Singular-value-like decomposition for complex matrix triples

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August 19, 2007

Dedicated to William B. Gragg on the occasion of his 70th birthday

Abstract

The classical singular value decomposition for a matrix $A \in \mathbb{C}^{m\times n}$ is a canonical form for A that also displays the eigenvalues of the Hermitian matrices AA^* and A^*A. In this paper, we develop a corresponding decomposition for A that provides the Jordan canonical forms for the complex symmetric matrices AA^T and A^TA. More generally, we consider the matrix triple (A, G_1, G_2), where $G_1 \in \mathbb{C}^{m\times m}, G_2 \in \mathbb{C}^{n\times n}$ are invertible and either complex symmetric and complex skew-symmetric, and we provide a canonical form under transformations of the form $(A, G_1, G_2) \mapsto (X^TAY, X^TG_1X, Y^TG_2Y)$, where X, Y are nonsingular.

Keywords singular value decomposition, canonical form, complex bilinear forms, complex symmetric matrix, complex skew-symmetric matrix, Hamiltonian matrix, Takagi factorization.
AMS subject classification. 65F15, 65L80, 65L05, 15A21, 34A30, 93B40.

1 Introduction

In [3] Bunse-Gerstner and Gragg derived an algorithm for computing the Takagi factorization $A = U^T\Sigma U$, U unitary, for a complex symmetric matrix $A^T = A \in \mathbb{C}^{n\times n}$. The Takagi factorization is just a special case of the singular value decomposition and combines two important aspects: computation of singular values (i.e., eigenvalues of A^*A and AA^*) and exploitation of structure with respect to complex bilinear forms (here, the symmetry of A is exploited by choosing U and U^T as unitary factors for the singular value decomposition).

These two aspects can be combined in a completely different way. Instead of computing the singular values of a general matrix $A \in \mathbb{C}^{m\times n}$ and thus revealing the eigenvalues of AA^* and A^*A, we may ask for a canonical form for A that reveals the eigenvalues of the complex symmetric matrices AA^T and A^TA. In this paper, we compute such a form by solving a more general problem: instead of restricting ourselves to the matrix A, we consider a triple of matrices (A, G_1, G_2) with $A \in \mathbb{C}^{m\times n}$, $G_1 \in \mathbb{C}^{m\times m}$ and $G_2 \in \mathbb{C}^{n\times n}$, where G_1 and G_2

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[*]Partially supported by the *Deutsche Forschungsgemeinschaft* through the DFG Research Center MATHEON *Mathematics for key technologies* in Berlin.

are nonsingular and either complex symmetric or complex skew-symmetric. Then we derive canonical forms under transformations of the form

$$(A, G_1, G_2) \mapsto (A_{\mathrm{CF}}, G_{1,\mathrm{CF}}, G_{2,\mathrm{CF}}) := (X^T A Y, X^T G_1 X, Y^T G_2 Y), \tag{1.1}$$

with nonsingular matrices $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{n\times n}$. This canonical form will allow the determination of the eigenstructure of the pair of structured matrices

$$\mathcal{H} = G_2^{-1} A^T G_1^{-1} A, \qquad \hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T,$$

because we find that

$$Y^{-1}\mathcal{H}Y = (Y^{-1}G_2^{-1}Y^{-T})(Y^T A^T X)(X^{-1}G_1^{-1}X^{-T})(X^T A Y) = G_{2,\mathrm{CF}}^{-1} A_{\mathrm{CF}}^T G_{1,\mathrm{CF}}^{-1} A_{\mathrm{CF}}, \tag{1.2}$$

$$X^{-1}\hat{\mathcal{H}}X = (X^{-1}G_1^{-1}X^{-T})(X^T A Y)(Y^{-1}G_2^{-1}Y^{-T})(Y^T A^T X) = G_{1,\mathrm{CF}}^{-1} A_{\mathrm{CF}} G_{2,\mathrm{CF}}^{-1} A_{\mathrm{CF}}^T. \tag{1.3}$$

For the special case $G_1 = I_m$ and $G_2 = I_n$, we obtain $\mathcal{H} = A^T A$ and $\hat{\mathcal{H}} = A A^T$ and thus, an appropriate canonical form (1.1) will display the eigenvalues of $A^T A$ and $A A^T$ via the identities (1.2) and (1.3). In the general case, if $G_1^T = (-1)^s G_1$ and $G_2^T = (-1)^t G_2$ with $s, t \in \{0, 1\}$, then the matrices $\mathcal{H}$ and $\hat{\mathcal{H}}$ satisfy

$$\mathcal{H}^T G_2 = (-1)^s A^T G_1^{-1} A = (-1)^s G_2 \mathcal{H}, \quad \hat{\mathcal{H}}^T G_1 = (-1)^t A G_2^{-1} A^T = (-1)^t G_1 \hat{\mathcal{H}}, \tag{1.4}$$

i.e., $\mathcal{H}$ and $\hat{\mathcal{H}}$ are either selfadjoint or skew-adjoint with respect to the complex bilinear form induced by G_2 or G_1, respectively. Indeed, setting

$$\langle x, y\rangle_{G_1} = y^T G_1 x, \qquad \langle x, y\rangle_{G_2} = y^T G_2 x \tag{1.5}$$

for $x, y \in \mathbb{C}^n$, the identities (1.4) can be rewritten as

$$\langle \mathcal{H}x, y\rangle_{G_2} = (-1)^s \langle x, \mathcal{H}y\rangle_{G_2} \quad \text{and} \quad \langle \hat{\mathcal{H}}x, y\rangle_{G_1} = (-1)^t \langle x, \hat{\mathcal{H}}y\rangle_{G_1} \quad \text{for all } x, y \in \mathbb{C}^n.$$

Indefinite inner products and related structured matrices have been intensively studied in the last few decades with main focus on real bilinear or complex sesquilinear forms, see [1, 4, 10, 14] and the references therein and, in particular, [5]. In recent years, there has also been interest in matrices that are structured with respect to complex bilinear forms, because such matrices do appear in applications such as the frequency analysis of high speed trains [7, 11].

Besides revealing the eigenstructure of the matrices $\mathcal{H}$ and $\hat{\mathcal{H}}$, the canonical form (1.1) also allows to determine the eigenstructure of the double-sized structured matrix pencil

$$\lambda\mathcal{G} - \mathcal{A} = \lambda \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} - \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix},$$

because we have that

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}^T \left(\lambda \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} - \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \right) \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = \lambda \begin{bmatrix} G_{1,\mathrm{CF}} & 0 \\ 0 & G_{2,\mathrm{CF}} \end{bmatrix} - \begin{bmatrix} 0 & A_{\mathrm{CF}} \\ A_{\mathrm{CF}}^T & 0 \end{bmatrix}.$$

The idea of generalizing the concept of the singular value decomposition to indefinite inner products is not new and has been considered in [2] for the case of complex Hermitian forms. The canonical forms presented here are the analogue in the case of complex bilinear forms. This case is more involved, because one has to make a clear distinction between symmetric and skew-symmetric bilinear forms, in contrast to the sesquilinear case where Hermitian and skew-Hermitian forms are closely related. Indeed, an Hermitian matrix can be easily transformed into a skew-Hermitian matrix by scalar multiplication with the imaginary unit i, but this is not true for complex symmetric matrices. Therefore, we have to treat the three cases separately that G_1 and G_2 are both symmetric, both skew-symmetric, or that one of the matrices is symmetric and the another skew-symmetric.

The remainder of the paper is organized as follows. In Section 2 we recall the definition of several structured matrices and review their canonical forms. In Section 3 we develop structured factorizations that are needed for the proofs of the results in the following sections. In the Sections 4–6 we present the canonical forms for matrix triples (A, G_1, G_2). In Section 4 we consider the case that both G_1 and G_2 are complex symmetric, in Section 5 we assume that G_1 is complex symmetric and G_2 is complex skew-symmetric, and Section 6 is devoted to the case that both G_1 and G_2 are complex skew-symmetric.

Throughout the paper we use the following notation. I_n and 0_n denote the $n \times n$ identity and $n \times n$ zero matrices, respectively. The $m \times n$ zero matrix is denoted by $0_{m\times n}$ and e_j is the jth column of the identity matrix I_n, or, equivalently, the jth standard basis vector of $\mathbb{C}^n$. Moreover, we denote

$$R_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}, \quad \Sigma_n := \begin{bmatrix} (-1)^0 & & 0 \\ & \ddots & \\ 0 & & (-1)^{n-1} \end{bmatrix}, \quad J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

The transpose and conjugate transpose of a matrix A are denoted by A^T and A^*, respectively. We use $A_1 \oplus \ldots \oplus A_k$ to denote a block diagonal matrix with diagonal blocks $A_1, \ldots, A_k$. If $A = [a_{ij}] \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{\ell\times k}$, then $A \otimes B = [a_{ij}B] \in \mathbb{C}^{n\ell\times mk}$ denotes the Kronecker product of A and B. We use $\mathcal{J}_n(\lambda)$ to denote the $n \times n$ upper triangular Jordan block associated with the eigenvalue λ and we set $\Gamma_n = \Sigma_n R_n$, i.e., we have

$$\mathcal{J}_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad \Gamma_n = \begin{bmatrix} 0 & & (-1)^0 \\ & \iddots & \\ (-1)^{n-1} & & 0 \end{bmatrix}.$$

2 Matrices structured with respect to complex bilinear forms

Our general theory will cover and generalize results for the following classes of matrices.

Definition 2.1 *Let $G \in \mathbb{C}^{n\times n}$ be invertible and let $\mathcal{H}, \mathcal{K} \in \mathbb{C}^{n\times n}$ such that*

$$(G\mathcal{H})^T = G\mathcal{H} \quad \text{and} \quad (G\mathcal{K})^T = -G\mathcal{K}.$$

1. *If G is symmetric, then $\mathcal{H}$ is called* G-symmetric *and $\mathcal{K}$ is called* G-skew-symmetric.
2. *If G is skew-symmetric, then $\mathcal{H}$ is called* G-Hamiltonian *and $\mathcal{K}$ is called* G-skew-Hamiltonian.

Thus, G-symmetric and G-skew-Hamiltonian matrices are selfadjoint in the inner product induced by G while G-skew-symmetric and G-Hamiltonian matrices are skew-adjoint. Observe that transformations of the form

$$(\mathcal{M}, G) \mapsto (P^{-1}\mathcal{M}P, P^T G P), \quad P \in \mathbb{C}^{n\times n} \text{ invertible}$$

preserve the structure of $\mathcal{M}$ with respect to G, i.e., if, for example, $\mathcal{M} = \mathcal{H}$ is G-Hamiltonian, then $P^{-1}\mathcal{H}P$ is $P^T GP$-Hamiltonian as well. Thus, instead of working with G directly, one may first transform G to a simple form using Takagi's factorization for complex symmetric and complex skew-symmetric matrices, see [3, 8, 15]. This factorization is a special case of the well-known singular value decomposition.

Theorem 2.2 (Takagi's factorization) *Let $G \in \mathbb{C}^{n\times n}$ be complex symmetric. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$G = U\operatorname{diag}(\sigma_1, \ldots, \sigma_n)U^T, \quad \text{where } \sigma_1, \ldots, \sigma_n \geq 0.$$

There is a variant for complex skew-symmetric matrices (see [8]). This result is a just a special case of the Youla form [17] for general complex matrices.

Theorem 2.3 (Skew-symmetric analogue of Takagi's factorization) *Let $\mathcal{K} \in \mathbb{C}^{n\times n}$ be complex skew-symmetric. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$\mathcal{K} = U\left(\begin{bmatrix} 0 & r_1 \\ -r_1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & r_k \\ -r_k & 0 \end{bmatrix} \oplus 0_{n-2k}\right) U^T,$$

where $r_1, \ldots, r_n \in \mathbb{R} \setminus \{0\}$.

As immediate corollaries, we obtain the following well-known results.

Corollary 2.4 *Let $G \in \mathbb{C}^{n\times n}$ be complex symmetric and let* $\operatorname{rank} G = r$. *Then there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^T G X = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Corollary 2.5 *Let $G \in \mathbb{C}^{m\times m}$ be complex skew-symmetric and let* $\operatorname{rank} G = r$. *Then r is even and there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^T G X = \begin{bmatrix} J_{r/2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we review canonical forms for the classes of matrices defined in Definition 2.1. These canonical forms are closely related to the well-known canonical forms for pairs of matrices that are complex symmetric or complex skew-symmetric, see [16] for an overview on this topic. Proofs of the following results can be found, e.g., in [12].

Theorem 2.6 (Canonical form for G-symmetric matrices) *Let $G \in \mathbb{C}^{n\times n}$ be symmetric and invertible and let $\mathcal{H} \in \mathbb{C}^{n\times n}$ be G-symmetric. Then there exists an invertible matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^{-1}\mathcal{H}X = \mathcal{J}_{\xi_1}(\lambda_1) \oplus \ldots \oplus \mathcal{J}_{\xi_m}(\lambda_m), \quad X^T G X = R_{\xi_1} \oplus \ldots \oplus R_{\xi_m},$$

where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ are the (not necessarily pairwise distinct) eigenvalues of $\mathcal{H}$.

Theorem 2.7 (Canonical form for G-skew-symmetric matrices) *Let $G \in \mathbb{C}^{n\times n}$ be symmetric and invertible and let $\mathcal{K} \in \mathbb{C}^{n\times n}$ be G-skew-symmetric. Then there exists an invertible matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^{-1}\mathcal{K}X = \mathcal{K}_c \oplus \mathcal{K}_z, \quad X^T G X = G_c \oplus G_z,$$

where

$$\begin{array}{ll} \mathcal{K}_c = \mathcal{K}_{c,1} \oplus \cdots \oplus \mathcal{K}_{c,m_c}, & G_c = G_{c,1} \oplus \cdots \oplus G_{c,m_c}, \\ \mathcal{K}_z = \mathcal{K}_{z,1} \oplus \cdots \oplus \mathcal{K}_{z,m_o+m_e}, & G_z = G_{z,1} \oplus \cdots \oplus G_{z,m_o+m_e}, \end{array}$$

and where the diagonal blocks are given as follows:

1) *blocks associated with pairs $(\lambda_j, -\lambda_j)$ of nonzero eigenvalues of $\mathcal{K}$:*

$$\mathcal{K}_{c,j} = \begin{bmatrix} \mathcal{J}_{\xi_j}(\lambda_j) & 0 \\ 0 & -\mathcal{J}_{\xi_j}(\lambda_j) \end{bmatrix}, \quad G_{c,j} = \begin{bmatrix} 0 & R_{\xi_j} \\ R_{\xi_j} & 0 \end{bmatrix},$$

   where $\lambda_j \in \mathbb{C} \setminus \{0\}$ and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m_c$ when $m_c > 0$;

2) *blocks associated with the eigenvalue $\lambda = 0$ of $\mathcal{K}$:*

$$\mathcal{K}_{z,j} = \mathcal{J}_{\eta_j}(0), \quad G_{z,j} = \Gamma_{\eta_j},$$

   where $\eta_j \in \mathbb{N}$ is odd for $j = 1, \ldots, m_o$ when $m_o > 0$, and

$$\mathcal{K}_{z,j} = \begin{bmatrix} \mathcal{J}_{\eta_j}(0) & 0 \\ 0 & -\mathcal{J}_{\eta_j}(0) \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{\eta_j} \\ R_{\eta_j} & 0 \end{bmatrix},$$

   where $\eta_j \in \mathbb{N}$ is even for $j = m_o + 1, \ldots, m_o + m_e$ when $m_e > 0$.

The matrix $\mathcal{K}$ has the non-zero eigenvalues $\lambda_1, \ldots, \lambda_{m_c}, -\lambda_1, \ldots, -\lambda_{m_c}$ (not necessarily pairwise distinct), and the additional eigenvalue 0 if $m_o + m_e > 0$.

Theorem 2.8 (Canonical form for G-Hamiltonian matrices) *Let $G \in \mathbb{C}^{2n\times 2n}$ be complex skew-symmetric and invertible and let $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ be G-Hamiltonian. Then there exists an invertible matrix $X \in \mathbb{C}^{2n\times 2n}$ such that*

$$X^{-1}\mathcal{H}X = \mathcal{H}_c \oplus \mathcal{H}_z, \quad X^T G X = G_c \oplus G_z,$$

where

$$\begin{array}{ll} \mathcal{H}_c = \mathcal{H}_{c,1} \oplus \cdots \oplus \mathcal{H}_{c,m_c}, & G_c = G_{c,1} \oplus \cdots \oplus G_{c,m_c}, \\ \mathcal{H}_z = \mathcal{H}_{z,1} \oplus \cdots \oplus \mathcal{H}_{z,m_o+m_e}, & G_z = G_{z,1} \oplus \cdots \oplus G_{z,m_o+m_e}, \end{array}$$

and where the diagonal blocks are given as follows:

1) *blocks associated with pairs $(\lambda_j, -\lambda_j)$ of nonzero eigenvalues of $\mathcal{H}$:*

$$\mathcal{H}_{c,j} = \begin{bmatrix} \mathcal{J}_{\xi_j}(\lambda_j) & 0 \\ 0 & -\mathcal{J}_{\xi_j}(\lambda_j) \end{bmatrix}, \quad G_{c,j} = \begin{bmatrix} 0 & R_{\xi_j} \\ -R_{\xi_j} & 0 \end{bmatrix},$$

   where $\lambda_j \in \mathbb{C} \setminus \{0\}$ with $\arg(\lambda_j) \in [0, \pi)$ and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m_c$ when $m_c > 0$;

2) *blocks associated with the eigenvalue* $\lambda = 0$ *of* $\mathcal{H}$*:*

$$\mathcal{H}_{z,j} = \begin{bmatrix} \mathcal{J}_{\xi_j}(0) & 0 \\ 0 & -\mathcal{J}_{\xi_j}(0) \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{\xi_j} \\ -R_{\xi_j} & 0 \end{bmatrix},$$

where $\eta_j \in \mathbb{N}$ *is odd for* $j = 1, \ldots, m_o$ *when* $m_o > 0$*, and*

$$\mathcal{H}_{z,j} = \mathcal{J}_{\eta_j}(0), \quad G_{z,j} = \Gamma_{\eta_j}$$

where $\eta_j \in \mathbb{N}$ *is even for* $j = m_o + 1, \ldots, m_o + m_e$ *when* $m_e > 0$*.*

The matrix $\mathcal{H}$ *has the non-zero eigenvalues* $\lambda_1, \ldots, \lambda_{m_c}, -\lambda_1, \ldots, -\lambda_{m_c}$ *(not necessarily pairwise distinct), and the additional eigenvalue* 0 *if* $m_o + m_e > 0$*.*

Theorem 2.9 (Canonical form for G-skew-Hamiltonian matrices) *Let* $G \in \mathbb{C}^{2n\times 2n}$ *be complex skew-symmetric and invertible and let* $\mathcal{K} \in \mathbb{C}^{2n\times 2n}$ *be* G*-skew-Hamiltonian. Then there exists an invertible matrix* $X \in \mathbb{C}^{2n\times 2n}$ *such that*

$$X^{-1}\mathcal{K}X = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_m, \quad X^T G X = G_1 \oplus \cdots \oplus G_m,$$

where

$$\mathcal{K}_j = \begin{bmatrix} \mathcal{J}_{\xi_j}(\lambda_j) & 0 \\ 0 & \mathcal{J}_{\xi_j}(\lambda_j) \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 & R_{\xi_j} \\ -R_{\xi_j} & 0 \end{bmatrix}.$$

The matrix $\mathcal{K}$ *has the (not necessarily pairwise distinct) eigenvalues* $\lambda_1, \ldots, \lambda_m$*.*

The following lemma on existence and uniqueness of structured square roots of structured matrices will frequently be used.

Lemma 2.10 *Let* $G \in \mathbb{C}^{n\times n}$ *be invertible and let* $\mathcal{H} \in \mathbb{C}^{n\times n}$ *be invertible and such that* $\mathcal{H}^T\mathcal{G} = \mathcal{G}\mathcal{H}$*.*

1. *If* $G \in \mathbb{C}^{n\times n}$ *is complex symmetric (i.e.,* $\mathcal{H} \in \mathbb{C}^{n\times n}$ *is* G*-symmetric), then there exists a square root* $S \in \mathbb{C}^{n\times n}$ *of* $\mathcal{H}$ *that is a polynomial in* $\mathcal{H}$ *and that satisfies* $\sigma(S) \subseteq \{z \in \mathbb{C} : \arg(z) \in [0, \pi)\}$*. The square root is uniquely determined by these properties. In particular,* S *is* G*-symmetric.*

2. *If* $G \in \mathbb{C}^{n\times n}$ *if complex skew-symmetric (i.e.,* $\mathcal{H} \in \mathbb{C}^{n\times n}$ *is* G*-skew-Hamiltonian, then there exists a square root* $S \in \mathbb{C}^{n\times n}$ *of* $\mathcal{H}$ *that is a polynomial in* $\mathcal{H}$ *and that satisfies* $\sigma(S) \subseteq \{z \in \mathbb{C} : \arg(z) \in [0, \pi)\}$*. The square root is uniquely determined by these properties. In particular,* S *is* G*-skew-Hamiltonian.*

Proof. By the discussion in Chapter 6.4 in [9], we obtain for both cases that a square root S of $\mathcal{H}$ with $\sigma(S) \subseteq \{z \in \mathbb{C} : \arg(z) \in [0, \pi)\}$ exists, is unique, and can be expressed as a polynomial in $\mathcal{H}$. It is straightforward to check that a matrix that is a polynomial in $\mathcal{H}$ is again G-symmetric or G-skew-Hamiltonian, respectively. □

3 Structured factorizations

In this section, we develop basic factorizations that will be needed for computing the canonical forms in the Sections 4–6. We start with factorizations for matrices $B \in \mathbb{C}^{m\times n}$ satisfying $B^TB = I$ or $B^TB = 0$.

Lemma 3.1 *If $B \in \mathbb{C}^{m\times n}$ satisfies $B^TB = I_n$, then $m \geq n$ and there exists a nonsingular matrix $X \in \mathbb{C}^{m\times m}$ such that*

$$X^TB = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad X^TX = I_m.$$

Proof. By assumption B has full column rank. So there exists $\widetilde{B} \in \mathbb{C}^{m\times(m-n)}$ such that $\widetilde{X} = \begin{bmatrix} B & \widetilde{B} \end{bmatrix} \in \mathbb{C}^{m\times m}$ is invertible. Then

$$\widetilde{X}^T\widetilde{X} = \begin{bmatrix} I_n & B^T\widetilde{B} \\ \widetilde{B}^TB & \widetilde{B}^T\widetilde{B} \end{bmatrix},$$

and with

$$X_1 = \begin{bmatrix} I_n & -B^T\widetilde{B} \\ 0 & I_{m-n} \end{bmatrix},$$

we have

$$(\widetilde{X}X_1)^T(\widetilde{X}X_1) = \begin{bmatrix} I_n & 0 \\ 0 & \widetilde{B}^T(I - BB^T)\widetilde{B} \end{bmatrix}.$$

Since $\widetilde{X}X_1$ is nonsingular, so is the complex symmetric matrix $\widetilde{B}^T(I - BB^T)\widetilde{B}$. By Corollary 2.4, there exists a nonsingular matrix X_2 such that

$$X_2^T\big(\widetilde{B}^T(I - BB^T)\widetilde{B}\big)X_2 = I_{m-n}.$$

With

$$X = \widetilde{X}X_1\begin{bmatrix} I_n & 0 \\ 0 & X_2 \end{bmatrix}$$

we then obtain $X^TX = I_m$. Note that

$$X = \begin{bmatrix} B & \widetilde{B} \end{bmatrix}\begin{bmatrix} I_n & -B^T\widetilde{B} \\ 0 & I_{m-n} \end{bmatrix}\begin{bmatrix} I_n & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} B & (I - BB^T)\widetilde{B}X_2 \end{bmatrix},$$

and hence $X^TX = I_m$ implies

$$X^TB = \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \quad \square$$

Lemma 3.2 *If $B \in \mathbb{C}^{m\times n}$ satisfies* $\operatorname{rank} B = n$ *and $B^TB = 0$, then $m \geq 2n$ and there exists a unitary matrix $X \in \mathbb{C}^{m\times m}$ such that*

$$X^TB = \begin{bmatrix} B_0 \\ 0_n \\ 0 \end{bmatrix}, \quad X^TX = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_{m-2n} \end{bmatrix},$$

where $B_0 \in \mathbb{C}^{n\times n}$ is upper triangular invertible.

Proof. We present a constructive proof which allows to determine the matrix X numerically. We may assume that $m \geq 2$, otherwise the result holds trivially. Let

$$Be_1 = u_1 + iv_1, \quad u_1, v_1 \in \mathbb{R}^m.$$

Then (using e.g. a Householder transformation, see [6]) there exists an orthogonal matrix $Q_1 \in \mathbb{R}^{m\times m}$ such that $Q_1^T u_1 = \alpha_1 e_1$ and $0 \leq \alpha_1 \in \mathbb{R}$. Let $\widetilde{v}_1$ be the vector formed by the trailing $m-1$ components of $Q_1^T v_1$. Then (using e.g. a QR-decomposition, see [6]) there exists an orthogonal matrix $Q_2 \in \mathbb{R}^{(m-1)\times(m-1)}$ such that $Q_2^T \widetilde{v}_1 = \beta_1$ and $0 \leq \beta_1 \in \mathbb{R}$. With $U_1 = Q_1(1 \oplus Q_2)$, then

$$U_1^T B = \begin{bmatrix} \alpha_1 + iv_{11} & b_1 \\ i\beta_1 & b_2 \\ 0 & B_1 \end{bmatrix},$$

where $B_1 \in \mathbb{C}^{(m-2)\times(n-1)}$, $b_1, b_2 \in \mathbb{C}^{1\times(n-1)}$, and $v_{11} \in \mathbb{R}$. Since U_1 is real orthogonal, we have

$$(U_1^T B)^T (U_1^T B) = B^T B = 0,$$

and hence,

$$(\alpha_1 + iv_{11})^2 - \beta_1^2 = 0, \quad (\alpha_1 + iv_{11})b_1 + i\beta_1 b_2 = 0, \quad B_1^T B_1 + b_1^T b_1 + b_2^T b_2 = 0_{n-2}. \tag{3.1}$$

From the first identity in (3.1), it follows that $v_{11} = 0$ and $\alpha_1 = \beta_1$. Since $\alpha_1, \beta_1 \geq 0$ we have that $\alpha_1 = \beta_1 > 0$, because otherwise we would have that $\operatorname{rank} B \leq n-1$, which is a contradiction. With this, the last two identities in (3.1) imply that $b_1 = -ib_2$, $B_1^T B_1 = 0$, and thus,

$$U_1^T B = \begin{bmatrix} \alpha_1 & -ib_2 \\ i\alpha_1 & b_2 \\ 0 & B_1 \end{bmatrix}, \qquad B_1 \in \mathbb{C}^{(m-2)\times(n-1)}.$$

One can easily verify that $\operatorname{rank} B_1 = n-1$.

Applying the same procedure inductively to B_1 we obtain the existence of a real orthogonal matrix U_2 such that

$$U_2^T B_1 = \begin{bmatrix} \alpha_2 & -ib_3 \\ i\alpha_2 & b_3 \\ 0 & B_2 \end{bmatrix}, \qquad B_2 \in \mathbb{C}^{(m-4)\times(n-2)}.$$

Similarly, as above we can show that $\alpha_2 > 0$ and $\operatorname{rank} B_2 = n-2$.

Continuing the procedure, we finally obtain a real orthogonal matrix U such that

$$UB = \begin{bmatrix} \alpha_1 & -ib_{12} & \dots & -ib_{1n} \\ i\alpha_1 & b_{12} & \dots & b_{1n} \\ & \alpha_2 & \dots & -ib_{2n} \\ & i\alpha_2 & \dots & b_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \\ & & & -i\alpha_n \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{bmatrix}.$$

and from this we obtain that $m \geq 2n$. Moreover, we see that every other row of UB is a multiple by i of the preceding row. Thus, setting

$$Z_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \qquad Z = \underbrace{Z_1 \oplus \ldots \oplus Z_1}_{n} \oplus I_{m-2n},$$

letting P be a permutation matrix for which premultiplication has the effect of re-arranging the first $2n$ rows of a matrix in the order of $1, 3, \ldots, 2n-1, 2, 4, \ldots, 2n$, and introducing the unitary matrix $X = (PZU)^T$, we then have

$$X^T B = \sqrt{2}\begin{bmatrix} \alpha_1 & -ib_{12} & \ldots & -ib_{1n} \\ & \alpha_2 & \ldots & -ib_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{bmatrix}.$$

and we obtain furthermore that

$$ZZ^T = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{n} \oplus I_{m-2n} \quad \text{and} \quad X^T X = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_{m-2n} \end{bmatrix},$$

using the fact that U is real orthogonal, i.e., $U^T U = I$. □

Proposition 3.3 *Let $B \in \mathbb{C}^{m\times n}$ and suppose that* $\operatorname{rank} B = n$, $\operatorname{rank} B^T B = n_0 \leq n$, *and that $\delta_0 = n - n_0$ is the dimension of the null space of $B^T B$. Then there exists a nonsingular $X \in \mathbb{C}^{m\times m}$ such that*

$$X^T B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \begin{matrix} m-n \\ n \end{matrix}, \quad X^T X = I_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & I_{n_0} & 0 \\ I_{\delta_0} & 0 & 0 \end{bmatrix},$$

where $B_0 \in \mathbb{C}^{n\times n}$ is nonsingular and $n_1 = m - n - \delta_0$.

Proof. Since $B^T B$ is complex symmetric, by the assumption and by Corollary 2.4, there exists a nonsingular matrix $Y \in \mathbb{C}^{n\times n}$ such that

$$Y^T B^T B Y = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix}.$$

Let $\widetilde{B} \in \mathbb{C}^{m\times n_0}$ be the matrix formed by the leading n_0 columns of BY. By Lemma 3.1 there exists $X_1 \in \mathbb{C}^{m\times m}$ such that

$$X_1^T \widetilde{B} = \begin{bmatrix} I_{n_0} \\ 0 \end{bmatrix}, \quad X_1^T X_1 = I_m$$

and we obtain that

$$X_1^T BY = \begin{bmatrix} I_{n_0} & B_{12} \\ 0 & B_1 \end{bmatrix}.$$

Since

$$(X_1^T BY)^T (X_1^T BY) = Y^T B^T BY = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix},$$

we have that

$$B_{12} = 0, \quad B_1^T B_1 = 0_{\delta_0}.$$

By assumption, B has full column rank, so this also holds for $B_1 \in \mathbb{C}^{(m-n_0)\times\delta_0}$. By Lemma 3.2 there exists a nonsingular matrix $X_2 \in \mathbb{C}^{(m-n_0)\times(m-n_0)}$ such that

$$X_2^T B_1 = \begin{bmatrix} T \\ 0_{\delta_0} \\ 0 \end{bmatrix}, \quad X_2^T X_2 = \begin{bmatrix} 0 & I_{\delta_0} & 0 \\ I_{\delta_0} & 0 & 0 \\ 0 & 0 & I_{n_1} \end{bmatrix},$$

where $T \in \mathbb{C}^{\delta_0\times\delta_0}$ is nonsingular and $n_1 = m-n_0-2\delta_0 = m-n-\delta_0$. With $X_3 = X_1(I_{n_0} \oplus X_2)$ we then have

$$X_3^T BY = \begin{bmatrix} I_{n_0} & 0 \\ 0 & T \\ 0 & 0_{\delta_0} \\ 0 & 0 \end{bmatrix}, \quad X_3^T X_3 = I_{n_0} \oplus \begin{bmatrix} 0 & I_{\delta_0} \\ I_{\delta_0} & 0 \end{bmatrix} \oplus I_{n_1}.$$

Let P be the permutation that rearranges the block rows of $X_3^T BY$ in the order $4, 3, 1, 2$ and let $X = X_3 P^T$. Then

$$X^T BY = \begin{bmatrix} 0 & 0 \\ 0 & 0_{\delta_0} \\ I_{n_0} & 0 \\ 0 & T \end{bmatrix}, \quad X^T X = I_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & I_{n_0} & 0 \\ I_{\delta_0} & 0 & 0 \end{bmatrix}.$$

Post-multiplying Y^{-1} to the first of these two equations and setting

$$B_0 = \begin{bmatrix} I_{n_0} & 0 \\ 0 & T \end{bmatrix} Y^{-1},$$

we have the asserted form. □

In the previous results we have obtained factorizations for matrices B such that $B^T B$ is the identity or zero. We get similar results if $B^T J_m B = J_n$ or $B^T J_m B = 0$.

Lemma 3.4 *If $B \in \mathbb{C}^{2m\times 2n}$ satisfies $B^T J_m B = J_n$, then $m \geq n$ and there exists a nonsingular matrix $X \in \mathbb{C}^{2m\times 2m}$ such that*

$$X^T B_1 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_n \\ 0 & 0 \end{bmatrix}, \quad X^T J_m X = J_m.$$

Proof. The proof is similar to that for Lemma 3.1 and is hence omitted. □

Lemma 3.5 *Let $b \in \mathbb{C}^{2m}$. Then there is a unitary matrix $X \in \mathbb{C}^{2m\times 2m}$ such that*

$$X^T b = \alpha e_1, \qquad X^T J_m X = J_m.$$

Proof. We again present a constructive proof that can be implemented into a numerical algorithm. Let $b = [b_1^T, b_2^T]^T$ with $b_1, b_2 \in \mathbb{C}^m$ and let $H_2 \in \mathbb{C}^{m\times m}$ be a unitary matrix (e.g. a Householder matrix) such that

$$H_2^T b_2 = \beta e_1.$$

With $H_2^{-1} b_1 = [b_{11}, \ldots, b_{m1}]^T$ one then can determine (e.g. via a QR factorization) a unitary matrix

$$G = \frac{1}{\widetilde{b}_{11}} \begin{bmatrix} \bar{b}_{11} & -\beta \\ \bar{\beta} & b_{11} \end{bmatrix}, \quad \widetilde{b}_{11} = \sqrt{|b_{11}|^2 + |\beta|^2}, \quad \text{such that } G^T \begin{bmatrix} b_{11} \\ \beta \end{bmatrix} = \begin{bmatrix} \widetilde{b}_{11} \\ 0 \end{bmatrix}.$$

Note that $G^T J_2 G = J_2$. Next, determine a unitary matrix $H_1 \in \mathbb{C}^{m\times m}$ such that

$$H_1^T [\widetilde{b}_{11}, b_{21}, \ldots, b_{m1}]^T = \alpha e_1.$$

Finally, let

$$X = \begin{bmatrix} H_2^{-T} & 0 \\ 0 & H_2 \end{bmatrix} \hat{G} \begin{bmatrix} H_1 & 0 \\ 0 & H_1^{-T} \end{bmatrix},$$

where $\hat{G} \in \mathbb{C}^{2m\times 2m}$ is the unitary matrix obtained by replacing the $(1,1)$, $(1,m+1)$, $(m+1,1)$, and $(m+1,m+1)$ elements of the identity matrix I_{2m} with the corresponding elements of G, respectively. It is easily verified that X is unitary and satisfies $X^T b = \alpha e_1$ and $X^T J_m X = J_m$. □

Lemma 3.6 *If $B \in \mathbb{C}^{2m\times n}$ satisfies $\operatorname{rank} B = n$ and $B^T J_m B = 0$, then $m \geq n$ and there exists a unitary matrix $X \in \mathbb{C}^{2m\times 2m}$ such that*

$$X^T B = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad X^T J_m X = J_m,$$

where $B_0 \in \mathbb{C}^{n\times n}$ is upper triangular invertible.

Proof. By Lemma 3.5, there exists a unitary matrix X_1 such that

$$X_1^T B = \begin{bmatrix} b_{11} & b_1^T \\ 0 & B_{22} \\ 0 & b_3^T \\ 0 & B_{24} \end{bmatrix}, \quad X_1^T J_m X_1 = J_m,$$

where $b_1, b_3 \in \mathbb{C}^{n-1}$. Since $\operatorname{rank} B = n$, we have $b_{11} \neq 0$ and from

$$(X_1 B)^T J_m (X_1 B) = B^T J_m B = 0,$$

it follows that

$$b_3 = 0, \quad \begin{bmatrix} B_{22} \\ B_{24} \end{bmatrix}^T J_{m-1} \begin{bmatrix} B_{22} \\ B_{24} \end{bmatrix} = 0.$$

Applying the same procedure inductively to $\begin{bmatrix} B_{22} \\ B_{24} \end{bmatrix}$, we obtain a unitary matrix X such that

$$X^T B = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} \begin{matrix} n \\ 2m-n \end{matrix}, \quad X^T J_m X = J_m,$$

where $B_0 \in \mathbb{C}^{n\times n}$ is upper triangular and invertible. □

Proposition 3.7 *Let $B \in \mathbb{C}^{2m\times n}$. Suppose that* $\operatorname{rank} B = n$, $\operatorname{rank} B^T J_m B = 2n_0 \le n$, *i.e., $\delta_0 = n - 2n_0$ is the dimension of the null space of $B^T J_m B$. Then there exists an invertible matrix $X \in \mathbb{C}^{2m\times 2m}$ such that*

$$X^T B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \begin{matrix} 2m-n \\ n \end{matrix}, \quad X^T J_m X = J_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & J_{n_0} & 0 \\ -I_{\delta_0} & 0 & 0 \end{bmatrix},$$

where $B_0 \in \mathbb{C}^{n\times n}$ is nonsingular and $n_1 = m - n_0 - \delta_0$.

Proof. Since $B^T J_m B$ is complex skew-symmetric, by the assumption and Corollary 2.5 there exists a nonsingular matrix $Y \in \mathbb{C}^{n\times n}$ such that

$$Y^T B^T J_m B Y = \begin{bmatrix} J_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix}.$$

Let $B_1 \in \mathbb{C}^{2m\times 2n_0}$ be the matrix formed by the leading $2n_0$ columns of BY. By Lemma 3.4 there exists a nonsingular $X_1 \in \mathbb{C}^{2m\times 2m}$ such that

$$X_1^T B_1 = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \\ 0 & I_{n_0} \\ 0 & 0 \end{bmatrix}, \quad X_1^T J_m X_1 = J_m.$$

We have

$$X_1^T B Y = \begin{bmatrix} I_{n_0} & 0 & B_{13} \\ 0 & 0 & B_{23} \\ 0 & I_{n_0} & B_{33} \\ 0 & 0 & B_{43} \end{bmatrix}.$$

Since $X_1^T J_m X_1 = J_m$ also implies $X_1 J_m X_1^T = J_m$, then from

$$(X_1^T B Y)^T J_m (X_1^T B Y) = Y^T B^T J_m B Y = \begin{bmatrix} J_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix},$$

we obtain that

$$B_{13} = 0, \quad B_{33} = 0, \quad \begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix}^T J_{m-n_0} \begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix} = 0_{\delta_0}.$$

Since B has full column rank, so does $\begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix}$. By Lemma 3.6, there exists an invertible $X_2 \in \mathbb{C}^{(2m-2n_0)\times(2m-2n_0)}$ such that

$$X_2^T \begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix} = \begin{bmatrix} \widetilde{B}_0 \\ 0 \end{bmatrix}, \quad X_2^T J_{m-n_0} X_2 = J_{m-n_0},$$

where $\widetilde{B}_0 \in \mathbb{C}^{\delta_0 \times \delta_0}$ is invertible. Let P_1 be a permutation that interchanges the second and third block rows of $X_1^T BY$ and set $X_3 = X_1 P_1^T (I_{2n_0} \oplus X_2)$. Then

$$X_3^T BY = \begin{bmatrix} I_{2n_0} & 0 \\ 0 & \widetilde{B}_0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} 2n_0 \\ \delta_0 \\ n_1 \\ \delta_0 \\ n_1 \end{matrix}, \qquad X_3^T J_m X_3 = J_{n_0} \oplus J_{m-n_0},$$

where $n_1 = m - n_0 - \delta_0$. (For convenience, we have split the zero block row of $X_3^T BY$ into three block rows.) Let P be a permutation that changes the block rows of $X_3^T BY$ to the order $3, 5, 4, 1, 2$ by pre-multiplication, and let $X = X_3 P^T (I_{2n_1} \oplus (-I_{\delta_0}) \oplus I_{2n_0+\delta_0})$. Then

$$X^T BY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{2n_0} & 0 \\ 0 & \widetilde{B}_0 \end{bmatrix} \begin{matrix} 2n_1 \\ \delta_0 \\ 2n_0 \\ \delta_0 \end{matrix}, \quad X^T J_m X = J_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & J_{n_0} & 0 \\ -I_{\delta_0} & 0 & 0 \end{bmatrix}.$$

Post-multiplying Y^{-1} to the first equation and setting $B_0 = (I_{2n_0} \oplus \widetilde{B}_0) Y^{-1}$, we have the asserted form. □

In this section we have presented preliminary factorizations that will form the basis in determining the canonical forms in the following sections.

4 Canonical form for G_1, G_2 complex symmetric

In this section we derive the canonical form for the matrix triple (A, G_1, G_2) for the case that G_1, G_2 are complex symmetric. We start with the special case that the matrix A under consideration is square and nonsingular.

Theorem 4.1 *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and let $G_1, G_2 \in \mathbb{C}^{n\times n}$ be complex symmetric and nonsingular. Then there exist nonsingular matrices $X, Y \in \mathbb{C}^{n\times n}$ such that*

$$\begin{array}{rcl} X^T AY &=& \mathcal{J}_{\xi_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}(\mu_m), \\ X^T G_1 X &=& R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \\ Y^T G_2 Y &=& R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \end{array} \tag{4.1}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi)$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Moreover, for the G_2-symmetric matrix $\mathcal{H} = G_2^{-1} A^T G_1^{-1} A$ and for the G_1-symmetric matrix $\hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T$ we have that

$$\begin{array}{rcl} Y^{-1} \mathcal{H} Y &=& \mathcal{J}_{\xi_1}^2(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m), \\ X^{-1} \hat{\mathcal{H}} X &=& \mathcal{J}_{\xi_1}^2(\mu_1)^T \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m)^T. \end{array} \tag{4.2}$$

Moreover, the form (4.1) *is unique up to the simultaneous permutation of blocks in the right hand side of* (4.1).

Proof. By Lemma 2.10, $\mathcal{H}$ has a unique G_2-symmetric square root $S \in \mathbb{C}^{n\times n}$ satisfying $\sigma(S) \subseteq \{\mu \in \mathbb{C} \setminus \{0\} : \arg(\mu) \in [0, \pi)\}$. Then by Theorem 2.6, there exists a nonsingular

matrix $\widetilde{Y} \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} S_{\mathrm{CF}} := \widetilde{Y}^{-1} S \widetilde{Y} &= \mathcal{J}_{\xi_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}(\mu_m), \\ G_{1,\mathrm{CF}} := \widetilde{Y}^T G_2 \widetilde{Y} &= R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \\ \mathcal{H}_{\mathrm{CF}} := \widetilde{Y}^{-1} \mathcal{H} \widetilde{Y} &= \mathcal{J}^2_{\xi_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}^2_{\xi_m}(\mu_m), \end{aligned}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi)$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. (Here, the third line immediately follows from $\mathcal{H} = S^2$). Using $G_1^{-1} A \mathcal{H} = \hat{\mathcal{H}} G_1^{-1} A$ and the fact that $G_1^{-1} A$ is nonsingular, we find that $\mathcal{H}$ and $\hat{\mathcal{H}}$ are similar. Since the canonical form of G-symmetric matrices in Theorem 2.6 is uniquely determined by the Jordan canonical form, we obtain from Theorem 2.6 that the canonical forms of the pairs $(\mathcal{H}, G_2)$ and $(\hat{\mathcal{H}}, G_1)$ coincide. In particular, this implies the existence of a nonsingular matrix $\widetilde{X} \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} \mathcal{H}_{\mathrm{CF}} = \widetilde{X}^{-1} \hat{\mathcal{H}} \widetilde{X} &= \mathcal{J}^2_{\xi_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}^2_{\xi_m}(\mu_m), \\ G_{1,\mathrm{CF}} = \widetilde{X}^T G_1 \widetilde{X} &= R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}. \end{aligned}$$

Finally setting $X = G_1^{-1} \widetilde{X}^{-T}$ and $Y = A^{-1} G_1 \widetilde{X} S_{\mathrm{CF}}$, we obtain

$$\begin{aligned} X^T A Y &= \widetilde{X}^{-1} G_1^{-1} A A^{-1} G_1 \widetilde{X} S_{\mathrm{CF}} = S_{\mathrm{CF}} \\ X^T G_1 X &= \widetilde{X}^{-1} G_1^{-1} G_1 G_1^{-1} \widetilde{X}^{-T} = (\widetilde{X}^T G_1 \widetilde{X})^{-1} = G_{1,\mathrm{CF}}^{-1} = G_{1,\mathrm{CF}} \\ Y^T G_2 Y &= S_{\mathrm{CF}}^T \widetilde{X}^T G_1 A^{-T} G_2 A^{-1} G_1 \widetilde{X} S_{\mathrm{CF}} \\ &= S_{\mathrm{CF}}^T \widetilde{X}^T G_1 \widetilde{X} \widetilde{X}^{-1} \hat{\mathcal{H}}^{-1} \widetilde{X} S_{\mathrm{CF}} \\ &= S_{\mathrm{CF}}^T G_{1,\mathrm{CF}} (\mathcal{H}_{\mathrm{CF}})^{-1} S_{\mathrm{CF}} = G_{1,\mathrm{CF}} S_{\mathrm{CF}} (\mathcal{H}_{\mathrm{CF}})^{-1} S_{\mathrm{CF}} = G_{1,\mathrm{CF}} \end{aligned}$$

as desired, where we used that S_{CF} is $G_{1,\mathrm{CF}}$-symmetric and that $S_{\mathrm{CF}}^2 = \mathcal{H}_{\mathrm{CF}}$. It is now easy to check that $Y^{-1} \mathcal{H} Y$ and $X^{-1} \hat{\mathcal{H}} X$ have the claimed forms. Concerning uniqueness, we note that the form (4.1) is already uniquely determined by the Jordan structure of $\mathcal{H}$ and by the restriction $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi)$. □

The canonical form for the case that A is singular or rectangular is more involved, because then the matrices $\mathcal{H}$ and $\hat{\mathcal{H}}$ may be singular as well. The key idea in the proof of Theorem 4.1 was the construction of a G_2-symmetric square root of $\mathcal{H}$, but if $\mathcal{H}$ is singular, then such a square root need not exists. (For example, the R_n-symmetric nilpotent matrix $\mathcal{J}_n(0)$ does not have any square root let alone a R_n-symmetric one.) A second difficulty comes from the fact that the Jordan structures of $\mathcal{H}$ and $\hat{\mathcal{H}}$ may be different. For example, if

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; G_1 = R_2 \oplus R_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; G_2 = R_1 \oplus R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

then we obtain that

$$\mathcal{H} = G_2^{-1} A^T G_1^{-1} A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here $\mathcal{H}$ has a 1×1 and a 3×3 Jordan block associated with the eigenvalue zero, while $\hat{\mathcal{H}}$ has two 2×2 Jordan blocks associated with zero. In general, we obtain the following result.

Theorem 4.2 *Let $A \in \mathbb{C}^{m\times n}$ and let $G_1 \in \mathbb{C}^{m\times m}, G_2 \in \mathbb{C}^{n\times n}$ be complex symmetric and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} X^T A Y &= A_c \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4}, \\ X^T G_1 X &= G_c \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4}, \\ Y^T G_2 Y &= \widetilde{G}_c \oplus \widetilde{G}_{z,1} \oplus \widetilde{G}_{z,2} \oplus \widetilde{G}_{z,3} \oplus \widetilde{G}_{z,4}. \end{aligned} \tag{4.3}$$

Moreover, for the G_2-symmetric matrix $\mathcal{H} = G_2^{-1}A^T G_1^{-1} A \in \mathbb{C}^{n\times n}$ and for the G_1-symmetric matrix $\hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T \in \mathbb{C}^{m\times m}$ we have that

$$\begin{aligned} Y^{-1}\mathcal{H}Y &= \mathcal{H}_c \oplus \mathcal{H}_{z,1} \oplus \mathcal{H}_{z,2} \oplus \mathcal{H}_{z,3} \oplus \mathcal{H}_{z,4}, \\ X^{-1}\hat{\mathcal{H}}X &= \hat{\mathcal{H}}_c \oplus \hat{\mathcal{H}}_{z,1} \oplus \hat{\mathcal{H}}_{z,2} \oplus \hat{\mathcal{H}}_{z,3} \oplus \hat{\mathcal{H}}_{z,4}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\mathcal{H}$ and $\hat{\mathcal{H}}$:*
$A_c, G_c, \widetilde{G}_c$ have the forms as in (4.1) and $\mathcal{H}_c, \hat{\mathcal{H}}_c$ have the forms as in (4.2);

1) *one block corresponding to n_0 Jordan blocks of size 1×1 of $\mathcal{H}$ and m_0 Jordan blocks of size 1×1 of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$A_{z,1} = 0_{m_0\times n_0}, \quad G_{z,1} = I_{m_0}, \quad \widetilde{G}_{z,1} = I_{n_0}, \quad \mathcal{H}_{z,1} = 0_{n_0}, \quad \hat{\mathcal{H}}_{z,1} = 0_{m_0};$$

2) *blocks corresponding to a pair of $j\times j$ Jordan blocks of $\mathcal{H}$ and $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$\begin{aligned} A_{z,2} &= \bigoplus_{i=1}^{\ell_1} \mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{\ell_2} \mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_\nu} \mathcal{J}_{2\nu}(0), \\ G_{z,2} &= \bigoplus_{i=1}^{\ell_1} R_2 \oplus \bigoplus_{i=1}^{\ell_2} R_4 \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_\nu} R_{2\nu}, \\ \widetilde{G}_{z,2} &= \bigoplus_{i=1}^{\ell_1} R_2 \oplus \bigoplus_{i=1}^{\ell_2} R_4 \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_\nu} R_{2\nu}, \\ \mathcal{H}_{z,2} &= \bigoplus_{i=1}^{\ell_1} \mathcal{J}_2^2(0) \oplus \bigoplus_{i=1}^{\ell_2} \mathcal{J}_4^2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_\nu} \mathcal{J}_{2\nu}^2(0), \\ \hat{\mathcal{H}}_{z,2} &= \bigoplus_{i=1}^{\ell_1} \mathcal{J}_2^2(0)^T \oplus \bigoplus_{i=1}^{\ell_2} \mathcal{J}_4^2(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_\nu} \mathcal{J}_{2\nu}^2(0)^T, \end{aligned}$$

where $\ell_1, \ldots, \ell_\nu \in \mathbb{N}\cup\{0\}$; thus, $\mathcal{H}_{z,2}$ and $\hat{\mathcal{H}}_{z,2}$ both have each $2\ell_j$ Jordan blocks of size $j\times j$ for $j = 1, \ldots, \nu$;

3) *blocks corresponding to a $j\times j$ Jordan blocks of $\mathcal{H}$ and a $(j+1)\times(j+1)$ Jordan block*

of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:

$$
\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} I_1 \\ 0 \end{bmatrix}_{2\times 1} \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} I_2 \\ 0 \end{bmatrix}_{3\times 2} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\nu-1}} \begin{bmatrix} I_{\nu-1} \\ 0 \end{bmatrix}_{\nu\times(\nu-1)}, \\
G_{z,3} &= \bigoplus_{i=1}^{m_1} R_2 \quad \oplus \quad \bigoplus_{i=1}^{m_2} R_3 \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{m_{\nu-1}} R_\nu \,, \\
\widetilde{G}_{z,3} &= \bigoplus_{i=1}^{m_1} R_1 \quad \oplus \quad \bigoplus_{i=1}^{m_2} R_2 \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{m_{\nu-1}} R_{\nu-1} \,, \\
\mathcal{H}_{z,3} &= \bigoplus_{i=1}^{m_1} \mathcal{J}_1(0) \quad \oplus \quad \bigoplus_{i=1}^{m_2} \mathcal{J}_2(0) \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{m_{\nu-1}} \mathcal{J}_{\nu-1}(0) \,, \\
\hat{\mathcal{H}}_{z,3} &= \bigoplus_{i=1}^{m_1} \mathcal{J}_2(0)^T \quad \oplus \quad \bigoplus_{i=1}^{m_2} \mathcal{J}_3(0)^T \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{m_{\nu-1}} \mathcal{J}_\nu(0)^T,
\end{aligned}
$$

where $m_1, \ldots, m_{\nu-1} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,3}$ has m_j Jordan blocks of size $j \times j$ and $\hat{\mathcal{H}}_{z,3}$ has m_j Jordan blocks of size $(j+1) \times (j+1)$ for $j = 1, \ldots, \nu-1$;

4) *blocks corresponding to a $(j+1) \times (j+1)$ Jordan blocks of $\mathcal{H}$ and a $j \times j$ Jordan block of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & I_1 \end{bmatrix}_{1\times 2} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & I_2 \end{bmatrix}_{2\times 3} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\nu-1}} \begin{bmatrix} 0 & I_{\nu-1} \end{bmatrix}_{(\nu-1)\times\nu}, \\
G_{z,4} &= \bigoplus_{i=1}^{n_1} R_1 \quad \oplus \quad \bigoplus_{i=1}^{n_2} R_2 \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{n_{\nu-1}} R_{\nu-1} \,, \\
\widetilde{G}_{z,4} &= \bigoplus_{i=1}^{n_1} R_2 \quad \oplus \quad \bigoplus_{i=1}^{n_2} R_3 \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{n_{\nu-1}} R_\nu \,, \\
\mathcal{H}_{z,4} &= \bigoplus_{i=1}^{n_1} \mathcal{J}_2(0) \quad \oplus \quad \bigoplus_{i=1}^{n_2} \mathcal{J}_3(0) \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{n_{\nu-1}} \mathcal{J}_\nu(0) \,, \\
\hat{\mathcal{H}}_{z,4} &= \bigoplus_{i=1}^{n_1} \mathcal{J}_1(0)^T \quad \oplus \quad \bigoplus_{i=1}^{n_2} \mathcal{J}_2(0)^T \quad \oplus \cdots \oplus \quad \bigoplus_{i=1}^{n_{\nu-1}} \mathcal{J}_{\nu-1}(0)^T,
\end{aligned}
$$

where $n_1, \ldots, n_{\nu-1} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,4}$ has n_j Jordan blocks of size $(j+1) \times (j+1)$ and $\hat{\mathcal{H}}_{z,4}$ has n_j Jordan blocks of size $j \times j$ for $j = 1, \ldots, \nu-1$;

The matrices $\mathcal{H}$ and $\hat{\mathcal{H}}$ have $2\ell_j + m_j + n_{j-1}$ respectively $2\ell_j + m_{j-1} + n_j$ Jordan of size $j \times j$ for $j = 1, \ldots, \nu$, where $m_\nu = n_\nu = 0$ and where ν is the maximum of the nilpotency index of $\mathcal{H}$ and the nilpotency index of $\hat{\mathcal{H}}$.

Moreover, the form (4.3) *is unique up to simultaneous block permutation of the blocks in the diagonal blocks of the right hand side of* (4.3).

Proof. Clearly, there exist invertible matrices $\widetilde{X}$ and $\widetilde{Y}$ such that $\widetilde{X}^T G_1 \widetilde{X} = I_m$ and $\widetilde{Y}^T G_2 \widetilde{Y} = I_n$. Thus, replacing A by $\widetilde{X}^T A \widetilde{Y}$ otherwise, we may assume that $G_1 = I_m$ and $G_2 = I_n$. The proof then proceeds by repeatedly applying Proposition 3.3 to the factors of the full rank decomposition $A = B_1 C_1$, $B_1 \in \mathbb{C}^{m\times r}$, $C_1 \in \mathbb{C}^{r\times n}$, where $r = \operatorname{rank} A$. In this way, a staircase-like form is obtained in a stepwise reduction procedure. This form can then further be reduced until the canonical form has been reached. This reduction procedure (although straightforward) is very technical and tedious and is very similar to the corresponding procedure for the case of complex sesquilinear forms. Therefore, we omit the proof here and refer the reader to [13].

Concerning uniqueness, in view of Theorem 4.1 it remains to show that the numbers ℓ_j, m_j, n_j are uniquely determined. Note that there exists a unique sequence of subspaces

$$\operatorname{Eig}_\nu(\mathcal{H},0) \subseteq \operatorname{Eig}_{\nu-1}(\mathcal{H},0) \subseteq \cdots \subseteq \operatorname{Eig}_1(\mathcal{H},0) = \ker \mathcal{H}$$

where $\operatorname{Eig}_j(\mathcal{H},0)$ contains all eigenvectors of $\mathcal{H}$ associated with zero that can be extended to a Jordan chain of length at least j. Define $\kappa_\nu = \dim\big(\operatorname{Eig}_\nu(\mathcal{H},0) \cap \ker A\big)$ and

$$\kappa_j = \dim\big(\operatorname{Eig}_j(\mathcal{H},0) \cap \ker A\big) - \dim\big(\operatorname{Eig}_{j+1}(\mathcal{H},0) \cap \ker A\big), \quad j = 1, \ldots, \nu-1.$$

Then any eigenvector of $\mathcal{H}$ that is associated with a Jordan block of size $j \times j$ in the canonical form and that is also in the kernel of A contributes to κ_j. Similarly, we define $\hat{\kappa}_\nu = \dim\big(\operatorname{Eig}_\nu(\hat{\mathcal{H}},0) \cap \ker A^T\big)$ and

$$\hat{\kappa}_j = \dim\big(\operatorname{Eig}_j(\hat{\mathcal{H}},0) \cap \ker A^T\big) - \dim\big(\operatorname{Eig}_{j+1}(\hat{\mathcal{H}},0) \cap \ker A^T\big), \quad j = 1, \ldots, \nu-1.$$

Then elementary counting yields

$$\kappa_j = \ell_j + n_{j-1} \quad \text{and} \quad \hat{\kappa}_j = \ell_j + m_{j-1}, \quad j = 1, \ldots, \nu.$$

If p_j respectively $\hat{p}_j$ denote the number of Jordan blocks of size $j \times j$ in the canonical form of $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively, we also have that

$$p_j = 2\ell_j + m_j + n_{j-1} \quad \text{and} \quad \hat{p}_j = 2\ell_j + m_{j-1} + n_j, \quad j = 1, \ldots, \nu.$$

Hence, we obtain

$$p_j - \kappa_j - \hat{\kappa}_j = m_j - m_{j-1}, \quad \text{and} \quad \hat{p}_j - \kappa_j - \hat{\kappa}_j = n_j - n_{j-1}, \quad j = 1, \ldots, \nu,$$

from which we can successively compute m_j, n_j, $j = \nu-1, \ldots, 0$ using $m_\nu = n_\nu = 0$. We furthermore obtain that

$$\ell_j = \frac{1}{2}(p_j - m_j - n_{j-1})$$

for $j = 1, \ldots, \nu$. Thus, the numbers ℓ_j, m_j, n_j are uniquely determined by the invariant numbers $p_j, \hat{p}_j, \kappa_j, \hat{\kappa}_j$, $j = 1, \ldots, \nu$. □

5 Condensed forms for G_1 complex symmetric, G_2 complex skew-symmetric

In this section we study the canonical forms for the case that G_1 is complex symmetric and G_2 complex skew-symmetric. Again, we start with the canonical form for the case that A is quadratic and nonsingular.

Theorem 5.1 *Let $A, G_1, G_2 \in \mathbb{C}^{2n\times 2n}$ be nonsingular and let G_1 be complex symmetric and G_2 be complex skew-symmetric. Then there exists nonsingular matrices $X, Y \in \mathbb{C}^{2n\times 2n}$ such that*

$$\begin{aligned} X^T A Y &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\mu_m) \end{bmatrix}, \\ X^T G_1 X &= \begin{bmatrix} 0 & R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ R_{\xi_m} & 0 \end{bmatrix}, \\ Y^T G_2 Y &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix}, \end{aligned} \tag{5.1}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi/2)$, *and* $\xi_j \in \mathbb{N}$ *for* $j = 1, \ldots, m$. *Moreover, for the* G_2*-Hamiltonian matrix* $\mathcal{H} = G_2^{-1} A^T G_1^{-1} A$ *and for the* G_1*-skew-symmetric matrix* $\hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T$ *we have that*

$$\begin{aligned} Y^{-1}\mathcal{H}Y &= \begin{bmatrix} -\mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -\mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}, \\ X^{-1}\hat{\mathcal{H}}X &= \begin{bmatrix} \mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & -\mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & -\mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}^T. \end{aligned} \tag{5.2}$$

Proof. By Theorem 2.8, there exists a nonsingular matrix $\mathcal{Y} \in \mathbb{C}^{n \times n}$ such that

$$\begin{aligned} \mathcal{Y}^{-1}\mathcal{H}\mathcal{Y} &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\lambda_1) & 0 \\ 0 & -\mathcal{J}_{\xi_1}(\lambda_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\lambda_m) & 0 \\ 0 & -\mathcal{J}_{\xi_m}(\lambda_m) \end{bmatrix}, \\ \mathcal{Y}^T G_2 \mathcal{Y} &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix}, \end{aligned}$$

where $\lambda_j \in \mathbb{C} \setminus \{0\}$, $\arg(\lambda_j) \in [0, \pi)$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Next construct the matrix $\widetilde{S}$ such that

$$\mathcal{Y}^{-1}\widetilde{S}\mathcal{Y} = \begin{bmatrix} \mathcal{J}_{\xi_1}(\lambda_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\lambda_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\lambda_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\lambda_m) \end{bmatrix}$$

It is easily verified that $\widetilde{S}$ is G_2-skew-Hamiltonian, that it satisfies $\widetilde{S}^2 = \mathcal{H}^2$, and that we have $\sigma(\widetilde{S}) \subseteq \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [0, \pi)\}$. Thus, by the uniqueness property of Lemma 2.10, we obtain that $\widetilde{S}$ is a polynomial in $\mathcal{H}^2$. Moreover, applying Lemma 2.10 once more, we obtain that $\widetilde{S}$ has a unique square root $S \in \mathbb{C}^{n \times n}$ satisfying $\sigma(S) \subseteq \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [0, \pi)\}$ and

$$\mathcal{Y}^{-1} S \mathcal{Y} = \begin{bmatrix} \mathcal{J}_{\xi_1}(\lambda_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\lambda_1) \end{bmatrix}^{\frac{1}{2}} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\lambda_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\lambda_m) \end{bmatrix}^{\frac{1}{2}}.$$

In fact, we must have

$$\sigma(S) \subseteq \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [0, \pi/2)\},$$

because otherwise $\widetilde{S}$ would have eigenvalues λ_j with $\arg(\lambda_j) \in [\pi, 2\pi)$. Clearly, being a polynomial in $\widetilde{S}$ and thus in $\mathcal{H}^2$, it follows that S is also a polynomial in $\mathcal{H}$. Let $\mu_j^2 = \lambda_j$ and $\arg(\mu_j) \in [0, \pi/2)$. We then obtain by Theorem 2.9 that there exists a nonsingular matrix $\widetilde{Y} \in \mathbb{C}^{n \times n}$ such that

$$\begin{aligned} S_{\text{CF}} := \widetilde{Y}^{-1} S \widetilde{Y} &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\mu_m) \end{bmatrix}, \\ G_{\text{CF}}^{(2)} := \widetilde{Y}^T G_2 \widetilde{Y} &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix}. \end{aligned}$$

Moreover, using $G_1^{-1} A \mathcal{H} = \hat{\mathcal{H}} G_1^{-1} A$ and the fact that $G_1^{-1} A$ is nonsingular, we find that $\mathcal{H}$ and $\hat{\mathcal{H}}$ are similar. Thus, by Theorem 2.7 there exists a nonsingular matrix $\widetilde{X} \in \mathbb{C}^{n \times n}$ such that

$$\begin{aligned} \mathcal{H}_{\text{CF}} = \widetilde{X}^{-1} \hat{\mathcal{H}} \widetilde{X} &= \begin{bmatrix} -\mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -\mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}, \\ G_{\text{CF}}^{(1)} = \widetilde{X}^T G_1 \widetilde{X} &= \begin{bmatrix} 0 & R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ R_{\xi_m} & 0 \end{bmatrix}. \end{aligned}$$

Indeed, since $\hat{\mathcal{H}}$ is similar to $\mathcal{H}$, it has the eigenvalues $\lambda_j = \mu_j^2$ with partial multiplicities ξ_j, $j = 1, \ldots, m$. Since the canonical form of G-skew-symmetric matrices in Theorem 2.7 is uniquely determined by the Jordan canonical form, we find that the pairs $(\hat{\mathcal{H}}, G_1)$ and $(\mathcal{H}_{\rm CF}, G_{1,\rm CF})$ must have the same canonical form. Observe that $S_{\rm CF}$ is $G_{\rm CF}^{(1)}$-symmetric, but not a square root of $\mathcal{H}_{\rm CF}$. Instead, it is easy to check that

$$S_{\rm CF}(\mathcal{H}_{\rm CF})^{-1} S_{\rm CF} = \begin{bmatrix} -I_{\xi_1} & 0 \\ 0 & I_{\xi_1} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -I_{\xi_m} & 0 \\ 0 & I_{\xi_m} \end{bmatrix}.$$

Using this identity and setting $X = G_1^{-1} \widetilde{X}^{-T}$ and $Y = A^{-1} G_1 \widetilde{X} S_{\rm CF}$, we obtain

$$\begin{aligned}
X^T A Y &= \widetilde{X}^{-1} G_1^{-1} A A^{-1} G_1 \widetilde{X} S_{\rm CF} = S_{\rm CF}, \\
X^T G_1 X &= \widetilde{X}^{-1} G_1^{-1} G_1 G_1^{-1} \widetilde{X}^{-T} = (\widetilde{X}^T G_1 \widetilde{X})^{-1} = (G_{\rm CF}^{(1)})^{-1} = G_{\rm CF}^{(1)}, \\
Y^T G_2 Y &= S_{\rm CF}^T \widetilde{X}^T G_1 A^{-T} G_2 A^{-1} G_1 \widetilde{X} S_{\rm CF} \\
&= S_{\rm CF}^T \widetilde{X}^T G_1 \widetilde{X} \widetilde{X}^{-1} \hat{\mathcal{H}}^{-1} \widetilde{X} S_{\rm CF} \\
&= S_{\rm CF}^T G_{\rm CF}^{(1)} (\mathcal{H}_{\rm CF})^{-1} S_{\rm CF} = G_{\rm CF}^{(1)} S_{\rm CF} (\mathcal{H}_{\rm CF})^{-1} S_{\rm CF} = G_{\rm CF}^{(2)}.
\end{aligned}$$

It is now straightforward to check that $Y^{-1}\mathcal{H}Y$ and $X^{-1}\hat{\mathcal{H}}X$ have the claimed forms. Concerning uniqueness, we note that the form (5.1) is already uniquely determined by the Jordan structure of $\mathcal{H}$ and by the restriction $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi/2)$. □

Theorem 5.2 *Let $A \in \mathbb{C}^{m\times 2n}$, let $G_1 \in \mathbb{C}^{m\times m}$ be complex symmetric and nonsingular and let $G_2 \in \mathbb{C}^{2n\times 2n}$ be complex skew-symmetric and nonsingular. Then there exists nonsingular matrices $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{2n\times 2n}$ such that*

$$\begin{aligned}
X^T A Y &= A_c \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4} \oplus A_{z,5} \oplus A_{z,6}, \\
X^T G_1 X &= G_c \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4} \oplus G_{z,5} \oplus G_{z,6}, \\
Y^T G_2 Y &= \widetilde{G}_c \oplus \widetilde{G}_{z,1} \oplus \widetilde{G}_{z,2} \oplus \widetilde{G}_{z,3} \oplus \widetilde{G}_{z,4} \oplus \widetilde{G}_{z,5} \oplus \widetilde{G}_{z,6}.
\end{aligned} \tag{5.3}$$

Moreover, for the G_2-Hamiltonian matrix $\mathcal{H} = G_2^{-1} A^T G_1^{-1} A \in \mathbb{C}^{2n\times 2n}$ and for the G_1-skew-symmetric matrix $\hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T \in \mathbb{C}^{m\times m}$ we have that

$$\begin{aligned}
Y^{-1}\mathcal{H}Y &= \mathcal{H}_c \oplus \mathcal{H}_{z,1} \oplus \mathcal{H}_{z,2} \oplus \mathcal{H}_{z,3} \oplus \mathcal{H}_{z,4} \oplus \mathcal{H}_{z,5} \oplus \mathcal{H}_{z,6}, \\
X^{-1}\hat{\mathcal{H}}X &= \hat{\mathcal{H}}_c \oplus \hat{\mathcal{H}}_{z,1} \oplus \hat{\mathcal{H}}_{z,2} \oplus \hat{\mathcal{H}}_{z,3} \oplus \hat{\mathcal{H}}_{z,4} \oplus \hat{\mathcal{H}}_{z,5} \oplus \hat{\mathcal{H}}_{z,6}.
\end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\mathcal{H}$ and $\hat{\mathcal{H}}$:*
$A_c, G_c, \widetilde{G}_c$ have the forms as in (5.1) and $\mathcal{H}_c, \hat{\mathcal{H}}_c$ have the forms as in (5.2);

1) *one block corresponding to $2n_0$ Jordan blocks of size 1×1 of $\mathcal{H}$ and m_0 Jordan blocks of size 1×1 of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$A_{z,1} = 0_{m_0\times 2n_0}, \quad G_{z,1} = I_{m_0}, \quad \widetilde{G}_{z,1} = J_{n_0}, \quad \mathcal{H}_{z,1} = 0_{2n_0}, \quad \hat{\mathcal{H}}_{z,1} = 0_{m_0};$$

2) *blocks corresponding to a pair of $j \times j$ Jordan blocks of $\mathcal{H}$ and $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,2} &= \bigoplus_{i=1}^{\ell_1} \mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{\ell_2} \mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_{2\nu+1}} \mathcal{J}_{4\nu+2}(0)\,, \\
G_{z,2} &= \bigoplus_{i=1}^{\ell_1} R_2 \oplus \bigoplus_{i=1}^{\ell_2} R_4 \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_{2\nu+1}} R_{4\nu+2}\,, \\
\widetilde{G}_{z,2} &= \bigoplus_{i=1}^{\ell_1} (-\Gamma_2) \oplus \bigoplus_{i=1}^{\ell_2} (-\Gamma_4) \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_{2\nu+1}} (-\Gamma_{4\nu+2})\,, \\
\mathcal{H}_{z,2} &= \bigoplus_{i=1}^{\ell_1} 0_2 \oplus \bigoplus_{i=1}^{\ell_2} \Sigma\mathcal{J}_4^2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_{2\nu+1}} \Sigma\mathcal{J}_{4\nu+2}^2(0)\,, \\
\hat{\mathcal{H}}_{z,2} &= \bigoplus_{i=1}^{\ell_1} 0_2 \oplus \bigoplus_{i=1}^{\ell_2} \Sigma\mathcal{J}_4^2(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{\ell_{2\nu+1}} \Sigma\mathcal{J}_{4\nu+2}^2(0)^T\,,
\end{aligned}
$$

where $\ell_1, \ldots, \ell_\nu \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,2}$ and $\hat{\mathcal{H}}_{z,2}$ both have each $2\ell_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, 2\nu+1$;

3) *blocks corresponding to a $2j \times 2j$ Jordan block of $\mathcal{H}$ and a $(2j+1) \times (2j+1)$ Jordan block of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_2} \begin{bmatrix} I_2 \\ 0 \end{bmatrix}_{3\times 2} \oplus \bigoplus_{i=1}^{m_4} \begin{bmatrix} I_4 \\ 0 \end{bmatrix}_{5\times 4} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu}} \begin{bmatrix} I_{2\nu} \\ 0 \end{bmatrix}_{(2\nu+1)\times 2\nu}, \\
G_{z,3} &= \bigoplus_{i=1}^{m_2} R_3 \oplus \bigoplus_{i=1}^{m_4} R_5 \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu}} R_{2\nu+1}\,, \\
\widetilde{G}_{z,3} &= \bigoplus_{i=1}^{m_2} (-\Gamma_2) \oplus \bigoplus_{i=1}^{m_4} (-\Gamma_4) \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu}} (-\Gamma_{2\nu})\,, \\
\mathcal{H}_{z,3} &= \bigoplus_{i=1}^{m_2} \Sigma\mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{m_4} \Sigma\mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu}} \Sigma\mathcal{J}_{2\nu}(0)\,, \\
\hat{\mathcal{H}}_{z,3} &= \bigoplus_{i=1}^{m_2} \Sigma\mathcal{J}_3(0)^T \oplus \bigoplus_{i=1}^{m_4} \Sigma\mathcal{J}_5(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu}} \Sigma\mathcal{J}_{2\nu+1}(0)^T,
\end{aligned}
$$

where $m_2, m_4, \ldots, m_{2\nu} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,3}$ has m_{2j} Jordan blocks of size $2j \times 2j$ and $\hat{\mathcal{H}}_{z,3}$ has m_{2j} Jordan blocks of size $(2j+1) \times (2j+1)$ for $j = 1, \ldots, \nu$;

4) *blocks corresponding to two $(2j-1) \times (2j-1)$ Jordan blocks of $\mathcal{H}$ and two $2j \times 2j$ Jordan*

blocks of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:

$$
\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & I_1 \\ 0 & 0 \\ I_1 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2} \oplus \bigoplus_{i=1}^{m_3} \begin{bmatrix} 0 & I_3 \\ 0 & 0 \\ I_3 & 0 \\ 0 & 0 \end{bmatrix}_{8\times 6} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu-1}} \begin{bmatrix} 0 & I_{2\nu-1} \\ 0 & 0 \\ I_{2\nu-1} & 0 \\ 0 & 0 \end{bmatrix}_{4\nu\times(4\nu-2)}, \\
G_{z,4} &= \bigoplus_{i=1}^{m_1} R_4 \oplus \bigoplus_{i=1}^{m_3} R_8 \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu-1}} R_{4\nu}, \\
\widetilde{G}_{z,4} &= \bigoplus_{i=1}^{m_1} (-\Gamma_2) \oplus \bigoplus_{i=1}^{m_3} (-\Gamma_6) \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu-1}} (-\Gamma_{4\nu-2}), \\
\mathcal{H}_{z,4} &= \bigoplus_{i=1}^{m_1} 0_2 \oplus \bigoplus_{i=1}^{m_3} \begin{bmatrix} \Sigma\mathcal{J}_3(0) & 0 \\ 0 & -\Sigma\mathcal{J}_3(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu-1}} \begin{bmatrix} \Sigma\mathcal{J}_{2\nu-1}(0) & 0 \\ 0 & -\Sigma\mathcal{J}_{2\nu-1}(0) \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,4} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} \mathcal{J}_2(0) & 0 \\ 0 & -\mathcal{J}_2(0) \end{bmatrix}^T \oplus \bigoplus_{i=1}^{m_3} \begin{bmatrix} \Sigma\mathcal{J}_4(0) & 0 \\ 0 & -\Sigma\mathcal{J}_4(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\nu-1}} \begin{bmatrix} \Sigma\mathcal{J}_{2\nu}(0) & 0 \\ 0 & -\Sigma\mathcal{J}_{2\nu}(0) \end{bmatrix}^T,
\end{aligned}
$$

where $m_1, m_3, \ldots, m_{2\nu-1} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,4}$ has $2m_{2j-1}$ Jordan blocks of size $(2j-1)\times(2j-1)$ and $\hat{\mathcal{H}}_{z,4}$ has $2m_{2j-1}$ Jordan blocks of size $2j\times 2j$ for $j = 1, \ldots, \nu$;

5) *blocks corresponding to a $2j\times 2j$ Jordan block of $\mathcal{H}$ and a $(2j-1)\times(2j-1)$ Jordan block of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,5} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & I_1 \end{bmatrix}_{1\times 2} \oplus \bigoplus_{i=1}^{n_3} \begin{bmatrix} 0 & I_3 \end{bmatrix}_{3\times 4} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu-1}} \begin{bmatrix} 0 & I_{2\nu-1} \end{bmatrix}_{(2\nu-1)\times 2\nu}, \\
G_{z,5} &= \bigoplus_{i=1}^{n_1} R_1 \oplus \bigoplus_{i=1}^{n_3} R_3 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu-1}} R_{2\nu-1}, \\
\widetilde{G}_{z,5} &= \bigoplus_{i=1}^{n_1} (-\Gamma_2) \oplus \bigoplus_{i=1}^{n_3} (-\Gamma_4) \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu-1}} (-\Gamma_{2\nu}), \\
\mathcal{H}_{z,5} &= \bigoplus_{i=1}^{n_1} \Sigma\mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{n_3} \Sigma\mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu-1}} \Sigma\mathcal{J}_{2\nu}(0), \\
\hat{\mathcal{H}}_{z,5} &= \bigoplus_{i=1}^{n_1} 0_1 \oplus \bigoplus_{i=1}^{n_3} \big(\Sigma\mathcal{J}_3(0)\big)^T \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu-1}} \big(\Sigma\mathcal{J}_{2\nu-1}(0)\big)^T,
\end{aligned}
$$

where $n_1, n_3 \ldots, n_{2\nu-1} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,5}$ has n_{2j-1} Jordan blocks of size $2j\times 2j$ and $\hat{\mathcal{H}}_{z,5}$ has n_{2j-1} Jordan blocks of size $(2j-1)\times(2j-1)$ for $j = 1, \ldots, \nu$;

6) *blocks corresponding to two $(2j+1)\times(2j+1)$ Jordan blocks of $\mathcal{H}$ and two $2j\times 2j$ Jordan blocks of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,6} &= \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & 0 & 0 & I_2 \\ 0 & I_2 & 0 & 0 \end{bmatrix}_{4\times 6} \oplus \bigoplus_{i=1}^{n_4} \begin{bmatrix} 0 & 0 & 0 & I_4 \\ 0 & I_4 & 0 & 0 \end{bmatrix}_{8\times 10} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu}} \begin{bmatrix} 0 & 0 & 0 & I_{2\nu} \\ 0 & I_{2\nu} & 0 & 0 \end{bmatrix}_{4\nu\times(4\nu+2)}, \\
G_{z,6} &= \bigoplus_{i=1}^{n_2} R_4 \oplus \bigoplus_{i=1}^{n_4} R_8 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu}} R_{4\nu}, \\
\widetilde{G}_{z,6} &= \bigoplus_{i=1}^{n_2} (-\Gamma_6) \oplus \bigoplus_{i=1}^{n_4} (-\Gamma_{10}) \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu}} (-\Gamma_{4\nu+2}), \\
\mathcal{H}_{z,6} &= \bigoplus_{i=1}^{n_2} \begin{bmatrix} \Sigma\mathcal{J}_3(0) & 0 \\ 0 & -\Sigma\mathcal{J}_3(0) \end{bmatrix} \oplus \bigoplus_{i=1}^{n_4} \begin{bmatrix} \Sigma\mathcal{J}_5(0) & 0 \\ 0 & -\Sigma\mathcal{J}_5(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu}} \begin{bmatrix} \Sigma\mathcal{J}_{2\nu+1}(0) & 0 \\ 0 & -\Sigma\mathcal{J}_{2\nu+1}(0) \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,6} &= \bigoplus_{i=1}^{n_2} \begin{bmatrix} \Sigma\mathcal{J}_2(0)^T & 0 \\ 0 & -\Sigma\mathcal{J}_2(0)^T \end{bmatrix} \oplus \bigoplus_{i=1}^{n_4} \begin{bmatrix} \Sigma\mathcal{J}_4(0)^T & 0 \\ 0 & -\Sigma\mathcal{J}_4(0)^T \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\nu}} \begin{bmatrix} \Sigma\mathcal{J}_{2\nu}(0)^T & 0 \\ 0 & -\Sigma\mathcal{J}_{2\nu}(0)^T \end{bmatrix},
\end{aligned}
$$

where $n_2, n_4, \ldots, n_{2\nu} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,6}$ has $2n_{2j}$ Jordan blocks of size $(2j+1)\times(2j+1)$ and $\hat{\mathcal{H}}_{z,6}$ has $2n_{2j}$ Jordan blocks of size $2j \times 2j$ for $j = 1, \ldots, \nu$;

The matrices $\mathcal{H}$ and $\hat{\mathcal{H}}$ have $2\ell_{2j} + m_{2j} + n_{2j-1}$ respectively $2\ell_{2j} + 2m_{2j-1} + 2n_{2j}$ Jordan blocks of size $2j \times 2j$ for $j = 1, \ldots, \nu$ and $2\ell_{2j+1} + 2m_{2j+1} + 2n_{2j}$ respectively $2\ell_{2j+1} + m_{2j} + n_{2j+1}$ Jordan blocks of size $(2j+1) \times (2j+1)$ for $j = 0, \ldots, \nu$. Here $m_{2\nu+1} = n_{2\nu+1} = 0$ and $2\nu+1$ is the smallest odd number that is larger or equal to the maximum of the nilpotency indices of $\mathcal{H}$ and $\hat{\mathcal{H}}$.

Moreover, the form (4.3) *is unique up to simultaneous block permutation of the blocks in the diagonal blocks of the right hand side of* (4.3).

Proof. Without loss of generality we may assume that $G_1 = I_m$ and $G_2 = J_n$. The canonical form is then obtained by repeatedly applying Propositions 3.3 and 3.7 to the factors of the full rank decomposition $A = B_1 C_1$, $B_1 \in \mathbb{C}^{m\times r}$, $C_1 \in \mathbb{C}^{r\times n}$, where $r = \operatorname{rank} A$. Again, we omit the detailed proof here and refer the reader to [13]. Uniqueness of the form is proved as in the proof of Theorem 4.2. □

6 Canonical forms for G_1, G_2 complex skew-symmetric

In this section we finally treat that case that both G_1 and G_2 are complex skew-symmetric.

Theorem 6.1 *Let $A \in \mathbb{C}^{2n\times 2n}$ be nonsingular and let $G_1, G_2 \in \mathbb{C}^{2n\times 2n}$ be nonsingular and complex skew-symmetric. Then there exists nonsingular matrices $X, Y \in \mathbb{C}^{2n\times 2n}$ such that*

$$\begin{aligned} X^T A Y &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\mu_m) \end{bmatrix}, \\ X^T G_1 X &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix}, \\ Y^T G_2 Y &= \begin{bmatrix} 0 & -R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & -R_{\xi_m} \\ R_{\xi_m} & 0 \end{bmatrix}, \end{aligned} \tag{6.1}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi)$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Furthermore, for the G_2-skew-Hamiltonian matrix $\mathcal{H} = G_2^{-1} A^T G_1^{-1} A$ and for the G_1-skew-Hamiltonian matrix $\hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T$ we have that

$$\begin{aligned} Y^{-1}\mathcal{H}Y &= \begin{bmatrix} \mathcal{J}^2_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}^2_{\xi_1}(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}^2_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}^2_{\xi_m}(\mu_m) \end{bmatrix}, \\ X^{-1}\hat{\mathcal{H}}X &= \begin{bmatrix} \mathcal{J}^2_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}^2_{\xi_1}(\mu_1) \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}^2_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}^2_{\xi_m}(\mu_m) \end{bmatrix}^T. \end{aligned} \tag{6.2}$$

Proof. The proof proceeds completely analogous to the proof of Theorem 4.1 using Lemma 2.10 and Theorem 2.9. □

Theorem 6.2 *Let $A \in \mathbb{C}^{2m\times 2n}$ and let $G_1 \in \mathbb{C}^{2m\times 2m}$, $G_2 \in \mathbb{C}^{2n\times 2n}$ be complex skew-symmetric and nonsingular. Then there exists nonsingular matrices $X \in \mathbb{C}^{2m\times 2m}$ and $Y \in \mathbb{C}^{2n\times 2n}$ such that*

$$\begin{aligned} X^T A Y &= A_c \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4}, \\ X^T G_1 X &= G_c \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4}, \\ Y^T G_2 Y &= \widetilde{G}_c \oplus \widetilde{G}_{z,1} \oplus \widetilde{G}_{z,2} \oplus \widetilde{G}_{z,3} \oplus \widetilde{G}_{z,4}. \end{aligned} \tag{6.3}$$

Moreover, for the G_2-skew-Hamiltonian matrix $\mathcal{H} = G_2^{-1}A^TG_1^{-1}A \in \mathbb{C}^{2n\times 2n}$ and for the G_1-skew-Hamiltonian matrix $\hat{\mathcal{H}} = G_1^{-1}AG_2^{-1}A^T \in \mathbb{C}^{2m\times 2m}$ we have that

$$\begin{aligned} Y^{-1}\mathcal{H}Y &= \mathcal{H}_c \oplus \mathcal{H}_{z,1} \oplus \mathcal{H}_{z,2} \oplus \mathcal{H}_{z,3} \oplus \mathcal{H}_{z,4}, \\ X^{-1}\hat{\mathcal{H}}X &= \hat{\mathcal{H}}_c \oplus \hat{\mathcal{H}}_{z,1} \oplus \hat{\mathcal{H}}_{z,2} \oplus \hat{\mathcal{H}}_{z,3} \oplus \hat{\mathcal{H}}_{z,4}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\mathcal{H}$ and $\hat{\mathcal{H}}$:*
$A_c, G_c, \widetilde{G}_c$ have the forms as in (6.1) and $\mathcal{H}_c, \hat{\mathcal{H}}_c$ have the forms as in (6.2);

1) *one block corresponding to $2n_0$ Jordan blocks of size 1×1 of $\mathcal{H}$ and $2m_0$ Jordan blocks of size 1×1 of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$A_{z,1} = 0_{2m_0\times 2n_0}, \quad G_{z,1} = J_{m_0}, \quad \widetilde{G}_{z,1} = J_{n_0}, \quad \mathcal{H}_{z,1} = 0_{2n_0}, \quad \hat{\mathcal{H}}_{z,1} = 0_{2m_0};$$

2) *blocks corresponding to a pair of $j\times j$ Jordan blocks of $\mathcal{H}$ and $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$\begin{aligned} A_{z,2} &= \bigoplus_{i=1}^{\ell_1} \mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{\ell_2} \mathcal{J}_4(0) \oplus\cdots\oplus \bigoplus_{i=1}^{\ell_\nu} \mathcal{J}_{2\nu}(0), \\ G_{z,2} &= \bigoplus_{i=1}^{\ell_1} \Gamma_2 \oplus \bigoplus_{i=1}^{\ell_2} \Gamma_4 \oplus\cdots\oplus \bigoplus_{i=1}^{\ell_\nu} \Gamma_{2\nu}, \\ \widetilde{G}_{z,2} &= \bigoplus_{i=1}^{\ell_1} \Gamma_2 \oplus \bigoplus_{i=1}^{\ell_2} \Gamma_4 \oplus\cdots\oplus \bigoplus_{i=1}^{\ell_\nu} \Gamma_{2\nu}, \\ \mathcal{H}_{z,2} &= \bigoplus_{i=1}^{\ell_1} 0_2 \oplus \bigoplus_{i=1}^{\ell_2} \mathcal{J}_4^2(0) \oplus\cdots\oplus \bigoplus_{i=1}^{\ell_\nu} \mathcal{J}_{2\nu}^2(0), \\ \hat{\mathcal{H}}_{z,2} &= \bigoplus_{i=1}^{\ell_1} 0_2 \oplus \bigoplus_{i=1}^{\ell_2} \mathcal{J}_4^2(0)^T \oplus\cdots\oplus \bigoplus_{i=1}^{\ell_\nu} \mathcal{J}_{2\nu}^2(0)^T, \end{aligned}$$

where $\ell_1,\ldots,\ell_\nu \in \mathbb{N}\cup\{0\}$; thus, $\mathcal{H}_{z,2}$ and $\hat{\mathcal{H}}_{z,2}$ both have each $2\ell_j$ Jordan blocks of size $j\times j$ for $j = 1,\ldots,\nu$;

3) *blocks corresponding to two $j\times j$ Jordan blocks of $\mathcal{H}$ and two $(j+1)\times(j+1)$ Jordan blocks of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$\begin{aligned} A_{z,4} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & I_1 \\ 0 & 0 \\ I_1 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2} \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} 0 & I_2 \\ 0 & 0 \\ I_2 & 0 \\ 0 & 0 \end{bmatrix}_{6\times 4} \oplus\cdots\oplus \bigoplus_{i=1}^{m_\nu} \begin{bmatrix} 0 & I_{\nu-1} \\ 0 & 0 \\ I_{\nu-1} & 0 \\ 0 & 0 \end{bmatrix}_{2\nu\times(2\nu-2)}, \\ G_{z,4} &= \bigoplus_{i=1}^{m_1} \Gamma_4 \oplus \bigoplus_{i=1}^{m_2} \Gamma_6 \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\nu-1}} \Gamma_{2\nu}, \\ \widetilde{G}_{z,4} &= \bigoplus_{i=1}^{m_1} (-1)^2\Gamma_2 \oplus \bigoplus_{i=1}^{m_2} (-1)^3\Gamma_4 \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\nu-1}} (-1)^\nu\Gamma_{2\nu-2}, \\ \mathcal{H}_{z,4} &= \bigoplus_{i=1}^{m_1} 0_2 \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} \mathcal{J}_2(0) & 0 \\ 0 & -\mathcal{J}_2(0) \end{bmatrix} \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\nu-1}} \begin{bmatrix} \mathcal{J}_{\nu-1}(0) & 0 \\ 0 & -\mathcal{J}_{\nu-1}(0) \end{bmatrix}, \\ \hat{\mathcal{H}}_{z,4} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} -\mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix}^T \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} -\mathcal{J}_3(0) & 0 \\ 0 & \mathcal{J}_3(0) \end{bmatrix}^T \oplus\cdots\oplus \bigoplus_{i=1}^{m_{\nu-1}} \begin{bmatrix} -\mathcal{J}_\nu(0) & 0 \\ 0 & \mathcal{J}_\nu(0) \end{bmatrix}^T, \end{aligned}$$

where $m_1, \ldots, m_{\nu-1} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,4}$ has $2m_j$ Jordan blocks of size $j \times j$ and $\hat{\mathcal{H}}_{z,4}$ has $2m_j$ Jordan blocks of size $(j+1) \times (j+1)$ for $j = 1, \ldots, \nu-1$;

4) *blocks corresponding to two $(j+1) \times (j+1)$ Jordan blocks of $\mathcal{H}$ and two $j \times j$ Jordan blocks of $\hat{\mathcal{H}}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,6} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & 0 & 0 & I_1 \\ 0 & I_1 & 0 & 0 \end{bmatrix}_{2\times 4} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & 0 & 0 & I_2 \\ 0 & I_2 & 0 & 0 \end{bmatrix}_{4\times 6} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\nu-1}} \begin{bmatrix} 0 & 0 & 0 & I_{\nu-1} \\ 0 & I_{\nu-1} & 0 & 0 \end{bmatrix}_{(2\nu-2)\times 2\nu}, \\
G_{z,6} &= \bigoplus_{i=1}^{n_1} \Gamma_2 \oplus \bigoplus_{i=1}^{n_2} \Gamma_4 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\nu-1}} \Gamma_{2\nu-2}, \\
\widetilde{G}_{z,6} &= \bigoplus_{i=1}^{n_1} (-1)^2 \Gamma_4) \oplus \bigoplus_{i=1}^{n_2} (-1)^3 \Gamma_6 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\nu-1}} (-1)^\nu \Gamma_{2\nu}, \\
\mathcal{H}_{z,6} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} \mathcal{J}_2(0) & 0 \\ 0 & -\mathcal{J}_2(0) \end{bmatrix} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} \mathcal{J}_3(0) & 0 \\ 0 & -\mathcal{J}_3(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\nu-1}} \begin{bmatrix} \mathcal{J}_\nu(0) & 0 \\ 0 & -\mathcal{J}_\nu(0) \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,6} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} -\mathcal{J}_1(0) & 0 \\ 0 & \mathcal{J}_1(0) \end{bmatrix}^T \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} -\mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\nu-1}} \begin{bmatrix} -\mathcal{J}_{\nu-1}(0) & 0 \\ 0 & \mathcal{J}_{\nu-1}(0) \end{bmatrix}^T,
\end{aligned}
$$

where $n_1, \ldots, n_{\nu-1} \in \mathbb{N} \cup \{0\}$; thus, $\mathcal{H}_{z,6}$ has $2n_j$ Jordan blocks of size $(j+1) \times (j+1)$ and $\hat{\mathcal{H}}_{z,6}$ has $2n_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, \nu-1$;

Then the matrices $\mathcal{H}$ and $\hat{\mathcal{H}}$ have $2\ell_j + 2m_j + 2n_{j-1}$ respectively $2\ell_j + 2m_{j-1} + 2n_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, \nu$. Here ν is the maximum of the nilpotency indices of $\mathcal{H}$ and $\hat{\mathcal{H}}$.

Moreover, the form (6.3) *is unique up to simultaneous block permutation of the blocks in the diagonal blocks of the right hand side of* (6.3)*.*

Proof. Without loss of generality we may assume that $G_1 = J_m$ and $G_2 = J_n$. The canonical form is then obtained by repeatedly applying Proposition 3.7 to the factors of the full rank decomposition $A = B_1 C_1$, $B_1 \in \mathbb{C}^{2m\times r}$, $C_1 \in \mathbb{C}^{r\times 2n}$, where $r = \operatorname{rank} A$. Again, we omit the detailed proof here and refer the reader to [13]. Uniqueness of the form is proved as in the proof of Theorem 4.2. □

7 Conclusion

We have presented canonical forms for matrix triples (A, G_1, G_2) where G_1, G_2 are complex symmetric or complex skew-symmetric and nonsingular. The canonical form for A can be interpreted as a variant of the singular value decomposition, because the form also displays the Jordan canonical forms of the structured matrices $\mathcal{H} = G_2^{-1} A^T G_1^{-1} A$ and $\hat{\mathcal{H}} = G_1^{-1} A G_2^{-1} A^T$.

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