may in turn be relatively small at the corresponding opposite values k_2 resp. k_1. On the other side, there is a climate risk neutral agent such as a bank (b) whose income H^b is a function of the stock market evolution alone. Trading climate risk for these agents can be viewed in the following way. f wants to hedge fluctuations caused by the external factor and signs a contract with b to transfer part of this risk. b's interest in the contract could be based on the wish to diversify its portfolio. The main example of the global ENSO risk provides a number of further relevant risk functionals for different, eventually complementary groups of agents treated in the mathematical parts, but not in the simulations below. For example, the exposure to ENSO for a big agent such as an insurance or a re-insurance i company will be a functional of the type

$$H^i = g(\tau, K_\tau), \quad \text{or} \quad H^i = g(\tau, K_\tau, X_1(\tau)),$$

if τ is the time ENSO strikes, which is realized by some entrance time for the process K.

As indicated above, we realize mathematically the idea of market completion by the design of a second security with price process X_2, which, besides X_1, can be uniquely traded by the agents in I. Suppose the market has been completed in this way so that the climate risk can be traded. The insurance asset is modelled by writing down a reasonable candidate for its price process X_2 in terms of a simple SDE. It is parametrized by a process θ_2 describing the market price of external risk. Given θ, each individual agent a maximizes his expected exponential utility from terminal wealth composed of the random risky income H^a and the terminal value of his portfolio in (X_1, X_2) obtained with his individual trading strategy.

We next compute the density of the unique probability measure equivalent to the underlying historical measure P such that the process (X_1, X_2) is a martingale. It will be given by a martingale measure Q^θ indexed by the market price of risk θ. The utility maximizing terminal wealth and the market clearing condition can be expressed in terms of this density and mathematically lead to a constraint for the process parameter θ. Therefore our next task will be to determine θ in such a way that the partial market clearing condition is satisfied. Mathematically, this leads to a Backward Stochastic Differential Equation (BSDE) the solution of which yields a unique θ^* and therefore a unique probability measure Q^{θ^*} with associated second security price process X_2^* such that (X_1^*, X_2^*) is a $Q^{\theta^*}-$martingale. The second security selected in this way satisfies the partial equilibrium condition (see [16]). The choice of income functionals for model traders in our market used in particular for the simulations will imply that the mathematical treatment is possible in the framework of forward-backward stochastic differential equations of Markovian character. Via the generalized Feynman-Kac formalism, this implies that we are able to associate to each individual problem a system of linear or semi-linear parabolic partial differential equations. The solution process of our problem of enlarging the market in equilibrium by an asset making risk tradable in a general setting will be given in terms of viscosity solutions, much as in Chaumont [6]. In the concrete situations we consider, the solutions of the associated pde will be classical. We will use numerical schemes for non-linear pde developed in [6] to approximate and simulate these classical solutions for the Ornstein-Uhlenbeck or bistable diffusion climate processes, and the risk functionals for fishermen, farmers

and bank sketched above. Notably, we shall be able to simulate the expected price of the risk trading asset X_2 which indicates the cumulative appreciation of trading the external risk by the affected agents, the temporal evolution of the optimal utility for the agents in dependence on the level of the temperature process, and the shape of the optimal investments of the agents into the additional security. This way we obtain first information on the dynamics of such a market which will, if not quantitatively, be of interest at least for qualitative issues.

The paper is organized as follows. In section 1 we give a more formal and detailed account of our market model, including in particular the formal links to the theory of semi-linear parabolic pde via generalized Feynman-Kac formulas, as well as proofs for existence, uniqueness and regularity for the pde expressing the Markovian optimal control problem derived from our utility maximization problems on completed markets under the equilibrium constraint. We shall also explain the concrete elementary models for temperature processes and risk functionals used in the simulations. Section 2 is devoted to exhibiting and explaining the numerical approximation schemes and convergence results. In section 3 we present our simulation results for the optimal allocation of risk given the particular temperature processes and risk exposure functionals for fisher, farmer and bank, and interpret the findings intuitively.

1 Model and concrete examples

In this section we describe formally the equations governing our model. The key to our simulation results presented in section 2 is a crucial link between stochastic forward and backward differential equations on the one hand and nonlinear PDE, possibly with solutions in the viscosity sense, on the other hand. It is provided by a nonlinear extension of the Feynman-Kac formula and will be explained in subsection 1.1. All relevant PDEs for our analysis will be derived from this formula. In subsection 1.2, we shall make precise the market and climate processes we use, in subsection 1.3. this will be done for the agents on the market. A particular emphasis is on their random incomes, in which their exposure to climate is reflected. We further specify utility functions according to which the agents maximize their utilities from terminal wealth which is obtained through investment in the financial market and random climate affected income. Utility maximization for the agents on the stochastic side may be achieved by a duality approach resting upon Legendre transforms of the utility functions which provides explicit formulas for maximal utility in terms of martingale measure densities of the asset price process. From the analytical point of view, with the help of a nonlinear Feynman-Kac formula, it leads to a linear backward PDE in terms of the infinitesimal generator of the diffusion described by the asset price and the climate process. The analytical description of the optimal investment policies of the different agents into the asset price process of the market X_1 and the insurance asset X_2 leads to an optimal control problem in terms of a nonlinear Hamilton-Jacobi-Bellman equation. The effect of the equilibrium (market clearing) condition determining X_2 on the analytical description of our utility maximization problem will be exhibited in subsection 1.3. In stochastic terms, utility maximization under equilibrium leads to a

nonlinear backwards stochastic differential equation. Its access in analytical terms is again guaranteed by the nonlinear Feynman-Kac formula which this time produces a nonlinear backwards PDE with solutions in the viscosity sense. In subsection 1.5 we give some formulas, based again on a linear PDE, by which we can compute moments of the insurance asset X_2 in particular situations. In the final subsection 1.6 we discuss concrete examples of risk exposure functionals which depict some of the situations alluded to in the introduction. Existence and uniqueness questions for the different PDEs governing the analytical description in this section will be discussed in section 2.

1.1 The link to PDE: Feynman-Kac formulas

In this section we recall the basic link between forward and backward stochastic differential equations and partial differential equations, which will be one of the crucial tools in our paper.

Let $n \geq 1$ and $d \geq 1$. Let $\mathcal{O}$ be an open subset of $]0,T[\times\mathbb{R}^n$. Let $t \in [0,T]$ be an arbitrary time, representing time of initial action. For a $d-$dimensional Brownian motion W, and $x \in \mathbb{R}^n$, we construct the process $X^{t,x}$ as the solution of the following SDE

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \quad t \leq s \leq \tau \\ X_t^{t,x} = x \in \{y \,|(t,y) \in \mathcal{O}\}, \end{cases} \tag{1}$$

and we define $\tau = \inf\{s \in]t,T] \,|(s, X_s^{t,x}) \notin \mathcal{O}\}$ the first exit time of $X^{t,x}$ from the domain $\mathcal{O}$.

In this section, we will denote by L the infinitesimal generator associated with $X^{t,x}$, i.e. for a regular (C^2) function ϕ,

$$L\phi(s,x) = b(s,x)D\phi(s,x) + \frac{1}{2}\text{trace}\left[\sigma\sigma^*(s,x)D^2\phi(s,x)\right], \quad (s,x) \in \mathcal{O},$$

where $D\phi$ stands for the gradient and $D^2\phi$ the Hessian matrix of ϕ.

In all the following, we will suppose that the drift b and the diffusion matrix σ are C^∞ functions of the state variable with linear growth at infinity, and that L is uniformly elliptic.

We first recall the well-known classical Feynman-Kac formula for linear problems, which can be found in [20].

Theorem 1.1 *Suppose that the coefficients f and h are Lipschitz functions of the state variable with linear growth at infinity. Assume further that h is bounded, g is continuous with polynomial growth in the state variable. We define the function v on $\overline{\mathcal{O}}$ by*

$$v(t,x) = \mathbb{E}_{t,x}\left[\int_t^\tau f(s, X_s^{t,x})e^{-\int_s^\tau h(r,X_r^{t,x})dr}ds + g(\tau, X_\tau)e^{-\int_t^\tau h(s,X_s)ds}\right]. \tag{2}$$

Then v is a classical solution of the following backward linear system

$$\begin{cases} -\dfrac{\partial v}{\partial s} - Lv + f - hv = 0 \text{ in } \mathcal{O}, \\ u(s,x) = g(s,x) \text{ on } \partial\mathcal{O}. \end{cases} \tag{3}$$

There is a similar formula for forward PDEs with an initial condition instead of a terminal one.

Proof:
The simplest way to prove that the function v solves (3) is to prove that there exists a classical solution to the PDE (3). Once this is guaranteed, we just have to apply Itô's formula in a well known manner to read off the PDE. So it is enough to quote a classical existence and uniqueness result. It is valid under certain hypotheses for the coefficients which will be seen to be satisfied in all the applications we have in mind below. To complete the proof we therefore recall a result which can be found for example in [11] or [13].□

Theorem 1.2 *Under the assumptions of Theorem 1.1, the system (3) has a unique classical solution.*

We now recall the nonlinear Feynman-Kac formula for BSDEs.

Theorem 1.3 *Suppose that $\sigma\sigma^*$ is uniformly elliptic, and $\mathcal{O} =]0,T[\times\mathbb{R}^n$. In addition to the family $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^n}$ given by (1) consider two additional processes Y and Z defined by the following BSDE*

$$\begin{cases} -dY_s^{t,x} = F(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds - Z_s^{t,x}dW_s, \quad t \le s \le \tau \\ Y_\tau^{t,x} = g(\tau, X_\tau^{t,x}). \end{cases} \tag{4}$$

Assume that $F : [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is C^∞ and $g \in C([0,T]\times C^1(\mathbb{R}^n))$. Then, for every $t \le s \le \tau$, we have

$$\begin{cases} Y_s^{t,x} = u(s, X_s^{t,x}) \\ Z_s^{t,x} = \sigma^* Du(s, X_s^{t,x}), \end{cases}$$

where u is the unique classical solution of the PDE

$$\begin{cases} -\dfrac{\partial u}{\partial s} - Lu - F(s,x,u,\sigma^*(t,x)Du) = 0 \text{ in } \mathcal{O}, \\ u(s,x) = g(s,x) \text{ on } \partial\mathcal{O}. \end{cases} \tag{5}$$

Proof:
Again, we shall invoke a classical existence, regularity and uniqueness result, in order to prove that the generally existing solution in the viscosity sense of (5) is in fact a unique regular classical solution. Once this is guaranteed, the proof of the existence may be completed by an appeal to Itô's formula (see [21], p. 581) in a well known manner. □

The theorem alluded to above which guarantees the existence, uniqueness and regularity of classical solutions for (5) is taken from Taylor [28].

Theorem 1.4 *Under the assumptions of theorem 1.3, system (5) has a unique classical solution* $w \in C\left([0,T], C^1\left(\mathbb{R}^n\right)\right) \cap C^\infty\left([0,T[\times\mathbb{R}^n\right)$.

Proof:
The proof is given in Taylor [28], in Proposition 15.1.1 on p.273. Note first that we may and do assume that the infinitesimal generator L is in divergence form, and thus self adjoint as a linear operator. This can be achieved by shuffling the drift part as well as an additional drift containing $D\sigma\sigma^*$ to the function F in Taylor's Proposition. This is possible due to the regularity assumptions on b and σ. With F and L thus modified, we next have to make sure that under the given assumptions the hypotheses of this Proposition are satisfied. For convenience, we recall these hypotheses. For any integer $r \geq 0$, they claim

$$e^{tL} : C^{r+1}([0,T] \times \mathbb{R}^n) \to C^{r+1}([0,T] \times \mathbb{R}^n) \qquad \text{is a strongly continuous semigroup, for} \quad t \geq 0, \tag{6}$$

$$\Phi : \begin{array}{ccc} C^{r+1}([0,T] \times \mathbb{R}^n) & \to & C^r([0,T] \times \mathbb{R}^n) \\ \varphi & \mapsto & F(\varphi, D\varphi) \end{array} \qquad \text{is a locally Lipschitz map,} \tag{7}$$

$$e^{tL} : C^r([0,T] \times \mathbb{R}^n) \to C^{r+1}([0,T] \times \mathbb{R}^n), \tag{8}$$

and, for some $\gamma < 1$,

$$||e^{tL}||_{\mathcal{L}(C^r([0,T]\times\mathbb{R}^n), C^{r+1}([0,T]\times\mathbb{R}^n))} \leq Ct^{-\gamma}. \tag{9}$$

The condition on F is evidently satisfied. To verify the conditions on the semigroup of L, we refer to Davies [7]. Strong continuity is due to [7], Theorem 1.4.1, p.22. the smoothing property is related to [7], Theorem 5.2.1, p. 149, and the large time asymptotic property can be obtained from [7], Theorem 2.3.6, p. 73. □

1.2 Market and climate models

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with two independent Brownian motions W_1 (m-dimensional) and W_2 (n-dimensional), indexed by the finite time interval $[0,T]$, where $T > 0$ is a deterministic time horizon. Let $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ be the completion of the natural filtration of $W = (W_1, W_2)$ by the sets of measure 0.

We consider a simple financial market model composed of $m+1$ securities, consisting of one bond with null interest rate

$$X_{0,t} = 1, \quad \text{for all} \quad t \in [0,T],$$

and p stocks. We assume that the stock price vector process X_1 is given by a Markovian SDE, i.e. :

$$\begin{cases} dX_{1,s} = X_{1,s}\left(b_1(s, X_{1,s})ds + \sigma_1(s, X_{1,s})dW_{1,s}\right), & t \leq s \leq T, \\ X_{1,t} = x_1 \in \mathbb{R}^p. \end{cases}$$

The coefficients $b_1 : [0,T] \times I\!R^m \mapsto I\!R^m$, $\sigma_1 : [0,T] \times I\!R^m \mapsto I\!R^m \times I\!R^m$ are supposed to satisfy Lipschitz conditions in the state variables.

We also consider an n-dimensional climate process, the dynamics of which is also described by an SDE of the form

$$\begin{cases} dK_s = b_K(s, K_s)ds + \sigma_K(s, K_s)dW_{2,s}, & t \leq s \leq T, \\ K_t = k \in I\!R^d. \end{cases}$$

The coefficients $b_K : [0,T] \times I\!R^n \mapsto I\!R^n$ and $\sigma_K : [0,T] \times I\!R^n \mapsto I\!R^n \times I\!R^n$ are again Lipschitz functions of the state variables.

Market completion is achieved by adding a security X_2 whose market price of risk process θ parametrizes the completion and thus the valuation of risky claims. Equilibrium is obtained via a market clearing condition for this additional security. The condition determines uniquely an equilibrium market price for trading exposures to climate risk. More details about the equilibrium construction will follow in subsection 1.4. See also [16]. X_2 will be determined again as the solution of an SDE of the form

$$\begin{cases} dX_{2,s} = X_{2,s}\left(b_{2,s}ds + \sigma_{2,s}dW_{2,s}\right), & t \leq s \leq T, \\ X_{2,t} = x_2 \in I\!R^d. \end{cases}$$

We will note $\theta_1 = b_1\left({\sigma_1}^{-1}\right)$ and $\theta_2 = b_2\left({\sigma_2}^{-1}\right)$. The local equilibrium probability Q^θ, under which (X_1, X_2) is a martingale, is given by Girsanov's formula

$$\left.\frac{dQ^\theta}{dI\!P}\right|_{\mathcal{F}_s} = Z_s^\theta = \mathcal{E}\left(-\int_0^s \theta_t dW_t\right) = \exp\left(-\int_0^s \theta_t dW_t - \frac{1}{2}\int_0^s ||\theta_t||^2 dt\right), \quad s \in [0,T]. \tag{10}$$

The main goal of this paper consists in computing θ_2, and consequently the optimal proportions each trader invests in both the market and the insurance asset. We will obtain this process as a function of the system state $(t, X_{1,t}, K_t)$ at a time $t \in [0,T]$. X_2 can then be constructed by taking the coefficients $\sigma_2 = \mathrm{id}$ and $b_2 = \theta_2$.

1.3 Modelling of the agents

From this section on, we shall take $n = m = 1$ for greater simplicity. Assume that the market has been completed by the introduction of the security X_2, which is completely determined by θ_2. We shall assume in the sequel, and justify in the computations later, that θ_2 can be written as a regular function of the state of the system, i.e.

$$\theta_{2,s} = \widetilde{\theta}_2(s, X_{1,s}, K_s) \text{ with } \widetilde{\theta}_2 \in C^2.$$

In the sequel we shall use the same symbol θ_2 for both the random process and the real valued regular function of $(s, x_1, k) \in [t,T] \times I\!R \times I\!R$, since it will be clear from the context which object we are dealing with. In fact, under the conditions we impose on the coefficients of our equations, it will be seen that $\widetilde{\theta}_2$ is even C^∞.

Let I be the finite set of agents active on the market. Each agent $a \in I$ is supposed to be endowed with an initial capital $v_0^a \geq 0$. He invests in the market including the insurance asset and uses an admissible trading strategy $\pi = (\pi_1, \pi_2)$.

Its wealth process is given by

$$V_s^a = v_0^a + \int_0^s \pi_{1,s} \frac{dX_{1,s}}{X_{1,s}} + \pi_{2,s} \frac{dX_{2,s}}{X_{2,s}}, \ s \in [0,T] \tag{11}$$

("not investing" means investing in X_0, i.e. choosing the strategy $\pi = 0$).

1.3.1 Income

At the end T of the trading interval, each agent receives a stochastic income H^a, describing the profits the company he represents obtains, which can depend on the market and on the climate. We assume it has the form

$$H^a = g^a(\tau, X_{1,\tau}, K_\tau) + \int_0^\tau \varphi^a(t, X_{1,t}, K_t) dt,$$

where g^a and φ^a are real valued bounded smooth (C^∞) functions, with

$$\tau = \inf \{ s \in]t,T] \, | (s, X_{1,s}, K_s) \notin \mathcal{O} \}$$

the entrance time of some critical set $\mathcal{O}$, an open subset of $]0,T[\times \mathbb{R}^p \times \mathbb{R}^d$.

1.3.2 Utility maximization

Each agent $a \in I$, by acting on its trading strategy π, wants to maximize the expected utility of the sum of the terminal wealth V_T^a and the income H^a. We will use the family of exponential utility functions

$$U^a(x) = -\exp(-\alpha_a x), \ x \in \mathbb{R},$$

with individual risk aversion coefficient $\alpha_a > 0$. In mathematical terms, every agent wants to attain

$$J^a = \sup_{\pi \ admissible} \mathbb{E}\left[U^a\left(V_T^a + H^a\right)\right]. \tag{12}$$

We can compute this expectation by solving a linear PDE. To see this, we start from the stochastic representation of maximal utility obtained via duality and Legendre transforms of U derived in Karatzas, Lehoczky, Shreve [18] or Kramkov, Schachermayer [22]. The formula valid in our setting is derived in [16], Proposition 10. We have

$$J^a = \mathbb{E}\left[-\frac{\lambda_a}{\alpha_a} Z_T^\theta\right] = -\frac{\lambda_a}{\alpha_a}, \tag{13}$$

since Z^θ is a $\mathbb{P}-$martingale, where λ_a is defined by

$$\log(\lambda_a) = \log(\alpha_a) - \alpha_a v_0^a + \mathbb{E}^\theta\left[-\log\left(Z_T^\theta\right) - \alpha_a H^a\right].$$

Now define for $t \in [0,T], x_1 \in I\!R, k \in I\!R$

$$\chi(t,x_1,k) = I\!E^\theta \left[-\log\left(Z_T^\theta\right) - \alpha_a H^a \right]$$

$$= I\!E^\theta \left[\int_t^T \left(\left|\left| -\frac{1}{2}\theta\left(s, X_{1,s}, K_s\right) \right|\right|^2 - \alpha_a \varphi^a\left(s, X_{1,s}, K_s\right) \right) ds - \alpha_a g^a\left(T, X_{1,T}, K_T\right) \right].$$

An appeal to the backward version of Theorem 1.1 translates the stochastic utility maximization formula into a linear backward PDE.

Corollary 1.1 *Let $\widetilde{L}$ be the infinitesimal generator of the diffusion (X_1, K) under Q^θ, determined for a regular function ϕ by*

$$\widetilde{L}\phi = \left(b_K - \theta_2 \sigma_K\right) \frac{\partial \phi}{\partial k} + \frac{1}{2} trace \left\{ \begin{pmatrix} x_1^2\sigma_1^2 & 0 \\ 0 & \sigma_K^2 \end{pmatrix} D^2\phi \right\}.$$

Then χ is the unique classical solution of the following backward PDE

$$\left\{ \begin{array}{rl} -\dfrac{\partial \chi}{\partial t} - \widetilde{L}\chi - \dfrac{1}{2}||\theta||^2 - \alpha_a \varphi^a = & 0 \\ \chi(T,x_1,k) = & -\alpha_a g^a(x_1,k). \end{array} \right. \tag{14}$$

Proof:
The result follows from Theorem 1.1 in dimension $n = p+d = 2$ with $b = \begin{pmatrix} 0 \\ b_K - \theta_2\sigma_K \end{pmatrix}$, $\sigma\sigma^* = \begin{pmatrix} x_1^2\sigma_1^2 & 0 \\ 0 & \sigma_K^2 \end{pmatrix}$, $f = -\frac{1}{2}||\theta||^2 - \alpha_a\varphi^a$, $g = -\alpha_a g^a$ and $h = 0$. Obviously, f and h are Lipschitz continuous and possess linear growth at infinity, g is continuous and bounded. There is one small gap here, which can be easily overcome. The diffusion matrix $\sigma\sigma^*$ is not uniformly elliptic, due to the appearance of x_1^2 in the first diagonal entry. But since the generated diffusion does not visit the boundary $x_1 = 0$, we may argue by using a logarithmic coordinate change in x_1 at the beginning of the analysis (see the proof of Corollary 1.3). By this change, the diffusion matrix becomes constant in the first diagonal entry, and thus uniformly elliptic. The drift is modified, but stays Lipschitz with linear growth at infinity. The change of variable being a regular bijection of the domain, existence and uniqueness of solutions in the two coordinate systems are equivalent. □

If, as usual, the initial time of action is 0, we have

$$J^a = -\exp\left(-\alpha_a v_0^a + \chi(0,x_1,k)\right).$$

1.3.3 Optimal control problem

While Corollary 1.1 offers a convenient possibility of describing the optimal utility, an analytic access to the actual optimal portfolio strategies (π_1, π_2), the quantities of (X_1, X_2) to be invested, requires to dig a little deeper. We have to invoke the basic results of stochastic control theory (see for example [23] or [5]).

Suppose as before that the trading period begins at a time $t \in]0,T]$, each agent starting with an initial capital v_t^a (and $X_{1,t} = x_1$ and $K_t = k$). The agents want to attain their value function

$$J^a(t, x_1, k, v_t^a) = \sup_{\pi \ admissible} I\!E_{t,x_1,k,v_t^a} \left[U^a \left(V_T^a + H^a \right) \right].$$

By the same calculus we have

$$J^a(t, x_1, k, v_t^a) = -\exp\left(-\alpha_a v_t^a + \chi(t, x_1, k) \right), \tag{15}$$

where χ is the classical solution of the PDE of Corollary 1.1.

Let us rewrite the wealth process V^a defined in (11) in terms of the proportions $p = (p_1, p_2)$ to be invested in (X_1, X_2). Formally $\frac{\pi_i}{V^a} = p_i$, and we now have $p_0^a + p_1^a + p_2^a = 1$. In these terms we may write

$$\frac{dV_s^a}{V_s^a} = p_{1,s} \left(b_1 ds + \sigma_1 dW_{1,s} \right) + p_{2,s} \left(\theta_2 ds + dW_{2,s} \right),$$

so that the coefficients of this SDE controlled by p do not depend on X_2.

Equation (15) yields that the function J^a is C^2, as is the function χ. So, using theorem 1.3.1, p. 25, in [6] this implies that J^a solves the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} \dfrac{\partial J^a}{\partial t}(s, x_1, k, v) + \sup\limits_p \left\{ \mathcal{L}^p J^a(s, x_1, k, v) \right\} = 0 \text{ for } (s, x_1, k) \in \mathcal{O} \\ J^a(s, x_1, k, v) = U^a(v + g^a(x_1, k)) \text{ for } (s, x_1, k) \text{ on } \partial\mathcal{O}, \end{cases} \tag{16}$$

where $\mathcal{O}$ is the open set from 1.3.1, $\mathcal{L}^p$ is the infinitesimal generator of the diffusion $s \mapsto (X_{1,s}, K_s, V_s^a,)$, i.e. the differential operator determined by its value for a regular function ϕ by

$$\mathcal{L}^p \phi(s, x_1, k, v) = \begin{pmatrix} x_1 b_1(s, x_1) \\ b_K(s, k) \\ v\left(p_1 b_1 + p_2 \theta_2\right) \end{pmatrix} \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial k} \\ \frac{\partial \phi}{\partial v} \end{pmatrix}$$

$$+ \frac{1}{2} \text{trace} \left\{ \begin{pmatrix} x_1^2 \sigma_1^2 & 0 & v p_1 x_1 \sigma_1^2 \\ 0 & \sigma_K^2 & v p_2 \sigma_K \\ v p_1 x_1 \sigma_1^2 & v p_2 \sigma_K & v^2 \left(p_1^2 \sigma_1^2 + p_2^2 \right) \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \phi}{\partial x_1^2} & \frac{\partial^2 \phi}{\partial x_1 k} & \frac{\partial^2 \phi}{\partial x_1 v} \\ \frac{\partial^2 \phi}{\partial x_1 k} & \frac{\partial^2 \phi}{\partial k^2} & \frac{\partial^2 \phi}{\partial k v} \\ \frac{\partial^2 \phi}{\partial x_1 v} & \frac{\partial^2 \phi}{\partial k v} & \frac{\partial^2 \phi}{\partial v^2} \end{pmatrix} \right\}. \tag{17}$$

If the optimal control process exists, it is given in *feedback form*, i.e. as a function of the state of the system by

$$p^a(s, X_{1,s}, K_s, V_s^a) = \arg\max_p \mathcal{L}^p J^a(s, X_{1,s}, K_s, V_s^a). \tag{18}$$

Formulas of this type have been derived for example in Fleming, Soner [10] p.170, in a general setting, and also in [5] and [23]. As soon as this process p^a is well-defined, it coincides with the optimal strategy. In our case, existence problems for p^a are covered by (14) which guarantees the existence of a classical solution of the HJB equation.

Using (15), we can express the optimal proportions p^a in terms of the function χ defined by system (14) of Corollary 1.1. This will allow us to justify the existence of the optimal control.

Let us omit for the moment the superscript a. We then have to find p_1 which maximizes

$$p_1 \left(v b_1 \frac{\partial J^a}{\partial v} + v x_1 \sigma_1 \frac{\partial^2 J^a}{\partial v \partial x_1} \right) + \frac{1}{2} (p_1)^2 \left(v^2 \sigma_1^2 \frac{\partial^2 J^a}{\partial v^2} \right)$$

and, independently, p_2 which maximizes

$$p_2 \left(v \theta_2 \frac{\partial J^a}{\partial v} + v \sigma_K \frac{\partial^2 J^a}{\partial v \partial k} \right) + \frac{1}{2} (p_2)^2 \left(v^2 \frac{\partial^2 J^a}{\partial v^2} \right).$$

This is seen by applying (17) to J^a, separating the p_1- and the p_2-terms from the resulting polynomial in (p_1, p_2) and separately maximizing these. By (15), we have $\frac{\partial J^a}{\partial v} = -\alpha_a J^a$, hence $\frac{\partial^2 J^a}{\partial v^2} = (\alpha_a)^2 J^a$ and $\frac{\partial^2 J^a}{\partial k v} = -\alpha_a \frac{\partial J^a}{\partial k}$. Therefore, to compute p_2, we have to maximize the expression

$$-p_2 \alpha_a v \left(\theta_2 J^a + \sigma_K \frac{\partial J^a}{\partial k} \right) + \frac{1}{2} (p_2)^2 \left(v^2 (\alpha_a)^2 J \right).$$

Now, again by (15), $\frac{\partial J^a}{\partial k} = \frac{\partial \chi}{\partial k} J^a$. Moreover, by definition of the utility functions, it is clear that $J^a \leq 0$. We are therefore led to the problem of minimizing

$$-p_2 \alpha_a v \left(\theta_2 + \sigma_K \frac{\partial \chi}{\partial k} \right) + \frac{1}{2} (p_2)^2 v^2 (\alpha_a)^2.$$

The result is easily obtained by minimizing the given polynomial of degree 2 and, together with the analogous calculation for p_1 leads to the following formulas.

Corollary 1.2 *Let $a \in I$. Let χ be a solution of (1.1), define J^a by (15), and let X_2 and therefore θ_2 be given according to subsection 1.2. Then the solution (p_1, p_2) of the optimal control problem (18) at $(X_{1,s}, K_s, V_s^a) = (x_1, k, v)$ is given by*

$$p_1^a = \frac{b_1 + x_1 \sigma_1 \dfrac{\partial \chi}{\partial x_1}}{v \sigma_1^2 \alpha_a},$$

$$p_2^a = \frac{\theta_2 + \sigma_K \dfrac{\partial \chi}{\partial k}}{v \alpha_a}.$$

Accordingly, the quantity

$$\pi_{2,s}^a = V_s^a p_{2,s}^a = \frac{1}{\alpha_a} \left(\theta_2(s, X_{1,s}, K_s) + \sigma_K(s, K_s) \frac{\partial \chi}{\partial k}(s, X_{1,s} K_s) \right)$$

is the optimal amount of money to be invested in X_2 by agent a at time $s \in [t, T]$.

1.4 Local Equilibrium Measure

According to [16], the insurance security X_2 is chosen in such a way that the market is in local equilibrium. This means that the total investment $(\pi_2^a)_{a\in I}$ in this security satisfies the market clearing condition

$$\sum_{a\in I}\pi_2^a = 0 \ a.s. \tag{19}$$

As a consequence, the process θ_2, which determines completely the structure of the security X_2 and the unique martingale measure Q^θ, can be computed as the solution of a BSDE. We shall briefly recall how this can be seen. The structure result for the optimal utility of agent a reflected for example in (13) combines with (10) to produce for any $a \in I$ the following formula for the optimal income from trading in (X_1, X_2) including the income due to risk exposure

$$\begin{aligned} & -\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\lambda_a Z_T^\theta) - H^a \qquad (20)\\ &= -\frac{1}{\alpha_a}\log\left(\frac{\lambda_a}{\alpha_a}\right) + \frac{1}{\alpha_a}\int_0^T(\theta_{1,t}dW_{1,t}+\theta_{2,t}dW_{2,t}) + \frac{1}{2\alpha_a}\int_0^T(\theta_{1,t}^2+\theta_{2,t}^2)dt - H^a. \end{aligned}$$

On the other hand, taking into account the market clearing condition, the total optimal income of all agents on the market due to their trading strategies (π_1^a, π_2^a) amounts to the following quantity

$$\begin{aligned} &\sum_{a\in I}(B^a - H^a)\\ &= \sum_{a\in I} v_0^a + \int_0^T(\sum_{a\in I}\pi_{1,t}^a)dX_{1,t} + \int_0^T(\sum_{a\in I}\pi_{2,t}^a)dX_{2,t} \qquad (21)\\ &= \sum_{a\in I} v_0^a + \int_0^T(\sum_{a\in I}\pi_{1,t}^a)\,\sigma_{1,t}X_{1,t}\,(dW_{1,t}+\theta_{1,t}dt)\\ &\qquad\qquad + \int_0^T(\sum_{a\in I}\pi_{2,t}^a)\,\sigma_{2,t}(dW_{2,t}+\theta_{2,t}dt)\\ &= \sum_{a\in I} v_0^a + \int_0^T(\sum_{a\in I}\pi_{1,t}^a)\,\sigma_{1,t}X_{1,t}\,(dW_{1,t}+\theta_{1,t}dt). \end{aligned}$$

We sum (20) in $a \in I$ and equate the result to (21). The equation thus obtained is interpreted as an equation for the unknown process θ_2 with given parameter θ_1 and given risk exposure functionals H^a. With the abbreviations

$$\overline{\alpha} = \left(\sum_{\alpha\in I}\frac{1}{\alpha_a}\right)^{-1},$$

$$\begin{aligned}
\overline{H} &= \sum_{\alpha\in I} H^a + \frac{1}{2\overline{\alpha}}\int_0^\tau \theta_{1,t}^2 dt,\\
z_1 &= \theta_1 - \overline{\alpha}\sigma_1 \sum_{a\in I}\pi_1^a,\\
z_2 &= \theta_2,
\end{aligned}$$

we are thus led to a nonlinear BSDE of the form

$$h_s = \overline{\alpha}\overline{H} - \int_s^T z_t dW_t - \int_s^T \theta_{1,t} z_{1,t} dt - \frac{1}{2}\int_s^T z_{2,t}^2 dt \tag{22}$$

to be solved for the process (h, z_1, z_2).

Again, we shall associate a PDE with the stochastic object from (22). To simplify its derivation, let us further abbriviate

$$\begin{aligned}
g &= \overline{\alpha}\sum_{\alpha\in I} g^a,\\
\varphi &= \overline{\alpha}\sum_{\alpha\in I}\varphi^a + \frac{1}{2}\theta_1^2,\\
R_s &= \int_0^s \varphi(t, X_{1,t}, K_t)dt.
\end{aligned}$$

In these terms, we obtain $\overline{\alpha}\overline{H} = g(\tau, X_{1,\tau}, K_\tau) + R_\tau$ and we can rewrite the BSDE (22) as

$$-Y_s = h_s - R_s = g(\tau, X_{1,\tau}, K_\tau) - \int_s^T z_t dW_t - \int_s^T \theta_{1,t} z_{1,t} dt - \frac{1}{2}\int_s^T z_{2,t}^2 dt + \int_s^T \varphi_t dt. \tag{23}$$

Now using the nonlinear Feynman-Kac formula in its version of Theorem 1.3, we see that z can be obtained by computing the function u, which is the classical solution of a backward nonlinear PDE, provided the coefficient and risk functions satisfy the following regularity hypotheses.

(H1) the system state domain $\mathcal{O}$ is given by the cylinder $]0,T[\times]0,\infty)\times I\!R$ (there is no stopping time, no Dirichlet condition).

(H2) the terminal income g is a $C^1\left(]0,\infty)\times I\!R\right)$ function and all the other coefficients $b_1, \sigma_1, b_K, \sigma_K, \varphi$ are $C^\infty\left([0,T]\times]0,\infty)\times I\!R\right)$ functions.

(H3) σ_1^2, σ_K^2 are bounded below by positive constants.

Corollary 1.3 *Assume that the domain and coefficient functions satisfy the hypotheses* **(H1), (H2), (H3).** *Let u be a classical solution of the nonlinear PDE*

$$\left\{\begin{array}{l}
-\dfrac{\partial u}{\partial t} - b_K\dfrac{\partial u}{\partial K} - \dfrac{1}{2}\left(x_1^2\sigma_1^2\dfrac{\partial^2 u}{\partial x_1^2} + \sigma_K^2\dfrac{\partial^2 u}{\partial K^2}\right) + \dfrac{1}{2}\left(\sigma_K\dfrac{\partial u}{\partial K}\right)^2 - \varphi(t,x_1,k) = 0 \quad \textit{in } \mathcal{O},\\
u = -g \ \textit{on } \partial\mathcal{O}.
\end{array}\right. \tag{24}$$

Then by setting

$$\begin{array}{rcl} Y_s & = & R_s - h_s = u(s, X_{1,s}, K_s), \\ z_s & = & \begin{pmatrix} X_{1,s}\sigma_1 & 0 \\ 0 & \sigma_K \end{pmatrix} Du(s, X_{1,s}, K_s) \end{array} \tag{25}$$

we obtain the unique solution of BSDE (23).

Proof:

We shall prove that our system, under a regular change of variables, can be written in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - Lu - F(t, x, u, \gamma^*(t, x)Du) = 0 \text{ in } \mathcal{O}, \\ u(t, x) = g(t, x) \text{ on } \partial\mathcal{O}, \end{cases} \tag{26}$$

with

$$Lu = \frac{1}{2}\left(\gamma_1^2 \frac{\partial^2 u}{\partial x^2} + \gamma_2^2 \frac{\partial^2 u}{\partial y^2}\right),$$

and coefficients γ_1, γ_2 whose squares are bounded below by positive constants. Let us begin formally. Suppose that $\widetilde{u}$ is a solution, in some sense, of (24). Consider a function $\widetilde{w}$ defined by

$$\widetilde{w}(t, x, y) = \widetilde{u}(T - t, e^x, y) \text{ on } [0, T] \times I\!R^2.$$

It is straightforward to see that $\widetilde{w}$ is associated with the system (26) with terminal condition

$$f(x, y) = -g(T, e^x, y), \tag{27}$$

coefficients

$$\gamma_1(x) = \sigma_1(e^x), \quad \gamma_2(y) = \sigma_K(y), \quad x, y \in I\!R,$$

and generator

$$F(t, (x, y), w, (w_x, w_y)) = -\frac{1}{2}\gamma_1^2 w_x + \gamma_2 w_y - \frac{1}{2}\gamma_2^2 w_y^2 + \varphi(T - t, e^x, y). \tag{28}$$

Due to **(H2)**, f and F are regular functions, and **(H3)** guarantees the uniform ellipticity of the operator L. Hence the assumptions of Theorem 1.4 hold. There exists a unique classical solution $w \in C\left([0, T], C^1\left(I\!R^2\right)\right) \cap C^\infty\left([0, T[\times I\!R^2\right)$ of the system (26),(27) and (28). Now we can define rigorously u by setting

$$u(t, x, k) = w\left(T - t, \log(x), k\right) \text{ for } (t, x, k) \in [0, T]\times]0, \infty) \times I\!R.$$

This function has clearly the announced regularity. Finally, using (25) and Itô's formula, it easy to check that u solves (24) in the classical sense.□

Recall that θ_2 is defined as a partial derivative of the function u in (29). The preceding result allows us to justify this definition, moreover we obviously have

$$\theta_2 \in C\left([0, T], C\left(]0, \infty)^p \times I\!R^d\right)\right) \cap C^\infty\left([0, T[\times]0, \infty)^p \times I\!R^d\right).$$

In particular θ_2 is a Lipschitz continuous function, so the process X_2 is well-defined by (29).

Recalling the definitions of z above and of θ in subsection 1.2, we can use Corollary 1.3 to compute explicitly θ_2 and thus the insurance asset process X_2 through the following formulas

$$\begin{aligned} \theta_{2,s} &= \sigma_K \frac{\partial u}{\partial K}(s, X_{1,s}, K_s), & (29) \\ dX_{2,s} &= X_{2,s}\left(\theta_{2,s} ds + dW_{2,s}\right). & (30) \end{aligned}$$

1.5 Formulas for X_2

In this subsection, we derive some formulas enabling us to compute the moments of the insurance asset process X_2 in case σ_K is invertible. Under this hypothesis, we can write

$$dW_{2,s} = \frac{dK_s - b_K(s, K_s)ds}{\sigma_K(s, K_s)}.$$

This leads us to an integral expression for X_2 in terms of the trajectories of the process K, given by

$$\log\left(X_{2,t}\right) = \log\left(x_2\right) + \int_0^t \left[\theta_2(s, X_{1,s}, K_s) - \frac{1}{2} - \frac{b_K(s, K_s)}{\sigma_K(s, K_s)}\right] ds + \int_0^t \frac{1}{\sigma_K(s, K_s)} dK_s.$$

If σ_K is even constant which is the case if for example K is an Ornstein-Uhlenbeck process, we have

$$\log\left(X_{2,t}\right) = \log\left(x_2\right) + \int_0^t \left[\theta_2(s, X_{1,s}, K_s) - \frac{1}{2} - \frac{b_K(s, K_s)}{\sigma_K}\right] ds + \frac{K_s - K_0}{\sigma_K}.$$

In this case the expectation of $X_{2,t}$ with the initial conditions $X_{1,0} = x_1, X_{2,0} = x_2, K_0 = k$ may be expressed by the formula

$$\begin{aligned} I\!E_{x_1,x_2,k}[X_{2,t}] &= x_2 e^{-\frac{k}{\sigma_K}} I\!E_{x_1,k}\left[e^{\frac{K_s}{\sigma_K}} \exp\left(\int_0^t \left[\theta_2(s, X_{1,s}, K_s) - \frac{1}{2} - \frac{b_K(s, K_s)}{\sigma_K}\right] ds\right)\right] \\ &= x_2 e^{-\frac{k}{\sigma_K}} f(t, x_1, k). \end{aligned} \quad (31)$$

In this case again, we may translate its computation into analysis by associating with this expectation a PDE possessing a simple derivation from the forward linear Feynman-Kac formula in Theorem 1.1.

Corollary 1.4 *Suppose $\sigma_K \neq 0$ is constant. Define*

$$f(s, x_1, k) = \frac{1}{x_2} e^{\frac{k}{\sigma_K}} I\!E_{x_1,x_2,k}[X_{2,t}], \quad s \in [t, T], x_1, x_2, k \in I\!R.$$

Let L be the infinitesimal generator of the diffusion (X_1, K), i.e.

$$Lf = \begin{pmatrix} x_1 b_1 \\ b_K \end{pmatrix} Df + \frac{1}{2} trace \left\{ \begin{pmatrix} x_1\sigma_1 & 0 \\ 0 & \sigma_K \end{pmatrix} D^2 f \right\}.$$

Then f is the solution of the forward linear PDE

$$\begin{cases} \frac{\partial f}{\partial t} - Lf - \left(\theta_2 - \frac{1}{2} - \frac{b_K}{\sigma_K}\right) f &= 0 \\ f(0, x_1, k) &= \exp\left(\frac{k}{\sigma_K}\right). \end{cases} \tag{32}$$

Remark that the implicit dependence on x_2 in the definition of f above can indeed be suppressed, since X_2 depends only in a multiplicative way on its initial condition x_2.

Proof:

The result is directly given by the forward form of theorem 1.1 with $b = \begin{pmatrix} x_1 b_1 \\ b_K \end{pmatrix}$, $\sigma = \begin{pmatrix} x_1\sigma_1 & 0 \\ 0 & \sigma_K \end{pmatrix}$, $f = 0$, $g = \exp\left(\frac{k}{\sigma_K}\right)$, and $h = \theta_2 - \frac{1}{2} - \frac{b_K}{\sigma_K}$. For obtaining uniform ellipticity of the diffusion part, a procedure as in the proof of Corollary 1.3, based on a logarithmic coordinate change in x_1, again applies. □

With the same technique, we can compute any moment of X_2. For all $n \in \mathbb{N}$,

$$\mathbb{E}_{x_1,x_2,k}[X_{2,t}^n] = x_2 \exp\left(-n\frac{k}{\sigma_K}\right) f_n(t, x_1, k),$$

where f_n is the solution of

$$\begin{cases} \frac{\partial f_n}{\partial t} - Lf_n - n\left(\theta_2 - \frac{1}{2} - \frac{b_K}{\sigma_K}\right) f_n &= 0, \\ f_n(0, x_1, k) &= \exp\left(n\frac{k}{\sigma_K}\right). \end{cases}$$

1.6 Examples

We now specify some climate processes, stock price models, and risk exposure functionals we shall investigate in our numerical simulations in section 3.

1.6.1 Temperature models

Our climate process affecting the agents on our market will model the local temperature (of air, of ocean water) evolution as a random function of time. It is therefore usually modelled as a one-dimensional stochastic process, also for the simplicity of qualitative numerical simulations. The reduced physical models they come from usually lead to finite dimensional stochastic equations and describe some nonlinear interaction between finitely many physical quantities including the local temperature. For example, there are ENSO models consisting in nonlinear two-dimensional stochastic differential equations coupling the thermocline depth in some area of the South Pacific with the sea surface temperature. The system turns out to be an autonomous nonlinear stochastic oscillator which in some parameter regimes acts as a stochastically perturbed bistable

differential equation with an intrinsically defined periodicity. For our purposes, we take a one-dimensional SDE driven by a Brownian motion. It describes the motion of a state variable travelling through a bi-stable potential landscape, with an explicit periodic dependence of the potential shape creating a non-autonomous stochastic system. Another example comes from a 15-dimensional linear SDE of the Ornstein-Uhlenbeck type with a $15 \times 15-$matrix with non-trivial rotational part and entries determined by satellite measurements which is used in linear prediction models for ENSO. It creates a diffusion with non-trivial rotation numbers implying random periodicity for the sea surface temperature contained in the model. For our qualitative problems we may describe the temperature curve as a simple mean-reverting linear sde with an additional deterministic periodic forcing. The following concrete examples can be studied.

1. **Ornstein-Uhlenbeck**. A simple model for a temperature process fluctuating around an average value $K_a \in \mathbb{R}$ is given by an Ornstein-Uhlenbeck process (centered in K_a), determined by
$$dK_s = C(K_a - K_s)ds + \sigma_K dW_{2,s},$$
where $C > 0$ is the strength of restoring force to K_a, and $\sigma_K > 0$ the volatility. We use this process in our simulations (cf. model A in section 3).

2. **Ornstein-Uhlenbeck with periodic term**. This is a rudimentary version of the temperature part of the model used for ENSO prediction. It is obtained by modifying the preceding example in adding a periodical perturbation
$$dK_s = \left[C(K_a - K_s) + C' \sin\left(\frac{2\pi}{T_0}s\right)\right] ds + \sigma_K dW_{2,s},$$
where $C' > 0$ is the amplitude and T_0 the period of the sinusoidal periodic term.

3. **Multidimensional Ornstein-Uhlenbeck with intrinsic periodicity**. This is another less rudimentary version of the physicists' model for ENSO prediction. We take dimension to be 2, but could also consider a $d-$dimensional version. The temperature will be represented by the first component K_1 of the following two-dimensional process K, defined through the autonomous linear SDE
$$dK_s = C \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K_s ds + \begin{pmatrix} \sigma_{K_1} \\ \sigma_{K_2} \end{pmatrix} dW_{2,s}.$$

4. **Periodically forced bi-stable temperature**. This is a phenomenological version of the stochastic oscillator model for ENSO sketched above, where intrinsic periodicity is replaced by a non-autonomous periodic dependence of the bi-stable function U. U is a double-well potential function, for example $U(k) = \frac{k^4}{4} - \frac{k^2}{2}, k \in \mathbb{R}$. The diffusion process K given by the SDE
$$dK_s = U'(K_s)ds + Q.K_s. \sin\left(\frac{2\pi}{T_0}s\right) + \sqrt{\varepsilon} dW_{2,s}$$
models temperature in a bi-stable environment. For ε chosen appropriately, the trajectories of K are almost periodic. This phenomenon is investigated under the name *stochastic resonance*. See [15] for a review. We use this process in our simulations (cf. model B and C in section 3).

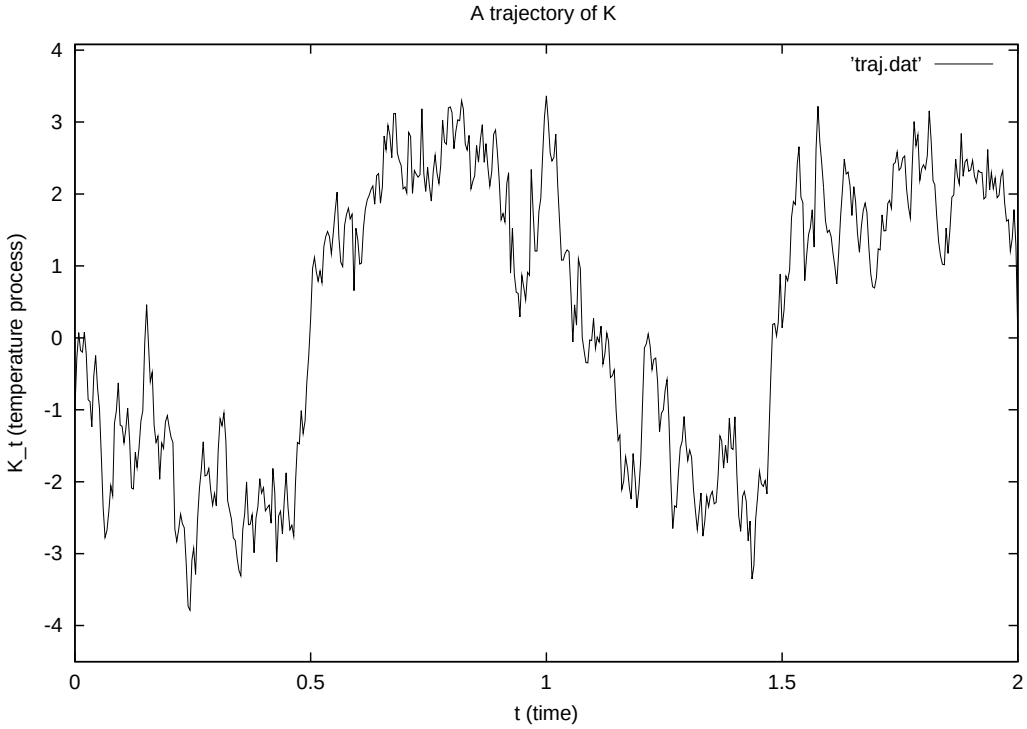

Figure 1: A sample path of the bi-stable temperature process K.

1.6.2 Asset price process

The stock price model, for simplicity, is just a simple Black-Scholes model of one risky asset.

(a) **Black-Scholes**. We will use a simple geometrical Brownian motion to describe the share price, i.e.

$$dX_{1,s} = X_{1,s}\left(b_1 ds + \sigma_1 dW_{1,s}\right), \quad s \in [t, T], \tag{33}$$

where $b_1 > 0$ is the rate and $\sigma_1 > 0$ the volatility of the asset.

1.6.3 Risk exposure of the agents

Three typical qualitative risk exposures will be considered: the one of a fisher describing profits from fishing whose efficiency depends on the surface temperature of the ocean and is optimal at some fixed temperature value while it drops off as temperature deviates from this optimum. A rice farmer's risk exposure functional may be quite similar, his interests, however, rather complementary to the fisher's. Think of the sea surface temperature process possessing two metastable equilibria, a low and a high one. As explained earlier, the fisher may have his temperature of optimal income near the lower equilibrium, while the farmer might profit more from higher precipitation rates at the higher ENSO

temperature equilibrium. This in particular means that the fisher profits from temperature values under which the farmer suffers most, and vice versa. The exposure of a bank may not directly dependent on climate risk.

(a) **Fisher**. Let $\tau = T$, the final time of the trading interval. Let K be a local sea surface temperature, and imagine a fishing company $f \in I$ that makes most profits if the temperature is near an optimal value k_1. We can describe the income H^f of this company on the period $[0, T]$ qualitatively by

$$H^f = \int_0^T \varphi^f(K_s)ds,$$

where φ^f is a positive function taking its global maximum in k_1, for example

$$\varphi^f(k) = e^{-(k-k_1)^2}.$$

(b) **Farmer**. The (rice) farmer or farming company we imagine as an example may have an exposure quite of the same type as the fisher. The optimal income still being a function of the sea surface temperature is just obtained at another value k_2, which typically is higher than k_1, and may be given by the second meta-stable point of a bi-stable random temperature. The income of the farmer may therefore be described by

$$H^r = \int_0^T \varphi^r(K_s)ds,$$

where φ^r is a positive function taking its global maximum in k_2, for example

$$\varphi^f(k) = e^{-(k-k_2)^2}.$$

If we work with a bi-stable randomly periodic sea surface temperature process, we see immediately that farmer and fisher have complementary interests, and therefore are likely to profit from trading the climate risk among each other.

(c) **Bank**. As an additional agent, we can consider a bank $b \in I$ whose profits only come from its portfolio management from investment on the financial market, and which participates in the climate risk share only by investing in the insurance security X_2. So its exposure functional will be the trivial $H^b = 0$.

2 Numerical approximations results

As explained in section 1, one of the main methods of our approach consists in translating the key stochastic equations appearing in our utility maximization problem into linear or non-linear PDE. The main equations we obtained this way are given by

- the backward linear PDE (14) describing the value function and providing the optimal strategy for any agent on the market.
- the forward linear PDE (32) computing the moments of X_2.
- the backward non-linear PDE (24) providing the coefficient θ_2 which determines the insurance asset X_2.

In this section we describe the construction of numerical schemes approximating the solutions of these parabolic PDEs and prove their convergence. In subsection 2.1 we shall employ a method initiated by Barles and Souganidis [3] based on the well known stability results for viscosity solutions (see [9], [2] for a general presentation) to derive a basic convergence result which will be applicable to our schemes. In the following subsections 2.2 and 2.3 we shall explain the numerical approximation schemes we use for our simulations, starting in the linear case, and ending with the case of a non-linear equation with quadratic terms.

2.1 Convergence

To state our convergence result in a fairly general framework, let $\mathcal{O}$ be an open subset of $]0,T[\times\mathbb{R}^n$, and let us consider a general possibly non-linear PDE of the second order written in the forward form

$$\begin{cases} \frac{\partial v}{\partial t} + G(t,x,v,Dv,D^2v) &= 0 \text{ in } \mathcal{O}, \\ v &= \Psi \text{ on } \partial\mathcal{O}. \end{cases} \tag{34}$$

Here G and Ψ are scalar functions, respectively continuous on $\mathcal{O}\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n$ and $\partial\mathcal{O}$, and $\mathcal{S}$ denotes the set of symmetric $n\times n-$matrices. Let $\varepsilon>0$. We consider time-explicit schemes of the form

$$\begin{cases} v_\varepsilon(t+\varepsilon,x) &= S(\varepsilon)v_\varepsilon(t,x) \quad \text{if} \quad (t,x)\in\mathcal{O}, \\ v_\varepsilon(t+\varepsilon,x) &= \Psi(t+\varepsilon,x) \quad \text{in any other case,} \end{cases} \tag{35}$$

where, for all $\varepsilon>0$, $S(\varepsilon)$ is an operator defined on $L^\infty(\mathcal{O})$ with values in $L^\infty(\mathcal{O})$.

We assume that the following assumptions hold.

Monotonicity :
For any $\varepsilon>0$, and any function $u,v\in L^\infty(\mathcal{O})$,

$$S(\varepsilon)u \le S(\varepsilon)v \text{ if } u\le v \text{ in } \mathcal{O}. \tag{36}$$

Let us note that this assumption can be relaxed (see [3] remark 2.1 p. 276), this inequality needs only to hold within up to $o(\varepsilon)$ terms.

Commutation with constants :
For any $\xi\in\mathbb{R}$,

$$S(\varepsilon)(u+\xi) = S(\varepsilon)u + \xi. \tag{37}$$

Stability :

$$\begin{array}{c}\text{There exists a sequence } (v_\varepsilon)_{\varepsilon>0} \text{ of solutions to the scheme (35)}\\ \text{which are locally uniformly bounded in } L^\infty(\overline{\mathcal{O}}).\end{array} \tag{38}$$

Consistency :
For any $(t,x) \in \mathcal{O}$ and any test function $\phi \in C_b^\infty(\overline{\mathcal{O}})$,

$$\lim_{\substack{\varepsilon \to 0 \\ (s,y) \to (t,x)}} \frac{\phi(s,y) - S(\varepsilon)\phi(s,y)}{\varepsilon} = G(t,x,\phi(t,x), D\phi(t,x), D^2\phi(t,x)). \tag{39}$$

We also assume that a **strong comparison result** holds for the equation (34) (see [2], [3]), i.e.

$$\begin{array}{c}\text{If } u \text{ is a bounded viscosity subsolution to (34)}\\ \text{and } v \text{ is a bounded viscosity supersolution to (34),}\\ \text{then } u \leq v \text{ on } \overline{\mathcal{O}}.\end{array} \tag{40}$$

Under these conditions, we have the following convergence result derived in [3], Theorem 2.1, p. 275, and also in [6], Theorem 2.4.5, page 81.

Theorem 2.1 *Under the assumptions (36), (37), (38), (39) and (40), the solution v_ε of the scheme (35) converges locally uniformly as $\varepsilon \to 0$ to the unique viscosity solution of PDE (34).*

We note that a unique classical solution of (34) coincides with the viscosity solution.

2.2 Approximation schemes for linear equations

Let us first treat a general backward parabolic linear PDE of the second order. Note that equations (14) and (32) are of this form:

$$\left\{\begin{array}{rll} -\dfrac{\partial u}{\partial t} - b.Du - \dfrac{1}{2}\text{trace}\left[\sigma\sigma^* D^2 u\right] &= 0 \quad \text{in} \quad \mathcal{O}, \\ u &= \Psi \quad \text{on} \quad \partial\mathcal{O}. \end{array}\right. \tag{41}$$

We here assume that $b : \mathcal{O} \to I\!R^n$ and $\sigma : \mathcal{O} \to I\!R^{n\times n}$ are Lipschitz continuous, Ψ is continuous and also that $a = \sigma\sigma^*$ is a diagonal dominant matrix, i.e.

$$\text{for all } j,\ \sigma\sigma^*_{i,i} \geq \sum_{j\neq i} \left|\sigma\sigma^*_{i,j}\right|.$$

We use a time-explicit upwind finite differences scheme (see [6], section 2.4, page 65, and [24]). Let $\Delta t = \varepsilon > 0$ and $\Delta x = \Delta x(\varepsilon) > 0$ be the mesh size of a space-time grid. We denote by $\mathcal{V}_{\Delta x}$ the set of neighboring points of $x = 0$ on the space grid of mesh size Δx.

Let us describe our scheme in the particular case $\mathcal{O} =]0,T[\times I\!R^n$.

Scheme 2.1 *Given $\Delta t > 0$ and $\Delta x > 0$, we construct a function $\overline{u}$ such that*

$$\overline{u}(T,x) = \Psi(x),$$

and

$$\overline{u}(t-\Delta t,x) = \sum_{h\in\mathcal{V}_{\Delta x}} p(x,h)\overline{u}(t,x+h) = S(\Delta t,\Delta x)_{\overline{u}(t,.)}(x), \tag{42}$$

with :

$$\begin{aligned}
p(x,0) &= 1 - \frac{\Delta t}{\Delta x}\sum_{i=1}^{d}|b_i|(x) - \frac{\Delta t}{(\Delta x)^2}\sum_{i=1}^{d}\left(a_{ii} - \sum_{j\neq i}|a_{ij}|\right)(x),\\
p(x,\pm e_i\Delta x) &= \frac{\Delta t}{(\Delta x)^2}(b_i)^{\pm}(x) + \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\left(a_{ii} - \sum_{j\neq i}|a_{ij}|\right)(x),\\
p(x,(e_i\pm e_j)\Delta x) &= p(x,-(e_i\pm e_j)\Delta x) = \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}(a_{ij})^{\pm}(x),\\
p(x,h) &= 0 \quad \textit{in any other case.}
\end{aligned}$$

Consistency, monotonicity and commutation with constants of the scheme are straightforward. Under the C.F.L. condition

$$\Delta t \leq (\Delta x)^2\left\{\sum_{i=1}^{d}\left(\Delta x|b_i| + a_{ii} - \sum_{j\neq i}|a_{ij}|\right)\right\}^{-1},$$

the scheme is stable, because $p(x,h)\geq 0$ and $\sum_{h\in\mathcal{V}_{\Delta x}} p(x,h) = 1$, and S is a contraction. In fact, we have

$$|\overline{u}(t-\Delta t,x)| \leq ||\overline{u}(t,.)||_\infty \leq ... \leq ||\Psi||_\infty.$$

Moreover, the classical uniqueness result of theorem 1.2 implies that a strong comparison result holds for equations (14) and (32) (see also theorem 3.3 p.18 in [9]).

Then, for both these equations, theorem 2.1 proves that $\overline{u}$ converges locally uniformly to the unique continuous viscosity solution, thus the unique classical solution, as Δt and Δx converge to 0.

2.3 Approximation schemes for non-linear equations with quadratic terms

We now consider a more complicated equation, the following general semilinear PDE with quadratic terms which generalizes (24):

$$\left\{\begin{array}{rll}
-\dfrac{\partial u}{\partial t} - b.Du - \dfrac{1}{2}\mathrm{trace}\left[\sigma\sigma^* D^2u\right] + ||MDu||^2 &= 0 \quad \text{in} \quad \mathcal{O},\\
u &= \Psi \quad \text{on} \quad \partial\mathcal{O},
\end{array}\right. \tag{43}$$

where M is a $n\times n-$matrix.

This kind of equation has been studied in the viscosity solutions framework in Kobylanski [21]. In particular, if we assume that the coefficients b, σ and M are Lipschitz functions of the state variable with linear growth at infinity, theorem 3.3.2 p. 582 in [21] states that a strong comparison result holds for (43).

Since $||MDu||^2 = MDu \cdot MDu = \text{trace}\,[MDuDu^*M^*] = \text{trace}\,[M^*MDuDu^*] = (M^*MDu).Du$, we can rewrite (43) as

$$\left\{ \begin{array}{rrcl} -\dfrac{\partial u}{\partial t} - (b - M^*MDu).Du - \dfrac{1}{2}\text{trace}\left[\sigma\sigma^* D^2 u\right] & = 0 & \text{in} & \mathcal{O}, \\ u & = \Psi & \text{on} & \partial\mathcal{O}. \end{array} \right. \tag{44}$$

This gives us a simple idea for defining an approximating scheme which we again describe in the case $\mathcal{O} =]0, T[\times I\!R^n$.

Scheme 2.2 *Given $\Delta t > 0$ and $\Delta x > 0$, we construct a function $\overline{u}$ such that*

$$\overline{u}(T, x) = \Psi(x),$$

and

$$\overline{u}(t - \Delta t, x) = \sum_{h \in \mathcal{V}_{\Delta x}} \widetilde{p}(x, h)\overline{u}(t, x + h) = \widetilde{S}(\Delta t, \Delta x)_{\overline{u}(t,.)}(x). \tag{45}$$

This time the transition coefficients $\widetilde{p}$ depend on $\overline{u}$ in the following way

$$\begin{array}{rcl} \widetilde{p}(x, 0) & = & p(x, 0) - \dfrac{\Delta t}{(\Delta x)^2} \displaystyle\sum_{i=1}^{d} |\left(M^*M\delta^{\Delta x}\overline{u}(t, x)\right)_i|, \\ \widetilde{p}(x, \pm e_i\Delta x) & = & p(x, \pm e_i\Delta x) + \dfrac{\Delta t}{(\Delta x)^2} \left(\left(M^*M\delta^{\Delta x}\overline{u}(t, x)\right)_i\right)^{\pm}, \\ \widetilde{p}(x, (e_i \pm e_j)\Delta x) & = & \widetilde{p}(x, -(e_i \pm e_j)\Delta x) = p(x, (e_i \pm e_j)\Delta x), \\ \widetilde{p}(x, h) & = & 0 \quad \textit{in any other case,} \end{array}$$

where

$$\delta^{\Delta x}\overline{u}(t, x) = \left(\begin{array}{c} \overline{u}(t, x + e_1\Delta x) - \overline{u}(t, x) \\ \overline{u}(t, x + e_2\Delta x) - \overline{u}(t, x) \\ \cdots \\ \overline{u}(t, x + e_d\Delta x) - \overline{u}(t, x) \end{array} \right).$$

It is straightforward to check the consistency of this scheme with (43), and the property of commutation with constants. Under the C.F.L. condition

$$\Delta t \leq (\Delta x)^2 \left\{ \sum_{i=1}^{d} \left(\Delta x|b_i| + 2||\Psi||_\infty ||M||_\infty^2 + a_{ii} - \sum_{j \neq i} |a_{ij}| \right) \right\}^{-1}, \tag{46}$$

the scheme is also stable : we have $\widetilde{p}(x, h) \in [0, 1]$ for all (x, h) and $\displaystyle\sum_h \widetilde{p}(x, h) = 1$ for all x. So $\widetilde{S}$ is a contraction and thus $||\overline{u}||_\infty \leq ||\Psi||_\infty$.

To check the monotonicity condition, let us further assume that

$$\Delta t \leq (\Delta x)^3. \tag{47}$$

For any functions u and v such that $u \leq v$, we obtain the following chain of inequalities, denoting $\hat{p} = \widetilde{p} - p$ and adding the subscript p_u to indicate that the transition matrix belongs to u etc:

$$\begin{aligned}
\widetilde{S}(\Delta t, \Delta x)(u) - \widetilde{S}(\Delta t, \Delta x)(v) &= \sum_{h \in \mathcal{V}_{\Delta x}} (\widetilde{p}_u(x,h)u(t,x+h) - \widetilde{p}_v(x,h)v(t,x+h)) \\
&= \sum_{h \in \mathcal{V}_{\Delta x}} p(x,h)\left(u(t,x+h) - v(t,x+h)\right) \\
&\quad + \sum_{h \in \mathcal{V}_{\Delta x}} (\hat{p}_u(x,h)u(t,x+h) - \hat{p}_v(x,h)v(t,x+h)) \\
&\leq \sum_{h \in \mathcal{V}_{\Delta x}} (\hat{p}_u(x,h)u(t,x+h) - \hat{p}_v(x,h)v(t,x+h)) \\
&\leq \frac{\Delta t}{(\Delta x)^2} C \leq \Delta x C.
\end{aligned}$$

Here, thanks to stability, the constant C depends only on bounds on Ψ, M, b, σ and on the Lispschitz constants of b and σ.

Moreover, according to [21], we have a strong comparison result for the PDE (43).

Hence, Theorem 2.1 allows us to conclude that $\overline{u}$ converges locally uniformly to the unique continuous viscosity solution u of (43), as Δt and Δx converge to 0.

3 Simulations and their interpretations

We now choose different types of simple toy agents and different temperature models (as given in section 1.6.3). We concentrate on simulating the expectation of the additional security X_2 (in subsection 3.1), the maximal expected utility J^a for each agent (in subsection 3.2) and the optimal strategy of investment in X_2 (in subsection 3.3). We use the following concrete models :

Model A

The time horizon is chosen to be $T = 2$. We use an Ornstein-Uhlenbeck process to describe the climate process K, with the following coefficients :

$$dK_s = -K_s + \frac{1}{2} dW_{2,s}, \quad s \in [0,T].$$

Here we consider only two model agents, a fisher and a bank as described in section 1.6.1. The fisher's random income function is

$$H^f = \int_0^T \varphi^f(K_s) ds,$$

with $\varphi^f(k) = 5\exp(-10k^2)$, for all $k \in \mathbb{R}$. This means that the optimal temperature for the fisher is normalized to be 0. The bank has no risky income, i.e. $H^b = 0$. We assume that each agent uses the risk aversion coefficient $\alpha^f = \alpha^b = 1$.

Model B
The temperature is now modelled by a periodically forced bi-stable temperature process with coefficients

$$dK_s = -8(K_s^3 - K_s) - \sin(2\pi s) + 4.5dW_{2,s}, \quad s \in [0, T].$$

See Figure 1 for a sample path of this process. Again, we choose $T = 2$ for the time horizon, i.e. 2 periods of the temperature process. This process is close to the high temperature value $k_2 = 2.5$ for $t \in [0; 0.5] \cup [1; 1.5]$ and symmetrically close to the low value $k_1 = -2.5$ for $t \in [0.5; 1] \cup [1.5; 2]$. Again we consider only two agents, a fisher and a farmer with respective income

$$H^1 = \int_0^T 5\exp\left(-10(K_s - k_1)^2\right) ds \text{ and } H^2 = \int_0^T 5\exp\left(-10(K_s - k_2)^2\right) ds,$$

where the optimal temperature is $k_1 = -2.5$ for the fisher and $k_2 = 2.5$ for the farmer, which coincide with the bistable states of the temperature process. We again assume that each agent uses the risk aversion coefficient $\alpha^1 = \alpha^2 = 1$.

Model C
This model uses the same characteristics as model B except for the time horizon, which is now chosen to be $T = 3/2$, i.e. 3 half-periods for the temperature process K. This gives an advantage to the farmer, since the temperature spends 1 unit of time i.e. 2/3 of the trading interval near the meta-stable state favorable for the farmer, and only 0.5 units of time near its low meta-equilibrium favorable for the fisher.

In all the models, the share price is a geometrical Brownian motion given by (33) with very strong coefficients $b_1 = 1$ and $\sigma_1 = 1$.

3.1 Expectation of X_2

Here we exhibit the expectation of $X_{2,t}$ at the same time $t = 1.5$ for each model, as a function of the initial condition (x_1, k) at time $t = 0$. X_2 is starting from 1 at time $t = 0$.

3.1.1 Model A

We observe that $\mathbb{E}[X_2]$ has a minimum if the temperature starts from the value 0 which is optimal for the fisher. Indeed, in this case, the fisher's income is maximal, since the temperature will only slightly oscillate around 0. So there is no need to transfer risk from the fisher to the bank : the expectation of X_2 (starting from $k = 0$) almost stays

at the initial value 1. This can serve as an indication that we could interpret the size of X_2 as an appreciation rate for the trading of climate risk among the affected agents.

If, on the other hand, the initial temperature is far from 0, the fact that the expectation of X_2 grows with time indicates that the fisher has an interest to invest in X_2. In this case the growth of X_2 compensates the smaller income of the fisher.

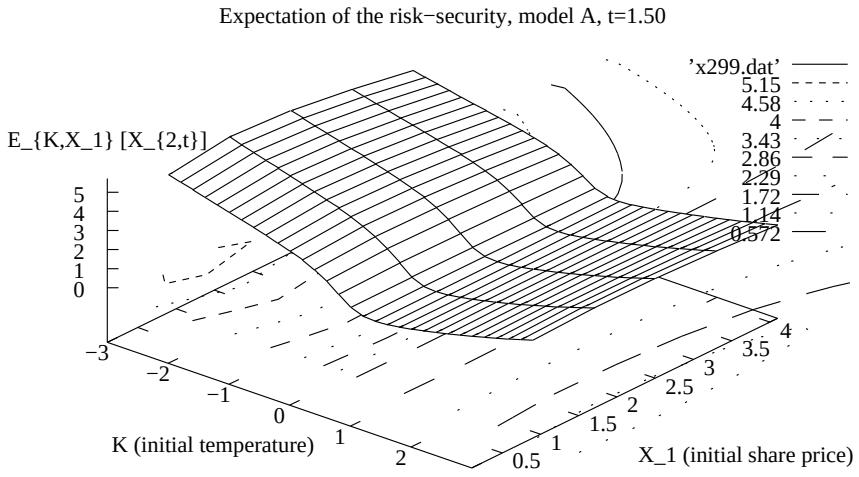

Figure 2: The expectation of $X_{2,t}$ (model A).

3.1.2 Model B

The dependence on K seems reversed in this model as compared to model A. We now see that the expectation of X_2 is maximal when starting from $k = 0$, i.e. in the middle between the optimal temperatures. At this temperature obviously both agents like to trade risk, since on the scale between -2.5 and 2.5 it corresponds to the worst situation for the totality of the affected agents. This is why X_2 is expected to be higher.

3.1.3 Model C

This case is very similar to the preceding one. We just observe that the maximum of the expectation has been translated to lower temperatures, to account for the difference of exposition of the agents.

3.2 Optimal value J^a

We now turn to numerical simulations of the underlying optimal control problem. First we will show the value J^a, the optimal utility, for both agents involved, at different times $t \in [0, 1]$, as a function of the current value of the temperature k at time t. Due to simplicity of our model for the share price X_1, and since the climate affected agents are chosen to have incomes not depending on X_1, J^a does not depend on $X_{1,t}$.

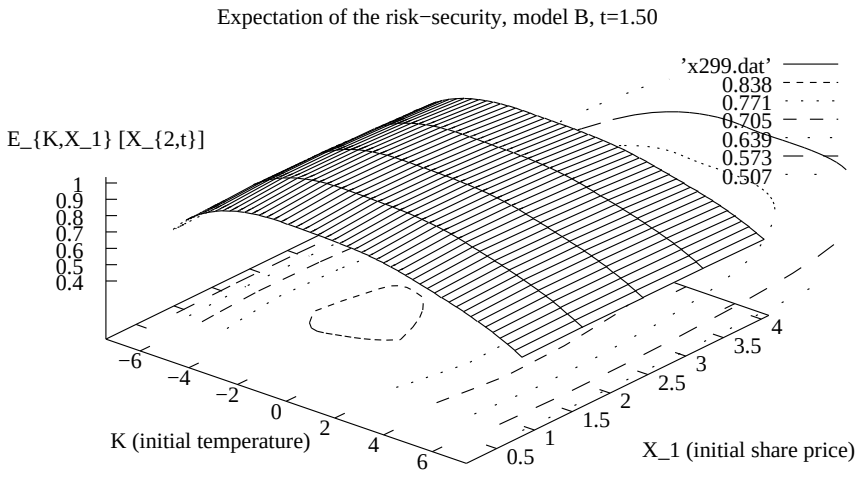

Figure 3: The expectation of $X_{2,t}$ (model B).

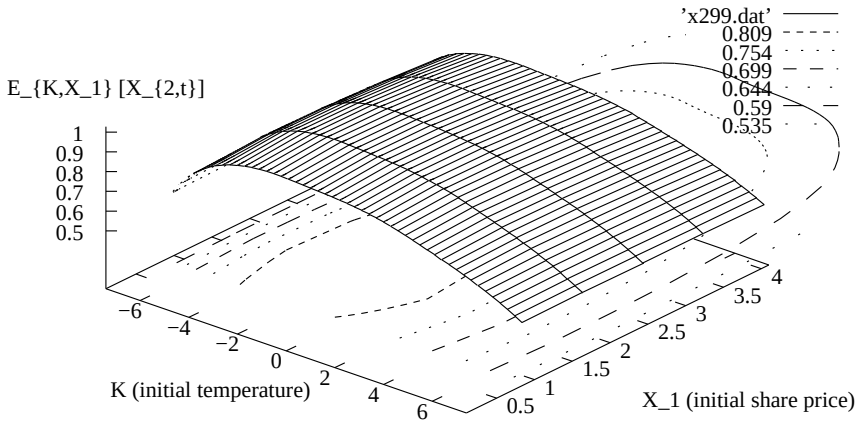

Figure 4: The expectation of $X_{2,t}$ (model C).

In a real situation, the wealth process of an agent with initial capital $v_0^a = 1$ should increase with time. Here we assume that the initial capital of each agent is normed by $v_t^a = 1$ at time t. This is why the expected terminal value J^a we simulated is decreasing with time. Indeed we have $J_T^a = 1$. This is not a limitation, since J^a depends in a multiplicative way on the initial value of the wealth process of agent a. Also, in our simulations we are more interested in exhibiting the dependence of J^a on k at different times.

3.2.1 Model A

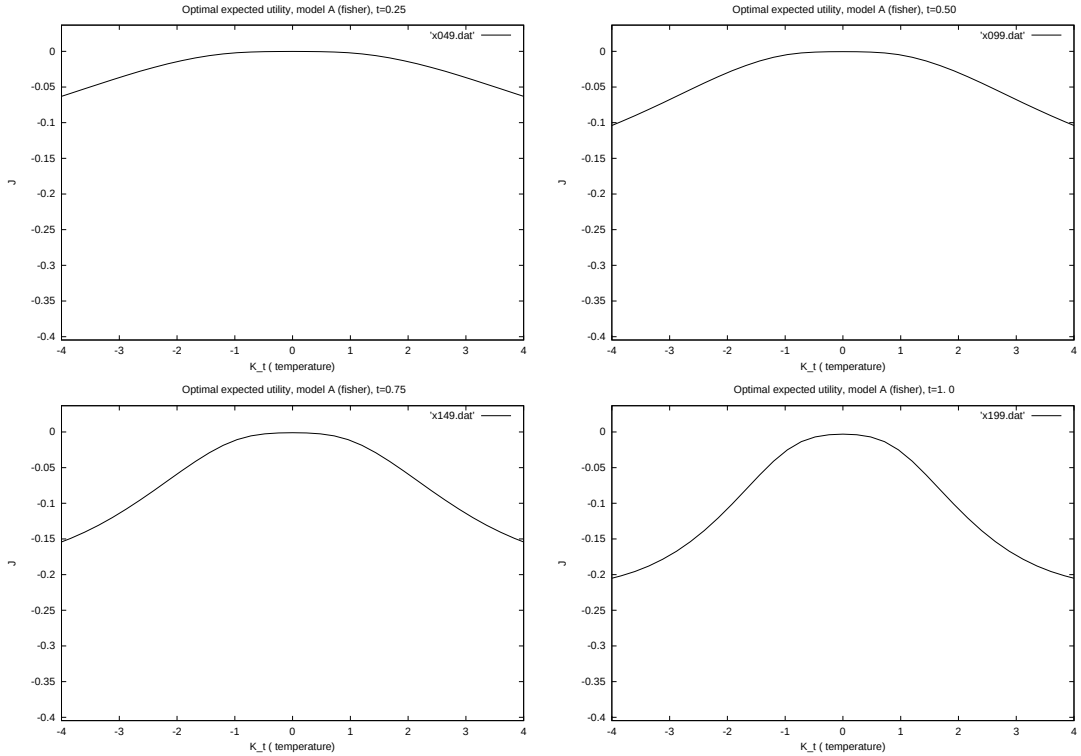

Figure 5: The maximal expected utility J for the fisher (model A).

Let us recall that if the fisher does not invest in X_2, his only benefits will be given by H^f. The dependence of this random income on K shows a narrow peak around the optimal temperature 0. The fisher benefits a lot when the temperature is near 0 and almost nothing not very far from there. We clearly observe that investing in X_2 reduces the fisher's risk exposure. The optimal utility curve exhibited by the simulations at different times has a very wide maximal zone around 0.

The bank's optimal utility curve as a function of temperature shows the following features. The bank's situation is best if the temperature is in a neighborhood of 0, but not too close to 0. Indeed, if it is very close to 0, it is not interesting for the fisher to invest in X_2: there is no risk to transfer. If temperature changes a little, both agents clearly have an interest in the exchange of X_2. If the temperature is too far from 0, then

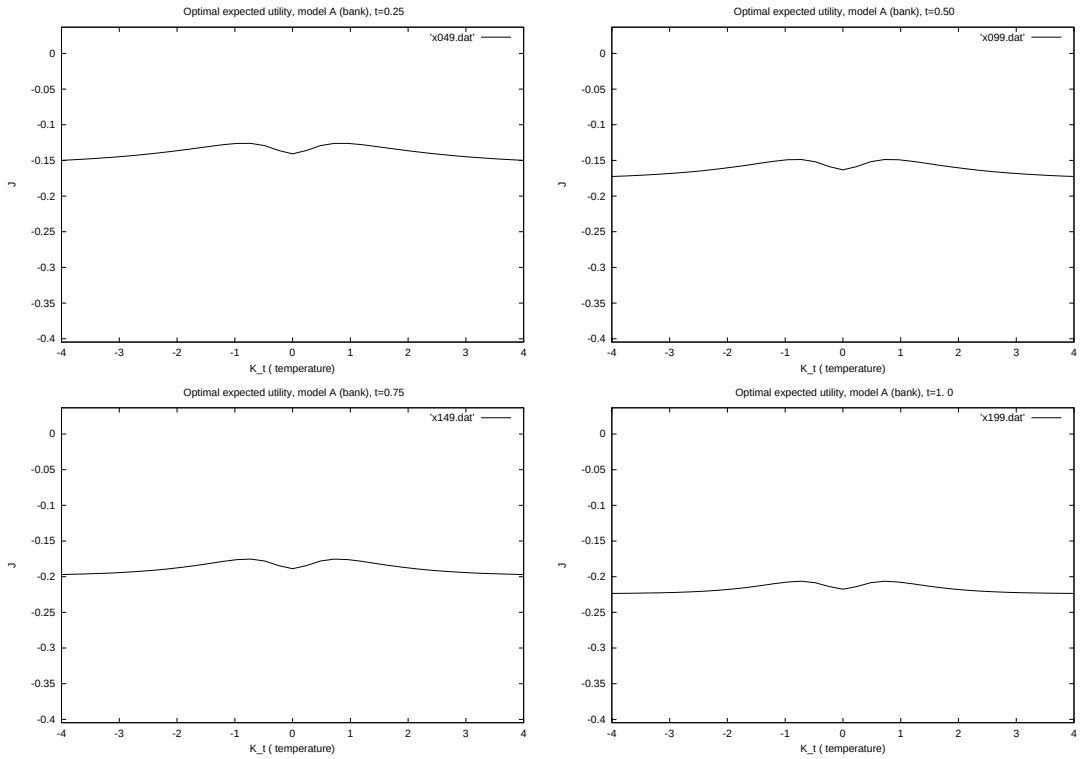

Figure 6: The maximal expected utility J for the bank (model A).

of course the situation is bad for both agents: the fisher has not much money to invest. The latter situation is, however, very unlikely to happen. The Ornstein-Uhlenbeck process used here reaches ± 2 before time 1 only with a very small probability.

3.2.2 Models B and C

We just show diagrams from the farmer's point of view for theses models, since there is symmetry in the exposure of the agents. The optimal expected utility for the farmer seems a very flat curve, which is maximal around the optimal temperature. This may indicate that trading on the risk asset brought security to the agents. There is no real qualitative difference in the shape of the curves between model B and model C. We just observe that model C reflects, of course, a better situation for the farmer.

3.3 Optimal strategies

We finally describe the optimal amount of money to be invested in X_2 by each agent during the trading interval, i.e. the strategy of investment which allow the agents to attain maximal expected utility J^a.

Since only two agents are active on the market, the local equilibrium condition (19) implies that at each time t the entire quantity of X_2 sold by one agent is bought by the other, i.e.

$$\pi^f_{2,t} = -\pi^b_{2,t} \text{ in model A, or } \pi^r_{2,t} = -\pi^f_{2,t} \text{ in models B and C.}$$

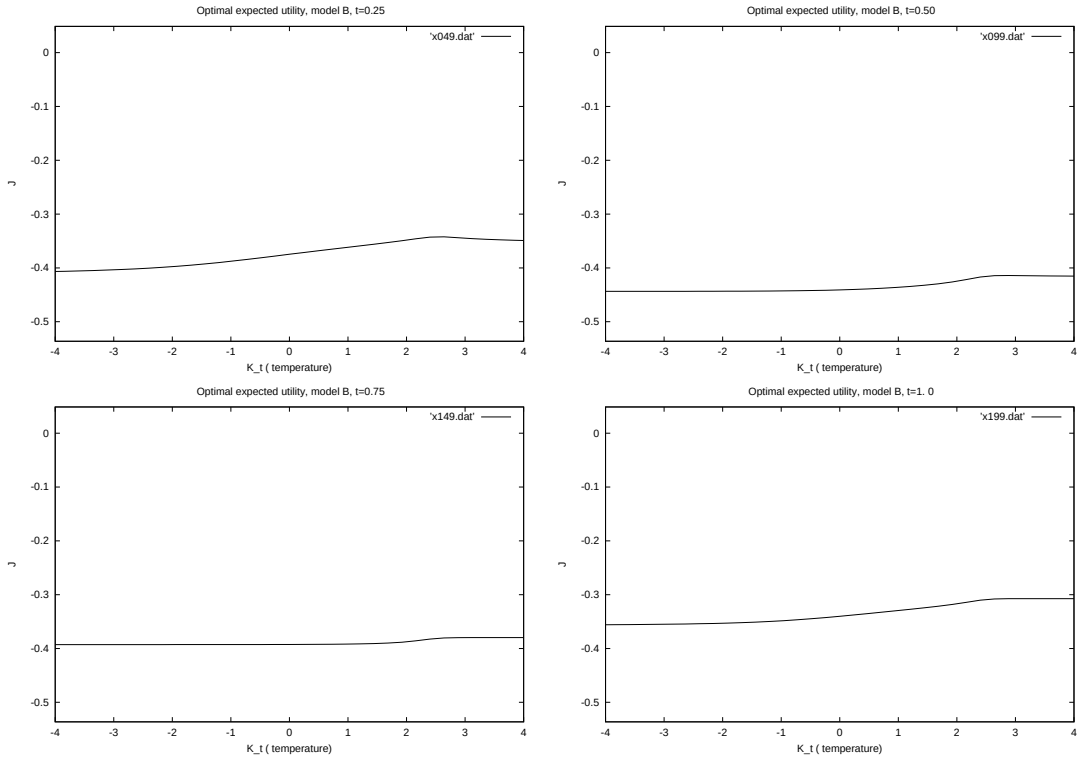

Figure 7: The maximal expected utility J for the farmer (model B).

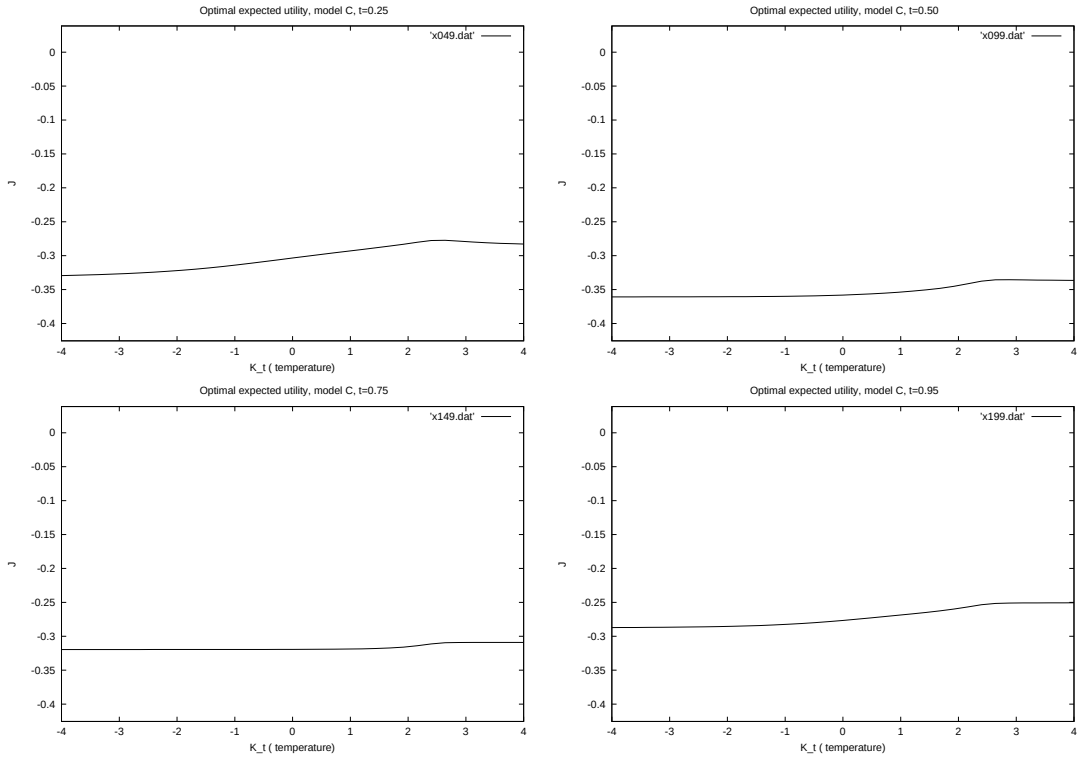

Figure 8: The maximal expected utility J for the farmer (model C).

Therefore it will be enough to show diagrams of the strategy of one agent (fisher in model A and farmer in models B and C). Since we are able to approximate numerically the strategies of both agents, we remark that the local equilibrium condition may be used to check the accuracy of our schemes.

We show the optimal strategies as functions of t (on the period [0,1]) and the current temperature K_t. As in the preceding subsection, in our simple example this strategy does not depend on $X_{1,t}$. The diagrams also display the optimal amount of money to be exchanged between the agents, from the selected agent's point of view.

3.3.1 Model A

Here we only show the fisher's optimal strategy π_2^f. At the optimal temperature for fishing $K_t = 0$, the fisher makes his maximal profit, and we observe that there is no exchange of risk trading money. As soon as the temperature grows a little, the fisher has to buy a certain quantity of X_2 from the bank. This exchange will bring security to the fisher and profits to the bank.

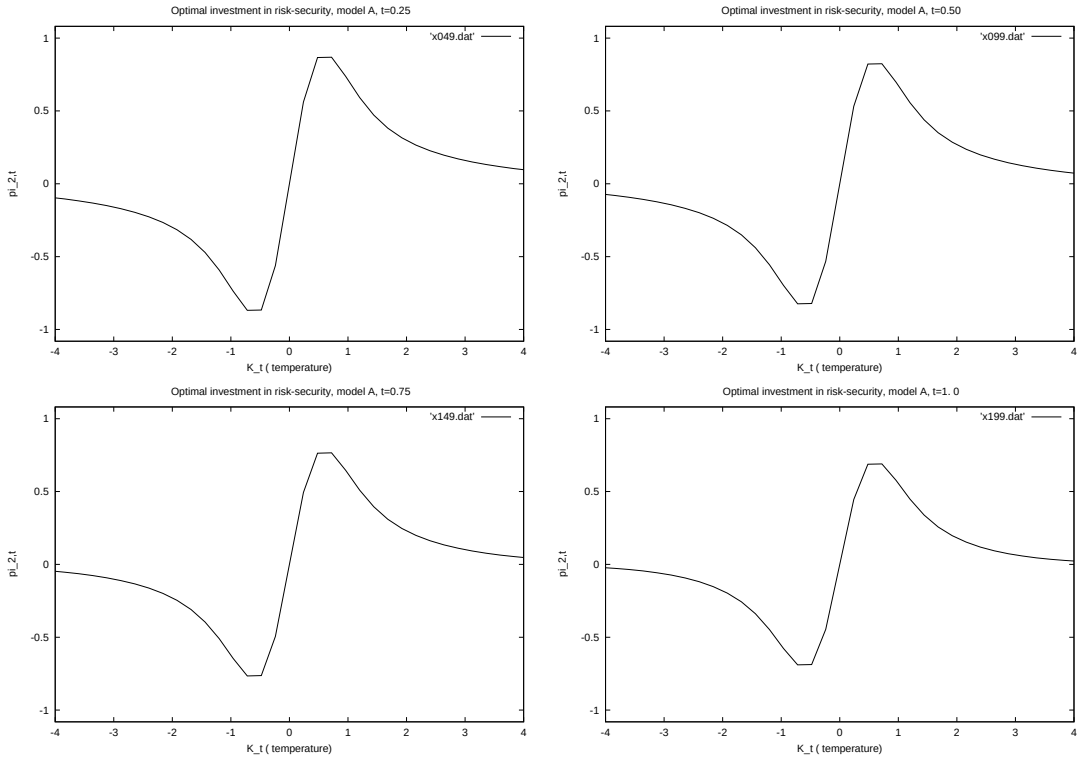

Figure 9: The optimal strategy for the fisher (model A).

3.3.2 Model B and C

We only show the farmer's optimal strategy π_2^r. We can first notice, by taking into account the estimates for the expectation of X_2 in subsection 3.1, that the appreciation

of risk trading is very low compared to model A.

On theses diagrams, we see that the farmer invests in X_2 when the temperature is high in the first half period [0,0.5], i.e. an interval that favors him, and sells X_2 (to the fisher) when the temperature is low, for $t \in [0.5, 1]$, i.e. when he needs money. This reflects the intuition that the agents have an interest to share their risks by exchanging money this way.

Again, the qualitative difference between models B and C is not big. We just observe that the farmer invests a little more than the fisher.

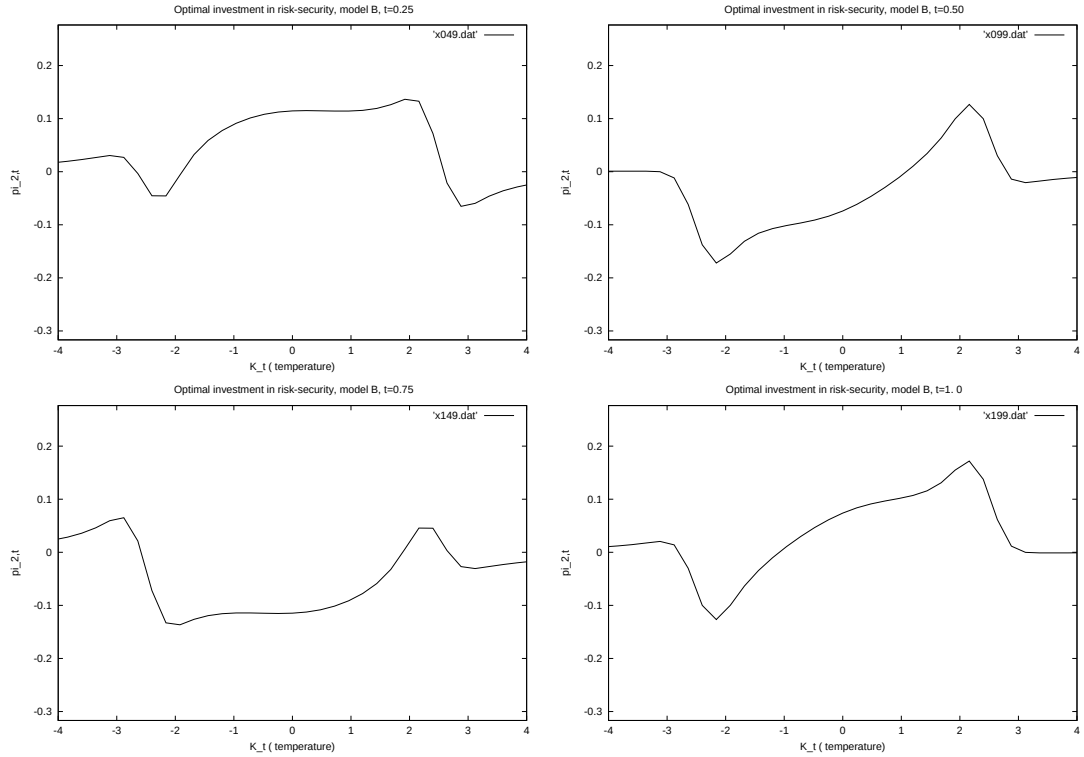

Figure 10: The optimal strategy for the farmer (model B).

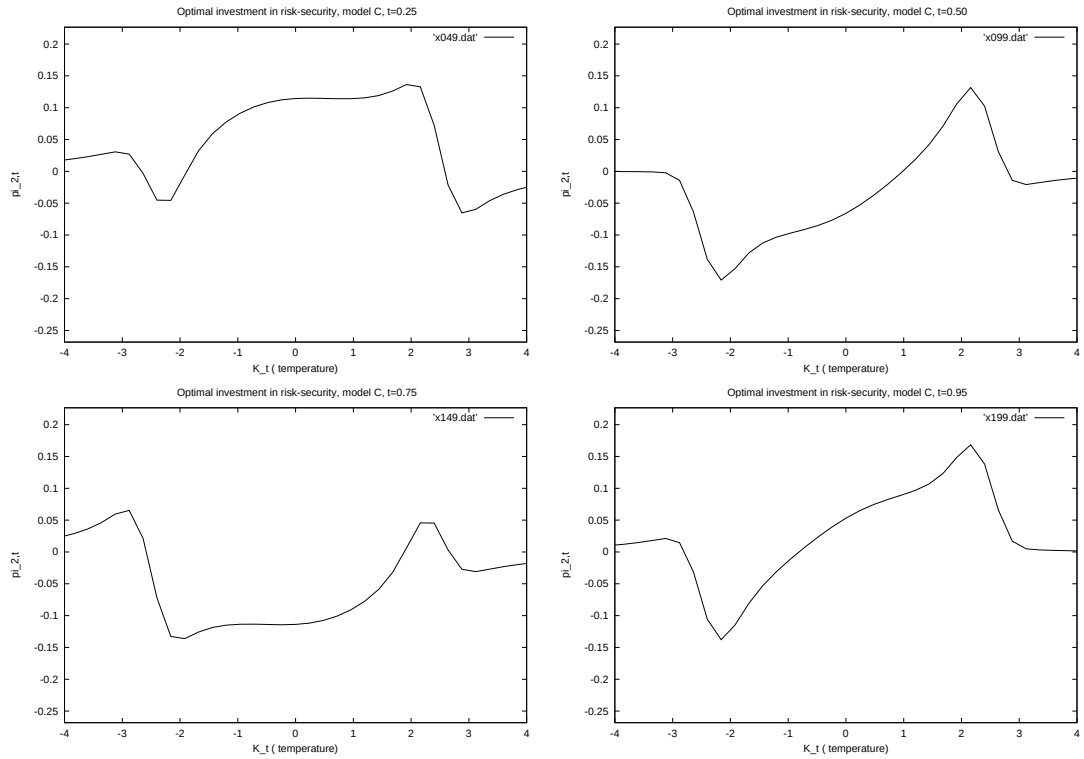

Figure 11: The optimal strategy for the farmer (model C).

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