POSITIVITY PRESERVING BALANCED TRUNCATION

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Abstract. We propose a model reduction method for positive systems that ensures the positivity of the reduced model. For both, continuous-time and discrete-time systems, our approach is based on constructing diagonal solutions of Lyapunov inequalities. These are linear matrix inequalities (LMIs), which are shown to be feasible. Stability is preserved and an error bound in the $\mathcal{H}_\infty$-norm is provided.

Key words. Positive systems; structured model reduction; linear matrix inequalities (LMI); balanced truncation; singular perturbation balanced truncation; continuous-time systems; discrete-time systems; error bound;

1. Introduction. We consider linear time-invariant systems in continuous-time

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t),\ x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

and in discrete-time

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t),\ x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{p\times p}$ are real constant coefficient matrices. In the continuous-time case, the state x, input u and output y are real-valued vector functions. In the discrete-time case x, u and y are real-valued vector sequences. We focus on (internally) positive systems. These are systems whose state and output variables take only nonnegative values at all times t for any nonnegative initial state and any nonnegative input, [1–3].

Positive systems arise naturally in many applications such as pollutant transport, chemotaxis, pharmacokinetics, Leontief input-output models, population models and compartmental systems, [1, 2, 4–8]. In these models, the variables represent concentrations, population numbers of bacteria or cells or, in general, measures that are per se nonnegative. Positive systems are subject to ongoing research by many authors, [1, 2, 9–16]. Recent advances on control theoretical issues have been made especially in the positive discrete-time case. Yet, there are still many open problems, in particular for standard positive systems in continuous-time. One such problem in the continuous-time as well as in the discrete-time case is model reduction which preserves the positivity of a system. In the present paper, we generalize the model reduction methods of standard balanced truncation and singular perturbation balanced truncation such that positivity is preserved. Our technique uses a linear matrix inequality (LMI) approach and we show that, stability is preserved and an error bound in the H_∞-norm is provided.
This paper is organized as follows. In Section 2, we introduce the notation, define the positivity concept and give a well-known characterization. We review the basics on

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balanced truncation in Section 2.1. Based on these results, we introduce positivity-preserving standard balanced truncation and singular perturbation balanced truncation in Section 3. Before this work is concluded, the applicability of the proposed methods is demonstrated by means of several examples in Section 4.

2. Preliminaries. Let the matrix quadruple $[\,A\,,\,B\,,\,C\,,\,D]$ denote the system (1.1) or (1.2), respectively. The function $G(\lambda) = C(\lambda I - A)^{-1}B + D$ is called *transfer function* and λ is called *frequency variable*. Conversely, $[\,A\,,\,B\,,\,C\,,\,D]$ is called *realization* of G. Note that the frequency variable is denoted by s in continuous-time and by z in discrete-time. Let $H_{\infty,c}$ be the space of all transfer functions that are analytic and bounded in the open right half-plane $\mathbb{C}^+$ and let $H_{\infty,d}$ be the space of all transfer functions that are analytic and bounded on $\mathbb{C}\backslash\mathbb{D}$, where $\mathbb{D}$ is the closed unit ball around the origin. The continuous-time and discrete-time H_∞*-norms* are defined by

$$\|G\|_{\infty,c} = \sup_{s\in\mathbb{C}^+} \|G(s)\|_2, \qquad \|G\|_{\infty,d} = \sup_{z\in\mathbb{C}\backslash\mathbb{D}} \|G(z)\|_2, \tag{2.1}$$

respectively, where $\|\cdot\|_2$ denotes the spectral matrix norm.
A matrix $A \in \mathbb{R}^{n\times n}$ is called *c-stable* if all its eigenvalues are located in the open left complex half-plane and it is called *d-stable* if the moduli of all eigenvalues are less than one. We call a realization of a continuous-time (discrete-time) system $[\,A\,,\,B\,,\,C\,,\,D]$ stable, if A is c-stable (d-stable).
A matrix $A \in \mathbb{R}^{m\times n}$ is called *nonnegative (positive)* and we write $A \geq 0$ $(A > 0)$ if all entries are nonnegative (positive). A matrix $A \in \mathbb{R}^{n\times n}$ is called *Z-matrix* if its off-diagonal entries are non-positive. In the literature, a matrix for which $-A$ is a Z-matrix sometimes is called L-matrix, Metzler matrix or essentially positive matrix, see, e.g., [1, 2, 17, 18]. Throughout this paper we will use the term $-Z$-matrix. Let $B \in \mathbb{R}^{n\times n}$ $B \geq 0$ with spectral radius $\rho(B)$. A matrix A of the form $A = \alpha I - B$, with $\alpha > 0$, and $\alpha \geq \rho(B)$ is called *M-matrix*. If $\alpha > \rho(B)$ then A is a *nonsingular* M-matrix, if $\alpha = \rho(B)$ then A is a *singular* M-matrix. The class of M-matrices is a subclass of the Z-matrices. Accordingly, a matrix $A \in \mathbb{R}^{n,n}$ for which $-A$ is an M-matrix is called a $-M$-matrix. Note that for a nonsingular M-matrix A, we have $A^{-1} \geq 0$ [18].

A symmetric matrix A is called *positive (semi)definite* and we write $(A \succeq 0)$ $A \succ 0$ if for all $x \neq 0$ we have $(x^TAx \geq 0)$ $x^TAx > 0$. If this holds for $-A$ then A is called *negative (semi)definite* and we write $(A \preceq 0)$ $A \prec 0$. For matrices A, B we write $(A \preceq B)$ $A \prec B$ if $(B - A \succeq 0)$ $B - A \succ 0$.

Next we define the class of positive systems.

Definition 2.1. *A continuous-time system (1.1) is called* positive *if for all $t \in \mathbb{R}^+$ the state $x(t)$ and the output $y(t)$ are nonnegative for any nonnegative initial state x_0 and any measurable input function $u : \mathbb{R} \to \mathbb{R}^p$ with $u(\tau) \geq 0$ for all $\tau \in [0, t]$. A discrete-time system (1.2) is called positive if for all $t \in \mathbb{N}_0$ the state $x(t)$ and the output $y(t)$ are nonnegative for any nonnegative initial state x_0 and any input sequence $u : \mathbb{N}_0 \to \mathbb{R}^p$ with $u(\tau) \geq 0$ for all $\tau \in \mathbb{N}_0$.* The following theorem states a well-known characterization of positive systems, see, e.g., [1, 2].

Theorem 2.2. *The continuous-time system (1.1) is positive if and only if A is a $-Z$-matrix and $B, C, D \geq 0$. The discrete-time system (1.2) is positive if and only if $A, B, C, D \geq 0$.*

2.1. Balanced truncation. In this section we review the properties of standard balanced truncation and singular perturbation balanced truncation established in [19, 20].

Continuous-time case. Consider the standard continuous-time system (1.1) with transfer function $G(s) = C(sI - A)^{-1}B + D$. Assume that A is c-stable. Let $\mathcal{P}, \mathcal{Q} \succeq 0$ be the solutions of the continuous-time Lyapunov equations

$$\begin{aligned} A\mathcal{P} + \mathcal{P}A^T + BB^T &= 0, \\ A^T\mathcal{Q} + \mathcal{Q}A + C^TC &= 0. \end{aligned} \tag{2.2}$$

The system (1.1) is said to be *balanced* if

$$\mathcal{P} = \begin{bmatrix} \Sigma & & & \\ & \Sigma_c & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} \Sigma & & & \\ & 0 & & \\ & & \Sigma_o & \\ & & & 0 \end{bmatrix}, \tag{2.3}$$

are partitioned accordingly with $\Sigma_c \succ 0$, $\Sigma_o \succ 0$ and

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k) \text{ for some } \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0. \tag{2.4}$$

The numbers $\sigma_1 \ldots \sigma_k$ are called *Hankel singular values.* Consider a partition of the balanced system

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{2.5}$$

where $A_{11} \in \mathbb{R}^{r\times r}$ and either $r = k$ or $r < k$ such that $\sigma_{r+1} < \sigma_r$. The matrices B and C are partitioned accordingly. By means of balanced realizations, reduced order models

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u_r(t), \\ y_r(t) &= C_r x_r(t) + D_r u_r(t), \end{aligned} \tag{2.6}$$

can now be constructed. In the method of standard balanced truncation [19], the matrices A_r, B_r, C_r, D_r are defined by

$$A_r = A_{11}, \quad B_r = B_1, \quad C_r = C_1, \quad D_r = D. \tag{2.7}$$

An alternative method for the construction of reduced order models is *singular perturbation balanced truncation* [20]. There, the reduced order model is defined by

$$\begin{aligned} A_r &= A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad & B_r &= B_1 - A_{12}A_{22}^{-1}B_2, \\ C_r &= C_1 - C_2A_{22}^{-1}A_{21}, & D_r &= D - C_2A_{22}^{-1}B_2. \end{aligned} \tag{2.8}$$

For the reduced order models, we have the following result, [20]:

LEMMA 2.3. *Let* $[\,A\,,\,B\,,\,C\,,\,D\,]$ *be a realization of* $G(s)$ *that is c-stable, balanced with* Σ *and partitioned as in (2.5) and let* $[\,A\,,\,B\,,\,C\,,\,D\,]$ *be the realization that is either constructed by (2.7) or (2.8). Then, the system (2.6) is balanced with* $\mathcal{P} = \mathcal{Q} =$

$\mathrm{diag}(\sigma_1,\ldots,\sigma_r)$. *For the corresponding transfer function* $G_r(s) = C_r(sI_r - A_r)^{-1}B_r + D_r$*, we have*

$$\|G - G_r\|_{\infty,c} \le 2 \sum_{i=r+1}^{k} \sigma_i. \tag{2.9}$$

The main difference between the discussed truncation methods is that standard balanced truncation is exact for $s = \infty$ meaning that $G(\infty) = G_r(\infty)$, whereas singular perturbation balanced truncation is exact at $s = 0$.

Discrete-time case. Consider a discrete-time system (1.2) with transfer function $G(z) = C(zI - A)^{-1}B + D$ and assume that A is d-stable. Then there exist $n \times n$ matrices $\mathcal{P}, \mathcal{Q} \succeq 0$ that solve the the discrete-time Lyapunov equations

$$\begin{aligned} A\mathcal{P}A^T - \mathcal{P} + BB^T &= 0,\\ A^T\mathcal{Q}A - \mathcal{Q} + C^TC &= 0. \end{aligned} \tag{2.10}$$

In accordance with the continuous-time case, (1.2) is called *balanced* if $\mathcal{P}$ and $\mathcal{Q}$ are as in (2.3), where $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_k$ are the *Hankel singular values.* Consider a partition of the balanced system as in (2.5) and assume that either $r = k$ or $r < k$ with $\sigma_{r+1} < \sigma_r$. Standard balanced truncation leads to a reduced order model

$$\begin{aligned} x_r(t+1) &= A_r x_r(t) + B_r u_r(t),\\ y_r(t) &= C_r x_r(t) + D_r u_r(t), \end{aligned} \tag{2.11}$$

that is constructed via (2.7). In the singular perturbation balanced truncation technique, the matrices in (2.11) are given by

$$\begin{aligned} A_r &= A_{11} + A_{12}(I_{n-r} - A_{22})^{-1}A_{21},\\ B_r &= B_1 + A_{12}(I_{n-r} - A_{22})^{-1}B_2,\\ C_r &= C_1 + C_2(I_{n-r} - A_{22})^{-1}A_{21},\\ D_r &= D + C_2(I_{n-r} - A_{22})^{-1}B_2. \end{aligned} \tag{2.12}$$

For the reduced order models, we have an analogous result as in Lemma 2.3, [19, 20]:

Lemma 2.4. *Let* $[\,A\,,\,B\,,\,C\,,\,D\,]$ *be a realization of* $G(z)$ *that is stable, balanced with* Σ *and partitioned as in (2.5) and let* $[\,A\,,\,B\,,\,C\,,\,D\,]$ *be the realization that is either constructed by (2.7) or (2.12). Then, the system (2.11) is balanced with* $\mathcal{P} = \mathcal{Q} = \mathrm{diag}(\sigma_1,\ldots,\sigma_r)$*. For the corresponding transfer function* $G_r(z) = C_r(zI_r - A_r)^{-1}B_r + D_r$*, we have*

$$\|G - G_r\|_{\infty,d} \le 2 \sum_{i=r+1}^{k} \sigma_i. \tag{2.13}$$

3. Model reduction for positive systems. In this section we generalize the model reduction methods reviewed in Section 2.1 to positive systems of ordinary differential equations. To this end, note that the results in Lemma 2.3 and Lemma 2.4 still hold if the Lyapunov equations in (2.2) and (2.10) are replaced by Lyapunov inequalities in continuous-time:

$$\begin{aligned} A\mathcal{P} + \mathcal{P}A^T + BB^T &\preceq 0,\\ A^T\mathcal{Q} + \mathcal{Q}A + C^TC &\preceq 0, \end{aligned} \tag{3.1}$$

or in discrete-time by

$$
\begin{aligned}
A\mathcal{P}A^T - \mathcal{P} + BB^T &\preceq 0, \\
A^T\mathcal{Q}A - \mathcal{Q} + C^TC &\preceq 0,
\end{aligned}
\tag{3.2}
$$

respectively. For standard balanced truncation, the proofs can be found in [21] for continuous-time and in [22] for discrete-time. Note that the results for singular perturbation balanced truncation can be shown analogously. We show that for a transfer function of any positive system there exists a positive realization with $\mathcal{P}$ and $\mathcal{Q}$ as in (2.3) that solve the inequalities (3.1) or (3.2), respectively.

THEOREM 3.1. *Consider the c-stable continuous-time positive system (1.1). Then, there exists a diagonal positive definite matrix T such that the positive system $[\,\tilde{A}\,,\,\tilde{B}\,,\,\tilde{C}\,,\,D\,]$ given by*

$$
\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B \quad \textit{and} \quad \tilde{C} = CT, \tag{3.3}
$$

is balanced in the sense that there exist matrices $\mathcal{P} \succeq 0$, $\mathcal{Q} \succeq 0$ as in (2.3) with diagonal and positive definite Σ, such that the following Lyapunov inequalities hold:

$$
\begin{aligned}
\tilde{A}\mathcal{P} + \mathcal{P}\tilde{A}^T + \tilde{B}\tilde{B}^T &\preceq 0, \\
\tilde{A}^T\mathcal{Q} + \mathcal{Q}\tilde{A} + \tilde{C}^T\tilde{C} &\preceq 0.
\end{aligned}
\tag{3.4}
$$

Proof. It is well known that a $-M$-matrix is diagonally stable, i.e., there exist diagonal positive definite matrices X, Y such that

$$
AX + XA^T \prec 0 \quad \text{and} \quad A^TY + YA \prec 0,
$$

see, e.g. [17, 23]. In particular, there exist positive semi-definite diagonal matrices X, Y such that

$$
AX + XA^T + BB^T \preceq 0, \quad A^TY + YA + C^TC \preceq 0.
$$

Take a permutation matrix Π such that

$$
\Pi^T X \Pi = \begin{bmatrix} X_{11} & & & \\ & X_{22} & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad \Pi Y \Pi^T = \begin{bmatrix} Y_{11} & & & \\ & 0 & & \\ & & Y_{33} & \\ & & & 0 \end{bmatrix}
$$

with the additional property that $X_{11} = \operatorname{diag}(x_1, \ldots, x_k)$, $Y_{11} = \operatorname{diag}(y_1, \ldots, y_k)$ satisfy

$$
x_1y_1 \geq x_2y_2 \geq \ldots \geq x_ky_k > 0.
$$

Now defining $\bar{T} = \operatorname{diag}((X_{11}Y_{11}^{-1})^{\frac{1}{4}}, I, I, I)$ and $T = \Pi\bar{T}$, we have that $\mathcal{P} = T^{-1}XT^{-T}$, $\mathcal{Q} = T^TYT$ have the desired form. The transformed system is given by $[\,\tilde{A}\,,\,\tilde{B}\,,\,\tilde{C}\,,\,D\,]$ as defined in (3.3). Since $\tilde{A}$ is a $-Z$-matrix and $B, C, D \geq 0$, the transformed system is positive by Theorem 2.2. ◻

In the discrete-time case, we have an analogous result, which we state without proof.

THEOREM 3.2. *Consider a d-stable discrete-time positive system* $[\,A\,,\,B\,,\,C\,,\,D\,]$, *i.e.,* $A,B,C,D \geq 0$ *and* $\rho(A) < 1$. *Then, there exists a positive definite diagonal matrix* T *such that the system* $[\,\tilde{A}\,,\,\tilde{B}\,,\,\tilde{C}\,,\,D\,]$, *where* $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$ *and* $\tilde{C} = CT^{-1}$, *is balanced in the sense that there exists a positive definite diagonal matrix* Σ *such that the following Lyapunov inequalities hold:*

$$\begin{aligned} &\tilde{A}\Sigma\tilde{A}^T - \Sigma + \tilde{B}\tilde{B}^T \preceq 0, \\ &\tilde{A}^T\Sigma\tilde{A} - \Sigma + \tilde{C}^T\tilde{C} \preceq 0. \end{aligned} \tag{3.5}$$

Theorem 3.1 and Theorem 3.2 guaranty the existence of a positive balanced realization. Once we have a positive balanced realization, standard balanced truncation and singular perturbation balanced truncation can be applied.

In the continuous-time case, the reduced system defined in (2.7) is again a positive system, since $B_r \geq 0$, $C_r \geq 0$, $B_r \geq 0$ and A_r is a $-M$-matrix as a submatrix of a $-M$-matrix. The positivity of the reduced system defined in (2.8) can be seen as follows: The $-M$-matrix property of A_r is preserved, since it is a Schur complement of A [18]. Furthermore, since A_{22} is also a $-M$-matrix, we have $A_{22}^{-1} \leq 0$, and hence, $B_r, C_r, D_r \geq 0$.

In the discrete-time case, the reduced system defined by (2.7) is positive, since A_r, B_r, C_r, D_r are submatrices of positive matrices. Furthermore, (2.12) is also a positive system, which can be observed as follows. By the stability assumption, we have that $\rho(A_{22}) \leq \rho(A) < 1$. Hence, $I_{n-r} - A_{22}$ is an M-matrix and $(I_{n-r} - A_{22})^{-1} \geq 0$. Therefore, we obtain $A_r, B_r, C_r, D_r \geq 0$.

Note that for the computation of positive reduced order models, there is no need to compute a balanced realization explicitely. Instead, for diagonal solutions $\mathcal{P} = \operatorname{diag}(p_1, \ldots, p_n)$ and $\mathcal{Q} = \operatorname{diag}(q_1, \ldots, q_n)$ of (2.2) or (2.10), indices $\{i_1, \ldots, i_n\}$ have to be found such that

$$p_{i_1}q_{i_1} \geq \ldots \geq p_{i_r}q_{i_r} > p_{i_{r+1}}q_{i_{r+1}} \geq \ldots \geq p_{i_n}q_{i_n}.$$

For a general matrix $F = [a_{ij}]_{i=1,\ldots,n,j=1,\ldots,m} \in \mathbb{R}^{n\times m}$ and some vectors $I = [i_1, \ldots, i_k] \in \{1, \ldots, n\}^k$, $J = [j_1, \ldots, j_l] \in \{1, \ldots, m\}^l$, we define the matrix $F(I, J) \in \mathbb{R}^{k\times l}$ by

$$F(I, J) = [a_{i_r j_s}]_{r=1,\ldots,k,s=1,\ldots,l}.$$

Then, reduced order models (2.6) (or (2.11)) can be obtained in the following way: Let $I_1 = [i_1, \ldots, i_r]$, $I_2 = [i_{r+1}, \ldots, i_n]$ and

$$\begin{aligned} \bar{A}_{11} &= A(I_1, I_1), & \bar{A}_{12} &= A(I_1, I_2), \\ \bar{A}_{21} &= A(I_2, I_1), & \bar{A}_{22} &= A(I_2, I_2), \\ \bar{B}_1 &= B(I_1, [1, \ldots, p]), & \bar{B}_2 &= B(I_2, [1, \ldots, p]), \\ \bar{C}_1 &= B([1, \ldots, q], I_1), & \bar{C}_2 &= B([1, \ldots, q], I_2). \end{aligned}$$

Then the following properties hold:

(i) The continuous-time (discrete-time) system $[\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D]$ is positive and has the same transfer function as the r-th order system obtained by positive standard balanced truncation.

(ii) The continuous-time system

$$\begin{aligned} [\,\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\,,\ \bar{B}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2\,, \\ \bar{C}_1 - \bar{C}_2\bar{A}_{22}^{-1}\bar{A}_{21}\,,\ D - \bar{C}_2\bar{A}_{22}^{-1}\bar{B}_2\,] \end{aligned}$$

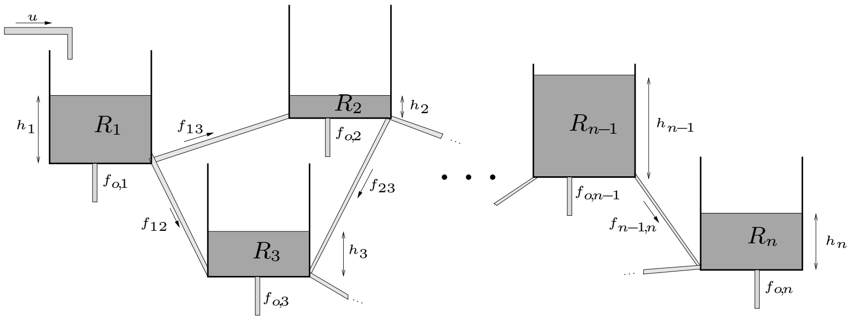

Fig. 1: System of n water reservoirs

is positive and has the same transfer function as the r-th order system obtained by positive singular perturbation balanced truncation.

(iii) The discrete-time system

$$[\,\bar{A}_{11}+\bar{A}_{12}(I_{n-r}-\bar{A}_{22})^{-1}\bar{A}_{21}\,,\ \bar{B}_1+\bar{A}_{12}(I_{n-r}-\bar{A}_{22})^{-1}\bar{B}_2\,,$$
$$\bar{C}_1+\bar{C}_2(I_{n-r}-\bar{A}_{22})^{-1}\bar{A}_{21}\,,\ D+\bar{C}_2(I_{n-r}-\bar{A}_{22})^{-1}\bar{B}_2\,]$$

is positive and has the same transfer function as the r-th order system obtained by positive singular perturbation balanced truncation.

Let us finally give a remark on the Lyapunov inequalities (3.1) and (3.2). It is clear that their solutions are not unique and one should look for solutions $\mathcal{P} = \operatorname{diag}(p_1,\ldots,p_n)$, $\mathcal{Q} = \operatorname{diag}(q_1,\ldots,q_n)$ such that $\sqrt{\mathcal{P}\mathcal{Q}}$ has a large number of small diagonal elements. This yields components of the state which are candidates to truncate. A good heuristic for this is the minimization of the trace of $\mathcal{P}$ and $\mathcal{Q}$. For getting even sharper bounds, the Lyapunov inequalities can be solved once more while now minimizing the sum of those diagonal elements of $\mathcal{P}$ and $\mathcal{Q}$ corresponding to the candidates for truncation.

4. Examples. In this section we present two numerical examples to demonstrate the properties of the discussed model reduction approaches for positive systems. The numerical tests were run in MATLAB® Version 7.4.0 on a PC with an Intel(R) Pentium(R) 4 CPU 3.20GHz processor.

Example 4.1 (Continuous-time). *Consider a system of n water reservoirs such as schematically shown in Figure 1. All reservoirs $R_1,\ldots,R_n$ are assumed to be located on the same level. The base area of R_i and its fill level are denoted by a_i and h_i, respectively. The first reservoir R_1 has an inflow u which is the input of the system, and for each $i \in \{1,\ldots,n\}$, R_i has an outflow $f_{o,i}$ through a pipe with diameter $d_{o,i}$. The output of the system is assumed to be the sum of all outflows. Furthermore, each R_i and R_j are connected by a pipe with diameter $d_{ij} = d_{ji} \geq 0$. The direct flow from R_i to R_j is denoted by f_{ij}. We assume that the flow depends linearly on the difference between the pressures on both ends. This leads to the equations*

$$f_{ij}(t) = d_{ij}^2 \cdot c \cdot (h_i(t) - h_j(t)), \quad f_{a,i}(t) = d_{o,i}^2 \cdot c \cdot (h_i(t) - h_j(t)),$$

where c is a constant that depends on the viscosity and density of the medium and gravity. The fill level of R_i thus satisfies the following differential equation

$$\dot{h}_i = \frac{c}{a_i}\left(-d_{o,i}^2 h_i(t) + \sum_{j=1}^{n} d_{ij}^2 (h_j(t) - h_i(t))\right) + \frac{1}{a_i}\delta_{1i}u(t),$$

where δ_{1i} denotes the Kronecker symbol, that is $\delta_{1i} = 1$ if $i = 1$ and zero otherwise. Then, we obtain system (1.1) with $D = 0$ and matrices $A = [a_{ij}]_{i,j=1,\ldots,n}$, $B = [b_{i1}]_{i=1,\ldots,n}$, $C = [c_{1j}]_{j=1,\ldots,n}$ with $b_{i1} = \frac{\delta_{1i}}{a_1}$, $c_{i1} = c \cdot d_{o,i}^2$ and

$$a_{ij} = \frac{c}{a_i} \cdot \begin{cases} -d_{o,i}^2 - \sum_{k=1}^{n} d_{ik}^2 & i = j, \\ d_{ij}^2 & i \neq j, \end{cases}$$

where we define $d_{ii} = 0$.

For our illustrative computation, we have constructed the presented compartment model with ten states. We assume that we have two well connected substructures each consisting of five reservoirs, where each reservoir is connected with every other reservoir by a pipe of diameter 1. The substructures are connected with each other by a pipe of diameter 0.1 between reservoirs one and ten. For simplicity reasons, we also set all base areas of the reservoirs to 1 and also $c = 1$. We show only the results obtained with singular perturbation balanced truncation. However, qualitatively similar results were obtained via standard balanced truncation. The reduced model with five states is again positive with

$$\begin{aligned} A_r &= \begin{bmatrix} -5.01 & 1.35 & 1.35 & 1.35 & 1.35 \\ 0.76 & -5.00 & 1.00 & 1.00 & 1.00 \\ 0.76 & 1.00 & -5.00 & 1.00 & 1.00 \\ 0.76 & 1.00 & 1.00 & -5.00 & 1.00 \\ 0.76 & 1.00 & 1.00 & 1.00 & -5.00 \end{bmatrix}, \\ B_r &= \begin{bmatrix} 0.45 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\ C_r &= \begin{bmatrix} 2.22 & 2.90 & 2.90 & 2.90 & 2.90 \end{bmatrix}. \end{aligned}$$

The frequency responses, i.e. the transfer function at values $s = j\omega$, for $\omega \in [0, 10]$, of the original and of the reduced order models are depicted in the upper diagram of Figure 2. The lower diagram shows the frequency response of the error systems along with the mutual error bound 0.0167.

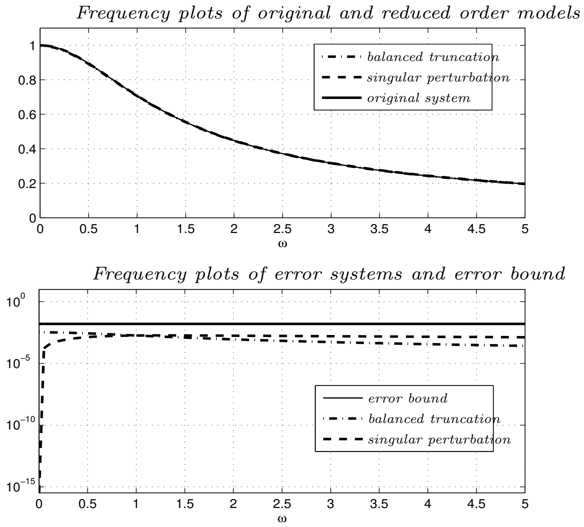

Fig. 2: Frequency plot showing original and reduced order models.

As an example in discrete-time, we consider the well-known Leslie model [24], which describes the time evolution of age-structured populations.

EXAMPLE 4.2 (Discrete-time). *Let the time $t \in \mathbb{N}_0$ describe the reproduction season (year) and let $x_i(t)$, $i = 1, \ldots, n$, represent the number of individuals of age i at time t. We assume constant survival rates s_i, $i = 0, \ldots, n-1$, i.e. the fraction of individuals of age i that survive for at least one year, and fertility rates f_i, $i = 1, \ldots, n$, i.e. the mean number of offspring born from an individual at age i. For purely illustrative purposes of this example, we use the estimated data given in [1, p. 118] for squirrel reproduction. Furthermore, we assume that immigration into the considered tribe can only happen at birth, i.e. the input is a positive multiple of the first unit vector, and as the output we take the total population, i.e. the sum of the population numbers over all ages. Thus, the aging process is described by the following difference equations*

$$x_i(t+1) = s_i x_i(t), \quad i = 1, \ldots, n-1, \tag{4.1}$$

and the first state equation takes into account reproduction and immigration

$$x_1(t+1) = s_0(f_1 x_1(t) + f_2 x_2(t) + \ldots + f_n x_n(t) + u(t)). \tag{4.2}$$

With singular perturbation balanced truncation the reduced model with five states is again positive with

$$\begin{aligned}
A_r &= \begin{bmatrix} 0.24 & 1.25 & 3.7441 & 5.33 & 10.43 \\ 0.09 & 0 & 0 & 0 & 0 \\ 0 & 0.16 & 0 & 0 & 0 \\ 0 & 0 & 0.23 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 \end{bmatrix}, \\
B_r &= \begin{bmatrix} 0.19 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
C_r &= \begin{bmatrix} 2.15 & 5.60 & 10.61 & 15.12 & 29.56 \end{bmatrix}.
\end{aligned}$$

The frequency responses, i.e. the transfer function at values $z = e^{j\omega}$, for $\omega \in [0, 2\pi]$, of the original and of the reduced order models are depicted in the upper diagram of Figure 3. The lower diagram shows the frequency response of the error systems along with the mutual error bound 0.0357. *The frequency responses, i.e. the transfer function at values $z = e^{j\omega}$, for $\omega \in [0, 2\pi]$, of the original and of the reduced order models are depicted in Figure 3.*

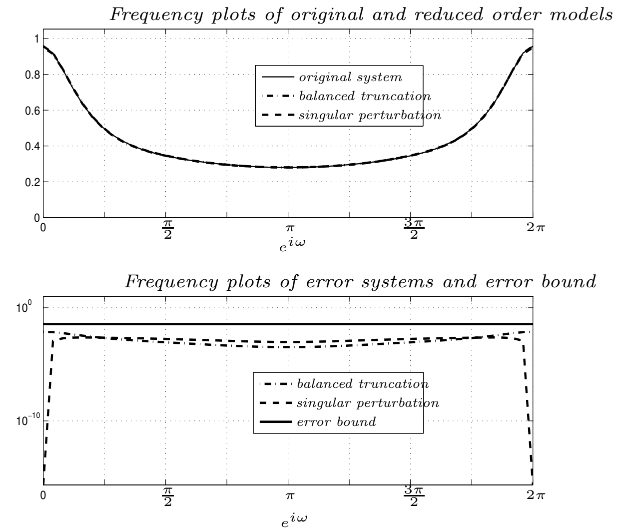

Fig. 2: Frequency plot showing original and reduced order models.

5. Conclusion. In this paper, we have presented a model reduction technique that preserves the positivity of a system in the continuous-time as well as in the discrete-time case. The proposed method is based on the existence of a diagonal solution of the corresponding Lyapunov inequalities, which may be obtained via LMI solution methods. The reduction then may be performed by standard balanced truncation or singular perturbation balanced truncation methods. It is shown that both methods preserve positivity. Furthermore, two numerical examples in the continuous-time and in the discrete-time case are provided and illustrate the functionality of the proposed algorithm.

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