Structured eigenvalue conditioning and backward error of a class of polynomial eigenvalue problems.*

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Abstract. Characterisations of simple eigenvalues of complex matrix polynomials with $\star$-even/odd and $\star$-palindromic/antipalindromic structures that have the same normwise condition number with respect to structure preserving and arbitrary perturbations are obtained. Here $\star$ denotes either the transpose T or the conjugate transpose $*$. In the process we obtain formulae for the normwise structured condition number of simple eigenvalues of $*$-palindromic/antipalindromic and T-even/odd polynomials and tight upper and lower bounds that localise the structured condition number of simple eigenvalues of T-palindromic/antipalindromic and $*$-even/odd polynomials. Moreover, conditions under which the normwise structured backward error of approximate eigenvalues of such polynomials is equal to the unstructured backward error are also derived. These lead to complete characterisations of approximate eigenvalues that have the same structured and unstructured backward errors for the $*$-even/odd and T-even/odd polynomials.

Keywords. Nonlinear eigenvalue problems, palindromic matrix polynomials, odd matrix polynomials, even matrix polynomials, structured eigenvalue condition number, structured backward error.

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1 Introduction

Given a matrix polynomial

$$P(\lambda) \quad := \quad \Sigma_{k=0}^{m} \lambda^k A_k, \;\; A_0, A_1, \ldots, A_m \in \mathbb{F}^{n\times n} \tag{1}$$

where $\mathbb{F}$ denotes the field $\mathbb{C}$ or $\mathbb{R}$, the polynomial eigenvalue problem consists of finding a vector x and a scalar λ such that $P(\lambda)x = 0$. We consider complex polynomials of the form (1) having $\star$-palindromic, $\star$-antipalindromic, $\star$-even and $\star$-odd structure where $\star$ denotes the transpose T or the conjugate transpose $*$. The name 'palindromic' refers to a word or a phrase which remains unchanged upon writing the letters in the reverse order. In the context of polynomials, $\star$-palindromic structure implies that we get back the original polynomial on reversing the order of its coefficient matrices and applying the transpose $\star$. On the other hand $\star$-even polynomials are such that the original polynomial may be obtained upon replacing the coefficient matrices by their transposes and λ by $-\lambda$. We make more precise definitions of these polynomials in section 2.

These polynomials came into focus in the work of Mackey, Mackey, Mehl and Mehrmann in [14] where they develop a systematic approach for constructing structure

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preserving linearizations of these polynomials and list some important applications. For instance, $*$-even polynomials arise in linear quadratic optimal control problems during the solution of the associated two point boundary value problems while $*$-palindromic polynomials arise in the context of solution of the discrete time optimal control problems. Complex T-palindromic polynomials arise in the vibration analysis of rail tracks excited by high speed trains [9, 10]. For some recent work on canonical forms of $\star$-palindromic polynomials, we refer to [22]. In this paper, we consider the condition number of simple eigenvalues as well as the backward error of approximate eigenvalues of these polynomials with respect to structure preserving perturbations, the setting of norms being identical to the one considered in [24]. In this context it may be mentioned that structured Hölder condition numbers of multiple eigenvalues of $\star$-palindromic/antipalindromic and $\star$-even/odd pencils have been recently investigated in [12] while formulae for structured backward errors of approximate eigenpairs of these pencils have been obtained in [1] for a setting of norms that is different from the one considered in this paper.

The main results of the paper are as follows. We obtain characterisations of simple eigenvalues of polynomials with $\star$-palindromic/antipalindromic or $\star$-even/odd structure that have equal structured and unstructured condition numbers. In particular, we derive formulae for the structured condition number of simple eigenvalues of $*$-palindromic/antipalindromic and T-even/odd polynomials and upper and lower bounds that localise the structured condition number of simple eigenvalues of T-palindromic/antipalindromic and $*$-even/odd polynomials. We then illustrate these bounds with numerical experiments which show them to be fairly tight. These bounds indicate that even if the eigenvalues have unequal structured condition numbers, the difference between them is very small except for a few cases where the structured condition number is known to be zero. We also investigate conditions under which approximate eigenvalues of these polynomials have the same structured and unstructured backward errors. In particular we derive sets of conditions that are necessary for the two backward errors to be equal and then show that some of these conditions are also sufficient for such an equality to occur. In the process we obtain complete characterisations of complex numbers that have the same structured and unstructured backward error for the $*$-even/odd and T-even/odd polynomials.

The eigenvalues of $*$-palindromic/antipalindromic polynomials and $*$-even/odd polynomials are symmetrically placed with respect to the unit circle and the imaginary axis respectively. Our results indicate that simple eigenvalues of $*$-palindromic/antipalindromic polynomials have the same structured and unstructured normwise condition numbers while the same holds for simple eigenvalues of $*$-even/odd polynomials if and only if they lie on the imaginary axis or have orthogonal left and right eigenvectors. We also observe structured and unstructured backward errors of all approximate eigenvalues of complex $*$-palindromic/antipalindromic polynomials lying on the unit circle are equal. The same is also true of approximate eigenvalues of complex $*$-even/odd polynomials lying on the imaginary axis.

Structure preserving perturbations of smallest size that cause $*$-palindromic and $*$-even polynomials to have eigenvalues on the unit circle and the imaginary axis respectively are very important for control theory applications as these may result in loss of spectral symmetry and uniqueness in the choice of the deflating subspace associated with the eigenvalues in the open unit disk (respectively the left half plane) [6, 16, 19, 20, 21]. It is well known that such information may be read off from the structured ϵ-pseudospectra of polynomials [26]. The results for the backward error show that the same information may be obtained from the corresponding unstructured ϵ-pseudospectra. A similar conclusion holds for the $*$-even polynomials with respect to the imaginary axis.

The eigenvalues of T-palindromic/antipalindromic polynomials are symmetrically placed either with respect to the number 1 or with respect to -1. Our results indicate that whenever 1 or -1 is a simple eigenvalue of such a polynomial, then it has the same structured and unstructured condition number except under certain conditions when the structured condition number is 0. We further show that the numbers 1 and -1 also have the same structured and unstructured backward error as approximate eigenvalues of such polynomials. These results are important because the presence of eigenvalues 1 and -1 may come in the way of finding structure preserving linearizations for T-palindromic/antipalindromic polynomials (see Section 6, [14]).

This paper is organized as follows. In section 2, we make preliminary definitions and set some notations. All results pertaining to the structured condition number are contained in section 3 while those corresponding to the structured backward error are contained in section 4.

2 Preliminaries

The space of all square matrices with real or complex entries are denoted by $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ respectively while real and complex vectors of length are respectively denoted by $\mathbb{R}^n$ and $\mathbb{C}^n$. The notation $\mathbb{F}$ is used to denote either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. All polynomials over real or complex square matrices of size n are denoted by $\mathcal{P}(\mathbb{F}^n)$. When the polynomials are structured, the corresponding space is denoted by $\mathcal{P}^S(\mathbb{F}^n)$ where S refers to any of the structures $\star$-palindromic, $\star$-antipalindromic, $\star$-even or $\star$-odd. Finally the notation P is used occasionally to denote the polynomial $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$.

For a concise description of the $\star$-palindromic, $\star$-antipalindromic, $\star$-even and $\star$-odd polynomials, considered in this paper, we make the following definitions.

Definition 2.1 *Given* $Q(\lambda) = \Sigma_{k=0}^m \lambda^k B_k$, $B_1, \ldots, B_m \in \mathbb{F}^{n\times n}$, $B_k \neq 0$, *we define*

$$Q^\star(\lambda) := \Sigma_{k=0}^m \lambda^k B_k^\star \text{ and } \mathrm{rev}Q(\lambda) := \lambda^m Q(1/\lambda) = \Sigma_{i=0}^m \lambda^{m-i} B_i$$

and refer to $Q^\star(\lambda)$ *as the adjoint and to* $\mathrm{rev}Q(\lambda)$ *as the reversal of* $Q(\lambda)$.

We refer to Table 2.2 0f [14] for the basic definitions of the structures and associated spectral symmetries where it is also established that the algebraic, geometric and partial multiplicities of the eigenvalues in each pair listed in table are all equal (for a proof see [14]).

2.1 Condition number and backward error

The condition number of an eigenvalue measures its rate of change with respect to change in the initial data and indicates its sensitivity to perturbations in the data. The backward error of a complex number z is a measure of the perturbation of smallest magnitude in the presence of which z becomes an eigenvalue of the perturbed problem. It gives a measure of the stability of a numerical method. It is well known that if the perturbation in the data is of the order of the backward error, then the product of the condition number and the backward error gives a first order error bound on the computed solution. These ideas are well developed in [27, 23, 7, 24]. However in order to analyse the performance of a structure preserving algorithm and derive error bounds on the computed solution, it is more useful to consider the condition number and backward error under the restriction that the perturbations preserve the structure of the problem. We refer to the corresponding

modified quantities as the structured condition number and the structured backward error respectively.

Given a simple eigenvalue λ of $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$, let x be a corresponding right eigenvector and y a corresponding left eigenvector so that we have $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$. The normwise condition number of λ is defined as

$$\begin{aligned}\kappa(\lambda, P) &:= \limsup_{\epsilon\to 0}\{\frac{|\Delta\lambda|}{\epsilon} : (P(\lambda+\Delta\lambda)+\Delta P(\lambda+\Delta\lambda))(x+\Delta x) = 0,\\ &\quad \Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k \in \mathcal{P}(\mathbb{F}^n), \|\Delta A_k\|_2 \le \epsilon, k = 0:m\}. \qquad (2)\end{aligned}$$

where $k = 0 : m$ implies that k takes all integer values from 0 to m. The above definition is also referred to as the absolute normwise condition number as it is a measure of the absolute change in λ under perturbation. For a simple eigenvalue λ, that is not zero or infinite, the relative change in λ may also be measured by dividing the ratio $\frac{|\Delta\lambda|}{\epsilon}$ by $|\lambda|$ in the above definition. However this does not matter for the purpose of this work as the main objective is to compare structured and unstructured condition numbers. Besides, our choice of the definition allows us to deal with the zero eigenvalue within the same framework without having to consider the condition number of the problem in homogeneous form as defined in [5] and [8]. A computable expression of the above condition number has been obtained in [24] as follows.

Theorem 2.2 *[24] Given a simple eigenvalue λ with corresponding left and right eigenvectors y and x respectively, the normwise condition number $\kappa(\lambda, P)$ is given by*

$$\kappa(\lambda, P) = \frac{\alpha\|y\|_2\|x\|_2}{|y^*P'(\lambda)x|}, \text{ where } \alpha = \Sigma_{k=0}^m |\lambda|^k.$$

Given $P(\lambda) \in \mathcal{P}^S(\mathbb{F}^n)$ we modify Definition (2) in the following obvious manner to obtain a corresponding structured normwise condition number.

$$\begin{aligned}\kappa^S(\lambda, P) &:= \limsup_{\epsilon\to 0}\left\{\frac{|\Delta\lambda|}{\epsilon} : (P(\lambda+\Delta\lambda)+\Delta P(\lambda+\Delta\lambda))(x+\Delta x) = 0\right.\\ &\quad \left.\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k \in \mathcal{P}^S(\mathbb{F}^n) \text{ and } \|\Delta A_k\|_2 \le \epsilon, k = 0:m\right\}. \qquad (3)\end{aligned}$$

Expanding the constraint $(P(\lambda+\Delta\lambda)+\Delta P(\lambda+\Delta\lambda))(x+\Delta x) = 0$ in the above definition, and neglecting second order terms we have,

$$\Delta\lambda P'(\lambda)x + P(\lambda)\Delta x + \Delta P(\lambda)x = \mathcal{O}(\epsilon^2)$$

If y be a left eigenvector of P corresponding to λ, then multiplying the above equation from the left by y^*, we have,

$$\Delta\lambda y^*P'(\lambda)x + y^*P(\lambda)\Delta x + y^*\Delta P(\lambda)x = \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \Delta\lambda = -\frac{y^*\Delta P(\lambda)x}{y^*P'(\lambda)x} + \mathcal{O}(\epsilon^2).$$

Therefore Definition (3) takes the form

$$\begin{aligned}\kappa^S(\lambda, P) &:= \limsup_{\epsilon\to 0}\left\{\frac{|y^*\Delta P(\lambda)x|}{\epsilon|y^*P'(\lambda)x|} : \Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k \in \mathcal{P}^S(\mathbb{F}^n) \text{ and}\right.\\ &\quad \left.\|\Delta A_k\|_2 \le \epsilon, k = 0:m\right\}. \qquad (4)\end{aligned}$$

It is evident from the definitions of $\kappa(\lambda, P)$ and $\kappa^S(\lambda, P)$ that in general we have

$$\kappa(\lambda, P) \geq \kappa^S(\lambda, P).$$

Given a complex number $\tilde{\lambda} \in \mathbb{C}$ and a vector $\tilde{x} \in \mathbb{C}^n$, the normwise backward error of the pair $(\tilde{\lambda}, \tilde{x})$ considered as an approximate eigenpair of $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ is the size of the smallest perturbation which when applied to $P(\lambda)$ causes $\tilde{\lambda}$ to become an eigenvalue of the perturbed polynomial with $\tilde{x}$ as a corresponding eigenvector. It is defined as follows.

$$\begin{aligned} \eta(\tilde{\lambda}, \tilde{x}, P) &:= \min\{\epsilon : (P(\tilde{\lambda}) + \Delta P(\tilde{\lambda}))\tilde{x} = 0, \Delta P(\lambda) := \Sigma_{k=0}^m \Delta A_k \in \mathcal{P}(\mathbb{F}^n), \\ &\quad \|\Delta A_k\|_2 \leq \epsilon, k = 0 : m\} \end{aligned} \tag{5}$$

The following explicit formula for computing the backward error was obtained in [24].

Theorem 2.3 *The normwise backward error $\eta(\tilde{\lambda}, \tilde{x}, P)$ is given by*

$$\eta(\tilde{\lambda}, \tilde{x}, P) = \frac{\|P(\tilde{\lambda})\tilde{x}\|_2}{\tilde{\alpha}\|\tilde{x}\|_2}$$

where $\tilde{\alpha} := \Sigma_{k=0}^m |\tilde{\lambda}|^k$.

When the eigenvectors are not under consideration, the backward error may be computed from the above formula for an approximate eigenvalue only by taking the infimum over all non-zero $\tilde{x} \in \mathbb{C}^n$ in the above expression [24]. Thus if $\tilde{\lambda}$ is not an eigenvalue of $P(\lambda)$, then

$$\eta(\tilde{\lambda}, P) = \frac{1}{\tilde{\alpha}\|[P(\tilde{\lambda})]^{-1}\|_2} \tag{6}$$

However, if the polynomial $P(\lambda)$ has some additional structure then as is the case with the condition number, we modify Definition (5) as follows to obtain the structured backward error.

$$\begin{aligned} \eta^S(\tilde{\lambda}, \tilde{x}, P) &:= \min\{\epsilon : (P(\tilde{\lambda}) + \Delta P(\tilde{\lambda}))\tilde{x} = 0, \Delta P(\lambda) := \Sigma_{k=0}^m \Delta A_k \in \mathcal{P}^S(\mathbb{F}^n), \\ &\quad \|\Delta A_k\|_2 \leq \epsilon, k = 0 : m\} \end{aligned} \tag{7}$$

Evidently, we have $\eta^S(\tilde{\lambda}, \tilde{x}, P) \geq \eta(\tilde{\lambda}, \tilde{x}, P)$.

Finally we state some important Theorems which play a crucial role in deriving our results. The first of these is an abridged version of Theorem 3.2 and Theorem 5.10 of [15].

Theorem 2.4 *Given a subspace $\tilde{S} \subset \mathbb{C}^{n\times n}$ and vectors $x, b \in \mathbb{C}^n$, the following table gives necessary and sufficient conditions for the existence of $A \in \tilde{S}$ such that $Ax = b$ when $\tilde{S}$ is the space of symmetric, skew-symmetric, Hermitian or skew-Hermitian matrices.*

$\tilde{S}$	*condition on x and b*
symmetric	*none*
skew-symmetric	$x^T b = 0$
Hermitian	*$x^* b$ is real*
skew-Hermitian	*$x^* b$ is purely imaginary*

Moreover for each of these spaces $\tilde{S}$, if the set $\mathcal{M}_{\tilde{S}} := \{A \in \tilde{S} : Ax = b\}$ is nonempty, then

$$\min_{A \in \mathcal{M}_{\tilde{S}}} \|A\|_2 = \frac{\|b\|_2}{\|x\|_2}.$$

The above result will be extensively used in the construction of structure preserving perturbations of the structured polynomials under consideration.

The next Theorem is an abridged version of Theorem 6.2 and Proposition 7.1 of [11].

Theorem 2.5 *Given $S \subset \mathbb{C}^{n\times n}$, and corresponding $a(\geq 0), b \in \mathbb{C}$ as given in the table below, the support function of the set*

$$K_S(x, y) := \{y^* A x : A \in S, \|A\|_2 \leq 1\}$$

is given by

$$\text{supp}_S(z) := \sqrt{a|z|^2 + Re(b\bar{z}^2)},\ z \in \mathbb{C}$$

so that $K_S(x, y) = \{e^{i\psi/2}(\sqrt{a + |b|}\xi_1 + i\sqrt{a - |b|}\xi_2) : \xi_1, \xi_2 \in \mathbb{R}, \xi_1^2 + \xi_2^2 \leq 1\}$. where $\psi = \arg(b)$.

S	a	b
$\mathbb{C}^{n\times n}$	$\|x\|_2^2\|y\|_2^2$	0
Hermitian	$\|x\|_2^2\|y\|_2^2 - \frac{1}{2}\|y^*x\|^2$	$\frac{1}{2}(y^*x)^2$
Complex symmetric	$\|x\|_2^2\|y\|_2^2$	0
Complex skew-symmetric	$\|x\|_2^2\|y\|_2^2 - \|x^T y\|^2$	0

Furthermore, if S denotes the set of skew-Hermitian matrices, then $K_S(x, y) = K_S(x, iy)$.

In view of this result, finding the structured condition number for the given polynomials is equivalent to maximising $|\Sigma_{k=0}^m \alpha_k \lambda_0^k|$ as each α_k varies over different sets of the form $K_S(x, y)$ (depending upon the given structure) for $k = 0 : m$.

3 Structured and unstructured condition numbers

In this section we consider the structured condition number of a simple eigenvalue of a complex matrix polynomial having any one of the structures listed in Table **??**. We denote the simple eigenvalue by λ_0 and corresponding left and right eigenvectors by y and x respectively. Additionally we assume that $\|x\|_2 = \|y\|_2 = 1$. Also we replace the S of the notation $\kappa^S(\lambda_0, P)$ introduced in the previous section by $* - pal$, $* - antipal$, $T - pal$, $T - antipal$, $* - even$, $* - odd$, $T - even$ and $T - odd$ to denote the structured condition number with respect to $*$-palindromic, $*$-antipalindromic, T-palindromic, T-antipalindromic, $*$-even, $*$-odd, T-even and T-odd perturbations respectively. The first result indicates that simple eigenvalues of $*$-palindromic/antipalindromic polynomials display the same sensitivity with respect to structure preserving and arbitrary perturbations.

Theorem 3.1 *Simple eigenvalues of $*$-palindromic and $*$-antipalindromic polynomials have the same structured and unstructured condition numbers.*

Proof: Let $P(\lambda)$ be a $*$-palindromic polynomial. From equation (4) and Theorem 2.2 it is evident the equality of the structured and unstructured condition numbers follow if there exists a $*$-palindromic perturbation polynomial $\Delta P(\lambda) = \Sigma_{k=0}^m \lambda^k \Delta A_k$ such that $\|\Delta A_k\|_2 \leq$

$\epsilon, k = 0 : m$ and $|y^*\Delta P(\lambda_0)x| = \epsilon\Sigma_{k=0}^m|\lambda_0|^k$. From Theorem 2.4 we know that exists a Hermitian matrix H satisfying $Hx = \text{sign}(y^*x)y$ with norm $\|H\|_2 = \|y\|_2/\|x\|_2 = 1$. Note that $\text{sign}(y^*x)$ denotes the sign function defined by $\text{sign}(z) = \bar{z}/|z|$ for a nonzero $z \in \mathbb{C}$ and $\text{sign}(0) = 1$. Let $\omega_0 := \text{sign}(\lambda_0^m)$ and $\xi_0 := e^{i\arg(\omega_0)/2}$. Then $\omega_0\bar{\xi_0} = \xi_0$. Setting $\Delta A_k := \epsilon\text{sign}(\lambda_0)^k\bar{\xi_0}H$, it is easy to see that $\Delta P(\lambda) = \Sigma_{k=0}^m\lambda^k\Delta A_k$ is $*$-palindromic, as

$$(\Delta A_{(m-k)})^* = \epsilon\bar{\omega_0}\xi_0\text{sign}(\lambda_0)^kH = \epsilon\bar{\xi_0}\text{sign}(\lambda_0)^kH = \Delta A_k, k = 0 : m.$$

Also $\|\Delta A_k\|_2 = \epsilon, k = 0 : m$ and $|y^*\Delta P(\lambda_0)x| = \epsilon\Sigma_{k=0}^m|\lambda_0|^k$.

When $P(\lambda)$ is $*$-antipalindromic, the proof follows by replacing ξ_0 by $\tilde{\xi_0} = e^{i((arg(\omega_0)+\pi)/2)}$ so that $\omega_0\overline{\tilde{\xi_0}} = -\tilde{\xi_0}$ and arguing exactly as above. □

Next we consider the T-palindromic and T-antipalindromic polynomials. Our first results characterise all simple eigenvalues of such polynomials that have the same structured and unstructured condition numbers. The proofs depend on the following Lemma.

Lemma 3.2 *Let $f : \mathbb{C}^{m+1} \Rightarrow \mathbb{C}$ defined by $f(u) = w^Tu$, where $w := [1, \lambda, \lambda^2, \ldots, \lambda^m]^T \in \mathbb{C}^{m+1}$ is fixed. Then,*

$$\max_{\substack{u\in\mathbb{C}^n \\ \|u\|_\infty\le 1}} |f(u)| = \Sigma_{k=0}^m|\lambda|^k \text{ and } f(u_0) = \Sigma_{k=0}^m|\lambda|^k \Leftrightarrow u_0 = \omega[1, \text{sign}(\lambda), \text{sign}(\lambda^2), \ldots, \text{sign}(\lambda^m)]^T$$

where $\omega \in \mathbb{C}$ such that $|\omega| = 1$.

Proof: It is clear that $\Sigma_{k=0}^m|\lambda|^k$ is an upper bound of $|f(u)|$ for $\|u\|_\infty \le 1$ which is attained for $\tilde{u} := [1, \text{sign}(\lambda), \text{sign}(\lambda^2), \ldots, \text{sign}(\lambda^m)]^T$. Therefore the equality $\max\limits_{\substack{u\in\mathbb{C}^n \\ \|u\|_\infty\le 1}} |f(u)| = \Sigma_{k=0}^m|\lambda|^k$ is immediate. Also it is easy to see that $f(u_0) = \Sigma_{k=0}^m|\lambda|^k$ for $u_0 = \omega[1, \text{sign}(\lambda), \text{sign}(\lambda^2), \ldots, \text{sign}(\lambda^m)]^T$.

We use induction on m to prove that if $|f(u)| = \Sigma_{k=0}^m|\lambda|^k$ then $u = u_0$. Setting $m = 1$, suppose that $|f(u)| = 1+|\lambda|$ where $u := [\alpha_0, \alpha_1]^T \in \mathbb{C}^2$. Since $|f(u)| \le |\alpha_0|+|\alpha_1||\lambda| < 1 + |\lambda|$ if either $|\alpha_0| < 1$ or $|\alpha_1| < 1$, we must have $\alpha_k = e^{i\theta_k}, k = 0, 1$. This gives

$$|f(u)| = \sqrt{1 + |\lambda|^2 + 2|\lambda|\cos(\phi - \theta_0 + \theta_1)}$$

where $\phi = \arg(\lambda)$. The fact that $|f(u)| = 1 + |\lambda|$ implies that

$$\phi - \theta_0 + \theta_1 = 2n\pi, n = 0, \pm1, \pm2, \ldots \Rightarrow \alpha_1 = e^{i\theta_1} = e^{i\theta_0}e^{-i\phi} = \alpha_0\text{sign}(\lambda).$$

Therefore the hypothesis is true for $m = 1$. Suppose that the hypothesis is true for $m = m_0$ and let $u := [\alpha_0, \alpha_1, \ldots, \alpha_{m_0}]^T \in \mathbb{C}^{m_0+1}$ such that $|f(u)| = \Sigma_{k=0}^{m_0}|\lambda|^k$. Let $z_1 := \Sigma_{k=0}^{m_0}\alpha_k\lambda^k$ and $z_2 := \alpha_{m_0+1}\lambda^{m_0+1}$. Then $|f(u)| = |z_1| + |\lambda|^{m_0+1}$ and $|z_1| = \Sigma_{k=0}^{m_0}|\lambda|^k$. But $|f(u)| = \{|z_1|^2 + |\alpha_{m_0+1}|^2|\lambda|^{2(m_0+1)} + 2|z_1||\alpha_{m_0+1}||\lambda|^{m_0+1}\cos(\arg(z_1) - \arg(z_2))\}^{1/2}$. This implies that $|\alpha_{m_0+1}| = 1$. Also for $n = 0, \pm1, \pm2, \ldots$ we must have

$$\begin{aligned}
& \arg(z_2) = \arg(z_1) - 2n\pi \\
\Leftrightarrow\quad & \arg(\alpha_{m_0+1}\lambda^{m_0+1}) = \arg\left(\Sigma_{k=0}^{m_0}\alpha_k\lambda^k\right) - 2n\pi \\
\Leftrightarrow\quad & \arg(\alpha_{m_0+1}) = \arg\left(\Sigma_{k=0}^{m_0}\alpha_k\lambda^k\right) - \arg(\lambda^{m_0+1}) - 2n\pi
\end{aligned}$$

Using the induction hypothesis, we have $\alpha_k = \omega\text{sign}(\lambda^k), k = 0 : m_0$ for some $\omega \in \mathbb{C}$ with $|\omega| = 1$. Therefore for $n = 0, \pm1, \pm2, \ldots,$

$$\alpha_{m_0+1} = \arg(\omega\Sigma_{k=0}^{m_0}|\lambda|^k) - \arg(\lambda^{m_0+1}) - 2n\pi = \arg(\omega) - \arg(\lambda^{m_0+1}) - 2n\pi.$$

Hence

$$\alpha_{m_0+1} = e^{i\arg(\alpha_{m_0+1})} = e^{i\arg(\omega)}e^{-i\arg(\lambda^{m_0+1})} = \omega\text{sign}(\lambda^{m_0+1})$$

and the proof follows. □

Theorem 3.3 *Let $P(\lambda)$ be a T-palindromic polynomial of degree m. A simple eigenvalue λ_0 of $P(\lambda)$ has the same structured and unstructured condition numbers if and only if one of the following conditions hold.*

(a) Either $\lambda_0 = 0$ or $\lambda_0 = \infty$.

(b) If $\phi = \arg(\lambda_0)$, then $|x^T y|^2 \leq \cos^2\left(\frac{m-2k}{2}\phi\right)$ for $k = 0 : \frac{m-1}{2}$.

(c) The eigenvalue λ_0 is a non zero real number if m is even and a positive real number if m is odd.

Proof: We give a proof for the case when m is odd as the proof for the other case is completely analogous. If $\lambda_0 = 0$, then the T-palindromic structure of $P(\lambda)$ has no effect on the condition number of λ_0 and the equality $\kappa^{T-pal}(\lambda_0, P) = \kappa(\lambda_0, P)$ holds trivially. In such a case ∞ is also a simple eigenvalue of $P(\lambda)$ which in turn implies that 0 is a simple eigenvalue of $\mathrm{rev}P(\lambda)$. Since $\mathrm{rev}P(\lambda)$ is also T-palindromic, we have $\kappa^{T-pal}(\infty, P) = \kappa(\infty, P)$ trivially. Therefore without loss of generality we suppose that λ_0 is a non zero finite simple eigenvalue of A. By definition we have, $\kappa^{T-pal}(\lambda_0, P) = \limsup_{\epsilon \to 0} \frac{|y^* \Delta P(\lambda_0) x|}{\epsilon |y^* P'(\lambda_0) x|}$ where $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k$ is any T-palindromic polynomial such that $\|\Delta A_k\|_2 \leq 1$, $k = 0 : m$. Thus if $B_k := \epsilon^{-1} \Delta A_k$, $k = 0 : m$, then $\kappa^{T-pal}(\lambda_0, P) = \limsup_{\epsilon \to 0} \frac{|\Sigma_{k=0}^m \alpha_k \lambda_0^k|}{|y^* P'(\lambda_0) x|}$ where $\alpha_k = y^* B_k x$, $k = 0 : m$. Evidently each α_k belongs to the unit disc in $\mathbb{C}$ which is a compact set. Therefore the equality of $\kappa^{T-pal}(\lambda_0, P)$ and $\kappa(\lambda_0, P)$ holds if and only if there exists a T-palindromic polynomial $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda_0^k \Delta A_k$ such that $\|\Delta A_k\|_2 \leq 1$ $k = 0 : m$, such that $|y^* \Delta P(\lambda_0) x| = \Sigma_{k=0}^m |\lambda_0|^k$. By Lemma 3.2 this is possible if and only if $y^* \Delta A_k x = \mathrm{sign} \lambda_0^k$, $k = 0 : m$. In view of the T-palindromic structure of $\Delta P(\lambda)$ this condition is equivalent to

$$y^* \Delta A_k x = \cos k\phi - i \sin k\phi \text{ and } y^* \Delta A_k^T x = \cos(m-k)\phi - i \sin(m-k)\phi,\ k = 0 : \frac{m-1}{2}.$$

But we have $\Delta A_k = \frac{\Delta A_k + \Delta A_k^T}{2} + \frac{\Delta A_k - \Delta A_k^T}{2}$ and $\Delta A_k^T = \frac{\Delta A_k + \Delta A_k^T}{2} - \frac{\Delta A_k - \Delta A_k^T}{2}$. By Theorem 2.5, $y^* \frac{\Delta A_k + \Delta A_k^T}{2} x$ and $y^* \frac{\Delta A_k - \Delta A_k^T}{2} x$ are elements of discs centred at the origin with radii 1 and $c := \sqrt{1 - |x^T y|^2}$ respectively. Thus for $k = 0 : \frac{m-1}{2}$ we have,

$$\begin{aligned} \xi_k + c\eta_k + i(\tilde{\xi}_k + c\tilde{\eta}_k) &= \cos k\phi - i \sin k\phi \quad (8) \\ \xi_k - c\eta_k + i(\tilde{\xi}_k - c\tilde{\eta}_k) &= \cos(m-k)\phi - i \sin(m-k)\phi \quad (9) \end{aligned}$$

where $\xi_k, \tilde{\xi}_k, \eta_k$, and $\tilde{\eta}_k$ are real numbers such that $\xi_k^2 + \tilde{\xi}_k^2 \leq 1$ and $\eta_k^2 + \tilde{\eta}_k^2 \leq 1$. Solving equations (8) and (9) we have,

$$\xi_k = \cos \frac{m-2k}{2}\phi \cos \frac{m}{2}\phi,\ \tilde{\xi}_k = -\sin \frac{m}{2}\phi \cos \frac{m-2k}{2}\phi.$$

$$c\eta_k = \sin \frac{m}{2}\phi \sin \frac{m-2k}{2}\phi,\ c\tilde{\eta}_k = \cos \frac{m}{2}\phi \sin \frac{m-2k}{2}\phi.$$

The above expressions indicate that the inequality $\xi_k^2 + \tilde{\xi}_k^2 \leq 1$ always holds for $k = 0 : \frac{m-1}{2}$. Thus equations (8) and (9) hold if and only if either $c = 0$ or $\eta_k^2 + \tilde{\eta}_k^2 \leq 1$ for $k = 0 : \frac{m-1}{2}$. If $c = 0$ then $e^{-ik\phi} = e^{-i(m-k)\phi}$ for $k = 0 : \frac{m-1}{2}$. Since $\lambda_0 \neq 0$ this implies that for $k = 0 : \frac{m-1}{2}$ we must have $\cos(m-2k)\phi = 1$ and $\sin(m-2k)\phi = 0$ which is possible if and only if λ_0 is a positive real number. Conversely if λ_0 is a positive real number, then by Theorem 2.4

there exists a symmetric matrix H such that $\|H\|_2 = 1$ and $Hx = y$ so that by setting $\Delta A_k := H$ for all $k = 0 : m$, we have $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k$ which is a T-palindromic polynomial satisfying $\|\Delta A_k\|_2 \leq 1$ for all $k = 0 : m$ and $|y^* \Delta P(\lambda_0) x| = \Sigma_{k=0}^m |\lambda_0|^k$. If $c \neq 0$, then equations (8) and (9) hold if and only if $|x^T y|^2 \leq \cos^2\left(\frac{m-2k}{2}\phi\right)$ which follows by using the expressions for η_k and $\tilde{\eta}_k$ in the inequality $\eta_k^2 + \tilde{\eta}_k^2 \leq 1$. Hence the proof. $\square$

The next result characterises simple eigenvalue of T-antipalindromic polynomials that have the same structured and unstructured condition numbers.

Theorem 3.4 *Let $P(\lambda)$ be a T-antipalindromic polynomial of degree m. If m is odd, then a simple eigenvalue λ_0 of $P(\lambda)$ has the same structured and unstructured condition numbers if and only if*

(a) Either $\lambda_0 = 0$ or $\lambda_0 = \infty$.

(b) If $\phi = \arg(\lambda_0)$, then $|x^T y|^2 \leq \sin^2\left(\frac{m-2k}{2}\phi\right)$ for $k = 0 : \frac{m-1}{2}$.

(c) The eigenvalue λ_0 is a negative real number if m is odd.

If m is even, then the equality of the structured and unstructured condition numbers hold if and only if either $\lambda_0 = 0$ or ∞ or $x^T y = 0$.

Proof: The proof follows on exactly the same lines as the proof for Theorem 3.3 except that we replace equations (8) and (9) by the following equations respectively

$$\xi_k + c\eta_k + i(\tilde{\xi}_k + c\tilde{\eta}_k) = \cos k\phi - i \sin k\phi \tag{10}$$

$$\xi_k - c\eta_k + i(\tilde{\xi}_k - c\tilde{\eta}_k) = -\cos(m-k)\phi + i \sin(m-k)\phi \tag{11}$$

where $\xi_k, \tilde{\xi}_k, \eta_k$ and $\tilde{\eta}_k$ are real numbers such that $\xi_k^2 + \tilde{\xi}_k^2 \leq 1$ and $\eta_k^2 + \tilde{\eta}_k^2 \leq 1$ and k varies from 0 to $\frac{m-1}{2}$ if m is odd and from 0 to $\frac{m}{2}$ if m is even. The characterisations for the case when m is even follow immediately by observing that the above equations hold for $k = m/2$ if and only if either $\lambda_0 = 0$ or $x^T y = 0$. $\square$

Note 3.5 *From Theorem 3.3 it is clear that if -1 is a simple eigenvalue of a T-palindromic polynomial $P(\lambda_0)$ of odd degree, then $\kappa^{T-pal}(-1, P) \neq \kappa(-1, P)$. In fact in such a case we have $\kappa^{T-pal}(-1, P) = 0$ since $P(-1)$ is a skew-symmetric matrix. The same holds for 1 if it is a simple eigenvalue of a T-antipalindromic polynomial and for -1 if it is a simple eigenvalue of a T-antipalindromic polynomial of even degree.*

Now given a simple eigenvalue λ_0 of a T-palindromic or T-antipalindromic polynomial such that its structured and unstructured condition numbers are not equal and neither does it satisfy any of the conditions mentioned in Note 3.5, we derive upper and lower bounds of the structured condition number of λ_0. We also perform numerical experiments to illustrate these bounds which suggest that even though the structured and unstructured condition numbers are not equal, they are very close to each other in most cases.

Theorem 3.6 *Let λ_0 be a simple eigenvalue of a T-palindromic polynomial $P(\lambda)$ with unequal structured and unstructured condition numbers. Also suppose that $\lambda_0 \neq -1$ if the degree of $P(\lambda)$ is odd. Setting $\phi := \arg(\lambda_0)$ and $c := \sqrt{1 - |x^T y|^2}$ let,*

$$S := \begin{cases} \{0 \leq k \leq \frac{m-1}{2} : c^2 < \sin^2\left(\frac{m-2k}{2}\phi\right)\} & \text{if } m \text{ is odd} \\ \{0 \leq k \leq \frac{m-2}{2} : c^2 < \sin^2\left(\frac{m-2k}{2}\phi\right)\} & \text{if } m \text{ is even} \end{cases}.$$

Then we have

$$\max\{l_1,l_2,l_3,l_4\}\kappa(\lambda_0,P) \le \kappa^{T-pal}(\lambda_0,P) \le \min\{1,u_1,u_2,u_3\}\kappa(\lambda_0,P)$$

where

$$l_1 := c\min_{k\in S}\left|\mathrm{cosec}\left(\frac{m-2k}{2}\phi\right)\right|,$$

$$l_2 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}((1-c)|\lambda_0^k+\lambda_0^{m-k}|+c^2|\lambda_0^k-\lambda_0^{m-k}|)+|\lambda_0|^{m/2}\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}((1-c)|\lambda_0^k+\lambda_0^{m-k}|+c^2|\lambda_0-\lambda_0^{m-k}|)\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is odd} \end{cases},$$

$$l_3 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(c|\lambda_0^k+\lambda_0^{m-k}|+c(1-c)|\lambda_0^k-\lambda_0^{m-k}|)+|\lambda_0|^{m/2}\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}(c|\lambda_0^k+\lambda_0^{m-k}|+c(1-c)|\lambda_0^k-\lambda_0^{m-k}|)\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is odd} \end{cases},$$

$$l_4 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}|\lambda_0^k+\lambda_0^{m-k}|+|\lambda_0|^{m/2}\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}|\lambda_0^k+\lambda_0^{m-k}|\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is odd} \end{cases},$$

$$u_1 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(2c|\lambda_0|^k+|\lambda_0^k+\lambda_0^{m-k}|)+|\lambda_0|^{m/2}\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}(2c|\lambda_0|^k+|\lambda_0^k+\lambda_0^{m-k}|)\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is odd} \end{cases}$$

$$u_2 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(2c|\lambda_0|^{m-k}+|\lambda_0^k+\lambda_0^{m-k}|)+|\lambda_0|^{m/2}\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}(2c|\lambda_0|^{m-k}+|\lambda_0^k+\lambda_0^{m-k}|)\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is odd} \end{cases}$$

$$u_3 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(|\lambda_0+\lambda_0^{m-k}|+c|\lambda_0-\lambda_0^{m-k}|+|\lambda_0|^{m/2})\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}(|\lambda_0+\lambda_0^{m-k}|+c|\lambda_0-\lambda_0^{m-k}|)\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is odd} \end{cases}$$

Proof: We establish the lower bounds by constructing three T-palindromic perturbations $P^{(i)}(\lambda)$, $i = 1, 2, 3$ and 4 to $P(\lambda)$ with coefficient matrices whose 2-norms are at most 1 such that $|y^*P^{(i)}(\lambda_0)x| \ge l_i\Sigma_{k=0}^{m}|\lambda_0|^k$, $i = 1, 2, 3, 4$. Observe that $S \neq \emptyset$ in view of the hypothesis. Assuming the degree m is odd and greater than 1 we choose $P^{(1)}(\lambda) = \Sigma_{k=0}^{m}\lambda^k A_k^{(1)}$ with $\|A_k^{(1)}\|_2 \le 1$, $k = 0 : m$ such that $y^*A_k^{(1)}x = \mathrm{sign}(\lambda_0^k)$ and $y^*A_{m-k}^{(1)}x = \mathrm{sign}(\lambda_0^{m-k})$ if $k \in \{j \in \mathbb{I} : 0 \le j \le \frac{m-1}{2}\} \setminus S$ whereas $y^*A_k^{(1)}x = c\,|\mathrm{cosec}\left(\frac{m-2k}{2}\phi\right)|\,\mathrm{sign}(\lambda_0^k)$ and $y^*A_{m-k}^{(1)}x = c\,|\mathrm{cosec}\left(\frac{m-2k}{2}\phi\right)|\,\mathrm{sign}(\lambda_0^{m-k})$ if $k \in S$.
Owing to the T-palindromic structure of $P^{(1)}(\lambda)$, we have $A_{m-k}^{(1)} = (A_k^{(1)})^T$, $k = 0 : \frac{m-1}{2}$. Moreover, there exist real numbers $\xi_k, \tilde{\xi}_k, \eta_k$ and $\tilde{\eta}_k$ satisfying $\xi_k^2 + \tilde{\xi}_k^2 \le 1$ and $\eta_k^2 + \tilde{\eta}_k^2 \le 1$ for which

$$y^*A_k^{(1)}x = \xi_k + c\eta_k + i(\tilde{\xi}_k + c\tilde{\eta}_k) \text{ and } y^*A_{m-k}^{(1)}x = \xi_k - c\eta_k + i(\tilde{\xi}_k - c\tilde{\eta}_k),\ k = 0 : \frac{m-1}{2}.$$

Also it is easy to see that

$$\begin{aligned} |y^*P^{(1)}(\lambda_0)x| &= \Sigma_{\substack{k=0 \\ k\notin S}}^{\frac{m-1}{2}}(|\lambda_0|^k+|\lambda_0|^{m-k}) + \Sigma_{k\in S}c\left|\mathrm{cosec}\left(\frac{m-2k}{2}\phi\right)\right|(|\lambda_0|^k+|\lambda_0|^{m-k}) \\ &\ge \min_{k\in S}c\left|\mathrm{cosec}\left(\frac{m-2k}{2}\phi\right)\right|\Sigma_{k=0}^{m}|\lambda_0|^k. \end{aligned} \tag{12}$$

which establishes the inequality $\kappa^{T-pal}(\lambda_0,P) \ge l_1\,\kappa(\lambda_0,P)$.

Before establishing the rest of the lower bounds we observe that given any T-palindromic polynomial $\hat{P}(\lambda) := \Sigma_{k=0}^{m}\lambda^k\hat{A}_k$, and vectors $\hat{x}$ and $\hat{y}$, for which $\hat{y}^*\hat{P}(\lambda)\hat{x}$ is defined, we have

$$\hat{y}^*\hat{P}(\lambda)\hat{x} = \Sigma_{k=0}^{m-1}(\lambda^k+\lambda^{m-k})\hat{y}^*\left(\frac{\hat{A}_k+\hat{A}_{m-k}}{2}\right)\hat{x} + (\lambda^k-\lambda^{m-k})\hat{y}^*\left(\frac{\hat{A}_k-\hat{A}_{m-k}}{2}\right)\hat{x}. \quad (13)$$

Now let $P^{(i)}(\lambda) = \Sigma_{k=0}^{m}\lambda^k A_k^{(i)}$ be T-palindromic polynomials such that $A_k^{(i)} = \alpha_i\{\text{sign}(\lambda_0^k + \lambda_0^{m-k})R + \beta_i\text{sign}(\lambda_0^k - \lambda_0^{m-k})S\}$, $k = 0:m$ where R and S are respectively symmetric and skew-symmetric matrices with $\|S\|_2 = \|R\|_2 = 1$ such that $Rx = y$ and $Sx = \hat{y}$ where $\hat{y} := \dfrac{y-(x^Ty)\bar{x}}{\sqrt{1-|x^Ty|^2}}$ and α_i, β_i are real numbers satisfying $\alpha_i + \beta_i = 1$ for $i = 2, 3, 4$ given by $\alpha_2 = 1-c, \alpha_3 = c$, and $\alpha_4 = 1$. We observe that such a choice of matrices R and S is possible in view of Theorem 2.4 since $\|y\|_2 = \|x\|_2 = \|\hat{y}\|_2 = 1$ and $x^T\hat{y} = 0$. This choice of the coefficient matrices also ensures that $\|A_k^{(i)}\|_2 \le 1$ for $i = 2, 3, 4$ and $k = 0:m$. By expanding each $y^*P^{(i)}(\lambda_0)x$ as in (13) and using the facts that $y^*Rx = 1$ and $y^*Sx = c$, it is now easy to see that $y^*P^{(i)}(\lambda_0)x = l_i\kappa(\lambda_0, P)$ for $i = 2, 3$ and 4 respectively.

Finally the upper bounds $u_i\kappa(\lambda_0, P)$, $i = 1, 2, 3$ follow by applying Theorem 2.5 to the right hand sides of the following inequalities respectively.

$$\begin{aligned}
|y^*P(\lambda_0)x| &\le \Sigma_{k=0}^{(m-1)/2}\left(|\lambda_0|^k|y^*(A_k - A_k^T)x| + |\lambda_0^k + \lambda_0^{m-k}||y^*A_k^Tx|\right)\\
|y^*P(\lambda_0)x| &\le \Sigma_{k=0}^{(m-1)/2}\left(|\lambda_0|^{m-k}|y^*(A_k - A_k^T)x| + |\lambda_0^k + \lambda_0^{m-k}||y^*A_kx|\right)\\
|y^*P(\lambda_0)x| &\le \Sigma_{k=0}^{(m-1)/2}|\lambda_0^k + \lambda_0^{m-k}|\left|y^*\left(\frac{A_k+A_k^T}{2}\right)x\right| + |\lambda_0^k - \lambda_0^{m-k}|\left|y^*\left(\frac{A_k-A_k^T}{2}\right)x\right|
\end{aligned}$$

□

The corresponding bounds for the structured condition number of simple eigenvalues of T-antipalindromic polynomials are given by the following theorem, the proof of which uses ideas similar to the ones used in the preceding proof.

Theorem 3.7 *Let λ_0 be a simple eigenvalue of a T-antipalindromic polynomial $P(\lambda)$ with unequal structured and unstructured condition numbers. Also suppose that $\lambda_0 \ne 1$ and $\lambda_0 \ne -1$ if the degree of $P(\lambda)$ is even. Setting $\phi := \arg(\lambda_0)$, and $c := \sqrt{1-|x^Ty|^2}$ let,*

$$S := \{0 \le k \le \frac{m-1}{2} : c^2 < \cos^2\left(\frac{m-2k}{2}\phi\right)\} \text{ if } m \text{ is odd.}$$

Then we have

$$\max\{l_1, l_2, l_3, l_4\}\kappa(\lambda_0, P) \le \kappa^{T-antipal}(\lambda_0, P) \le \min\{1, u_1, u_2, u_3\}\kappa(\lambda_0, P)$$

where

$$l_1 := \begin{cases} c & \text{if } m \text{ is even}\\ c\min\limits_{k\in S}\left|\sec\left(\dfrac{m-2k}{2}\phi\right)\right| & \text{if } m \text{ is odd}\end{cases}$$

$$l_2 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}((1-c)|\lambda_0^k - \lambda_0^{m-k}| + c^2|\lambda_0^k + \lambda_0^{m-k}|) + c|\lambda_0|^{m/2}\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is even}\\ \left(\Sigma_{k=0}^{(m-1)/2}((1-c)|\lambda_0^k - \lambda_0^{m-k}| + c^2|\lambda_0 + \lambda_0^{m-k}|)\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is odd}\end{cases},$$

$$l_3 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(c|\lambda_0^k - \lambda_0^{m-k}| + c(1-c)|\lambda_0^k + \lambda_0^{m-k}|) + c|\lambda_0|^{m/2}\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is even}\\ \left(\Sigma_{k=0}^{(m-1)/2}(c|\lambda_0^k - \lambda_0^{m-k}| + c(1-c)|\lambda_0^k + \lambda_0^{m-k}|)\right)/\left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is odd}\end{cases},$$

$$l_4 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}|\lambda_0^k - \lambda_0^{m-k}| + c|\lambda_0|^{m/2}\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}|\lambda_0^k - \lambda_0^{m-k}|\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^k\right) & \text{if } m \text{ is odd} \end{cases},$$

$$u_1 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(2c|\lambda_0|^k + |\lambda_0^k - \lambda_0^{m-k}|) + c|\lambda_0|^{m/2}\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}(2c|\lambda_0|^k + |\lambda_0^k - \lambda_0^{m-k}|)\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is odd} \end{cases}$$

$$u_2 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(2c|\lambda_0|^{m-k} + |\lambda_0^k - \lambda_0^{m-k}|) + c|\lambda_0|^{m/2}\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}(2c|\lambda_0|^{m-k} + |\lambda_0^k - \lambda_0^{m-k}|)\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is odd} \end{cases}$$

$$u_3 := \begin{cases} \left(\Sigma_{k=0}^{(m-2)/2}(|\lambda_0 - \lambda_0^{m-k}| + c|\lambda_0 + \lambda_0^{m-k}| + c|\lambda_0|^{m/2})\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is even} \\ \left(\Sigma_{k=0}^{(m-1)/2}(|\lambda_0 - \lambda_0^{m-k}| + c|\lambda_0 + \lambda_0^{m-k}|)\right) / \left(\Sigma_{k=0}^{m}|\lambda_0|^m\right) & \text{if } m \text{ is odd} \end{cases}$$

The following numerical examples performed in MATLAB demonstrate that the bounds given by Theorem 3.6 and Theorem 3.7 are fairly tight. The coefficient matrices in these examples have been generated by using the MATLAB `randn` command that produces random complex matrices A. Further skew-symmetric and symmetric coefficient matrices have been set to $A - A.'$ and $A + A.'$ respectively. In all the examples the bounds for the structured condition number have been computed for only one of the eigenvalues that constitute a pair $(\lambda, 1/\lambda)$ since the bounds for the other eigenvalue in the pair are identical. Also the columns ub and lb in the tables correspond to $\min\{1, u_1, u_2, u_3\}$ and $\max\{l_1, l_2, l_3, l_4\}$ respectively.

Example 3.8 The first example is that of a T-antipalindromic polynomial of degree 6 with coefficient matrices of size 4. In this case all the eigenvalues have unequal structured and unstructured condition numbers. The relevant computations are provided in Table 1.

Table 1: Relevant eigenvalues corresponding to Example 3.8 with corresponding values of lb and ub.

λ_0	lb	ub
1.5163 - 1.6967i	0.9581	1
-0.5206 - 1.4490i	0.8049	1
0.9390 + 1.2481i	0. 8684	1
-1.1075 + 0.8250i	0.9078	1
-1.2113 - 0.3204i	0.9836	1
-1.1599 + 0.0507i	0.9401	0.9651
-0.4770 - 1.0467i	0.9071	1
-0.3888 + 1.1182i	0.9722	1
0.2979 + 0.9204i	0.9445	1
1.1094 + 0.4585i	0.9025	1
0.6346 + 0.6884i	0.9063	1
1.2238 - 0.3258i	0.9127	1

Example 3.9 In this example we consider a T-antipalindromic pencil with coefficient matrices of size 10. It has only one pair of eigenvalues whose elements do not satisfy the criteria for equality of the structured and unstructured condition numbers. One of the eigenvalues of the pair is $5.3298 + 1.1972i$, the corresponding values of lb and ub being 0.9472 and 1 respectively.

Example 3.10 Next we consider a quadratic T-palindromic polynomial with coefficient matrices of size 13. This polynomial has three pairs of eigenvalues with elements that have different structured and unstructured condition numbers. The relevant results are given by Table 2.

Table 2: Relevant eigenvalues corresponding to Example 3.10 with corresponding values of lb and ub.

λ_0	lb	ub
-0.1565 - 1.2523i	0.9863	1
0.0847 + 1.1704i	0. 9946	1

Example 3.11 In the final example, we consider a T-palindromic polynomial of degree 3 with coefficient matrices of size 10. In this case there are 5 pairs of eigenvalues such that the elements constituting each pair have different structured and unstructured condition numbers. The upper and lower bounds with corresponding relevant eigenvalues are given by Table 3.

Table 3: Relevant eigenvalues corresponding to Example 3.11 with corresponding values of lb and ub.

λ_0	lb	ub
-3.2522 - 1.4096i	0.9498	1
0.7727 + 1.6458i	0.98	1
-1.5907 + 0.9999i	0. 9669	1
0.8603 - 1.2475i	0.9989	1
-1.292 + 0.1003i	0.9396	0.965
0.3332 - 0.9776i	0.9148	1

Next we consider the $*$-even/odd polynomials and characterise all simple eigenvalues that have the same structured and unstructured condition numbers.

Theorem 3.12 *If $P(\lambda)$ is $*$-even or $*$-odd, then $\kappa^S(\lambda_0, P) = \kappa(\lambda_0, P)$ if and only if either $x^*y = 0$ or λ_0 is purely imaginary. Here S stands for $*$-even when $P(\lambda)$ is $*$-even and for $*$-odd when $P(\lambda)$ is $*$-odd.*

Proof: Let $P(\lambda)$ be a $*$-even polynomial. Suppose that $\kappa^{*-even}(\lambda_0, P) = \kappa(\lambda_0, P)$. This is possible if and only if there exists a $*$-even perturbation $\Delta P(\lambda)$ such that $\|\Delta A_k\|_2 \leq 1, k = 0 : m$, for which $|y^*\Delta P(\lambda_0)x| = \Sigma_{k=0}^m |\lambda_0|^k$. But

$$|y^*\Delta P(\lambda_0)x| = \ |y^*(\Sigma_{k=0}^m \Delta A_k \lambda_0^k)x| = |\Sigma_{k=0}^m \lambda_0^k (y^*\Delta A_k x)|.$$

Setting $\beta_k := y^*\Delta A_k x, k = 0 : m$, in view of Theorem 2.5 we have,

$$\beta_k \in \begin{cases} E := \{(x,y) : c^2x^2 + y^2 \leq c^2\} & \text{for } k = 0 \text{ and even } k \\ \tilde{E} := \{(x,y) : x^2 + c^2y^2 \leq c^2\} & \text{for odd } k. \end{cases}$$

where $c := \sqrt{1 - |y^*x|^2}$. Observe that E and $\tilde{E}$ constitute sets whose boundaries are ellipses (touching the unit circle at the pairs of points $(1,0), (-1,0)$ and $(0,1), (0,-1)$ respectively)

except when $c = \pm 1$ in which case they become unit discs and $c = 0$ in which case $E = [-1, 1]$ and $\tilde{E} = iE$.

An application of Lemma 3.2 implies that $|y^*\Delta P(\lambda_0)x| = \Sigma_{k=0}^m |\lambda_0|^k$ if and only if $\beta_k = \omega \text{sign}(\lambda_0^k), k = 0 : m$ for some $\omega \in \mathbb{C}$ such that $|\omega| = 1$. In particular $\beta_0 = \omega$ and therefore $\omega \in E$. But this is possible if only if $c = \pm 1$ or $\omega = \pm 1$. The first condition holds if and only if $y^*x = 0$. So if this does not hold, then $\beta_1 \in \tilde{E}$ implies that $\text{sign}(\lambda_0) = \pm i$, or in other words, λ_0 is purely imaginary.

The proof for the $*$-odd polynomials follows by applying exactly similar arguments. □

Note 3.13 *The above Theorem brings up the important issue of the existence of simple eigenvalues of $*$-even or $*$-odd polynomials which satisfy the condition $x^*y = 0$. If the condition is satisfied, then since x and y are corresponding right and left eigenvectors of the eigenvalue 0 of the matrix $P(\lambda_0)$, it implies that 0 is a multiple eigenvalue of $P(\lambda_0)$ (page 11, [28]). Moreover as λ_0 is a simple eigenvalue of the polynomial $P(\lambda)$, the null space of 0 as an eigenvalue of $P(\lambda_0)$ must be of dimension 1. In other words, 0 must be a nonderogatory defective eigenvalue of $P(\lambda_0)$. Therefore instances of simple eigenvalues of $P(\lambda)$ with orthogonal left and right eigenvectors will not be very common. The following is an example where this happens. The $*$-even pencil $A_0 + \lambda A_1$ where*

$$A_0 = \begin{pmatrix} -1/6 & -(2/3 + i/6) \\ -2/3 + i/6 & 1/6 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} -i/3 & -(1/6 + i/3) \\ 1/6 - i/3 & i/3 \end{pmatrix},$$

has a simple eigenvalue $1 + i$ with corresponding right eigenvector $x := \begin{pmatrix} -1 \\ (1+2i)/5 \end{pmatrix}$ and left eigenvector $y := \begin{pmatrix} (-1+2i)/5 \\ -1 \end{pmatrix}$ which are clearly orthogonal.

In the next result we obtain bounds on the structured condition number $\kappa^S(\lambda_0, P)$ for simple eigenvalues of $*$-even/odd polynomials which have unequal structured and unstructured condition numbers. Numerical experiments are then performed to illustrate these bounds which show that although we have $\kappa^S(\lambda_0, P) \neq \kappa(\lambda_0, P)$, the difference between the two condition numbers is not very small.

Theorem 3.14 *Let $P(\lambda)$ be a $*$-even or $*$-odd polynomial. If λ_0 is not purely imaginary and/or $x^*y \neq 0$, then*

$$\frac{1}{\sqrt{2}}\kappa(\lambda_0, P) \leq \kappa^S(\lambda_0, P) < \kappa(\lambda_0, P) \text{ whenever } \lambda_0 \text{ is real.}$$

For all other eigenvalues, we have

$$\max\{l_1, l_2\}\kappa(\lambda_0, P) \leq \kappa^S(\lambda_0, P) < \kappa(\lambda_0, P) \tag{14}$$

where l_1 and l_2 are given by

$$l_1 := \begin{cases} 1 - (1-c)\max\limits_{1 \leq k \leq m/2}\{|\sin 2k\phi|, |\cos(2k-1)\phi|\} & P(\lambda) \text{ is } *-\text{even and } m \text{ is even} \\ 1 - (1-c)\max\limits_{1 \leq k \leq m/2}\{|\sin(2k-1)\phi|, |\cos 2k\phi|\} & P(\lambda) \text{ is } *-\text{odd and } m \text{ is even} \\ 1 - (1-c)\max\left\{\max\limits_{1 \leq k \leq (m-1)/2}|\sin 2k\phi|, \max\limits_{1 \leq k \leq (m+1)/2}|\cos(2k-1)\phi|\right\} & P(\lambda) \text{ is } *-\text{even and } m \text{ is odd} \\ 1 - (1-c)\max\left\{\max\limits_{1 \leq k \leq (m+1)/2}|\sin(2k-1)\phi|, \max\limits_{1 \leq k \leq (m-1)/2}|\cos 2k\phi|\right\} & P(\lambda) \text{ is } *-\text{odd and } m \text{ is odd} \end{cases}$$

$$l_2 := \begin{cases} \sqrt{2}\max\limits_{1\le k\le m/2}\{\sqrt{1+|\cos 2k\phi|},\sqrt{1+|\sin(2k-1)\phi|}\}-1 & P(\lambda)\ is\ *-\text{even}, m\ is\ even \\ \sqrt{2}\max\limits_{1\le k\le m/2}\{\sqrt{1+|\cos(2k-1)\phi|},\sqrt{1+|\sin 2k\phi|}\}-1 & P(\lambda)\ is\ *-\text{odd}, m\ is\ even \\ \sqrt{2}\max\left\{\max\limits_{1\le k\le (m-1)/2}\sqrt{1+|\cos 2k\phi|},\max\limits_{1\le k\le (m+1)/2}\sqrt{1+|\sin(2k-1)\phi|}\right\}-1 & P(\lambda)\ is\ *-\text{even}, m\ is\ odd \\ \sqrt{2}\max\left\{\max\limits_{1\le k\le (m+1)/2}\sqrt{1+|\cos(2k-1)\phi|},\max\limits_{1\le k\le (m-1)/2}\sqrt{1+|\sin 2k\phi|}\right\}-1 & P(\lambda)\ is\ *-\text{odd}, m\ is\ odd \end{cases}$$

Here $c := \sqrt{1-|y^*x|^2}$, *and* $\phi = arg(\lambda_0)$ *and it is assumed that terms involving* $\max\limits_{1\le k\le (m-1)/2}$ *are equal to* 0 *for* $m = 1$.

Proof: Let $P(\lambda)$ be a $*$-even polynomial of degree m. First consider the case that λ_0 is real. In view of Theorem 2.5 it is possible to choose Hermitian matrices ΔA_k with unit 2-norm such that $y^*\Delta A_k x = \text{sign}(\lambda_0^k)\text{e}^{i\arg(y*x)}$ for $k = 0$ and even values of k between 2 and m. For odd values of k between 1 and m we choose skew-Hermitian matrices ΔA_k with unit 2-norm such that $y^*\Delta A_k x = i\text{sign}(\lambda_0^k)\text{e}^{i\arg(y^*x)}$. Then given $\epsilon > 0$, $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k(\epsilon\Delta A_k)$ is such that

$$|y^*\Delta P(\lambda_0)x| = \begin{cases} \epsilon\left|\Sigma_{k=0}^{(m-1)/2}\left(|\lambda_0|^{2k}+i|\lambda_0|^{2k+1}\right)\right| & \text{when } m \text{ is odd} \\ \epsilon\left|1+\Sigma_{k=1}^{m/2}\left(|\lambda_0|^{2k}+i|\lambda_0|^{2k-1}\right)\right| & \text{when } m \text{ is even} \end{cases}$$

In either case we have $|y^*\Delta P(\lambda_0)x| \ge \epsilon(\Sigma_{k=0}^m|\lambda_0|^k)/\sqrt{2}$ and this establishes $\frac{1}{\sqrt{2}}\kappa(\lambda_0, P)$ as a lower bound of $\kappa^{*-even}(\lambda_0, P)$.

Next suppose that λ_0 has a non zero imaginary part and let $\phi = \arg(\lambda_0)$. For a given $\epsilon > 0$, we choose a $*$-even polynomial $\Delta P(\lambda) = \Sigma_{k=0}^m \lambda^k(\epsilon\Delta A_k)$ such that $\|\Delta A_k\|_2 = 1, k = 0 : m$, and $y^*\Delta A_k x = \text{e}^{i\arg(y^*x)}(\cos k\phi - ic\sin k\phi)$ for $k = 0$ and all even values of k between 1 and m, while $y^*\Delta A_k x = -i\text{e}^{i\arg(y^*x)}(\sin k\phi - ic\cos k\phi)$ for odd values of k between 1 and m. Then assuming that m is odd we have,

$$\begin{aligned} |y^*\Delta P(\lambda_0)x| &= \epsilon\left|\Sigma_{k=0}^{(m-1)/2}\{(\cos 2k\phi - ic\sin 2k\phi)\lambda_0^{2k} + (c\cos(2k+1)\phi - i\sin(2k+1)\phi)\lambda_0^{2k+1}\}\right| \\ &\ge \epsilon\left|\Sigma_{k=0}^m|\lambda_0|^k - (1-c)\left|\left\{\Sigma_{k=1}^{(m+1)/2}\lambda_0^{2k-1}\cos(2k-1)\phi - i\Sigma_{k=1}^{(m-1)/2}\lambda_0^{2k}\sin 2k\phi\right\}\right|\right| \end{aligned} \tag{15}$$

where it is assumed that the term involving $\Sigma_{k=1}^{(m-1)/2}$ is 0 when $m = 1$. Evidently,

$$\begin{aligned} &\left|\Sigma_{k=1}^{(m+1)/2}\{\lambda_0^{2k-1}\cos(2k-1)\phi - i\Sigma_{k=1}^{(m-1)/2}\{\lambda_0^{2k}\sin 2k\phi\}\right| \\ &\le \max\{\max_{1\le k\le (m+1)/2}|\cos(2k-1)\phi|, \max_{1\le k\le (m-1)/2}|\sin 2k\phi|\}\Sigma_{k=0}^m|\lambda_0|^k \end{aligned} \tag{16}$$

Therefore using inequality (16) in (15), we have the lower bound

$$|y^*\Delta P(\lambda_0)x| \ge \epsilon\left\{1-(1-c)\max\left\{\max_{1\le k\le (m+1)/2}|\cos(2k-1)\phi|, \max_{1\le k\le (m-1)/2}|\sin 2k\phi|\right\}\right\}\Sigma_{k=0}^m|\lambda_0|^k \tag{17}$$

which gives the lower bound l_1. We may also choose a $*$-even perturbation $\Delta P(\lambda) := \Sigma_{k=0}^m\lambda^k\epsilon\Delta A_k$ such that $\|\Delta A_k\|_2 = 1$, $k = 0 : m$, $y^*\Delta A_0 x = e^{i\arg(y^*x)}$, $y^*\Delta A_{2k}x = -\text{sign}(\cos 2k\phi)e^{i\arg(y^*x)}$ and $y^*\Delta A_{2k+1}x = i\text{sign}(\sin(2k+1)\phi)e^{i\arg(y^*x)}$ for $k = 1 : \frac{m-1}{2}$. Setting $\delta_0 := 1$, $\delta_{2k} := -\text{sign}(\cos 2k\phi)$ and $\delta_{2k+1} := \text{sign}(\sin(2k+1)\phi)$ for $k = 0 : \frac{m-1}{2}$ we

have $y^*\Delta P(\lambda_0)x := \epsilon\Sigma_{k=0}^{(m-1)/2}\left\{\delta_{2k}\lambda_0^{2k} + i\delta_{2k+1}\lambda_0^{2k+1}\right\}e^{i\arg(y^*x)}$ and

$$\begin{aligned}
\left|\epsilon\Sigma_{k=0}^{m}|\lambda_0|^k - |y^*\Delta P(\lambda_0)x|\right| &\leq \epsilon\left|\Sigma_{k=0}^{(m-1)/2}\left\{(\cos 2k\phi - i\sin 2k\phi)\lambda_0^{2k} + (\cos(2k+1)\phi - i\sin(2k+1)\phi)\lambda_0^{2k+1}\right\}\right. \\
&\quad \left.-\Sigma_{k=0}^{(m-1)/2}\left\{\delta_{2k}\lambda_0^{2k} + i\delta_{2k+1}\lambda_0^{2k+1}\right\}\right| \\
&= \epsilon\left|\Sigma_{k=1}^{(m-1)/2}(\cos 2k\phi - \delta_{2k} - i\sin 2k\phi)\lambda_0^{2k}\right. \\
&\quad \left.+\Sigma_{k=1}^{(m+1)/2}(\cos(2k-1)\phi - i(\sin(2k-1)\phi + \delta_{2k-1}))\lambda_0^{2k-1}\right| \\
&\leq \epsilon\Sigma_{k=1}^{(m-1)/2}\left\{\sqrt{(\cos 2k\phi - \delta_{2k})^2 + \sin^2 2k\phi}|\lambda_0|^{2k}\right\} \\
&\quad +\Sigma_{k=1}^{(m+1)/2}\left\{\sqrt{\cos^2(2k-1)\phi + (\sin(2k-1)\phi + \delta_{2k-1})^2}|\lambda_0|^{2k-1}\right\}
\end{aligned}$$

In view of our choice of δ_k this implies that

$$\begin{aligned}
\left|\Sigma_{k=0}^{m}|\lambda_0|^k - |y^*\Delta P(\lambda_0)x|\right| &\leq \epsilon\Sigma_{k=1}^{(m-1)/2}\left\{\sqrt{(|\cos 2k\phi| + 1)^2 + \sin^2 2k\phi}|\lambda_0|^{2k}\right\} \\
&\quad +\Sigma_{k=1}^{(m+1)/2}\left\{\sqrt{\cos^2(2k-1)\phi + (|\sin(2k-1)\phi| + 1)^2}|\lambda_0|^{2k-1}\right\} \\
&\leq \epsilon\Sigma_{k=1}^{(m-1)/2}\sqrt{2(1 + |\cos 2k\phi|}|\lambda_0|^{2k} + \Sigma_{k=1}^{(m+1)/2}\sqrt{2(1 + |\sin(2k-1)\phi|)}|\lambda_0|^{2k-1} \\
&\leq \epsilon\sqrt{2}\max\left\{\max_{1\leq k\leq(m-1)/2}\sqrt{1 + |\cos 2k\phi|}, \max_{1\leq k\leq(m+1)/2}\sqrt{1 + |\sin(2k-1)\phi|}\right\} \\
&\quad \times\Sigma_{k=0}^{m}|\lambda_0|^k
\end{aligned}$$

where once again it is assumed that the terms involving $\Sigma_{k=1}^{(m-1)/2}$ are 0 when $m = 1$. This gives

$$|y^*\Delta P(\lambda_0)x| \geq \epsilon\left\{\sqrt{2}\max\left\{\max_{1\leq(m-1)/2}\sqrt{1 + |\cos 2k\phi|}, \max_{1\leq(m+1)/2}\sqrt{1 + |\sin(2k-1)\phi|}\right\} - 1\right\}\Sigma_{k=0}^{m}|\lambda_0|^k. \tag{18}$$

Hence we have lower bound l_2. The proofs for all the other cases follow in an analogous manner. □

The following numerical experiments performed in MATLAB we observed that the lower bound given by (14) is fairly tight. In all these examples the coefficient matrices of the polynomials are produced by using the `randn` command to generate random complex matrices A and then setting the Hermitian and skew Hermitian coefficients to $A + A'$ and $A - A'$ respectively. Moreover the tables accompanying each example contain the lower bounds for only one of the eigenvalues forming a pair $(\lambda, -\bar{\lambda})$ as the bounds for the other eigenvalue are identical.

Example 3.15 We consider a $*$-odd matrix pencil $P(\lambda)$ where the coefficient matrices are of size 10. Table 4 gives the simple eigenvalues which neither real, nor purely imaginary such that $x^*y \neq 0$. Here $l_1 := 1 - (1 - c)|\sin\phi|$ and $l_2 := \sqrt{2(1 + \cos\phi)} - 1$.

Example 3.16 In this example $P(\lambda)$ is a $*$-even polynomial of degree 4 with coefficient matrices of size 4. It has 7 pairs of eigenvalues satisfying the criterion for bound (14) to hold. Table 5 shows these pairs with corresponding values of l_1 and l_2.

For the T-even and T-odd polynomials the following theorem provides exact expressions for $\kappa^S(\lambda_0, P)$ which suggest that the structured and unstructured condition numbers are not equal in general.

Table 4: Relevant eigenvalues corresponding to Example 3.15 with corresponding values of $lb1$ and $lb2$

λ_0	l_1	l_2
$1.83735 - 0.1843i$	0.9985	0.9976
$-0.3055 + 1.0349i$	0.8876	0.602
$-0.7852 + 0.1738i$	0.9666	0.9881
$-0.2657 - 0.2892i$	0.9952	0.8311

Table 5: Relevant eigenvalues of Example 3.16 with corresponding values of l_1 and l_2

λ_0	l_1	l_2
$-1.3684 - 0.4157i$	0.6828	0.9137
$0.2541 - 0.9345i$	0.9903	0.9824
$-0.6746 - 0.9928i$	0.691	0.9116
$-0.814 + 1.3599i$	0.6793	0.9277
$-0.9029 + 0.2782i$	0.8043	0.9113
$-0.6075 - 0.0214i$	0.7405	0.9988
$-0.5748 + 0.8537i$	0.9461	0.9129

Theorem 3.17 *Given a T-even polynomial $P(\lambda)$ we have*

$$\begin{aligned}\kappa^{T-even}(\lambda_0, P) &= \frac{\Sigma_{k=0}^{(m-1)/2}\left(|\lambda_0|^{2k} + \sqrt{1-|x^Ty|^2}|\lambda_0|^{2k+1}\right)}{|y^*P'(\lambda_0)x|} \text{ if } m \text{ is odd} \\ &= \frac{1+\Sigma_{k=1}^{m/2}\left(|\lambda_0|^{2k} + \sqrt{1-|x^Ty|^2}|\lambda_0|^{2k-1}\right)}{|y^*P'(\lambda_0)x|} \text{ if } m \text{ is even}\end{aligned}$$

If the polynomial $P(\lambda)$ is T-odd then we have

$$\begin{aligned}\kappa^{T-odd}(\lambda_0, P) &= \frac{\Sigma_{k=0}^{(m-1)/2}\left(|\lambda_0|^{2k+1} + \sqrt{1-|x^Ty|^2}|\lambda_0|^{2k}\right)}{|y^*P'(\lambda_0)x|} \text{ if } m \text{ is odd} \\ &= \frac{\sqrt{1-|x^Ty|^2} + \Sigma_{k=1}^{m/2}\left(|\lambda_0|^{2k-1} + \sqrt{1-|x^Ty|^2}|\lambda_0|^{2k}\right)}{|y^*P'(\lambda_0)x|} \text{ if } m \text{ is even}\end{aligned}$$

Proof: Let $P(\lambda)$ be a T-even polynomial of odd degree m. Theorem 2.5 states that $K_S(x,y) := \{y^*Ax : \|A\|_2 \leq 1\}$ is a disc about the origin of radius 1 if S is the set of complex symmetric matrices, while it is a disc of radius $\sqrt{1-|x^Ty|^2}$ if S denotes the set of complex skew-symmetric matrices. For an arbitrary $\epsilon > 0$, given any T-even perturbation $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k$ where $\|\Delta A_k\|_2 \leq \epsilon$, we have

$$\begin{aligned}|y^*\Delta P(\lambda_0)x| &= \left|\Sigma_{k=0}^m \lambda_0^k y^*\Delta A_k x\right| \\ &\leq \Sigma_{k=0}^{(m-1)/2}|\lambda_0^{2k}y^*\Delta A_{2k}x + \lambda_0^{2k+1}y^*\Delta A_{2k+1}x| \\ &\leq \Sigma_{k=0}^{(m-1)/2}\{|\lambda_0|^{2k}|y^*\Delta A_{2k}x| + |\lambda_0|^{2k+1}|y^*\Delta A_{2k+1}x|\}\end{aligned}$$

This implies that

$$\begin{aligned} |y^*\Delta P(\lambda_0)x| &\leq \epsilon\Sigma_{k=0}^{(m-1)/2}\left\{|\lambda_0|^{2k}\max_{\substack{S^T=S\\ \|S\|_2\leq\epsilon}}|y^*Sx| + |\lambda_0|^{2k+1}\max_{\substack{R^T=-R\\ \|R\|_2\leq\epsilon}}|y^*Rx|\right\} \\ &\leq \epsilon\Sigma_{k=0}^{(m-1)/2}\{|\lambda_0|^{2k} + \sqrt{1-|x^Ty|^2}|\lambda_0|^{2k+1}\} \qquad (19) \end{aligned}$$

It is evident that the above bound is attained by a T-even perturbation $\Delta P(\lambda) := \Sigma_{k=0}^{(m-1)/2}\lambda^k\epsilon\Delta A_k$ where $\Delta A_k = \text{sign}(\lambda_0^k)S$ when $k = 0$ or k is even and $\Delta A_k = \text{sign}(\lambda_0^k)SK$ when k is odd, S and SK being respectively symmetric and skew symmetric matrices with unit 2-norm for which $y^*Sx = 1$ and $y^*SKx = \sqrt{1-|x^Ty|^2}$. Since such a choice is possible by Theorem 2.5, this establishes the expression for $\kappa^{T-even}(\lambda_0, P)$ when m is odd. The expressions for the other structured condition numbers may be derived in a similar way. □

The above result yield the following immediate corollaries.

Corollary 3.18 *Let 0 and ∞ be simple eigenvalues of $P(\lambda)$.*

(i) If $P(\lambda)$ is T-even, then $\kappa(0,P) = \kappa^{T-even}(0,P) = 1$ while $\kappa^{T-even}(\infty,P) = \kappa(\infty,P) = 1$ if m is even and $\kappa^{T-even}(\infty,P) = 0$ if m is odd.

(ii) If $P(\lambda)$ is T-odd, then $\kappa^{T-odd}(0,P) = 0$ while $\kappa^{T-odd}(\infty,P) = 0$ if m is even and $\kappa^{T-odd}(\infty,P) = \kappa(\infty,P) = 1$ if m is odd.

Proof: If $P(\lambda)$ is T-even, then the proof of $\kappa(0,P) = \kappa^{T-even}(0,P) = 1$ follows immediately from Theorem 3.17. If $P(\lambda)$ is T-odd, then $P(0)$ is a skew-symmetric matrix so that 0 is always an eigenvalue of it. This implies that $\kappa^{T-odd}(0,P) = 0$.

If ∞ is a simple eigenvalue of $P(\lambda)$, then 0 must be a simple eigenvalue of $\text{rev}P(\lambda)$. Therefore the proofs of the relations between the structured and unstructured condition number of ∞ follow from the corresponding relations for 0 by using the fact that if $P(\lambda)$ is T-even (respectively, T-odd), then $\text{rev}P(\lambda)$ is T-odd (respectively T-even) if m is odd and T-even (respectively T-odd) when m is even. □

Corollary 3.19 *Given a T-even polynomial $P(\lambda)$, we have $\kappa^{T-even}(\lambda_0,P) = \kappa(\lambda_0,P)$ for a simple eigenvalue λ_0 if and only if either $\lambda_0 = 0$ or $x^Ty = 0$. If $P(\lambda)$ is T-odd then the corresponding equality holds if and only if $x^Ty = 0$.*

Note 3.20 *The above Corollary raises the issue of existence of simple eigenvalues λ_0 of T-even and T-odd polynomials whose right and left eigenvectors satisfy $x^Ty = 0$. The condition $x^Ty = 0$ is equivalent to $\bar{y}^*x = 0$. Now since the eigenvalues of such polynomial occur in pairs $(\lambda_0, -\lambda_0)$, and $\bar{y}$ and $\bar{x}$ are right and left eigenvectors of $-\lambda_0$ respectively, we will have $x^Ty = 0$ whenever there exists a vector in the null space $P(\lambda_0)$ which is orthogonal to a vector in the null space of $P(-\lambda_0)$. This happens for the following T-even pencil $A_0 + \lambda A_1$ where*

$$A_0 = \begin{pmatrix} 1+i & 0 \\ 0 & 2+i \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} -(3+i)/5 & 0 \\ 0 & 1 \end{pmatrix}.$$

It has eigenvalues $2+i$ and $-(2+i)$ with corresponding right eigenvectors $x_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ which are clearly orthogonal.

Note 3.21 *Theorem 3.17 also holds for simple real eigenvalues of real T-even and T-odd polynomials. In such cases the corresponding left and right eigenvectors are also real so that the matrices S and SK used in the proof of Theorem 3.17 to construct the coefficient matrices of the structure preserving perturbation that attain the upper bound (19) are also real in view of Theorem 2.4.*

4 Structured and unstructured backward error

In this section we consider structured backward errors of approximate eigenvalues of polynomials having any one of the structures listed in Table **??**. More specifically, we first determine sets of conditions that are necessary for the structured and unstructured backward errors of an approximate eigenvalue to be equal. Then we show that some of these conditions are also sufficient for the equality to hold. In certain cases, this results in necessary and sufficient conditions for the equality of the two types of backward errors. Throughout this section λ_0 denotes an approximate (but not exact) eigenvalue of $P(\lambda)$ and $\sigma_{\min}(P(\lambda_0))$ denotes the smallest singular value of $P(\lambda_0)$. Also we replace the S of the notation $\eta^S(\lambda_0, P)$ for structured backward error introduced in section 2 by $*-pal$, $*-antipal$, $T-pal$, $T-antipal$, $*-even$, $*-odd$, $T-even$ and $T-odd$ to denote the structured backward with respect to $*$-palindromic, $*$-antipalindromic, T-palindromic, T-antipalindromic, $*$-even, $*$-odd, T-even and T-odd perturbations respectively. We begin with the $*$-palindromic polynomials.

Theorem 4.1 *Let $P(\lambda)$ be a $*$-palindromic polynomial. If $\eta^{*-pal}(\lambda_0, P) = \eta(\lambda_0, P)$ and the degree m of $P(\lambda_0)$ is odd, then one of the following conditions hold.*

(a). There exist unit vectors u and v which are respectively left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ such that

$$|\langle v, u\rangle|^2 \left(1 + \sin^2\left(\frac{m\phi + 2\psi}{2}\right)\right) \leq 1$$

where $\psi = \arg(u^*v)$ and $\phi = \arg(\lambda_0)$.

(b). Either $\lambda_0 = 0$ or $\lambda_0 = \infty$.

(c). The equality $\langle v, u\rangle = \operatorname{sign}(\lambda_0^m)\langle u, v\rangle$ holds for some unit vectors u and v satisfying $P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u$.

Conversely if λ_0 is either 0 or ∞ or there exist unit singular vectors u and v such that $P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u$ and $\langle v, u\rangle = \operatorname{sign}(\lambda_0^m)\langle u, v\rangle$ then $\eta^{*-pal}(\lambda_0, P) = \eta(\lambda_0, P)$.

If $P(\lambda_0)$ is $*$-palindromic of even degree, then $\eta^{*-pal}(\lambda_0, P) = \eta(\lambda_0, P)$ if and only if either λ_0 is 0 or ∞ or there exist unit left and right singular vectors u and v such that $P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u$ and $\langle v, u\rangle = \operatorname{sign}(\lambda_0^m)\langle u, v\rangle$.

Proof: The equality of the structured and unstructured backward errors implies that $\eta^{*-pal}(\lambda_0, x, P) = \dfrac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^m |\lambda_0|^k}$ for some $x \in \mathbb{C}^n$ such that $\|x\|_2 = 1$. We prove by contradiction that x must be a right singular vector of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$. Suppose that x is not such a vector. Then, $\dfrac{\|P(\lambda_0)x\|_2}{\Sigma_{k=0}^m |\lambda_0|^k} > \dfrac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^m |\lambda_0|^k}$. However, $\eta(\lambda_0, x, P) = \dfrac{\|P(\lambda_0)x\|_2}{\Sigma_{k=0}^m |\lambda_0|^k}$ and $\eta(\lambda_0, x, P) \leq \eta^{*-even}(\lambda_0, x, P) = \dfrac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^m |\lambda_0|^k}$. But this contradicts the previous inequality. So there exists a right singular vector v such that $\|v\|_2 = 1$ and $x = v$.

This implies that there exists a $*$-palindromic polynomial $\Delta P(\lambda) = \Sigma_{k=0}^{m} \lambda^k \Delta A_k$, such that $\max\limits_{0\leq k\leq m} \|\Delta A_k\|_2 = \eta(\lambda_0, P) = \dfrac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^{m}|\lambda_0|^k}$ and

$$\Delta P(\lambda_0)v = P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u. \tag{20}$$

where u is a corresponding unit left singular vector. Equation (20) implies that

$$(\sigma_{\min}(P(\lambda_0)))^2 = \langle P(\lambda_0)v, P(\lambda_0)v\rangle = \sigma_{\min}(P(\lambda_0))\langle u, \Delta P(\lambda_0)v\rangle.$$

Since $\sigma_{\min}(P(\lambda_0)) \neq 0$, we have

$$u^*\Delta P(\lambda_0)v = \sigma_{\min}(P(\lambda_0)). \tag{21}$$

But we also have,

$$\begin{aligned} |u^*\Delta P(\lambda_0)v| &= \left|\Sigma_{k=0}^{m}\lambda_0^k u^*\Delta A_k v\right| \\ &= \left|\Sigma_{k=0}^{m}\lambda_0^k u^*\left(\frac{\Delta A_k}{\max\limits_{0\leq k\leq m}\|\Delta A_k\|_2}\right)v\right| \max_{0\leq k\leq m}\|\Delta A_k\|_2 \\ &\leq \Sigma_{k=0}^{m}|\lambda_0|^k \frac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^{m}|\lambda_0|^k} \end{aligned}$$

and the above inequality cannot be strict in view of equation (21). This implies that $\left|\Sigma_{k=0}^{m}\lambda_0^k u^*\left(\dfrac{\Delta A_k}{\max\limits_{0\leq k\leq m}\|\Delta A_k\|_2}\right)v\right| = \Sigma_{k=0}^{m}|\lambda_0|^k$. By Lemma 3.2 the latter equality holds if and only if

$$u^*\left(\frac{\Delta A_k}{\max\limits_{0\leq k\leq m}\|\Delta A_k\|_2}\right)v = \omega \text{sign}(\lambda_0)^k, k = 0:m,$$

where ω is any complex number such that $|\omega| = 1$. However, equation (21) implies that $\omega = 1$. Setting, $B_k := \dfrac{\Delta A_k}{\max\limits_{0\leq k\leq m}\|\Delta A_k\|_2}, k = 0:m$, in view of Theorem 2.5, we have,

$$u^*\left(\frac{B_k + B_k^*}{2}\right)v = e^{i\psi}\{\xi_k + ic\hat{\xi}_k\} \text{ and } u^*\left(\frac{B_k - B_k^*}{2}\right)v = -ie^{i\psi}\{\eta_k + ic\hat{\eta}_k\}, k = 0:m,$$

where $c = \sqrt{1 - |u^*v|^2}$, and $\xi_k, \hat{\xi}_k, \eta_k$, and $\hat{\eta}_k$ are real numbers satisfying, $\xi_k^2 + \hat{\xi}_k^2 \leq 1$, and $\eta_k^2 + \hat{\eta}_k^2 \leq 1$. This implies that for all $k = 0:m$,

$$u^*B_k v = e^{i\psi}\{\xi_k + c\hat{\eta}_k + i(c\hat{\xi}_k - \eta_k)\}, \text{ and } u^*B_k^* v = e^{i\psi}\{\xi_k - c\hat{\eta}_k + i(c\hat{\xi}_k + \eta_k)\},$$

so that,

$$e^{i\psi}\{\xi_k + c\hat{\eta}_k + i(c\hat{\xi}_k - \eta_k)\} = \cos k\phi - i\sin k\phi. \tag{22}$$

$$e^{i\psi}\{\xi_k - c\hat{\eta}_k + i(c\hat{\xi}_k + \eta_k)\} = \cos(m-k)\phi - i\sin(m-k)\phi \tag{23}$$

for all $k = 0:\frac{m-1}{2}$ if m is odd and $k = 0:\frac{m}{2}$ if m is even. Upon equating the real and imaginary parts in the above equations and solving them for $\xi_k, \hat{\xi}_k, \eta_k$ and $\hat{\eta}_k$, we have

$$\xi_k = \cos\frac{m\phi + 2\psi}{2}\cos\frac{(2k-m)\phi}{2},\ c\hat{\xi}_k = -\sin\frac{m\phi + 2\psi}{2}\cos\frac{(2k-m)\phi}{2}$$

$$\eta_k = \cos\frac{m\phi + 2\psi}{2}\sin\frac{(2k-m)\phi}{2},\ c\hat{\eta}_k = -\sin\frac{m\phi + 2\psi}{2}\sin\frac{(2k-m)\phi}{2}$$

under the assumption that $c \neq 0$.

Thus if $c \neq 0$, the above quantities satisfy the inequalities $\xi_k^2 + \hat{\xi}_k^2 \leq 1$ and $\eta_k + \hat{\eta}_k^2 \leq 1$. Using the above expressions for these quantities in terms of ϕ and ψ, these inequalities respectively imply that

$$\cos^2\left(\frac{m\phi+2\psi}{2}\right)\cos^2\left(\frac{m-2k}{2}\phi\right) + \frac{1}{c^2}\sin^2\left(\frac{m\phi+2\psi}{2}\right)\cos^2\left(\frac{m-2k}{2}\phi\right) \leq 1 \quad (24)$$

$$\cos^2\left(\frac{m\phi+2\psi}{2}\right)\sin^2\left(\frac{m-2k}{2}\phi\right) + \frac{1}{c^2}\sin^2\left(\frac{m\phi+2\psi}{2}\right)\sin^2\left(\frac{m-2k}{2}\phi\right) \leq 1 \quad (25)$$

Inequality (25) implies that

$$\begin{aligned}\cos^2\left(\frac{m\phi+2\psi}{2}\right) - \cos^2\left(\frac{m\phi+2\psi}{2}\right)\cos^2\left(\frac{m-2k}{2}\phi\right) &\leq 1 - \frac{1}{c^2}\sin^2\left(\frac{m\phi+2\psi}{2}\right) \\ &+ \frac{1}{c^2}\sin^2\left(\frac{m\phi+2\psi}{2}\right)\cos^2\left(\frac{m-2k}{2}\phi\right)\end{aligned}$$

so that upon using inequality (24) in the above relationship and rearranging the terms, we get $|u^*v|^2\left(1+\sin^2\left(\frac{m\phi+2\psi}{2}\right)\right) \leq 1$ which is condition (a).

If $c = 0$, then (22) and (23) imply that $e^{-i(\psi+k\phi)} = \overline{e^{-i(\psi+(m-k)\phi)}}$ which implies that either $\lambda_0 = 0$ or $e^{i(m\phi+2\psi)} = 1$ In case, the second relationship holds, then $e^{-i\psi} = e^{i(m\phi+\psi)}$ which implies that $\langle v, u\rangle = \text{sign}(\lambda_0)^m\langle u, v\rangle$.

If the degree m of $P(\lambda)$ is even, then $\Delta A_{m/2}$ must be Hermitian. Therefore the quantities $\eta_{m/2}$ and $\hat{\eta}_{m/2}$ must be zero. Using these conditions in (22) we have, $\xi_{m/2} + ic\hat{\xi}_{m/2} = e^{-i\left(\frac{m\phi+2\psi}{2}\right)}$. If $c \neq 1$, then the point on the left lies on an ellipse which touches the unit circle at 1 or -1. Therefore this relationship holds if and only if either $c = 1$ or $e^{-i\left(\frac{m\phi+2\psi}{2}\right)} = \pm 1$. The first condition implies that $u^*v = 0$ and the second implies that $e^{-i(m\phi+2\psi)} = 1$. In both cases we have, $\langle v, u\rangle = \text{sign}(\lambda_0)^m\langle u, v\rangle$.

Finally, we observe that in view of the eigenvalue pairing of $*$-palindromic polynomials, if $\eta^{*-pal}(\lambda_0, P) = \eta(\lambda_0, P)$, then we also have $\eta^{*-pal}(1/\bar{\lambda_0}, P) = \eta(1/\bar{\lambda_0}, P)$. Replacing λ_0 by $1/\bar{\lambda_0}$ in the above arguments, we obtain $1/\bar{\lambda_0} = 0$ or equivalently, $\lambda_0 = \infty$ as a possible consequence of this equality.

To complete the proof we prove that if λ_0 is either 0 or ∞ or there exist unit singular vectors u and v such that $P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u$ and $\langle v, u\rangle = \text{sign}(\lambda_0)^m\langle u, v\rangle$, then $\eta^{*-pal}(\lambda_0, P) = \eta(\lambda_0, P)$. In view of the $*$-palindromic structures of $P(\lambda_0$ and rev$P(\lambda)$, the backward errors of the number 0 when considered as an approximate eigenvalue of each of these polynomials are the same for both structure preserving and arbitrary perturbations. Therefore the equality of structured and unstructured backward errors trivially hold for these cases. So suppose that λ_0 is nonzero and finite. Let $\omega := (\text{sign}(\lambda_0))^{1/2}$. Then ω satisfies $\omega\overline{\text{sign}(\lambda_0^m)} = \bar{\omega}$ and $\langle v, u\rangle = \text{sign}(\lambda_0)^m\langle u, v\rangle$, implies that $\langle \omega P(\lambda_0)v, v\rangle$ is real. Invoking Theorem 2.4, we obtain a Hermitian matrix H such that

$$\bar{\omega}Hv = \frac{P(\lambda_0)v}{\Sigma_{k=0}^m|\lambda_0|^k} \text{ and } \|H\|_2 = \frac{\|P(\lambda_0)v\|_2}{\Sigma_{k=0}^m|\lambda_0|^k} = \frac{\sigma_{\min}(P(\lambda_0)v)}{\Sigma_{k=0}^m|\lambda_0|^k}.$$

Defining $\Delta A_k := \text{sign}(\lambda_0)^k\bar{\omega}H, k = 0 : m$, we observe that the polynomial $\Delta P(\lambda) = \Sigma_{k=0}^m\lambda^k\Delta A_k$ is $*$-palindromic as,

$$(\Delta A_{m-k})^* = \overline{\text{sign}(\lambda_0^m)}\omega\text{sign}(\lambda_0^k)H = \bar{\omega}\text{sign}\lambda_0^k H = \Delta A_k.$$

Also $\Delta P(\lambda_0)v = P(\lambda_0)v$ and $\|\Delta A_k\|_2 = \|H\|_2 = \eta(\lambda_0, P)$ for all $k = 0 : m$. This proves the equality of $\eta^{*-pal}(\lambda_0, P)$ and $\eta(\lambda_0, P)$. $\square$

Theorem 4.2 *Let $P(\lambda)$ be a $*$-antipalindromic polynomial and u and v be respectively unit left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$. If $\eta^{*-antipal}(\lambda_0, P) = \eta(\lambda_0, P)$ and $P(\lambda)$ is of odd degree, then one of the following conditions hold.*

(a). There exist unit vectors u and v which are respectively left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ such that

$$|\langle v, u\rangle|^2 \left(1 + \cos^2\left(\frac{m\phi + 2\psi}{2}\right)\right) \leq 1$$

where $\psi = \arg(u^*v)$ and $\phi = \arg(\lambda_0)$.

(b). Either $\lambda_0 = 0$ or $\lambda_0 = \infty$.

(c). The equality $\langle v, u\rangle = -\text{sign}(\lambda_0^m)\langle u, v\rangle$ holds for some unit vectors u and v satisfying $P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u$.

Conversely if λ_0 is either 0 or ∞ or there exist unit singular vectors u and v such that $P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u$ and $\langle v, u\rangle = -\text{sign}(\lambda_0^m)\langle u, v\rangle$ then $\eta^{*-antipal}(\lambda_0, P) = \eta(\lambda_0, P)$.

If $P(\lambda_0)$ is $*$-antipalindromic of even degree, then $\eta^{*-antipal}(\lambda_0, P) = \eta(\lambda_0, P)$ if and only if either λ_0 is 0 or ∞ or there exist unit left and right singular vectors u and v such that $P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u$ and $\langle v, u\rangle = -\text{sign}(\lambda_0^m)\langle u, v\rangle$.

Proof: The proof of this Theorem follows by using exactly the same ideas as those used in the previous proof except that we replace equations (22) and (23) by the following equations respectively.

$$e^{i\psi}\{\xi_k + c\hat{\eta}_k + i(c\hat{\xi}_k - \eta_k)\} = \cos k\phi - i\sin k\phi. \quad (26)$$

$$e^{i\psi}\{\xi_k - c\hat{\eta}_k + i(c\hat{\xi}_k + \eta_k)\} = -\cos(m-k)\phi + i\sin(m-k)\phi \quad (27)$$

In case m is even, the necessary and sufficient condition for the equality $\eta^{*-antipal}(\lambda_0, P) = \eta(\lambda_0, P)$ to hold follows by observing that $\Delta A_{m/2}$ must be a skew-Hermitian matrix. □

We note that the necessary condition (a) of Theorem 4.1 for the equality of $\eta^{*-pal}(\lambda_0, P) = \eta(\lambda_0, P)$ is satisfied by all complex numbers of modulus one. Likewise, condition (a) of Theorem 4.2 is also satisfied by such numbers. Next we consider polynomials with T-palindromic/antipalindromic structure.

Theorem 4.3 *Let $P(\lambda)$ be a T-palindromic polynomial. If $\eta^{T-pal}(\lambda_0, P) = \eta(\lambda_0, P)$ then one of the following conditions hold.*

(a) There exist unit vectors u and v which are respectively left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ such that $|u^Tv|^2 \leq \cos^2\left(\frac{m-2k}{2}\phi\right)$ for all $k = 0 : m$, where $\phi = \arg(\lambda_0)$.

(b) Either $\lambda_0 = 0$ or $\lambda_0 = \infty$.

(c) If m is odd, then λ_0 is a positive real number.

(d) If m is even, then λ_0 is a real number.

Conversely, if λ_0 is ∞ or a non-negative real number, and the degree m of $P(\lambda)$ is odd, then $\eta^{T-pal}(\lambda_0, P) = \eta(\lambda_0, P)$. If m is even, then the same equality holds whenever $\lambda_0 = \infty$ or λ_0 is a real number.

Proof: To prove that the given conditions are necessary, we suppose that $\eta^{T-pal}(\lambda_0, P) = \eta(\lambda_0, P)$. Then arguing as in Theorem 4.1, there exists a pair (u, v) of unit left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ and a T-palindromic perturbation $\Delta P(\lambda) = \Sigma_{k=0}^{m} \lambda^k \Delta A_k$ to $P(\lambda)$ satisfying

$$\Delta P(\lambda_0)v = P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u, \text{ and } \max_{0 \le k \le m} \|\Delta A_k\|_2 = \frac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^{m} |\lambda_0|^k}$$

such that $|u^* B_k v| = \text{sign}(\lambda_0)^k$ for all $k = 0 : m$, where $B_k := \dfrac{\Delta A_k}{\max\limits_{0 \le k \le m} \|\Delta A_k\|_2}, k = 0 : m.$
The necessity of the given conditions now follow by replacing x and y by u and v respectively in the arguments used in the proof of Theorem 3.3.

To prove the sufficiency of the given conditions, we observe that the equality of $\eta^{T-pal}(\lambda_0, P)$ and $\eta(\lambda_0, P)$ trivially holds if $\lambda_0 = 0$ or ∞ as structure preserving perturbations do not have any effect on the backward error of 0 as an approximate eigenvalue of $P(\lambda)$ or $\text{rev}(P(\lambda))$. So we assume that m is odd and that λ_0 is a positive real number. By Theorem 2.4, there exists a symmetric matrix S such that $Sv = \dfrac{P(\lambda_0)v}{\Sigma_{k=0}^{m} |\lambda_0|^k}$ and $\|S\|_2 = \dfrac{\|P(\lambda_0)v\|_2}{\Sigma_{k=0}^{m} |\lambda_0|^k} = \dfrac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^{m} |\lambda_0|^k}$. Let $\Delta P(\lambda) = \Sigma_{k=0}^{m} \lambda^k \Delta A_k$ where $\Delta A_k = \text{sign}(\lambda_0)^k S$ for all $k = 0 : m$. Then $\Delta P(\lambda)$ is evidently T-palindromic. It also satisfies $\Delta P(\lambda_0)v - P(\lambda_0)v = 0$ and $\|\Delta A_k\|_2 = \|S\|_2 = \eta(\lambda_0, P)$ for all $k = 0 : m$ thus implying the equality of $\eta(\lambda_0, P)$ and $\eta^{T-pal}(\lambda_0, P)$.

If m is even, then $\text{sign}(\lambda_0^{m-k}) = \text{sign}(\lambda_0^k)$ for all $k = 0 : m$ and any real number λ_0. Thus if we define $\Delta P(\lambda) := \Sigma_{k=0}^{m} \lambda^k \Delta A_k$ where $\Delta A_k = \text{sign} \lambda_0^k S, k = 0 : m$, S being a symmetric matrix which is chosen as above, then $\Delta P(\lambda)$ is T-palindromic such that $\Delta P(\lambda_0)v = P(\lambda_0)v$ and $\|\Delta A_k\|_2 = \eta(\lambda_0, P)$ for all $k = 0 : m$. This establishes the equality of $\eta^{T-pal}(\lambda_0, P)$ and $\eta(\lambda_0, P)$. □

Theorem 4.4 *Let $P(\lambda)$ be a T-antipalindromic polynomial. If $\eta^{T-antipal}(\lambda_0, P) = \eta(\lambda_0, P)$ and the degree of $P(\lambda)$ is odd, then one of the following conditions hold.*

(a) There exist unit vectors u and v which are respectively left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ such that $|u^T v|^2 \le \sin^2\left(\frac{m-2k}{2}\phi\right)$ for all $k = 0 : m$, where $\phi = \arg(\lambda_0)$.

(b) Either $\lambda_0 = 0$ or $\lambda_0 = \infty$.

(c) λ_0 is a negative real number.

Conversely, if λ_0 is ∞ or a non-positive real number, then $\eta^{T-antipal}(\lambda_0, P) = \eta(\lambda_0, P)$.
If m is even, then a necessary condition for the equality of $\eta(\lambda_0, P)$ and $\eta^{T-antipal}(\lambda_0, P)$ is that either $\lambda_0 = 0$ or ∞ or there exist unit left and right singular vectors u and v respectively of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ such that $u^T v = 0$. Conversely, if $\lambda_0 = \infty$ or λ_0 is a real number and $u^T v = 0$, then $\eta^{T-antipal}(\lambda_0, P) = \eta(\lambda_0, P)$.

Proof: The necessity of the given conditions follows by using arguments very similar to those used in Theorem 4.3. In case m is even, condition (a) takes the form $u^T v = 0$ as $m/2$ is an integer between 0 and m.

Similarly to prove the sufficiency of the conditions, we invoke Theorem 2.4 to obtain a symmetric matrix S (when m is odd) and a skew-symmetric matrix Q (when m is even)

both of which map v to $\frac{P(\lambda_0)v}{\Sigma_{k=0}^m|\lambda_0|^k}$ such that $\|S\|_2 = \|Q\|_2 = \eta(\lambda_0, P)$ and use these to construct T-antipalindromic polynomials $\Delta P(\lambda)$ in exactly the same way as defined in Theorem 4.3 to prove the equality of $\eta^{T-antipal}(\lambda_0, P)$ and $\eta(\lambda_0, P)$.

Note 4.5 *We observe that for any T-antipalindromic polynomial $P(\lambda)$, $P(1)$ is skew symmetric. Therefore 1 is always an eigenvalue of such polynomials so that $\eta^{T-antipal}(1, P) = \eta(1, P) = 0$. For the same reason we also have $\eta^{T-pal}(-1, P) = \eta(-1, P)$ whenever $P(\lambda)$ is T-palindromic of odd degree.*

These two cases were not considered in the above Theorem since it dealt with only those numbers λ_0 which are not eigenvalues of $P(\lambda)$.

The next result concerns the $*$-even/odd polynomials and completely characterises all approximate eigenvalues $\lambda_0 \in \mathbb{C}$ for which the structured and unstructured backward errors are equal.

Theorem 4.6 *Let $P(\lambda)$ be a $*$-even or $*$-odd polynomial. Given any $\lambda_0 \in \mathbb{C}$ which is not an eigenvalue of $P(\lambda)$ we have $\eta^S(\lambda_0, P) = \eta(\lambda_0, P)$ if and only if either λ_0 is purely imaginary or there exists a pair of unit vectors u and v which are respectively left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ satisfying $u^*v = 0$.*

Proof: Let $P(\lambda)$ be a $*$-even polynomial. To prove the necessity of the given conditions let $\lambda_0 \in \mathbb{C}$ be such that $\eta^{*-even}(\lambda_0, P) = \eta(\lambda_0, P)$. Once again by arguments similar to those in the preceding proofs, there exists a pair (u, v) of unit left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$ and a $*$-even perturbation $\Delta P(\lambda) = \Sigma_{k=0}^m \lambda^k \Delta A_k$ to $P(\lambda)$ satisfying

$$\Delta P(\lambda_0)v = P(\lambda_0)v = \sigma_{\min}(P(\lambda_0))u, \text{ and } \max_{0\le k\le m} \|\Delta A_k\|_2 = \frac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^m|\lambda_0|^k}$$

such that $|u^*B_kv| = \text{sign}(\lambda_0)^k$ for all $k = 0 : m$, where $B_k := \dfrac{\Delta A_k}{\max\limits_{0\le k\le m} \|\Delta A_k\|_2}, k = 0 : m.$
The necessity of the given conditions now follow by replacing x and y by u and v respectively in the arguments used in the proof of Theorem 3.12.

Conversely, suppose that there exist a pair of orthogonal unit left and right singular vectors u and v of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$. This implies that $v^*P(\lambda_0)v = 0$ and by Theorem 2.4, there exists a $*$-even perturbation $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k$ to $P(\lambda)$ such that

$$\Delta A_k v = \frac{\text{sign}(\lambda_0^k)P(\lambda_0)v}{\Sigma_{k=0}^m|\lambda_0|^k} \text{ and } \|\Delta A_k\|_2 = \frac{\|P(\lambda_0)v\|_2}{\Sigma_{k=0}^m|\lambda_0|^k} = \frac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^m|\lambda_0|^k} = \eta(\lambda_0, P), k = 0 : m$$

The equality of $\eta^{*-even}(\lambda_0, P)$ and $\eta(\lambda_0, P)$ now follows from the fact that $(\Delta P - P)(\lambda_0)v = P(\lambda_0)v - P(\lambda_0)v = 0$. .

If λ_0 is purely imaginary then $v^*(P(\lambda_0)v$ is real for all unit right singular vectors v of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$. An application of Theorem 2.4 implies the existence of a Hermitian matrix H such that $Hv = \frac{P(\lambda_0)v}{\Sigma_{k=0}^m|\lambda_0|^k}$ and $\|H\|_2 = \frac{\|P(\lambda_0)v\|_2}{\Sigma_{k=0}^m|\lambda_0|^k}$. The proof now follows by constructing the $*$-even perturbation $\Delta P(\lambda) = \Sigma_{k=0}^m \lambda^k (\text{sign}(\lambda_0))^k H, k = 0 : m$, to $P(\lambda)$. □

Note 4.7 *In view of the above Theorem, it is interesting to identify points $\lambda_0 \in \mathbb{C}$ which are not eigenvalues of a $*$-even or $*$-odd polynomial $P(\lambda)$ for which $P(\lambda_0)$ has orthogonal*

left and right singular vectors say, u and v, corresponding to its smallest singular value. Corollary 4.8 of [3] states that this is the case if $\sigma_{\min}(P(\lambda_0))$ is a simple singular value of $P(\lambda_0)$ and 0 is a common boundary point of components of the ϵ-pseudospectrum of $P(\lambda_0)$. If $\sigma_{\min}(P(\lambda_0))$ is not simple, then we still have $u^*v = 0$ if $P(\lambda_0))$ is a normal matrix having at least two eigenvalues equidistant from 0 [2]. This latter criterion holds for the following $*$-even pencil $A_0 + zA_1$ at the point $z_0 = -(1+i)$ where

$$A_0 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, A_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

The only singular value of $A_0 + z_0A_1$ is $\sqrt{2}$ and $v = [\frac{1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}]^T$ and $u = [1, -1]^T$ are a corresponding mutually orthogonal pair of right and left unit singular vectors respectively.

The next Theorem considers the T-even and T-odd polynomials and completely characterises all $\lambda_0 \in \mathbb{C}$ which are not eigenvalues of $P(\lambda)$ for which the structured and unstructured backward errors are same.

Theorem 4.8 *Given a polynomial T-even polynomial $P(\lambda)$, $\eta^{T-even}(\lambda_0, P) = \eta(\lambda_0, P)$ if and only if either $\lambda_0 = 0$ or there exists unit vectors u and v which are respectively left and right singular vectors of $P(\lambda_0)$ corresponding to the $\sigma_{\min}(P(\lambda_0))$ such that $u^Tv = 0$.*

If $P(\lambda)$ is T-odd, then the same equality holds if and only if $u^Tv = 0$.

Proof: Assuming that $P(\lambda_0)$ is a T-even polynomial of odd degree, we first prove the necessity of the given condition. Suppose that $\eta^{T-even}(\lambda_0, P) = \eta(\lambda_0, P)$. Then by arguing as in Theorem 4.1, we have $\eta^{T-even}(\lambda_0, P) = \eta^{T-even}(\lambda_0, v, P)$ for some unit right singular vector v of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$. Therefore there exists a T-even perturbation $\Delta P(\lambda) = \Sigma_{k=0}^m \lambda^k \Delta A_k$ such that $\max\limits_{0\le k\le m} \|\Delta A_k\|_2 = \eta(\lambda_0, P) = \dfrac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^m |\lambda_0|^k}, k = 0:m$, and

$$u^*\Delta P(\lambda_0)v = \sigma_{\min}(P(\lambda_0)). \tag{28}$$

However an application of Theorem 2.5 implies that

$$|u^*\Delta P(\lambda_0)v| \le \frac{\Sigma_{k=0}^{(m-1)/2}(|\lambda_0|^{2k} + \sqrt{1-|u^Tv|^2}|\lambda_0|^{2k+1})}{\Sigma_{k=0}^m |\lambda_0|^k}\sigma_{\min}(P(\lambda_0)) \tag{29}$$

which implies that $|u^*\Delta P(\lambda_0)v| < \sigma_{\min}(P(\lambda_0))$ unless $u^Tv = 0$ or $\lambda_0 = 0$. But this violates equation (28). Therefore the given conditions are necessary.

To prove the sufficiency of the conditions, suppose that $\lambda_0 = 0$. The T-even structure of $P(\lambda)$ implies that $P(0)$ is a symmetric matrix so that we have $u = v$. The equality $\eta^{T-even}(0, P) = \eta(0, P)$ now follows by choosing a T-even perturbation $\Delta P(\lambda_0) = \Sigma_{k=0}^m \lambda^k \Delta A_k$ such that $\Delta A_0 = \sigma_{\min} P(0) vv^T$ and $\|\Delta A_k\|_2 = \sigma_{\min}(P(0)), k = 1:m$.

Finally, suppose that $P(\lambda_0)$ has a pair of unit left and right singular vectors u and v respectively corresponding to $\sigma_{\min}(P(\lambda_0))$ such that $u^Tv = 0$. Since $\sigma_{\min}(P(\lambda_0)) \neq 0$, this implies that $v^TP(\lambda_0)v = 0$. The equality of the structured and unstructured backward errors now follow by using Theorem 2.4 to construct a T-even perturbation $\Delta P(\lambda) = \Sigma_{k=0}^m \lambda^k \Delta A_k$ to $P(\lambda)$ such that

$$\Delta A_k v = \frac{\text{sign}(\lambda_0)^k P(\lambda_0)v}{\Sigma_{k=0}^m |\lambda_0|^k} \text{ and } \|\Delta A_k\|_2 = \frac{\|P(\lambda_0)v\|_2}{\Sigma_{k=0}^m |\lambda_0|^k} = \frac{\sigma_{\min}(P(\lambda_0))}{\Sigma_{k=0}^m |\lambda_0|^k} = \eta(\lambda_0, P), k = 0:m.$$

Hence the given conditions are sufficient. □

Note 4.9 *If $P(\lambda)$ is T-odd, then $P(0)$ is evidently a complex skew symmetric matrix so that 0 is always an eigenvalue of it. Thus 0 is always an eigenvalues of a T-odd polynomial so that $\eta^{T-odd}(0,P) = \eta(0,P) = 0$. This case has not been considered in the above Theorem since it deals with only those λ_0 which are not eigenvalues of $P(\lambda)$.*

Note 4.10 *If $P(\lambda)$ is T-even or T-odd, then $\sigma_{\min}(P(\lambda_0)) = \sigma_{\min}(P(-\lambda_0))$. Also if u and v are respectively unit left and right singular vectors of $P(\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$, then $\bar{v}$ and $\bar{u}$ are corresponding left and right singular vectors of $P(-\lambda_0)$ corresponding to $\sigma_{\min}(P(\lambda_0))$. Thus the condition $u^T v = 0$ of the above Theorem is satisfied whenever the $P(\lambda_0)$ and $P(-\lambda_0)$ have mutually orthogonal right singular vectors corresponding to their common smallest singular value. This is fulfilled by the T-even polynomial $A_0 + zA_1$ when $z_0 = \dfrac{2+i}{5}$ where*

$$A_0 := \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1+2i \\ -(1+2i) & 0 \end{pmatrix}.$$

Indeed, $A_0 + z_0 A_1 = \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix}$, which has a double singular value at $\sqrt{5}$ and corresponding right and left unit singular vectors $v = [\frac{2+i}{\sqrt{5}}, 0]^T$ and $u = [0, 1]^T$ respectively which satisfy $u^T v = 0$.

Conclusion. We have obtained characterisations of simple eigenvalues of $\star$-palindromic, $\star$-antipalindromic, $\star$-even and $\star$-odd polynomials which have the same structured and unstructured condition numbers. In the process, we have obtained formulae for the structured condition number of simple eigenvalues of $*$-palindromic/antipalindromic and T-even/odd polynomials and tight bounds that localise the structured condition number for the T-palindromic/antipalindromic and $*$-even/odd polynomials. The bounds indicate that the structured and unstructured condition numbers are nearly equal except for a few exceptional cases. We have also investigated conditions under which structured and unstructured backward errors of approximate eigenvalues of these polynomials are equal. In the process we have obtained complete characterisations of approximate eigenvalues with equal structured and unstructured backward errors for the $\star$-even and $\star$-odd polynomials. The results for the structured condition numbers may be used to carry out a search for optimal structure preserving linearisations of these structured polynomials (on the lines of the one carried out in [8], for the unstructured polynomials) so that the condition numbers of the eigenvalues of the linearized problem are as close as possible to that of the original polynomial. This shall be the focus of future work.

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