Nonnegativity of Schur complements of nonnegative projectors*

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Abstract

In this note we prove that the Schur complement of a nonnegative idempotent matrix, i.e. a nonnegative projector, is again a nonnegative idempotent matrix for certain generalized inverses.

Key words. Nonnegative projectors; idempotent matrices; Schur complement; Moore-Penrose inverse; generalized inverse

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1 Introduction

Our study was motivated by the analysis of positive differential-algebraic equations (DAEs) or descriptor systems such as

$$\begin{aligned} E\dot{x} &= Ax + Bu,\ x(t_0) = x_0, \\ y &= Cx; \end{aligned}$$

or as a discrete time system

$$\begin{aligned} Ex_{k+1} &= Ax_k + Bu_k,\ x_0 \text{ given}, \\ y_k &= Cx_k; \end{aligned}$$

where E, A are real $n \times n$ matrices and $B \in \mathbb{R}^{n\times m}$. In the continuous-time case, the state x, input u and output y are real-valued vector functions. In the discrete-time case x, u and y are real-valued vector sequences. Positive systems are systems whose state and output variables take only nonnegative values at all times t for nonnegative input and nonnegative initial state, [3], [5], [7]. In the descriptor case, the choice of the right projector onto the deflating subspace that corresponds to the finite eigenvalues of the matrix pair (E, A) is crucial for the analysis [8]. It turned out that nonnegative projectors play an important role in the context of positive

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systems [12]. Note that in the linear algebra literature, projectors are also referred to as idempotent matrices. Finally, Schur complements constitute a fundamental tool in applications [14], in particular such as algebraic multigrid methods [13] or model reduction [6]. However, one has to ensure that properties such as nonnegativity are preserved.

Let $\langle n\rangle := \{1,\ldots,n\}$ and assume that $\alpha \subset \langle n\rangle, \alpha^c := \langle n\rangle\backslash\alpha, \beta \subset \langle n\rangle$ be three nonempty sets. For $A \in \mathbb{R}^{n\times n}$ denote by $A[\alpha,\beta]$ the submatrix of A composed of the rows an columns indexed by the set α and β respectively. Assume that $A[\alpha,\alpha]$ is invertible. Then the α Schur complement of A is given by

$$A(\alpha) := A[\alpha^c,\alpha^c] - A[\alpha^c,\alpha]A[\alpha,\alpha]^{-1}A[\alpha,\alpha^c]. \tag{1}$$

If $A[\alpha,\alpha]$ is not invertible we define

$$A_{\mathrm{ginv}}(\alpha) := A[\alpha^c,\alpha^c] - A[\alpha^c,\alpha]A[\alpha,\alpha]^{\mathrm{ginv}}A[\alpha,\alpha^c], \tag{2}$$

for some semi inverse $A[\alpha,\alpha]^{\mathrm{ginv}}$ [1]. The α Moore-Penrose Schur complement of A is defined as

$$A_{\dagger}(\alpha) := A[\alpha^c,\alpha^c] - A[\alpha^c,\alpha]A[\alpha,\alpha]^{\dagger}A[\alpha,\alpha^c],$$

where $A[\alpha,\alpha]^{\dagger}$ is a Moore-Penrose inverse, [2], [9], [10]. Assume that A is a nonnegative projector. We show here that we can always define a semi inverse $A[\alpha,\alpha]^{\mathrm{ginv}}$ such that $A_{\mathrm{ginv}}(\alpha)$ is a nonnegative projector. If $A[\alpha,\alpha]$ does not have zeros on its main diagonal then $A[\alpha,\alpha]^{\mathrm{ginv}} = A[\alpha,\alpha]^{\dagger}$. In the case that $A[\alpha,\alpha]$ has a zero on its main diagonal, $A_{\dagger}(\alpha)$ may fail to be nonnegative and therefore a suitable semi inverse has to be defined in a different way.

We now briefly survey the contents of the paper. In Section 2 we review some well-known facts about nonnegative matrices and nonnegative projectors. In Section 3 we first prove that if $A[\alpha,\alpha]$ does not have a zero on its man diagonal, then the Schur complement of a nonnegative projector constructed via the Moore-Penrose inverse is again a nonnegative projector. Then we give an example which shows that the above result is false in general if $A[\alpha,\alpha]$ has a zero on its main diagonal. Finally, we show that if A is a nonnegative projector then $A_{\mathrm{ginv}}(\alpha)$ is always a nonnegative projector for a suitably defined semi inverse.

2 Preliminaries

2.1 Nonnegative Matrices

Let $\mathbb{R}_+ = [0,\infty)$. For $x \in \mathbb{R}^n_+$ denote by $x > 0$ a vector whose all coordinates are positive. Let $A \in \mathbb{R}^{n\times n}_+$. Denote by $\rho(A)$ the spectral radius of A. Denote by $\Pi_n \subset \mathbb{R}^{n\times n}_+$ the set of permutation matrices of order n.

Assume that $\rho(A) > 0$. Then there exists a permutation matrix $Q \in \Pi_n \subset \{0,1\}^{n\times n}$ such that $B = QAQ^T$ is in Frobenius normal upper triangular form [4],

$$B = \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1k} \\ 0 & B_{22} & \ldots & B_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & B_{kk} \end{bmatrix}, \; B_{ij} \in \mathbb{R}^{m_i\times m_j}_+, \; i,j = 1,\ldots,k, \tag{3}$$

where $\rho(B_{11}) \geq \ldots \geq \rho(B_{kk})$. We assume the following.

1. If $\rho(B_{kk}) > 0$ then $B_{11}, \dots, B_{kk}$ are irreducible.

2. If $\rho(B_{kk}) = 0$, then $\rho(B_{(k-1)(k-1)}) > 0$ and $B_{11}, \dots, B_{(k-1)(k-1)}$ are irreducible.

In what follows it will be convenient to adopt the following notation. Let $\alpha = \{\alpha_1, \dots, \alpha_l\}$, $\beta = \{\beta_1, \dots, \beta_m\} \subseteq \langle n \rangle$ be two nonempty sets. We do not assume that the sequences $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ are arranged in decreasing order and define $A[\alpha, \beta] := [a_{\alpha_i \beta_j}]_{i,j=1}^{l,m} \in \mathbb{R}^{l \times m}$. So if $\alpha = \{\alpha_1, \dots, \alpha_n\} = \langle n \rangle$ then $A[\alpha, \alpha] = QAQ^T$ for some $Q \in \Pi_n$.

Recall a special case of Rothblum's theorem [11].

Theorem 2.1 *Let $B \in \mathbb{R}_+^{n \times n}, \rho(B) > 0$ be in the Frobenius normal form (3) satisfying the conditions 1-2. Assume that $\rho(B_{11}) = \ldots = \rho(B_{ll}) > \rho(B_{(l+1)(l+1)})$, where $1 \le l \le k$ and we define $\rho(B_{(k+1)(k+1)}) = 0$. Then $\rho(B)$ is geometrically simple eigenvalue, i.e. B has l linearly independent eigenvectors corresponding to $\rho(A)$, if and only if $B_{ij} = 0$ for $1 \le i < j \le l$.*

2.2 Nonnegative Projectors

A matrix $P \in \mathbb{R}^{n \times n}$ is called *projector* if $P^2 = P$. Note that in the linear algebra literature, a matrix that is a projector is called idempotent.

Theorem 2.2 *Let $B \in \mathbb{R}_+^{n \times n}, \rho(B) > 0$ be a projector in the Frobenius normal form (3) satisfying the conditions 1-2. Then, for the block structure in (3), we have*

$$\begin{aligned} &(i)\ B_{ii} = u_i v_i^T,\ 0 < u_i, v_i \in \mathbb{R}_+^{m_i},\ v_i^T u_i = 1,\ i = 1, \dots, l, \\ &(ii)\ B_{ij} = 0,\ 1 \le i < j \le l, \end{aligned} \tag{4}$$

and for the index l one of the following conditions holds:

1. $l = k$.

2. $l = k-1$. Then, $B_{kk} = 0_{m_k \times m_k}$ and $B_{ik} = u_i w_i^T, w_i \in \mathbb{R}_+^{m_k}$ for $i = 1, \dots, k-1$.

Proof. We consider the two cases in the Frobenius normal upper triangular form (3).
1. Suppose that $\rho(B_{ii}) > 0$ for $i = 1, \dots, k$. For a projector, this is equivalent to $\rho(B_{ii}) = 1$ for $i = 1, \dots, k$. Furthermore, each B_{ii} is irreducible, which implies that each B_{ii} has exactly one eigenvalue equal to 1 and all other eigenvalues are zero. Moreover, since any idempotent matrix is diagonable we deduce that $\operatorname{rank} B_{ii} = 1$. Hence, $B_{ii} = u_i v_i^T \in \mathbb{R}_+^{m_i \times m_i}$. Since B_{ii} is irreducible we can assume that $u_i, v_i > 0$. As $\rho(B_{ii}) = 1$ we deduce $v_i^T u_i = \operatorname{trace}(B_{ii}) = \operatorname{rank}(B_{ii}) = 1$. By Theorem 2.1, we obtain condition (ii) of (4).
2. Assume now that $\rho(B_{kk}) = 0$. Since $B_{kk}^2 = B_{kk}$ we deduce that $B_{kk} = 0$. By the proof of case 1 of the Frobenius normal upper triangular form, we obtain condition (4) for $l = k - 1$. Furthermore, the equality $B = B^2$ implies that $B_{ik} = B_{ii} B_{ik}$ for $i < k$. So $B_{ik} = u_i w_i^T$, where $w_i = B_{ik}^T v_i \in \mathbb{R}_+^{m_k}$. $\square$

3 Schur complements of nonnegative idempotents

Recall that for $A \in \mathbb{R}^{n\times n}$ a matrix $A^{\mathrm{ginv}} \in \mathbb{R}^{n\times n}$ is called a *semi inverse* [1] if

$$AA^{\mathrm{ginv}}A = A, \quad A^{\mathrm{ginv}}AA^{\mathrm{ginv}} = A^{\mathrm{ginv}}. \tag{5}$$

Denote by $A^{\dagger}$ the Moore-Penrose inverse of A that is uniquely defined by the following properties, see e.g. [2]:

$$\begin{aligned} AA^{\dagger}A &= A \\ A^{\dagger}AA^{\dagger} &= A^{\dagger} \\ (A^{\dagger}A)^T &= A^{\dagger}A \\ (AA^{\dagger})^T &= AA^{\dagger} \end{aligned} \tag{6}$$

The following lemma is well known, see e.g. [2].

Lemma 3.1 *Let $A = xy^T, 0 \neq x, y \in \mathbb{R}^n$. Then any $A^{\mathrm{ginv}} = zw^T,\ z, w \in \mathbb{R}^n$ such that $(y^Tz)(w^Tx) = 1$. In particular $A^{\dagger} = \frac{1}{(x^Tx)(y^Ty)}yx^T$.*

3.1 Nonnegativity of Moore-Penrose inverse Schur complement

We first prove a special case in Theorem 3.2, where we assume that $A[\alpha, \alpha]$ does not have zero diagonal entries. We show that in this case the Schur complement constructed via the Moore-Penrose inverse is again a nonnegative projector. Note that this includes the case when $A[\alpha, \alpha]$ is invertible. However, this result is false for the general case of the Moore-Penrose Schur complement. A counterexample is given in the next Section 3.2, in Example 3.9. .

Theorem 3.2 *Let $A \in \mathbb{R}_+^{n\times n}, A \neq 0$ be idempotent. Then for any nonempty $\alpha \subsetneqq \langle n\rangle$, such that $A[\alpha, \alpha]$ does not have zero diagonal elements, $A_{\dagger}(\alpha)$ is a nonnegative idempotent matrix.*

Proof. It is enough to prove the theorem for the corresponding matrix B that is in the Frobenius normal upper triangular form given by Theorem 2.2. Assume that α is chosen. Without loss of generality we can assume that $B_{kk} = 0$ and $B_{ii} = B[\beta_i, \beta_i],\ i = 1, \dots, k$, where $\beta_1, \dots, \beta_k$ is a disjoint partition of $\langle n\rangle$. Furthermore, there are three following possibilities :

(i) $\beta_i \subset \alpha^c$ for $i = k$ and $i = 1, \dots, q < k-1$, where $q \geq 0$, i.e. if $q \geq 1$ and $i \leq q$ then the whole nonzero block B_{ii} is chosen for $B[\alpha^c, \alpha^c]$;

(ii) $\alpha_i := \beta_i \cap \alpha \neq \emptyset$ and $\alpha_i^c := \beta_i \cap \alpha^c \neq \emptyset$ for $i = q+1 \dots, q+p \leq k-1$, where $p \geq 0$, i.e. if $p \geq 1$ and $q < i \leq q+p$ then the block B_{ii} is split between α and α^c;

(iii) $\beta_i \subset \alpha$ for $i = q+p+1, \dots, q+p+l = k-1$, where $l \geq 0$, i.e. if $l \geq 1$ and $q+p < i \leq q+p+l$ then the whole block B_{ii} is chosen for $B[\alpha, \alpha]$;

Note that $\beta_k \subset \alpha^c$ since we require that $B[\alpha, \alpha]$ does not have zero diagonal entries. For the blocks in (ii) define $a_i = u_i[\alpha_i^c],\ b_i = v_i[\alpha_i^c]$, i.e. the parts of the positive vectors u_i, v_i that correspond to the row indices in α_i^c and $x_i = u_i[\alpha_i],\ y_i = v_i[\alpha_i]$, i.e. the remaining parts of the vectors u_i, v_i, respectively. View

$$\alpha^c = (\cup_{i=1}^{q} \beta_i) \cup (\cup_{j=q+1}^{q+p} \alpha_j^c) \cup \beta_k,\ \alpha = (\cup_{j=q+1}^{q+p} \alpha_j) \cup (\cup_{i=p+q+1}^{k-1} \beta_i).$$

For the block matrix

$$C = \left[\begin{array}{c||c} B[\alpha^c, \alpha^c] & B[\alpha^c, \alpha] \\ \hline\hline B[\alpha, \alpha^c] & B[\alpha, \alpha] \end{array}\right], \quad C = QBQ^T \text{ for corresponding } Q \in \Pi_n, \tag{7}$$

by using Theorem 2.2 we obtain the following picture

$$\left[\begin{array}{c|c|c||c|c}
\begin{matrix} u_1 v_1^T & & \\ & \ddots & \\ & & u_q v_q^T \end{matrix} & 0 & \begin{matrix} u_1 w_1^T \\ \vdots \\ u_q w_q^T \end{matrix} & 0 & 0 \\ \hline
0 & \begin{matrix} a_1 b_1^T & & \\ & \ddots & \\ & & a_p b_p^T \end{matrix} & \begin{matrix} a_1 w_{q+1}^T \\ \vdots \\ a_p w_{q+p}^T \end{matrix} & \begin{matrix} a_1 y_1^T & & \\ & \ddots & \\ & & a_p y_p^T \end{matrix} & 0 \\ \hline
0 & 0 & 0 & 0 & 0 \\ \hline\hline
0 & \begin{matrix} x_1 b_1^T & & \\ & \ddots & \\ & & x_p b_p^T \end{matrix} & \begin{matrix} x_1 w_{q+1}^T \\ \vdots \\ x_p w_{q+p}^T \end{matrix} & \begin{matrix} x_1 y_1^T & & \\ & \ddots & \\ & & x_p y_p^T \end{matrix} & 0 \\ \hline
0 & 0 & \begin{matrix} u_{q+p+1} w_{q+p+1}^T \\ \vdots \\ u_{q+p+l} w_{q+p+l}^T \end{matrix} & 0 & \begin{matrix} u_{q+p+1} v_{q+p+1}^T & & \\ & \ddots & \\ & & u_{q+p+l} v_{q+p+l}^T \end{matrix}
\end{array}\right].$$

It is easy to see that the l blocks of category (iii) will be zeroed out in $B[\alpha, \alpha]^\dagger$ by the zero blocks in $B[\alpha^c, \alpha]$ and $B[\alpha, \alpha^c]$. Hence, without loss of generality, we assume that $l = 0$. Furthermore, the q irreducible blocks in category (i) remain unchanged in $B_\dagger(\alpha)$, also due to the corresponding zero blocks in $B[\alpha, \alpha^c]$ and $B[\alpha^c, \alpha]$. Hence, we can also assume $q = 0$. Thus $\alpha = (\cup_{i=1}^{p} \alpha_i) \cup \beta_k$, where $\alpha_i \subsetneqq \beta_i$ is a nonempty set for $i = 1, \ldots, p$. Then, the matrix $B[\alpha, \alpha]$ has block diagonal form

$$\operatorname{diag}(B[\alpha_1, \alpha_1], \ldots, B[\alpha_p, \alpha_p]),$$

where $B[\alpha_i, \alpha_i] = x_i y_i^T$. By Lemma 3.1, we have

$$B[\alpha, \alpha]^\dagger = \operatorname{diag}\left(\frac{y_1 x_1^T}{(x_1^T x_1)(y_1^T y_1)}, \ldots, \frac{y_p x_p^T}{(x_p^T x_p)(y_p^T y_p)} \right).$$

Hence $B[\alpha^c, \alpha] B[\alpha, \alpha]^\dagger$ is in generalized block diagonal form as $B[\alpha^c, \alpha]$:

$$B[\alpha^c, \alpha] B[\alpha, \alpha]^\dagger = \left[\begin{array}{c} \operatorname{diag}\left(\frac{a_1 x_1^T}{x_1^T x_1}, \ldots, \frac{a_p x_p^T}{x_p^T x_p} \right) \\ \hline 0 \end{array} \right]. \tag{8}$$

Thus, we have

$$B[\alpha^c, \alpha] B[\alpha, \alpha]^\dagger B[\alpha, \alpha^c] = \left[\begin{array}{c|c} \begin{matrix} a_1 b_1^T & & \\ & \ddots & \\ & & a_p b_p^T \end{matrix} & \begin{matrix} a_1 w_1^T \\ \vdots \\ a_p w_p^T \end{matrix} \\ \hline 0 & 0 \end{array} \right],$$

and therefore $B_\dagger(\alpha) = 0$. $\square$

The proof of Theorem 3.2 yields the following corollaries.

Corollary 3.3 *Let the assumptions of Theorem 3.2 hold. Assume the conditions (i-iii) in the proof of Theorem 3.2. Then $A_\dagger(\alpha)$ is a nonnegative idempotent matrix which has eigenvalue 1 of multiplicity q.*

Corollary 3.4 *Let $A \in \mathbb{R}_+^{n\times n}, A \neq 0$ be idempotent. If $\alpha \subsetneqq \langle n\rangle$ is chosen such that $A[\alpha,\alpha]$ is an invertible matrix, then $A[\alpha,\alpha]$ is diagonal.*

Proof. Note that the number l in the proof of Theorem 3.2 must be zero or the corresponding blocks B_{ii} must be positive 1×1 matrices. Furthermore, for the split blocks, we must also have that $x_i y_i^T \in \mathbb{R}^{1\times 1}$ since $x_i y_i^T$ is of rank 1. Therefore, $A[\alpha,\alpha]$ is diagonal. □

Corollary 3.5 *Let $A \in \mathbb{R}_+^{n\times n}, A \neq 0$ be idempotent. If $\alpha \subsetneqq \langle n\rangle$ is chosen such that $A[\alpha,\alpha]$ is a regular matrix, then the standard Schur complement (1) is nonnegative.*

Corollary 3.6 *Let $A \in \mathbb{R}_+^{n\times n}, A \neq 0$ be idempotent. Choose $\alpha \subsetneqq \langle n\rangle$, such that $A[\alpha,\alpha]$ does not have zero diagonal elements. Then, $\tilde{A}_\dagger(\alpha)$ defined by*

$$\tilde{A}_\dagger(\alpha) := A[\alpha^c,\alpha^c] + A[\alpha^c,\alpha](I - A[\alpha,\alpha])^\dagger A[\alpha,\alpha^c] \tag{9}$$

is a nonnegative idempotent matrix.

Proof. It is enough to prove the corollary for the corresponding matrix B that is in the Frobenius normal upper triangular form given by Theorem 2.2. Using the same argument as in the proof of Theorem 3.2, without loss of generality we can assume that $l = q = 0$. We have that $B[\alpha,\alpha] \geq 0$ with $\rho(B[\alpha,\alpha]) < 1$, since we have split positive irreducible blocks with spectral radius 1. Therefore, $(I - B[\alpha,\alpha])$ is a regular M-matrix and $(I - B[\alpha,\alpha])^\dagger = (I - B[\alpha,\alpha])^{-1} \geq 0$. By using this and the projector properties of B, we obtain

$$\begin{aligned}
\tilde{B}_\dagger(\alpha)\tilde{B}_\dagger(\alpha) =& B[\alpha^c,\alpha^c]B[\alpha^c,\alpha^c] + B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c]B[\alpha^c,\alpha^c] \\
&+ B[\alpha^c,\alpha^c]B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c] \\
&+ B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c]B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c] = \\
=& B[\alpha^c,\alpha^c]B[\alpha^c,\alpha^c] + B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c]B[\alpha^c,\alpha^c] \\
&+ B[\alpha^c,\alpha](I - B[\alpha,\alpha])(I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c] \\
&+ B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger (I - B[\alpha,\alpha])B[\alpha,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c] = \\
=& B[\alpha^c,\alpha^c]B[\alpha^c,\alpha^c] + B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c]B[\alpha^c,\alpha^c] \\
&+ B[\alpha^c,\alpha]B[\alpha,\alpha^c] + B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha]B[\alpha,\alpha^c] = \\
=& B[\alpha^c,\alpha^c] + B[\alpha^c,\alpha](I - B[\alpha,\alpha])^\dagger B[\alpha,\alpha^c] = \tilde{B}_\dagger(\alpha).
\end{aligned}$$

Hence $\tilde{B}_\dagger(\alpha)$ is idempotent. Furthermore, since $(I - B[\alpha,\alpha])$ is an M-matrix, we also conclude that $\tilde{B}_\dagger(\alpha) \geq 0$. □

3.2 An example

In this subsection we assume that $A[\alpha,\alpha]$ has a zero on its main diagonal. We give an example where $A_\dagger(\alpha)$ may fail to be nonnegative. To this end, we first start with the following known result.

Lemma 3.7 *Let $A \in \mathbb{R}^{n\times n}$ be a singular matrix of the following form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{bmatrix},\ A_{11} \in \mathbb{R}^{p\times p}, A_{12} \in \mathbb{R}^{p\times(n-p)},\ \textit{for some } 1 \le p < n.$$

Then $(A^\dagger)^T$ has the same block form as A.

Proof. Let $r = \operatorname{rank} A$. So $r \le p$. Then the reduced singular value decomposition of A is of the form $U_r\Sigma_r V_r^T$, where $U_r, V_r \in \mathbb{R}^{n\times r}, U_r^T U_r = V_r V_r^T = I_r$ and Σ_r is a diagonal matrix, whose diagonal entries are the positive singular values of A.

Clearly, $AA^T = \begin{bmatrix} A_{11}A_{11}^T + A_{12}A_{12}^T & 0 \\ 0 & 0 \end{bmatrix}$. Hence all eigenvectors of AA^*, corresponding to positive eigenvalues are of the form $(x^T, 0^T)^T, x \in \mathbb{R}^p$. Thus $U_r^T = [U_{r1}^T\ 0_{r\times(n-p)}]$ where $U_{r1} \in \mathbb{R}^{p\times r}$. Recall that $A^\dagger = V_r\Sigma_r^{-1}U_r^T$. The above form of U_r establishes the lemma. □

Lemma 3.8 *Let B satisfy condition 2 of Theorem 2.2. Denote $D := B^\dagger$ and assume that $D = [D_{ij}]_{i,j=1}^k$ has the same block partition as B. Then*

1. $D_{ii} = \frac{1}{(u_i^T u_i)(v_i^T v_i + w_i^T w_i)} v_i u_i^T$ *for $i = 1, \dots, k-1$.*
2. $D_{ki} = \frac{1}{(u_i^T u_i)(v_i^T v_i + w_i^T w_i)} w_i u_i^T$ *for $i = 1, \dots, k-1$.*
3. *All other D_{ij} are zero matrices.*

Proof. Use the proof of the previous lemma to find the positive singular values and the corresponding left singular vectors $x_i, i = 1, \dots, k-1$ of B. Note that $\operatorname{rank} B = k-1$. Then the right singular vectors of B are $\frac{1}{\sigma_i(B)} B^T x_i$ for $i = 1, \dots, k-1$. Now use the formula $B^\dagger = V_{k-1}\Sigma_{k-1}^{-1}U_{k-1}^T$ to deduce the lemma. □

Example 3.9 *Consider a nonnegative projector in block form as in (7)*

$$C = \left[\begin{array}{ccc|cc} u_1v_1^T & 0 & u_1s_1^T & u_1t_1^T & 0 \\ 0 & a_2b_2^T & a_2s_2^T & a_1t_2^T & a_2y_2^T \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & x_2b_2^T & x_2s_2^T & x_1t_2^T & x_2y_2^T \end{array}\right],$$

where the vectors w_i in (7) are partitioned as $w_i = \begin{bmatrix} s_i & t_i \end{bmatrix}^T$, $i = 1, 2$. By Lemma 3.8, we have that

$$B[\alpha,\alpha]^\dagger = \begin{bmatrix} 0 & \frac{t_2x_2^T}{(x_2^Tx_2)(t_2^Tt_2 + y_2^Ty_2)} \\ 0 & \frac{y_2x_2^T}{(x_2^Tx_2)(t_2^Tt_2 + y_2^Ty_2)} \end{bmatrix},$$

and

$$B[\alpha^c,\alpha]B[\alpha,\alpha]^\dagger B[\alpha,\alpha^c] = \begin{bmatrix} 0 & \frac{t_1^Tt_2u_1b_2^T}{t_2^Tt_2 + y_2^Ty_2} & \frac{t_1^Tt_2u_1s_2^T}{t_2^Tt_2 + y_2^Ty_2} \\ 0 & a_2b_2^T & a_2s_2^T \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $B_\dagger(\alpha)_{12} \le 0$ and the Moore-Penrose inverse Schur complement is not nonnegative if $t_1^Tt_2 > 0$.

3.3 Nonnegativity of semi inverse Schur complement

In this section we extend the results of Section 3.1 to special semi inverse Schur complements without any assumption on the diagonal of $A[\alpha,\alpha]$. We show that if $A[\alpha,\alpha]$ has a zero on its main diagonal, then we can always define a semi inverse for $A[\alpha,\alpha]$ that is not the Moore-Penrose inverse such that the corresponding Schur complement is nonnegative. If $A[\alpha,\alpha]$ does not have a zero on the diagonal, then we simply define the semi inverse to be the Moore-Penrose inverse. Theorem 3.11 states the general result.

We start with the following simple observation.

Proposition 3.10 *Let the assumptions of Lemma 3.7 hold. Suppose that*

$$A_{11}(A_{11})^{\dagger}A_{12} = A_{12}.$$

Then $A^{\mathrm{ginv}} = \begin{bmatrix} (A_{11})^{\dagger} & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{bmatrix}$ *is a semi inverse of A. In particular any principle submatrix of a projector B, satisfying condition 2 of Theorem 2.2, with at least one zero diagonal element has a semi inverse of this form.*

Proof. The proposition follows by checking the conditions in (5). □

Note that condition $A_{11}(A_{11})^{\dagger}A_{12} = A_{12}$ holds in general for projectors A of the form as in Lemma 3.8.

The following Theorem states the most general result of this paper.

Theorem 3.11 *Let $A \in \mathbb{R}_+^{n\times n}$ with $A^2 = A$. Assume that $\alpha \subsetneqq \langle n\rangle$ is nonempty. Then, there exists a semi inverse $A^{\mathrm{ginv}}[\alpha,\alpha]$ of $A[\alpha,\alpha]$ such that $A_{\mathrm{ginv}}(\alpha)$ as defined in (2) is a nonnegative projector. The rank of $A_{ginv}(\alpha)$ is equal to the multiplicity of the eigenvalue 1 in $A[\alpha^c,\alpha^c]$. In particular, if 1 is not an eigenvalue of $A[\alpha^c,\alpha^c]$ then $A_{\mathrm{ginv}}(\alpha) = 0$.*

Proof. In view of Theorem 3.2 it is enough to consider the following case. A is in the Frobenius normal upper triangular form given by Theorem 2.2. Furthermore $B_{kk} = 0$ and $B_{ii} = B[\beta_i,\beta_i]$, $i = 1,\dots,k$, where $\beta_1,\dots,\beta_k$ is a disjoint partition of $\langle n\rangle$ satisfying the three following possibilities:

- (i) $\beta_i \subset \alpha^c$ for $i = 1,\dots,q \le k-1$ where $q \ge 0$, i.e. if $q \ge 1$ and $i \le q$ then the whole nonzero block B_{ii} is chosen for $B[\alpha,\alpha]$;
- (ii) $\beta_i \cap \alpha \ne \emptyset$ and $\beta_i \cap \alpha^c \ne \emptyset$ for $i = q+1\dots,q+p \le k-1$, where $p \ge 0$, i.e. if $p \ge 1$ and $q < i \le q+p$ then the block B_{ii} is split between α and α^c;
- (iii) $\beta_i \subset \alpha$ for $i = q+p+1,\dots,q+p+l = k-1$, where $l \ge 0$, i.e. if $l \ge 1$ and $q+p < i \le q+p+l$ then the whole block B_{ii} is chosen for $B[\alpha,\alpha]$;

Furthermore, $\alpha \cap \beta_k \ne \emptyset$. Define $A[\alpha,\alpha]^{\mathrm{ginv}}$ as in Proposition 3.10. Then $A_{\mathrm{ginv}}(\alpha)$ is a nonnegative idempotent matrix of rank q, which is the multiplicity of the eigenvalue 1 in $A[\alpha,\alpha]$. The proof follows the lines of the proof of Theorem 3.2 by using Proposition 3.10. (Note that the rank one matrix $a_ib_i^T$ defined in the proof of Theorem 3.2 has spectral radius strictly less than 1.) □

4 Conclusions

We have shown that for a nonnegative projector, i.e. idempotent matrix, we have that the Schur complement constructed via a special semi inverse is again a nonnegative projector. In particular the nonnegativity also holds for the standard Schur complement if the corresponding part of the matrix is invertible.

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