Boundary optimal flow control with state constraints

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The numerical solution of the Dirichlet boundary optimal control problem of the Navier-Stokes equations in presence of pointwise state constraints is investigated. A Moreau-Yosida regularization of the problem is proposed to obtain regular multipliers. Optimality conditions are derived and the convergence of the regularized solutions towards the original one is presented. The paper is ended with a numerical experiment.

1 Problem statement

Consider a bounded regular domain $\Omega \subset \mathbb{R}^2$. Further, let Γ_1 be an open subset of the boundary Γ. Our aim is to find a solution of the following optimal control problem:

$$(\mathcal{P}) \qquad \begin{cases} \min\ J(y,u) := \frac{1}{2}\|y - z_d\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\,\|u\|^2_{\mathbf{H}^1_0(\Gamma_1)} \\ \text{subject to} \\ -\nu\Delta y + (y\cdot\nabla)y + \nabla p = f & \text{in } \Omega \\ \text{div } y = 0 & \text{in } \Omega \\ y|_\Gamma = g + \mathcal{B}u & \text{on } \Gamma \\ a \le y \le b \text{ a.e.} & \text{in } \Omega, \end{cases}$$

where $\alpha > 0,\ \nu > 0$ denotes the viscosity coefficient and z_d is the desired state. The operator $\mathcal{B}$ is the extension by zero operator. The functions $f \in \mathbf{L}^2(\Omega)$ and $g \in \mathbf{H}^1(\Gamma)$, with $\int_\Gamma g\cdot\vec{n}\ \ d\Gamma = 0$, are given. Moreover, the lower and upper bounds $a, b \in \mathbf{C}(\overline{\Omega})$ satisfy $a(x) < b(x)$ for all $x \in \overline{\Omega}$. The inequalities in the last line of $(\mathcal{P})$ have to be understood componentwise. We denote by $(\cdot,\cdot)_X$ the inner product in the Hilbert space X and by $\|\cdot\|_X$ the associated norm. The bold notation stands for the product of spaces. We introduce the solenoidal space $V = \{v \in \mathbf{H}^1_0(\Omega) : \text{div } v = 0\}$, the closed subspace $\mathbf{H} := \{v \in \mathbf{H}^1(\Omega) : \text{div } v = 0\}$ and the trilinear form $c : \mathbf{H}\times\mathbf{H}\times\mathbf{H} \to \mathbb{R}$ defined by

$$(1) \qquad c(u,w,v) = ((u\cdot\nabla)w, v)_{\mathbf{L}^2(\Omega)}.$$

We consider the control space $\mathcal{U} := \{v \in \mathbf{H}^1_0(\Gamma_1) : \int_{\Gamma_1} v\cdot\vec{n}\ \ d\Gamma = 0\}$ and assume that there exists at least a feasible point $(y,u) \in \mathbf{H}\times\mathcal{U}$ satisfying the state equation and the inequality state constraints in $(\mathcal{P})$. Hence, by standard arguments, there exists an optimal solution $(y^*, u^*) \in \mathbf{H}\times\mathcal{U}$ of $(\mathcal{P})$; cf. [1].

2 Moreau-Yosida regularization

We consider a Moreau-Yosida regularization technique for solving $(\mathcal{P})$. The basic idea of the regularization is to consider, alternatively to the state constrained problem, the following penalized control problem:

$$(\mathcal{P}_\gamma) \qquad \begin{cases} \min\ J(y,u) := \frac{1}{2}\|y - z_d\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\,\|u\|^2_{\mathbf{H}^1_0(\Gamma_1)} + \frac{\gamma}{2}\|\max(0, y-b)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\gamma}{2}\|\min(0, y-a)\|^2_{\mathbf{L}^2(\Omega)} \\ \text{subject to} \\ -\nu\Delta y + (y\cdot\nabla)y + \nabla p = f & \text{in } \Omega \\ \text{div } y = 0 & \text{in } \Omega \\ y|_\Gamma = g + \mathcal{B}u & \text{on } \Gamma. \end{cases}$$

This regularization approach has been utilized for state constrained control problems in [4]. Hereafter, the same penalization was applied to different constrained control problems (see [1, 3]).

Since the state constraints are replaced by a penalized cost functional, the analysis of the control problem follows the lines of [1]. In this manner existence of an optimal solution and Lagrange multipliers can be justified. The following optimality

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We impose a state constraint on the backward fluid flow in sector Ω_S depicted in Figure..... In this manner a substantial reduction of the recirculation after the step is obtained. Specifically, the state constraint is given by $y_1 \geq 0$. The resulting controlled state is depicted in Figure... The remaining parameters take the values $\alpha = 0.005,\ Re = 800,\ \gamma = 10^7$ The regularized problems are solved by means of a semi-smooth Newton method as developed in [2]. The algorithm is based on a reformulation of the complementarity problem as an operator equation involving the max and min functions. A main feature of this type of algorithms is its local superlinear convergent behavior. In our case, the SSN algorithm stops after 6 iterations with 320 grid points The resulting linear systems in each semi-smooth Newton iteration are solved exactly using MATLAB's sparse solver.

Fig. 1 Example 1: optimal control

system follows from the first order necessary conditions: there exist multipliers $\lambda \in \mathbf{H}^2(\Omega) \cap V$, $q \in H^1(\Omega) \cap L^2_0(\Omega)$ and $\sigma \in \mathbb{R}$ such that

$$\begin{aligned} -\nu\Delta y_\gamma + (y_\gamma \cdot \nabla)y_\gamma + \nabla p_\gamma &= f \\ \operatorname{div} y_\gamma &= 0 \\ y_\gamma|_\Gamma &= g + \mathcal{B}u_\gamma, \end{aligned} \tag{2}$$

$$\begin{aligned} -\nu\Delta\lambda - (y_\gamma \cdot \nabla)\lambda + (\nabla y_\gamma)^T\lambda + \nabla q &= z_d - y_\gamma - \mu \\ \operatorname{div} \lambda &= 0 \\ \lambda|_\Gamma &= 0, \end{aligned} \tag{3}$$

$$-\alpha\Delta_\Gamma u_\gamma = \mathcal{B}^\star\left(-\nu\frac{\partial\lambda}{\partial\vec{n}} + q\vec{n}\right) + \sigma\vec{n} \qquad \text{in } \mathbf{H}^1_0(\Gamma_1), \tag{4}$$

$$\mu = max(0, \gamma(y_\gamma - b)) + min(0, \gamma(y_\gamma - a)), \tag{5}$$

where Δ_Γ stands for the Laplace-Beltrami operator, hold in variational form.

In the following theorem a main result with respect to the regularization is stated which is followed from [5].

Theorem 2.1 *Let $\nu > \mathcal{M}(y^*) := \sup_{v\in V} \frac{|c(v,y^*,v)|}{\|v\|_V^2}$ hold for every optimal state y^*. If there exists a feasible solution for $(\mathcal{P})$, then the regularized solutions $(y_\gamma, p_\gamma, u_\gamma)$ converge strongly in $\mathbf{H}^{3/2}(\Omega) \times H^{1/2}(\Omega) \times \mathbf{H}^1_0(\Gamma_1)$ to an optimal solution of $(\mathcal{P})$.*

3 Numerical test

For the numerical experiment we utilize a forward facing step channel. The fuid flows from left to right with infow boundary condition of parabolic type and outflow "do nothing" condition. The domain is discretized with an homogeneous staggered grid. Also a first order upwind scheme is used for the approximation of the convective term.

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