STRUCTURED PSEUDOSPECTRA AND THE CONDITION OF A NONDEROGATORY EIGENVALUE

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Abstract. Let λ be a nonderogatory eigenvalue of $A \in \mathbb{C}^{n\times n}$. The sensitivity of λ with respect to matrix perturbations $A \rightsquigarrow A + \Delta, \Delta \in \mathbf{\Delta}$, is measured by the structured condition number $\kappa_{\mathbf{\Delta}}(A, \lambda)$. Here $\mathbf{\Delta}$ denotes the set of admissible perturbations. However, if $\mathbf{\Delta}$ is not a vector space over $\mathbb{C}$ then $\kappa_{\mathbf{\Delta}}(A, \lambda)$ provides only incomplete information about the mobility of λ under small perturbations from $\mathbf{\Delta}$. The full information is then given by a certain set $K_{\mathbf{\Delta}}(x, y) \subset \mathbb{C}$ which depends on $\mathbf{\Delta}$ and a pair of normalized right and left eigenvectors x, y. In this paper we study the sets $K_{\mathbf{\Delta}}(x, y)$ and obtain methods for computing them. In particular we show that $K_{\mathbf{\Delta}}(x, y)$ is an ellipse in some important cases.

Key words. eigenvalues, structured perturbations, pseudospectra, condition numbers

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Notation. The symbols $\mathbb{R}$, $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{K}^{m\times n}$ is the set of $m \times n$ matrices and $\mathbb{K}^n = \mathbb{K}^{n\times 1}$ is the set of column vectors of length n, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By $A^\top$, $\bar{A}$, A^*, $\Re A$, $\Im A$ we denote the transpose, the conjugate, the conjugate transpose, the real and the imaginary part of $A \in \mathbb{C}^{m\times n}$. Furthermore, I_n stands for the $n \times n$ unit matrix. Finally, $\underline{n} = \{1, \ldots, n\}$ for any positive integer n.

1. Introduction.

The subject of this paper are the sets

$$K_{\mathbf{\Delta}}(x, y) = \{\, y^*\Delta x;\ \ \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1 \,\}, \qquad x, y \in \mathbb{C}^{n\times n}, \tag{1.1}$$

where $\|\cdot\|$ is a norm on $\mathbb{C}^{n\times n}$ and $\mathbf{\Delta} \subseteq \mathbb{C}^{n\times n}$ is assumed to be a closed cone (the latter means, that $\Delta \in \mathbf{\Delta}$ implies $r\Delta \in \mathbf{\Delta}$ for all $r \ge 0$). Our motivation for considering these sets stems from eigenvalue perturbation analysis by means of pseudospectra. The sets $K_{\mathbf{\Delta}}(x, y)$ provide the full first order information about the sensitivity of a nonderogatory eigenvalue with respect to structured matrix perturbations. This is explained in some detail in the following discussion.

Let $\lambda \in \mathbb{C}$ be a nonderogatory eigenvalue of algebraic multiplicity m of $A \in \mathbb{C}^{n\times n}$. Let $x \in \mathbb{C}^n \setminus 0$ be a right eigenvector, i.e. $Ax = \lambda x$. Then there exists a unique left generalized eigenvector $\hat{y} \in \mathbb{C}^n \setminus 0$ satisfying

$$\hat{y}^*(A - \lambda I_n)^m = 0, \qquad \hat{y}^*(A - \lambda I_n)^{m-1} \neq 0, \qquad \hat{y}^*x = 1.$$

Let $y^* = \hat{y}^*(A - \lambda I_n)^{m-1}$ and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^{n\times n}$. Under a small perturbation of A of the form

$$A \rightsquigarrow A(\Delta) = A + \Delta, \qquad \Delta \in \mathbb{C}^{n\times n} \tag{1.2}$$

the eigenvalue λ splits into m eigenvalues $\lambda_1(\Delta), \ldots, \lambda_m(\Delta)$ of $A(\Delta)$ with the first order expansion [16]

$$\lambda_j(\Delta) = \lambda + \theta_j(\Delta) + \mathcal{O}(\|\Delta\|^{2/m}), \qquad j \in \underline{m}. \tag{1.3}$$

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where $\theta_1(\Delta), \ldots, \theta_m(\Delta)$ are the m^{th} roots of $y^*\Delta x \in \mathbb{C}$. Obviously,

$$|\theta_j(\Delta)| = |y^*\Delta x|^{1/m} = \mathcal{O}(\|\Delta\|^{1/m}), \qquad j \in \underline{m}.$$

We assume now that the perturbations Δ are elements of a nonempty closed cone $\mathbf{\Delta} \subseteq \mathbb{C}^{n\times n}$. Let

$$\kappa_{\mathbf{\Delta}}(A,\lambda) = \max\{\, |y^*\Delta x|^{1/m};\ \ \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1\,\}.$$

Then $\kappa_{\mathbf{\Delta}}(A,\lambda)$ is the smallest number κ such that

$$|\lambda_j(\Delta) - \lambda| \le \kappa\,\|\Delta\|^{1/m} + \mathcal{O}(\|\Delta\|^{2/m}) \qquad \text{for } \ \Delta \in \mathbf{\Delta}.$$

The quantity $\kappa_{\mathbf{\Delta}}(A,\lambda)$ is called the structured condition number of λ with respect to $\mathbf{\Delta}$ and the norm $\|\cdot\|$. It measures the sensitivity of the eigenvalue λ if the matrix A is subjected to perturbations from the class $\mathbf{\Delta}$. In recent years some work has been done in order to obtain estimates or computable formulas for $\kappa_{\mathbf{\Delta}}(A,\lambda)$ [3, 4, 5, 7, 13, 15, 16, 18, 19]. However, the condition number can not reveal how the eigenvalue moves into a specific direction under structured perturbations. For instance if λ is a simple real eigenvalue of a real matrix A and the perturbations Δ are also assumed to be real then the perturbed eigenvalues $\lambda(\Delta)$ remains on the real axis if $\|\Delta\|$ is small enough. Information of this kind can be obtained from the structured pseudospectrum $\sigma_{\mathbf{\Delta}}(A,\epsilon)$, which is defined as follows.

$$\sigma_{\mathbf{\Delta}}(A,\epsilon) = \{\, z \in \mathbb{C};\ \ z \text{ is an eigenvalue of } A+\Delta \text{ for some } \Delta \in \mathbf{\Delta},\ \ \|\Delta\| \le \epsilon\,\},\ \ \epsilon > 0.$$

Let $\mathcal{C}_{\mathbf{\Delta}}(A,\lambda,\epsilon)$ denote the connected component of $\sigma_{\mathbf{\Delta}}(A,\epsilon)$ that contains the eigenvalue λ. Then we have for sufficiently small ϵ that

$$\mathcal{C}_{\mathbf{\Delta}}(A,\lambda,\epsilon) = \{\lambda_j(\Delta);\ \ \Delta \in \mathbf{\Delta}, \|\Delta\| \le \epsilon,\ j \in \underline{m}\}.$$

We now consider the sets

$$K_{\mathbf{\Delta}}^{(m)}(x,y) = \{\, z \in \mathbb{C};\ \ z^m \in K_{\mathbf{\Delta}}(x,y)\,\}. \tag{1.4}$$

In words, $K_{\mathbf{\Delta}}^{(m)}(x,y)$ is the set of all m^{th} roots of the numbers $y^*\Delta x$, where $\Delta \in \mathbf{\Delta}$, $\|\Delta\| \le 1$. Consequently, the condition number $\kappa_{\mathbf{\Delta}}(A,\lambda)$ equals the radius of the smallest disk about 0 that contains $K_{\mathbf{\Delta}}^{(m)}(x,y)$. Moreover, (1.3) yields that

$$\lim_{\epsilon\to 0} \frac{\mathcal{C}_{\mathbf{\Delta}}(A,\lambda,\epsilon) - \lambda}{\epsilon^{1/m}} = K_{\mathbf{\Delta}}^{(m)}(x,y), \tag{1.5}$$

where the limit is taken with respect to the Hausdorff-metric. More explicitely, (1.5) states that to each $\delta > 0$ there exists an $\epsilon_0 > 0$ such that for all positive $\epsilon < \epsilon_0$,

(1) $\mathcal{C}_{\mathbf{\Delta}}(A,\lambda,\epsilon) \subset \lambda + \epsilon^{1/m}\,\mathcal{U}_\delta(K_{\mathbf{\Delta}}^{(m)}(x,y))$,

(2) $\lambda + \epsilon^{1/m} K_{\mathbf{\Delta}}^{(m)}(x,y) \subset \mathcal{U}_\delta(\mathcal{C}_{\mathbf{\Delta}}(A,\lambda,\epsilon))$,

where $\mathcal{U}_\delta(\mathcal{M}) = \{\, z \in \mathbb{C};\ \ |z-s| < \delta \text{ for some } s \in \mathcal{M}\,\}$ is a δ-neighborhood of $\mathcal{M} \subset \mathbb{C}$.

EXAMPLE 1.1. *The relation (1.5) is illustrated in Figure 1.1. The underlying norm in the following explanation is the spectral norm.*

The upper row of the figure deals with the case $m = 1$. The first two pictures in that row show the sets $\mathcal{C}_{\mathbb{R}^{3\times 3}}(A, \lambda, \epsilon)$ for the matrix

$$A = \begin{bmatrix} 2 & -5 & -5 \\ 3 & -4 & -4 \\ -2 & 2 & 2 \end{bmatrix}$$

and its simple eigenvalue $\lambda = i$. A corresponding pair of right and left eigenvectors satisfying $y^*x = 1$ is given by

$$x = [2 - i \quad 3 + 2i \quad -2 - 2i]^\top, \qquad y = (1/2)[1 \quad 2 - i \quad 2 - i]^\top.$$

The right picture in the upper row shows the set $K^{(1)}_{\mathbb{R}^{n\times n}}(x, y) = K_{\mathbb{R}^{n\times n}}(x, y)$. By (1.5) we have

$$\lim_{\epsilon\to 0} \frac{\mathcal{C}_{\mathbb{R}^{3\times 3}}(A, i, \epsilon) - i}{\epsilon} = K_{\mathbb{R}^{3\times 3}}(x, y).$$

The pictures indicate the convergence. The scalings have been chosen such that the displayed sets have approximately the same size. The plots of the pseudospectra components $\mathcal{C}_{\mathbb{R}^{3\times 3}}(A, i, \epsilon)$ have been generated using the formula

$$\sigma_{\mathbb{R}^{n\times n}}(A, \epsilon) = \{\, s \in \mathbb{C};\ \widetilde{\tau}_n(sI - A) \le \epsilon \,\}, \qquad A \in \mathbb{C}^{n\times n},\ \epsilon > 0.$$

Here $\widetilde{\tau}_n$ denotes the smallest real perturbation value of second kind [2], which is given by

$$\widetilde{\tau}_n(M) = \sup_{\gamma\in(0,1]} \sigma_{2n-1}\left(\begin{bmatrix} \Re M & -\gamma\,\Im M \\ \gamma^{-1}\,\Im M & \Re M \end{bmatrix}\right), \qquad M \in \mathbb{C}^{n\times n},$$

where σ_{2n-1} is the second smallest singular value. The set $K_{\mathbb{R}^{3\times 3}}(x, y)$ has been computed using Theorem 6.2.

The left pictures in the lower row of the figure show the real pseudospectra $\sigma_{\mathbb{R}^{3\times 3}}(J_3, \epsilon) = \mathcal{C}_{\mathbb{R}^{3\times 3}}(J_3, 0, \epsilon)$ for the 3 by 3 Jordan block

$$J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The right picture shows the limit set $K^{(3)}_{\mathbb{R}^{n\times n}}(e_1, e_3)$, where $e_1 = [1\ 0\ 0]^\top$, $e_3 = [0\ 0\ 1]^\top$. Note that e_1 is a right eigenvector and e_1^* is a left generalized eigenvector of J_3 satisfying $e_1^*e_1 = 1$, $e_1^*J_3^2 = e_3^*$. Hence, (1.5) yields,

$$\lim_{\epsilon\to 0} \frac{\mathcal{C}_{\mathbb{R}^{3\times 3}}(J_3, 0, \epsilon)}{\epsilon^{1/3}} = K^{(3)}_{\mathbb{R}^{3\times 3}}(e_1, e_3).$$

It is easily verified that the set $K_{\mathbb{R}^{3\times 3}}(e_1, e_3)$ equals the interval $[-1, 1]$. Thus,

$$K^{(3)}_{\mathbb{R}^{3\times 3}}(e_3, e_1) = [-1, 1] \cup e^{\pi i/3}[-1, 1] \cup e^{2\pi i/3}[-1, 1].$$

The aim of this paper is to provide methods for calculating the sets $K_{\mathbf{\Delta}}(x, y)$. In

doing so we concentrate on the following perturbation classes $\mathbf{\Delta}$:

$$\begin{aligned} &\mathbb{K}^{n\times n}, \\ &\mathrm{Sym}_{\mathbb{K}} \;=\; \{\,\Delta \in \mathbb{K}^{n\times n};\;\; \Delta^T = \Delta\,\}, \\ &\mathrm{Skew}_{\mathbb{K}} \;=\; \{\,\Delta \in \mathbb{K}^{n\times n};\;\; \Delta^T = -\Delta\,\}, \\ &\mathrm{Herm} \;=\; \{\,\Delta \in \mathbb{C}^{n\times n};\;\; \Delta^* = \Delta\,\}, \qquad\quad \mathbb{K} \in \{\mathbb{R},\mathbb{C}\}. \end{aligned} \tag{1.6}$$

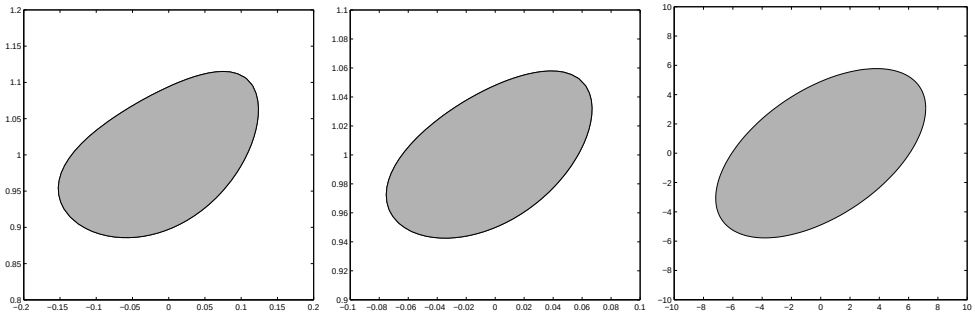
Our further considerations are based on two observations concerning $K_{\mathbf{\Delta}}(x,y)$, $\mathbf{\Delta} \subseteq \mathbb{C}^{n\times n}$:

(A) If $\Delta \in \mathbf{\Delta}$ implies that $z\Delta \in \mathbf{\Delta}$ for all $z \in \mathbb{C}$, then then $K_{\mathbf{\Delta}}(x,y)$ is a disk. The m^{th} root of the radius of that disk equals the condition number $\kappa_{\mathbf{\Delta}}(A,\lambda)$.

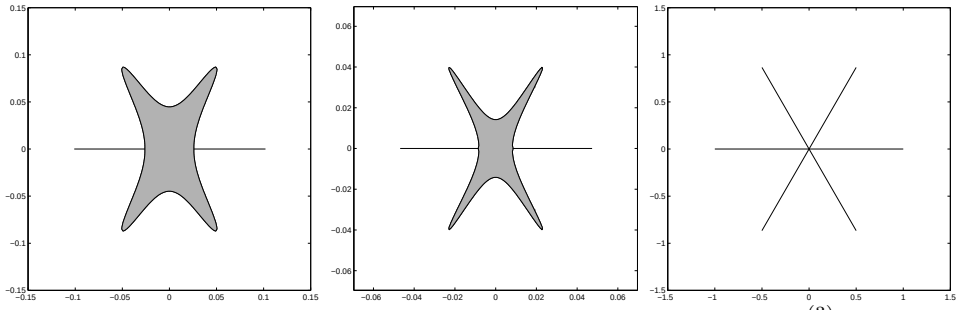
(B) If $\mathbf{\Delta}$ is convex then $K_{\mathbf{\Delta}}(x,y)$ is convex, too.

Statement (A) yields that $K_{\mathbf{\Delta}}(x,y)$ is a disk for $\mathbf{\Delta} \in \{\mathbb{C}^{n\times n}, \mathrm{Sym}_{\mathbb{C}}, \mathrm{Skew}_{\mathbb{C}}\}$. Observation (B) enables us to approximate $K_{\mathbf{\Delta}}(x,y)$ using its support function.

The organization of this paper is as follows. In Section 1 we recall some basic facts about convex sets and support functions and specialize them to the sets $K_{\mathbf{\Delta}}(x,y)$. In Section 2 we characterize the support function of $K_{\mathbf{\Delta}}(x,y)$ for the sets $\mathbf{\Delta}$ in (1.6) via dual norms and orthogonal projectors. The results are then applied to the cases that the underlying norm is of Hölder type (see Section 3) or unitarily invariant (Section 4). Section 5 deals with the spectral norm and Frobenius norm. The results obtained so far will be extended in Section 6 to classes of matrices which are self- or skew-adjoint with respect to an inner product.



the set $\mathcal{C}_{\mathbb{R}^{3\times3}}(A,i,\epsilon)$, $\epsilon = 0.02$ | the set $\mathcal{C}_{\mathbb{R}^{3\times3}}(A,i,\epsilon)$, $\epsilon = 0.01$ | the set $K_{\mathbb{R}^{3\times3}}(x,y)$



the set $\mathcal{C}_{\mathbb{R}^{3\times3}}(J_3,0,\epsilon)$, $\epsilon = 10^{-3}$ | the set $\mathcal{C}_{\mathbb{R}^{3\times3}}(J_3,0,\epsilon)$, $\epsilon = 10^{-4}$ | the set $K^{(3)}_{\mathbb{R}^{3\times3}}(e_1,e_3)$.

FIG. 1.1. *The sets defined in Example 1.1*

2. Characterization by support functions. Let K be a nonempty compact convex subset of $\mathbb{C}$. Then its support function $s_K : \mathbb{C} \to \mathbb{R}$ is defined by

$$s_K(z) = \max_{\xi \in K} \Re(\bar{z}\,\xi) = \max_{\xi \in K} z^T \xi, \tag{2.1}$$

where in the second equation the complex numbers $z = z_1 + iz_2, \xi = \xi_1 + i\xi_2$ have been identified with the corresponding vectors $[z_1,\, z_2]^\top$, $[\xi_1,\, \xi_2]^\top \in \mathbb{R}^2$. The set K is uniquely determined by its support function since we have [9, Corollary 3.1.2]

$$K = \{\xi \in \mathbb{C};\ \ \Re(\bar{z}\,\xi) \le s_K(z) \ \text{ for all } z \in \mathbb{C} \text{ with } |z| = 1\}. \tag{2.2}$$

Furthermore, the boundary of K is given as

$$\partial K = \{\xi \in \mathbb{C};\ \ \Re(\bar{z}\,\xi) = s_K(z) \ \text{ for some } z \in \mathbb{C} \text{ with } |z| = 1\}. \tag{2.3}$$

This follows from (2.2) and the compactness of the unit circle. Let $r_K = \max\{\,|\xi|;\ \ \xi \in K\,\}$. Then r_K is the radius of the smallest disk about 0 that contains K. It is easily seen that $r_K = \max\{s_K(z);\ z \in \mathbb{C},\ |z| = 1\,\}$. If $s_K(z) = r\,|z|$, for some $r \ge 0$, then K is a disk about 0 with radius $r = r_K$. We will also need the following fact.

PROPOSITION 2.1. *Assume the nonempty compact convex set $K \subset \mathbb{C}$ is point symmetric with respect to 0, i.e. $\xi \in K$ implies $-\xi \in K$. Assume further, that $s_K(z) = 0$ for some $z \in \mathbb{C}$ with $|z| = 1$. Then K is a line segment. Specifically,*

$$K = \{\,\theta\, iz;\ \ \theta \in \mathbb{R},\ \ |\theta| \le s_K(iz)\,\}.$$

Proof. From the point symmetry it follows that $s_K(z) = s_K(-z)$. Hence, if $s_K(z) = 0$ then $\Re(\bar{z}\,\xi) = 0$ for all $\xi \in K$. Thus $K \subset \mathbb{R}(iz)$. By compactness and convexity, $K = \{\,\theta\, iz;\ \ \theta \in \mathbb{R},\ |\theta| \le r\,\}$ for some $r \ge 0$. It is easily verified that $r = s_K(iz)$ if $|z| = 1$. □

The relations (2.2) and (2.3) can be used to approximate K via the following method [11, Section 1.5]. Let $z_j = e^{i\phi_j}, j \in \underline{N}$, where $0 = \phi_1 < \phi_2 < \ldots < \phi_N < 2\pi$. Let $\xi_j \in K, j \in \underline{N}$, be such that $\Re(\bar{z}_j\,\xi_j) = s(z_j)$. Then by (2.3) each ξ_j is a boundary point of K. Let K_1 denote the convex hull of these points, and let $K_2 = \{\xi \in \mathbb{C};\ \ \Re(\bar{z}_j\xi) \le s(z_j),\ j \in \underline{N}\,\}$. Then we have $K_1 \subseteq K \subseteq K_2$, where the latter inclusion follows from (2.2). The boundary of K_1 is a polygone with vertices $\xi_1, \xi_2, \ldots, \xi_N$.

The proposition below yields the basis for our further development.

PROPOSITION 2.2. *Let $\mathbf{\Delta}$ be a nonempty compact and convex subset of $\mathbb{C}^{n\times n}$. Then the following holds.*

(i) The set $K_{\mathbf{\Delta}}(x, y)$ defined in (1.1) is a compact convex subset of $\mathbb{C}$ with support function

$$s_{\mathbf{\Delta}}(z) = \max_{\substack{\Delta \in \mathbf{\Delta} \\ \|\Delta\| \le 1}} \Re(\bar{z}\, y^*\Delta x) = \max_{\substack{\Delta \in \mathbf{\Delta} \\ \|\Delta\| \le 1}} \Re \operatorname{tr}(\Delta^*(z\, yx^*)), \qquad z \in \mathbb{C}. \tag{2.4}$$

If $\mathbf{\Delta}$ is a cone then the maximum is attained for some $\Delta \in \mathbf{\Delta}$ with $\|\Delta\| = 1$.

(ii) Let $|z| = 1$ and let $\Delta_z \in \mathbf{\Delta}$ be a maximizer for (2.4). Then $y^*\Delta_z x$ is a boundary point of $K_{\mathbf{\Delta}}(x, y)$.

(iii) Suppose $\mathbf{\Delta}$ is a vector space over $\mathbb{R}$ and $s_{\mathbf{\Delta}}(z) = 0$ for some $z \in \mathbb{C}$ with $|z| = 1$. Then $K_{\mathbf{\Delta}}(x, y)$ is a line segment. Specifically,

$$K_{\mathbf{\Delta}}(x, y) = \{\,\theta\, iz;\ \ \theta \in \mathbb{R},\ \ |\theta| \le s_{\mathbf{\Delta}}(iz)\,\}.$$

Proof. The compactness and convexity of $K_{\mathbf{\Delta}}(x,y)$ is obvious. (2.4) is immediate from (2.1) and the relations

$$\bar{z}\, y^*\Delta x = \mathrm{tr}(\bar{z}\, y^*\Delta x) = \mathrm{tr}(\bar{z}\, xy^*\Delta) = \mathrm{tr}((z\, yx^*)^*\Delta) = \overline{\mathrm{tr}(\Delta^*(z\, yx^*))}.$$

(ii) follows from (2.3). (iii) is a consequence of Proposition 2.1. ∎

3. Dual norms and orthogonal projectors. The dual of a vector norm $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ is defined by

$$\|x\|' = \max_{\substack{y\,\in\,\mathbb{C}^n \\ \|y\|\,=\,1}} \Re(y^*x), \qquad x \in \mathbb{C}^n. \tag{3.1}$$

There is a natural extension of this definition to matrix norms.

DEFINITION 3.1. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^{m\times n}$. Then its dual is defined as*

$$\|X\|' := \max_{\substack{Y\,\in\,\mathbb{C}^{m\times n} \\ \|Y\|\,=\,1}} \Re\,\mathrm{tr}(Y^*X), \qquad X \in \mathbb{C}^{m\times n}. \tag{3.2}$$

This yields the following corollary to Proposition 2.2.

COROLLARY 3.2. *For any norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ the support function $s_{\mathbb{C}}$ of $K_{\mathbb{C}^{n\times n}}(x,y)$ is given by $s_{\mathbb{C}}(z) = |z|\,\|yx^*\|'$, $z \in \mathbb{C}$. Thus $K_{\mathbb{C}^{n\times n}}(x,y)$ is a disk of radius $\|yx^*\|'$.*

The map $(X,Y) \mapsto \Re\,\mathrm{tr}(Y^*X)$ is a positive definite symmetric $\mathbb{R}$-bilinear form on $\mathbb{C}^{n\times n}$. Thus for each subspace (over $\mathbb{R}$) $\mathbf{\Delta} \subseteq \mathbb{C}^{n\times n}$ we have the direct decomposition $\mathbb{C}^{n\times n} = \mathbf{\Delta} \oplus \mathbf{\Delta}^{\perp}$, where $\mathbf{\Delta}^{\perp} = \{\, X \in \mathbb{C}^{n\times n};\ \ \Re\,\mathrm{tr}(\Delta^*X) = 0 \text{ for all } \Delta \in \mathbf{\Delta}\,\}$. The orthogonal projector onto $\mathbf{\Delta}$ is the linear map $\mathcal{P}_{\mathbf{\Delta}} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ satisfying

$$\mathcal{P}_{\mathbf{\Delta}}(X_1 + X_2) = X_1 \quad \text{ for all } X_1 \in \mathbf{\Delta}, X_2 \in \mathbf{\Delta}^{\perp}.$$

We have for all $X, Y \in \mathbb{C}^{n\times n}$,

$$\Re\,\mathrm{tr}(\mathcal{P}_{\mathbf{\Delta}}(Y)^*X) = \Re\,\mathrm{tr}(\mathcal{P}_{\mathbf{\Delta}}(Y)^*\mathcal{P}_{\mathbf{\Delta}}(X)) = \Re\,\mathrm{tr}(Y^*\mathcal{P}_{\mathbf{\Delta}}(X)). \tag{3.3}$$

The table below gives the orthogonal projectors for the subspaces introduced in (1.6).

$\mathbf{\Delta}$	$\mathcal{P}_{\mathbf{\Delta}}(X)$
$\mathbb{C}^{n\times n}$	X
$\mathbb{R}^{n\times n}$	$\Re X$
Herm	$(X + X^*)/2$
$\mathrm{Sym}_{\mathbb{C}}$	$(X + X^{\top})/2$
$\mathrm{Skew}_{\mathbb{C}}$	$(X - X^{\top})/2$
$\mathrm{Sym}_{\mathbb{R}}$	$\Re(X + X^{\top})/2$
$\mathrm{Skew}_{\mathbb{R}}$	$\Re(X - X^{\top})/2$

(3.4)

The main results of this paper are based on the next lemma.

LEMMA 3.1. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^{n\times n}$ and let $\mathbf{\Delta} \subseteq \mathbb{C}^{n\times n}$ be a vector space over $\mathbb{R}$. Suppose the orthogonal projector onto $\mathbf{\Delta}$ is a contraction, i.e.*

$$\|\mathcal{P}_{\mathbf{\Delta}}(X)\| \le \|X\| \text{ for all } X \in \mathbb{C}^{n\times n}. \tag{3.5}$$

Then for all $M \in \mathbb{C}^{n\times n}$,

$$\max_{\substack{\Delta \in \mathbf{\Delta} \\ \|\Delta\| = 1}} \Re \operatorname{tr}(\Delta^* M) \;=\; \|\mathcal{P}_{\mathbf{\Delta}}(M)\|'. \tag{3.6}$$

Let $\Delta_0 \in \mathbb{C}^{n\times n}$ *be such that* $\|\Delta_0\| = 1$ *and* $\Re \operatorname{tr}(\Delta_0^* \mathcal{P}_{\mathbf{\Delta}}(M)) = \|\mathcal{P}_{\mathbf{\Delta}}(M)\|'$.

If $\mathcal{P}_{\mathbf{\Delta}}(M) \neq 0$ *then the matrix* $\Delta_1 = \mathcal{P}_{\mathbf{\Delta}}(\Delta_0)$ *is a maximizer for the left hand side of (3.6).*

Proof. Let L denote the left hand side of (3.6). For $\Delta \in \mathbf{\Delta}$ we have $\Re \operatorname{tr}(\Delta^* M) = \Re \operatorname{tr}(\Delta^* \mathcal{P}_{\mathbf{\Delta}}(M))$. This yields $L \leq \|\mathcal{P}_{\mathbf{\Delta}}(M)\|'$. We show the opposite inequality. For the matrix Δ_0 we have $\|\mathcal{P}_{\mathbf{\Delta}}(M)\|' = \Re \operatorname{tr}(\Delta_0^* \mathcal{P}_{\mathbf{\Delta}}(M)) = \Re \operatorname{tr}(\mathcal{P}_{\mathbf{\Delta}}(\Delta_0)^* \mathcal{P}_{\mathbf{\Delta}}(M))$. If $\mathcal{P}_{\mathbf{\Delta}}(\Delta_0) = 0$ then $\|\mathcal{P}_{\mathbf{\Delta}}(M)\|' = 0 = L$. Suppose $\mathcal{P}_{\mathbf{\Delta}}(\Delta_0) \neq 0$. By condition (3.5) we have $\|\mathcal{P}_{\mathbf{\Delta}}(\Delta_0)\| \leq \|\Delta_0\| = 1$. The matrix $\Delta_1 = \mathcal{P}_{\mathbf{\Delta}}(\Delta_0)/\|\mathcal{P}_{\mathbf{\Delta}}(\Delta_0)\|$ satisfies $\|\Delta_1\| = 1$ and $\Re \operatorname{tr}(\Delta_1^* \mathcal{P}_{\mathbf{\Delta}}(M)) = \|\mathcal{P}_{\mathbf{\Delta}}(M)\|'/\|\mathcal{P}_{\mathbf{\Delta}}(\Delta_0)\| \geq \|\mathcal{P}_{\mathbf{\Delta}}(M)\|'$. Thus $L \geq \|\mathcal{P}_{\mathbf{\Delta}}(M)\|'$. Consequently, $L = \|\mathcal{P}_{\mathbf{\Delta}}(M)\|'$ and $\|\mathcal{P}_{\mathbf{\Delta}}(\Delta_0)\| = 1$. □

From Proposition 2.4 and Lemma 3.1 (applied to the matrix $M = z\,yx^*$) we obtain

THEOREM 3.3. *Let* $\mathbf{\Delta} \subseteq \mathbb{C}^{n\times n}$ *be a vector space over* $\mathbb{R}$, *and let* $s_{\mathbf{\Delta}} : \mathbb{C} \to \mathbb{R}$ *denote the support function of* $K_{\mathbf{\Delta}}(x, y)$. *Suppose (3.5) holds for the underlying norm. Then*

(i) The support function satisfies

$$s_{\mathbf{\Delta}}(z) = \|\mathcal{P}_{\mathbf{\Delta}}(z\,yx^*)\|', \qquad z \in \mathbb{C}. \tag{3.7}$$

(ii) Let $|z| = 1$ *and let* $\Delta_0 \in \mathbb{C}^{n\times n}$ *be such that* $\|\Delta_0\| = 1$ *and* $\Re \operatorname{tr}(\Delta_0^* \mathcal{P}_{\mathbf{\Delta}}(z\,yx^*)) = s_{\mathbf{\Delta}}(z)$. *Then* $y^* \mathcal{P}_{\mathbf{\Delta}}(\Delta_0) x \in \mathbb{C}$ *is a boundary point of* $K_{\mathbf{\Delta}}(x, y)$. *If* $x^* \mathcal{P}_{\mathbf{\Delta}}(\Delta_0) y = 0$ *then* $K_{\mathbf{\Delta}}(x, y)$ *is a line segment.*

(iii) If $\mathbf{\Delta}$ *is a vector space over* $\mathbb{C}$, *then*

$$s_{\mathbf{\Delta}}(z) = \|\mathcal{P}_{\mathbf{\Delta}}(yx^*)\|'\,|z|, \qquad z \in \mathbb{C}. \tag{3.8}$$

Thus $K_{\mathbf{\Delta}}(x, y)$ *is a disk about 0 with radius* $\|\mathcal{P}_{\mathbf{\Delta}}(yx^*)\|'$.

Next, we consider norms that have one of the following properties (a)-(c) for all $X \in \mathbb{C}^{n\times n}$.

$$\text{(a) } \|X\| = \|\bar{X}\|, \qquad \text{(b) } \|X\| = \|X^*\|, \qquad \text{(c) } \|X\| = \|X^\top\|.$$

Note that two of these conditions imply the third.

LEMMA 3.2. *Condition (3.5) holds for the following cases:*

(i) The norm satisfies (a) and $\mathbf{\Delta} = \mathbb{R}^{n\times n}$.

(ii) The norm satisfies (b) and $\mathbf{\Delta} = \mathrm{Herm}$.

(iii) The norm satisfies (c) and $\mathbf{\Delta} \in \{\mathrm{Sym}_{\mathbb{C}}, \mathrm{Skew}_{\mathbb{C}}\}$.

(iv) The norm satisfies (a), (b) and (c) and $\mathbf{\Delta} \in \{\mathrm{Sym}_{\mathbb{R}}, \mathrm{Skew}_{\mathbb{R}}\}$.

Proof. (a) yields $\|\Re X\| = \|(X + \bar{X})/2\| \leq (\|X\| + \|\bar{X}\|)/2 = \|X\|$. (b) implies that $\|\mathcal{P}_{\mathrm{Herm}}(X)\| = \|(X + X^*)/2\| \leq (\|X\| + \|X^*\|)/2 = \|X\|$. The proofs of the other statements are analogous and left to the reader. □

THEOREM 3.4. *The following assertions hold for the support function* $s_{\mathbf{\Delta}} : \mathbb{C} \to \mathbb{R}$ *of* $K_{\mathbf{\Delta}}(x, y)$.

(i) If the norm $\|\cdot\|$ *satisfies condition (a) then* $s_{\mathbb{R}^{m\times n}}(z) = \|\Re(z\,yx^*)\|'$.

(ii) If the norm $\|\cdot\|$ *satisfies condition (b) then* $s_{\mathrm{Herm}}(z) = \|\mathcal{P}_{\mathrm{Herm}}(z\,yx^*)\|'$.

(iii) If the norm $\|\cdot\|$ satisfies condition (c) then

$$s_{\mathrm{Sym}_{\mathbb{C}}}(z) = \|\mathcal{P}_{\mathrm{Sym}_{\mathbb{C}}}(z\,yx^*)\|' = \|\mathcal{P}_{\mathrm{Sym}_{\mathbb{C}}}(yx^*)\|'\,|z|$$

and

$$s_{\mathrm{Skew}_{\mathbb{C}}}(z) = \|\mathcal{P}_{\mathrm{Skew}_{\mathbb{C}}}(z\,yx^*)\|' = \|\mathcal{P}_{\mathrm{Skew}_{\mathbb{C}}}(yx^*)\|'\,|z|$$

(iv) If the norm $\|\cdot\|$ satisfies (a), (b) and (c) then

$$s_{\mathrm{Sym}_{\mathbb{R}}}(z) = \|\mathcal{P}_{\mathrm{Sym}_{\mathbb{R}}}(z\,yx^*)\|' \quad \textit{and } s_{\mathrm{Skew}_{\mathbb{R}}}(z) = \|\mathcal{P}_{\mathrm{Skew}_{\mathbb{R}}}(z\,yx^*)\|'$$

In the next sections we specialize Theorem 3.4 to classes of norms for which the duals can be explicitely given.

4. Norms of Hölder type. The Hölder-p-norm of $x = [x_1,\ \ldots,\ x_n]^\top \in \mathbb{C}^n$ is defined by

$$\|x\|_p = \begin{cases} \left(\sum_{j\in\underline{n}} |x_j|^p\right)^{1/p} & \text{for } 1 \le p < \infty, \\ \max_{j\in\underline{n}} |x_j| & \text{for } p = \infty. \end{cases} \tag{4.1}$$

We consider the following matrix norms of Hölder type [8, page 717] defined by

$$\|X\|_{r|p} = \big|\big|\ [\ \|x_1^\top\|_r, \ldots, \|x_n^\top\|_r\ ]^\top\ \big|\big|_p\,, \qquad 1 \le p, r \le \infty, \tag{4.2}$$

where $x_1, \ldots, x_n$ denote the rows of $X \in \mathbb{C}^{n\times n}$. Note that $\|X\|_{1|\infty}$ is the row sum norm and

$$\|X\|_{p|p} = \begin{cases} \left(\sum_{j,k\in\underline{n}} |x_{jk}|^p\right)^{1/p} & \text{for } 1 \le p < \infty, \\ \max_{j,k\in\underline{m}} |x_{jk}| & \text{for } p = \infty, \end{cases} \tag{4.3}$$

where x_{jk} are the entries of X. In particular, $\|\cdot\|_{2|2}$ is the Frobenius norm.

As is well known the dual of the Hölder-p-norm is the Hölder-q-norm, where $\frac{1}{p} + \frac{1}{q} = 1$ if $1 \le p < \infty$ and $q = 1$ if $p = \infty$. Using this fact the next Proposition is easily verified.

PROPOSITION 4.1. *The dual of the norm $\|\cdot\|_{r|p}$ is $\|\cdot\|_{t|q}$, where*

$$\begin{aligned} &\tfrac{1}{r} + \tfrac{1}{t} = 1 \textit{ if } 1 \le r < \infty \textit{ and } t = 1 \textit{ if } r = \infty, \\ &\tfrac{1}{p} + \tfrac{1}{q} = 1 \textit{ if } 1 \le p < \infty \textit{ and } q = 1 \textit{ if } p = \infty. \end{aligned} \tag{4.4}$$

To a given $X \in \mathbb{C}^{n\times n}$ with rows $x_1, \ldots, x_n$ a matrix $Y_0 \in \mathbb{C}^{n\times n}$ satisfying $\|Y_0\|_{t|q} = 1$ and $\Re\,\mathrm{tr}(Y_0^* X) = \|X\|_{r|p}$ can be constructed via the following procedure. Let $\xi = [\ \|x_1^\top\|_r, \ldots, \|x_n^\top\|_r\ ]^\top$. Choose a nonnegative vector $\eta = [\eta_1, \ldots, \eta_n]^\top$ such that $\|\eta\|_q = 1$ and $\eta^\top\xi = \|\xi\|_p$. To each $j \in \underline{n}$ choose a $y_j \in \mathbb{C}^n$ with $\|y_j\|_t = \eta_j$ and $y_j^* x_j^\top = \eta_j \|x_j^\top\|_r$. Then $Y_0 = [\,y_1^\top, \ldots, y_n^\top\,]^\top$ has the required properties.

From Proposition 4.1 combined with Lemma 3.2 and Theorem 3.4 we get

COROLLARY 4.2. *Let $1 \le r, p \le \infty$, and let t, q be given by (4.4). Let $K_{\mathbf{\Delta}}(x, y) = \{\, y^*\Delta x;\ \ \Delta \in \mathbf{\Delta},\ \|\Delta\|_{r|p} \le 1\}$. Then*

(i) the set $K_{\mathbb{C}^{n\times n}}(x, y)$ is a disk of radius $\|yx^*\|_{t|q}$;

(ii) the support function of $K_{\mathbb{R}^{n\times n}}(x,y)$ *is*

$$s_{\mathbb{R}}(z) = \|\Re(z\,yx^*)\|_{t|q}, \qquad z \in \mathbb{C};$$

(iii) for the case $p = r$ *and* $\mathbf{\Delta} \in \{\mathrm{Herm}, \mathrm{Sym}_{\mathbb{C}}, \mathrm{Skew}_{\mathbb{C}}, \mathrm{Sym}_{\mathbb{R}}, \mathrm{Skew}_{\mathbb{R}}\}$ *the support function of* $K_{\mathbf{\Delta}}(x,y)$ *is* $s_{\mathbf{\Delta}}(z) = \|\mathcal{P}_{\mathbf{\Delta}}(z\,yx^*)\|_{q|q}$.

EXAMPLE 4.3. *Figure 4 shows the sets*

$$\begin{aligned} K_{\mathbb{R}^{n\times n}}(x,y) &= \{\, y^*\Delta x;\ \ \Delta \in \mathbb{R}^{n\times n},\ \|\Delta\|_{1|\infty} \le 1 \,\}, \\ K^{(3)}_{\mathbb{R}^{n\times n}}(x,y) &= \{\, z \in \mathbb{C};\ \ z^3 \in K_{\mathbb{R}^{n\times n}}(x,y) \,\}, \end{aligned} \tag{4.5}$$

where

$$x = [\,1+i,\ 5+4i,\ 3i,\ -1+3i\,]^\top, \qquad y = [\,3+4i,\ 3+3i,\ 2+2i,\ 5\,].$$

The plot of $K_{\mathbb{R}^{n\times n}}(x,y)$ *has been generated by computing boundary points using claim (ii) of Theorem 3.3 and Proposition 4.1.*

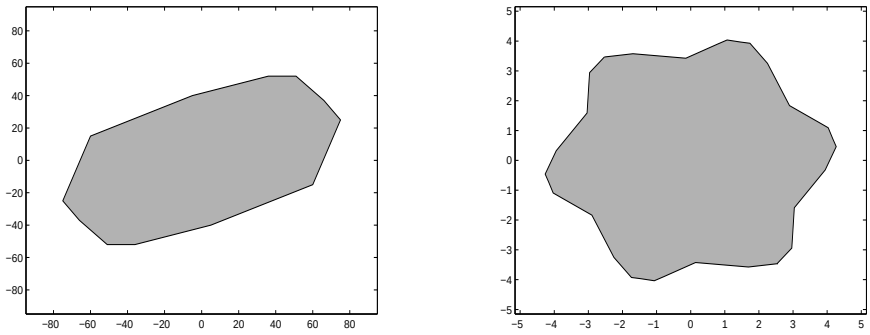


FIG. 4.1. *The sets* $K_{\mathbb{R}^{n\times n}}(x,y)$ *(left) and* $K^{(3)}_{\mathbb{R}^{n\times n}}(x,y)$ *(right) from Example 4.3.*

5. Unitarily invariant norms. In the sequel $\mathcal{U}_n$ denotes the set of all unitary $n\times n$ matrices. A norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ is said to be unitarily invariant if $\|UXV\| = \|X\|$ for all $X \in \mathbb{C}^{n\times n}$, $U, V \in \mathcal{U}_n$. There is a one to one correspondence between the unitarily invariant norms on $\mathbb{C}^{n\times n}$ and symmetric gauge functions [20, Section II.3]. A symmetric gauge function Φ is a symmetric and absolute norm on $\mathbb{R}^n$. The unitarily invariant norm $\|\cdot\|_\Phi$ associated with Φ is given by

$$\|X\|_\Phi := \Phi([\sigma_1(X),\ \sigma_2(X), \ldots,\ \sigma_n(X)]^\top), \tag{5.1}$$

where $\sigma_1(X) \ge \sigma_2(X) \ge \ldots \ge \sigma_n(X)$ denote the singular values of $X \in \mathbb{C}^{n\times n}$. The unitarily invariant norm induced by the Hölder-p-norm is called the Schatten-p-norm, which we denote by

$$\|X\|_{(p)} := \begin{cases} \left(\sum_{k\in\underline{n}} \sigma_k(X)^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \sigma_1(X) & \text{if } p = \infty. \end{cases}$$

Note that $\|X\|_{(\infty)}$ is the spectral norm and $\|X\|_{(2)} = \sqrt{\mathrm{tr}(X^*X)}$ is the Frobenius norm of X. In the following Φ' stands for the dual of the symmetric gauge function Φ, i.e.

$$\Phi'(\xi) = \max_{\substack{\eta \in \mathbb{R}^n \\ \Phi(\eta) = 1}} \eta^\top \xi, \qquad \xi \in \mathbb{R}^n. \tag{5.2}$$

Let $X = U\mathrm{diag}(\sigma)V^*$ be a singular value decomposition, where $U, V \in \mathcal{U}_n$ and $\sigma = [\sigma_1, \ldots, \sigma_n]^\top$ is the vector of singular values of $X \in \mathbb{C}^{n\times n}$. Let $\tau = [\tau_1, \ldots, \tau_n]^\top$ be a nonnegative vector such that $\Phi(\tau) = 1$ and $\tau^\top \sigma = \Phi(\sigma)$. Let $Y_0 = U\mathrm{diag}(\tau)V^*$. Then

$$\|X\|'_\Phi = \max_{\substack{Y \in \mathbb{C}^{n\times n} \\ \|Y\|_\Phi = 1}} \Re\,\mathrm{tr}(Y^*X) \geq \Re\,\mathrm{tr}(Y_0^*X) = \tau^\top\sigma = \Phi'(\sigma) = \|X\|_{\Phi'}. \tag{5.3}$$

It can be shown that the inequality in (5.3) is actually an equality. Hence we have the following result [1, Prop. IV.2.11].

PROPOSITION 5.1. *For any symmetric gauge function Φ the dual of the unitarily invariant norm $\|\cdot\|_\Phi$ is $\|\cdot\|_{\Phi'}$.*

From (5.1) it follows that unitarily invariant norms have the properties (a),(b) and (c). Thus, by combining Theorem 3.4 and Proposition 5.1 we get the result below.

THEOREM 5.2. *Let Φ be a symmetric gauge function on $\mathbb{R}^n$ and let $\mathbf{\Delta}$ be one of the sets in (3.4). Then the support function of*

$$K_{\mathbf{\Delta}}(x, y) = \{\, y^*\Delta x;\ \ \Delta \in \mathbf{\Delta},\ \|\Delta\|_\Phi \leq 1\}, \qquad x, y \in \mathbb{C}^n,$$

is given by

$$s_{\mathbf{\Delta}}(z) = \|\mathcal{P}_{\mathbf{\Delta}}(z\,yx^*)\|_{\Phi'} = \Phi'(\,[\sigma_1(z), \ldots, \sigma_n(z)]^\top\,), \qquad z \in \mathbb{C},$$

where $\sigma_1(z), \ldots, \sigma_n(z)$ denote the singular values of $\mathcal{P}_{\mathbf{\Delta}}(z\,yx^*)$.

6. Frobenius norm and spectral norm. In this section we provide explicite formulas for $K_{\mathbf{\Delta}}(x, y)$ for the case that $\mathbf{\Delta}$ is one of the sets in 3.4 and the underlying norm is the spectral norm or the Frobenius norm. First, we give a result on the support function of an ellipse.

PROPOSITION 6.1. *Let $K \subset \mathbb{C}$ be a nonempty compact convex set with support function*

$$s_K(z) = \sqrt{a\,|z|^2 + \Re(b\,\bar{z}^2)}, \qquad z, b \in \mathbb{C},\ a \geq |b|.$$

Then K is an ellipse (which may be degenerated to a line segment). Specifically,

$$K = \left\{\, e^{i\phi/2}(\sqrt{a + |b|}\,\xi_1 + \sqrt{a - |b|}\,\xi_2\, i);\quad \xi_1, \xi_2 \in \mathbb{R},\ \xi_1^2 + \xi_2^2 \leq 1 \right\}, \tag{6.1}$$

where $\phi = \arg(b)$.

Proof. Let E denote the set on the right hand side of (6.1), and let s_E denote its support function. Let

$$\alpha = \frac{1}{2}(\sqrt{a + |b|} + \sqrt{a - |b|})\, e^{i\phi/2}, \qquad \beta = \frac{1}{2}(\sqrt{a + |b|} - \sqrt{a - |b|})\, e^{i\phi/2}.$$

Then for $\xi_1, \xi_2 \in \mathbb{R}$,

$$e^{i\phi/2}[\sqrt{a+|b|}\,\xi_1 + \sqrt{a-|b|}\,\xi_2\, i] = \alpha\xi + \beta\bar{\xi}, \qquad \text{where } \xi = \xi_1 + \xi_2\, i \in \mathbb{C}.$$

Thus

$$\begin{aligned} s_E(z) &= \max_{|\xi|\le 1} \Re(\bar{z}\,(\alpha\,\xi + \beta\,\bar{\xi})) \\ &= \max_{|\xi|\le 1} \Re(\,(\alpha\,\overline{z} + \bar{\beta}\,z)\,\xi) \\ &= |\alpha\,\overline{z} + \bar{\beta}\,z| \\ &= \sqrt{(|\alpha|^2 + |\bar{\beta}|^2)|z|^2 + 2\Re(\bar{z}^2\,\alpha\beta)} \\ &= \sqrt{a\,|z|^2 + \Re(\bar{z}^2\,b)}. \end{aligned}$$

Thus $s_E = s_K$, and consequently $E = K$. □

Note that the set (6.1) is a disk if $b = 0$ and $a > 0$. It is a line segment if $a = |b| > 0$.

THEOREM 6.2. *Let $\|\Delta\|$ denote either the Frobenius norm or the spectral norm of $\Delta \in \mathbb{C}^{n\times n}$. Let $\mathbf{\Delta} \subseteq \mathbb{C}^{n\times n}$ and let $a \ge 0$, $b \in \mathbb{C}$ be as in the tables below. Then the support function of*

$$K_{\mathbf{\Delta}}(x,y) = \{\, y^*\Delta x;\ \ \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1\}$$

is given by

$$s_{\mathbf{\Delta}}(z) = \sqrt{a\,|z|^2 + \Re(b\,\bar{z}^2)}, \qquad z \in \mathbb{C}. \tag{6.2}$$

Hence, $K_{\mathbf{\Delta}}(x,y)$ equals the ellipse defined in (6.1).

Table for the Frobenius norm:

$\mathbf{\Delta}$	a	b
$\mathbb{C}^{n\times n}$	$\|x\|^2\|y\|^2$	0
$\mathbb{R}^{n\times n}$	$\frac{1}{2}\,\|x\|^2\|y\|^2$	$\frac{1}{2}(x^\top x)\,(\overline{y^\top y})$
Herm	$\frac{1}{2}\|x\|^2\|y\|^2$	$\frac{1}{2}(y^*x)^2$
$\mathrm{Sym}_{\mathbb{C}}$	$\frac{1}{2}(\|x\|^2\|y\|^2 + \|x^\top y\|^2)$	0
$\mathrm{Skew}_{\mathbb{C}}$	$\frac{1}{2}(\|x\|^2\|y\|^2 - \|x^\top y\|^2)$	0
$\mathrm{Sym}_{\mathbb{R}}$	$\frac{1}{4}(\|x\|^2\|y\|^2 + \|x^\top y\|^2)$	$\frac{1}{4}(\,(x^\top x)\,\overline{(y^\top y)} + (y^*x)^2)$
$\mathrm{Skew}_{\mathbb{R}}$	$\frac{1}{4}(\|x\|^2\|y\|^2 - \|x^\top y\|^2)$	$\frac{1}{4}(\,(x^\top x)\,\overline{(y^\top y)} - (y^*x)^2)$

Table for the spectral norm:

$\boldsymbol{\Delta}$	a	b
$\mathbb{C}^{n\times n}$	$\|x\|^2\|y\|^2$	0
$\mathbb{R}^{n\times n}$	$\frac{1}{2}\left[\|x\|^2\,\|y\|^2 + \sqrt{(\|x\|^4 - \|x^\top x\|^2)(\|y\|^4 - \|y^\top y\|^2)}\right]$	$\frac{1}{2}\,(x^\top x)(\overline{y^\top y})$
Herm	$\|x\|^2\|y\|^2 - \frac{1}{2}\|y^* x\|^2$	$\frac{1}{2}(y^* x)^2$
$\mathrm{Sym}_{\mathbb{C}}$	$\|x\|^2\|y\|^2$	0
$\mathrm{Skew}_{\mathbb{C}}$	$\|x\|^2\|y\|^2 - \|x^\top y\|^2$	0
$\mathrm{Skew}_{\mathbb{R}}$	$\frac{1}{2}(\,\|x\|^2\|y\|^2 - \|x^\top y\|^2 + \sqrt{\det(F^* F)}\,)$ $F = \begin{bmatrix} x & \bar{x} & y & \bar{y}\end{bmatrix}$	$\frac{1}{2}((x^\top x)(\overline{y^\top y}) - (y^* x)^2)$

Here and in the following $\|x\|$, $\|y\|$ denote the Euclidean norm of $x, y \in \mathbb{C}^n$.

REMARK 6.3. *Theorem 6.2 makes no statement about the case that $\boldsymbol{\Delta} = \mathrm{Sym}_{\mathbb{R}}$ and the underlying norm is the spectral norm. The associated sets $K_{\mathrm{Sym}_{\mathbb{R}}}(x, y)$ are in general no ellipses. Figure 6 gives two examples. It shows the sets $K_{\mathrm{Sym}_{\mathbb{R}}}(x_j, y_j)$, $j = 1, 2$, where*

$$\begin{aligned} x_1 &= [2+i \quad 2+i \quad 2]^\top, \quad & y_1 &= [-2 \quad -2 \quad 3i]^\top, \\ x_2 &= [1+2i \quad i \quad 2]^\top, & y_2 &= [i \quad -2+2i \quad 1+2i]^\top. \end{aligned} \tag{6.3}$$

REMARK 6.4. *Notice that Theorem 6.2 yields precise values for the structured condition numbers of a nonderogatory eigenvalue λ and the cases listed in the tables: According to the discussion in the introduction the condition number equals the radius r of the smallest disk about 0 that contains the set $K_{\boldsymbol{\Delta}}^{(m)}(x, y)$, where x, y form a normalized pair of eigenvectors. However, if $K_{\boldsymbol{\Delta}}(x, y)$ is an ellipse with support function (6.2) then $r = (a + |b|)^{1/(2m)}$.*

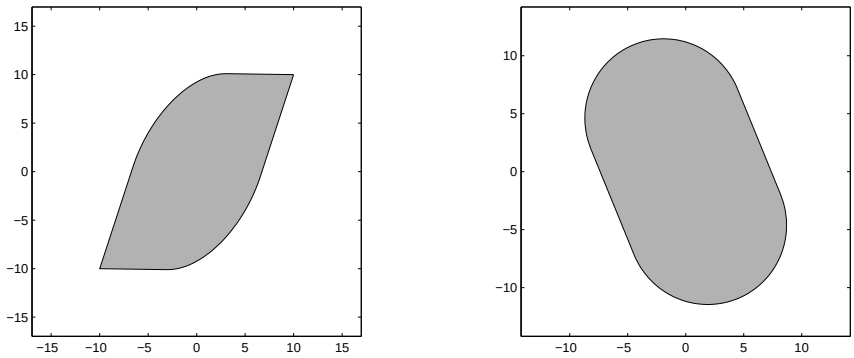


FIG. 6.1. *The sets $K_{\mathrm{Sym}_{\mathbb{R}}}(x_1, y_1)$ (left) and $K_{\mathrm{Sym}_{\mathbb{R}}}(x_2, y_2)$ (right) from Remark 6.3.*

The proof of Theorem 6.2 uses the lemma below.

LEMMA 6.1. *Let $M = a_1 b_1^* + a_2 b_2^*$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}^n$. Then the Frobenius*

norm and the Schatten-1-norm of M are given by

$$\|M\|_{(2)}^2 = \|a_1\|^2\|b_1\|^2 + \|a_2\|^2\|b_2\|^2 + 2\,\Re[\,(a_1^* a_2)\,(\overline{b_1^* b_2})\,],$$

$$\|M\|_{(1)}^2 = \|M\|_{(2)}^2 + 2\sqrt{(\,\|a_1\|^2\|a_2\|_2^2 - |a_1^* a_2|^2\,)\,(\,\|b_1\|^2\|b_2\|^2 - |b_1^* b_2|^2\,)}.$$

The Frobenius norms of the matrices $S_\pm = \frac{1}{2}(M \pm M^\top)$ are given by

$$\begin{aligned}\|S_\pm\|_{(2)}^2 = \frac{1}{2}\big(\,&\|a_1\|^2\|b_1\|^2 + \|a_2\|^2\|b_2\|^2 \pm |a_1^\top b_1|^2 \pm |a_2^\top b_2|^2\,\big)\\ &+ \Re\Big(\,(a_1^* a_2)\,(\overline{b_1^* b_2}) \,\pm\, (a_1^\top b_2)\,(\overline{a_2^\top b_1})\,\Big).\end{aligned}$$

The Schatten-1-norm of S_- satisfies

$$\|S_-\|_{(1)}^2 = 2\,\Big(\,\|S_-\|_{(2)}^2 + \sqrt{\det(A^* A)}\,\Big),$$

where $A = \begin{bmatrix} a_1 & a_2 & \bar{b}_1 & \bar{b}_2 \end{bmatrix} \in \mathbb{C}^{n\times 4}$.

Proof. See the appendix. □

Proof of Theorem 6.2. First, we treat the case that $\mathbf{\Delta} = \mathbb{R}^{n\times n}$. Let

$$M = 2\,\Re(z\,y\,x^*) = z\,y\,x^* + \bar{z}\,\bar{y}\,\bar{x}^*.$$

According to Proposition 5.1 the dual of the spectral norm is the Schatten-1-norm. Hence, by Theorem 5.2 the support function of $K_{\mathbb{R}^{n\times n}}(x,y)$ (with respect to spectral norm) is

$$\begin{aligned} s_{\mathbb{R}^{n\times n}}(z) &= \|\mathcal{P}_{\mathbb{R}^{n\times n}}(z\,yx^*)\|_{(1)} \qquad \text{(by Theorem 5.2)}\\ &= \|\Re(z\,yx^*)\|_{(1)}\\ &= \frac{1}{2}\|M\|_{(1)}\\ &= \frac{1}{2}\sqrt{\alpha_z + 2\sqrt{\beta_z}}, \qquad \text{(by Lemma 6.1)}\end{aligned}$$

where

$$\begin{aligned}\alpha_z &= \|M\|_{(2)}^2\\ &= \|z\,y\|^2\|x\|^2 + \|\bar{z}\,\bar{y}\|^2\|\bar{x}\|^2 + 2\,\Re[\,((z\,y)^*(\bar{z}\,\bar{y}))\,(\overline{(x^*\bar{x})}\,]\\ &= 2\,(|z|^2\|x\|^2\|y\|^2 + \Re[\bar{z}^2(x^\top x)\,(\overline{y^\top y})])\end{aligned}$$

$$\begin{aligned}\beta_z &= (\,\|z\,y\|^2\|\bar{z}\bar{y}\|_2^2 - |(z\,y)^*(\bar{z}\,\bar{y})|^2\,)\,(\,\|x\|^2\|\bar{x}\|^2 - |x^*\bar{x}|^2\,)\\ &= |z|^4(\|x\|^4 - |x^\top x|^2)(\|y\|^4 - |y^\top y|^2).\end{aligned}$$

If the underlying norm is the Frobenius norm then

$$s_{\mathbb{R}^{n\times n}}(z) = \|\Re(z\,yx^*)\|_{(2)} = \frac{1}{2}\|M\|_{(2)} = \frac{1}{2}\sqrt{\alpha_z}.$$

Next, we consider the real skew-symmetric case. Let $S_- = \frac{1}{2}(M - M^\top)$. The support function of $K_{\mathrm{Skew}_\mathbb{R}}(x,y)$ with respect to spectral norm norm is

$$\begin{aligned} s_{\mathrm{Skew}_\mathbb{R}}(z) &= \|\mathcal{P}_{\mathrm{Skew}_\mathbb{R}}(z\,yx^*)\|_{(1)} \qquad \text{(by Theorem 5.2)} \\ &= \frac{1}{2}\,\|S_-\|_{(1)} \\ &= \frac{1}{2}\sqrt{2\,\|S_-\|_{(2)}^2 + 2\,\sqrt{\det(A_z^*A_z)}}, \qquad \text{(by Lemma 6.1)} \\ &= \sqrt{\frac{1}{2}\,\|S_-\|_{(2)}^2 + \frac{1}{2}\,\sqrt{\det(A_z^*A_z)}} \end{aligned}$$

where

$$\begin{aligned} \|S_-\|_{(2)}^2 &= \frac{1}{2}\,\big(\,\|z\,y\|^2\|x\|^2 + \|\bar z\,\bar y\|^2\|\bar x\|^2 - |(z\,y)^\top x|^2 - |(\bar z\bar y)^\top \bar x|^2\,\big) \\ &\qquad + \Re\,\Big[\,((z\,y)^*(\bar z\,\bar y))\,(\overline{x^*\bar x}) \;-\; ((z\,y)^\top \bar x)\,(\overline{(\bar z\bar y)^\top x})\,\Big]\,. \\ &= |z|^2\,(\|x\|^2\|y\|^2 - |x^\top y|^2) \;+\; \Re[\bar z^2\,(\,(x^\top x)(\overline{y^\top y}) - (y^*x)^2\,)], \end{aligned}$$

$$A_z = \begin{bmatrix} z\,y & \bar z\,\bar y & x & \bar x \end{bmatrix} = \underbrace{\begin{bmatrix} y & \bar y & x & \bar x \end{bmatrix}}_{=A_1}\,\mathrm{diag}(z,\bar z,1,1).$$

We have $\det(A_z^*A_z) = |z|^4\det(A_1^*A_1) = |z|^4\det(F^*F)$, where $F = \begin{bmatrix} x & \bar x & y & \bar y \end{bmatrix}$.

The computions for the other cases is analogous. □

EXAMPLE 6.5. *Figure 6 shows the sets*

$$K_{\mathbf{\Delta}}(x,y) \;\;=\;\; \{\,y^*\Delta x;\;\; \Delta\in\mathbf{\Delta},\; \|\Delta\|_{(2)}\le 1\,\}, \tag{6.4}$$

where

$$x = [\,4+3i,\; -1,\; 1+5i,\; -i\,]^\top, \quad y \;=\; [\,4i,\; 4+3i,\; 4+3i,\; 4+i\,]^\top. \tag{6.5}$$

7. Self- and skew-adjoint perturbations. We now treat the case that $\mathbf{\Delta}$ is a set of matrices which are skew- or self-adjoint with respect to a scalar product on $\mathbb{C}^n$. Specifically we show that the associated sets $K_{\mathbf{\Delta}}(x,y)$ can be computed via the methods in the previous sections if the scalar product is induced by a unitary matrix and the underlying norm is unitarily invariant.

For nonsingular $\Pi\in\mathbb{C}^{n\times n}$ we consider the scalar products

$$\langle x,y\rangle_\Pi = x^\star\Pi y, \qquad x,y\in\mathbb{C}^n,\; \star\in\{*,\top\}.$$

Depending on whether $\star = \top$ or $\star = *$ the scalar product is a bilinear form or a sesquilinear form. We assume that Π satisfies a symmetry relation of the form

$$\Pi^\star = \epsilon_0\Pi, \text{ with } \epsilon_0 = -1 \text{ or } \epsilon_0 = 1. \tag{7.1}$$

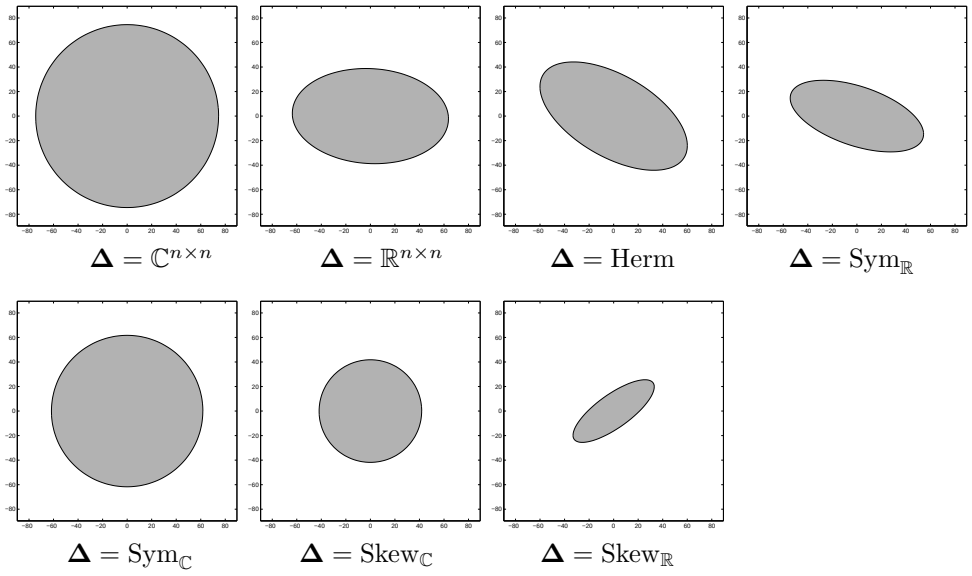


FIG. 6.2. *The sets $K_{\mathbf{\Delta}}(x, y)$ for the Frobenius norm and x, y defined in (6.5).*

A matrix $\Delta \in \mathbb{C}^{n\times n}$ is said to be self-adjoint (skew-adjoint) with respect to the scalar product $\langle \cdot, \cdot \rangle_\Pi$ if

$$\langle \Delta x, y\rangle_\Pi = \epsilon \, \langle x, \Delta y\rangle_\Pi \quad \text{for all } x, y \in \mathbb{C}^n, \tag{7.2}$$

and $\epsilon = 1$ ($\epsilon = -1$). The relation (7.2) is easily seen to be equivalent to

$$\Delta^\star \Pi = \epsilon \, \Pi \Delta. \tag{7.3}$$

We denote the sets of self- and skew-adjoint matrices by

$$\mathit{struct}(\Pi, \star, \epsilon) := \{ \; \Delta \in \mathbb{C}^{n\times n} \; ; \;\; \Delta^\star \Pi = \epsilon \, \Pi \Delta \; \}.$$

The relation (7.1) implies that (7.3) is equivalent to

$$(\Pi \Delta)^\star = \epsilon_0 \, \epsilon \, \Pi \Delta. \tag{7.4}$$

We thus have the lemma below.

LEMMA 7.1. *Let $\Pi, \Delta \in \mathbb{K}^{n\times n}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose $\Pi^\star = \epsilon_0 \Pi$ with $\epsilon_0 = -1$ or $\epsilon_0 = 1$. Then the following equivalences hold.*

$$\Delta \in \mathit{struct}(\Pi, \star, \epsilon) \Leftrightarrow \begin{cases} \Pi\Delta \in \mathrm{Herm} & \text{if } \epsilon_0 \epsilon = 1, \; \star = *, \\ \Pi\Delta \in \mathrm{Sym}_{\mathbb{K}} & \text{if } \epsilon_0 \epsilon = 1, \; \star = \top, \\ \Pi\Delta \in \mathrm{Skew}_{\mathbb{K}} & \text{if } \epsilon_0 \epsilon = -1, \; \star = \top, \\ i\,\Pi\Delta \in \mathrm{Herm} & \text{if } \epsilon_0 \epsilon = -1, \; \star = *. \end{cases}$$

In many applications Π is unitary. The most common examples are

$$\Pi \in \{\mathrm{diag}(I_k, -I_{n-k}), E_n, J_n\},$$

where

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{C}^{2n\times 2n}, \qquad E_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

PROPOSITION 7.1. *Suppose $\Pi \in \mathbb{C}^{n\times n}$ is unitary and satisfies $\Pi^\star = \epsilon_0\Pi$ with $\epsilon_0 = -1$ or $\epsilon_0 = 1$. Let struct = struct$(\Pi, \star, \epsilon)$. Then for any unitarily invariant norm,*

$$K_{struct}(x, y) \;=\; K_{\mathbf{\Delta}}(x, \Pi y),$$

where

$$\mathbf{\Delta} = \begin{cases} \text{Herm} & \text{if } \epsilon_0\epsilon = 1,\ \star = *, \\ \text{Sym}_{\mathbb{C}} & \text{if } \epsilon_0\epsilon = 1,\ \star = \top, \\ \text{Sym}_{\mathbb{R}} & \text{if } \epsilon_0\epsilon = 1,\ \star = \top,\ \text{and } \Pi \in \mathbb{R}^{n\times n} \\ \text{Skew}_{\mathbb{C}} & \text{if } \epsilon_0\epsilon = -1,\ \star = \top, \\ \text{Skew}_{\mathbb{R}} & \text{if } \epsilon_0\epsilon = -1,\ \star = \top \text{ and } \Pi \in \mathbb{R}^{n\times n}. \end{cases} \tag{7.5}$$

Furthermore, $K_{struct}(x, y) = K_{\text{Herm}}(x, i\Pi y)$ if $\epsilon_0\epsilon = -1$ and $\star = *$.

Proof. Using Lemma 7.1 and $\Pi^*\Pi = I_n$ we obtain for the sets in (7.5),

$$\begin{aligned} K_{struct}(x, y) &= \{\, y^*\Delta x;\ \ \Delta \in struct,\ \ \|\Delta\| \le 1 \,\} \\ &= \{\, (\Pi y)^*(\Pi\Delta)x;\ \ \Pi\Delta \in \mathbf{\Delta},\ \ \|\Pi\Delta\| \le 1 \,\} \\ &= K_{\mathbf{\Delta}}(x, \Pi y). \end{aligned}$$

The proof of the remaining statement is analogous. □

Appendix. We give the proof Lemma 6.1. To this end we need the following fact.

PROPOSITION 7.2. *Let $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n$ denote the singular values of $M = AB^*$, where $A, B \in \mathbb{C}^{n\times r}$. Then $\sigma_k = 0$ for $k > r$, and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$ are the eigenvalues of $(A^*A)(B^*B)$. In particular,*

$$\sum_{k=1}^{r} \sigma_k^2 = \operatorname{tr}((A^*A)(B^*B)), \qquad \prod_{k=1}^{r} \sigma_k^2 = \det((A^*A)(B^*B)).$$

Proof. Since $\operatorname{rank}(M) \le r$, we have $\sigma_k = 0$ for $k > r$. The squares of the singular values of M are the eigenvalues of $M^*M = XY$, where $X = B$, $Y = (A^*A)B^*$. As ist well known XY and $YX = (A^*A)(B^*B)$ have the same nonzero eigenvalues. □

Now, let σ_1, σ_2 denote the largest singular values of the matrix

$$M = a_1 b_1^* + a_2 b_2^* = [a_1\ a_2]\ [b_1\ b_2]^*, \qquad a_1, a_2, b_1, b_2 \in \mathbb{C}^n.$$

Since $\operatorname{rank}(M) \leq 2$ the other singular values of M are zero. Using Proposition 7.2 we obtain for the Frobenius norm and the Schatten-1-norm of M,

$$\begin{aligned}
\|M\|_{(2)}^2 &= \sigma_1^2 + \sigma_2^2 \\
&= \operatorname{tr}\left(\begin{bmatrix} \|a_1\|^2 & a_1^* a_2 \\ a_2^* a_1 & \|a_2\|^2 \end{bmatrix} \begin{bmatrix} \|b_1\|^2 & b_1^* b_2 \\ b_2^* b_1 & \|b_2\|^2 \end{bmatrix} \right) \\
&= \|a_1\|^2 \|b_1\|^2 + \|a_2\|^2 \|b_2\|^2 + 2\,\Re(\,(a_1^* a_2)\,(\overline{b_1^* b_2})\,).
\end{aligned}$$

$$\begin{aligned}
\|M\|_{(1)}^2 &= (\sigma_1 + \sigma_2)^2 \\
&= \sigma_1^2 + \sigma_2^2 + 2\sqrt{\sigma_1^2 \sigma_2^2} \\
&= \|M\|_{(2)}^2 + 2\sqrt{\beta},
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \det\left(\begin{bmatrix} \|a_1\|^2 & a_1^* a_2 \\ a_2^* a_1 & \|a_2\|^2 \end{bmatrix} \begin{bmatrix} \|b_1\|^2 & b_1^* b_2 \\ b_2^* b_1 & \|b_2\|^2 \end{bmatrix} \right) \\
&= (\,\|a_1\|^2 \|a_2\|^2 - |a_1^* a_2|^2)(\|b_1\|^2 \|b_2\|^2 - |b_1^* b_2|^2\,).
\end{aligned}$$

Next, we compute the norms of the symmetric and the skew-symmetric part of M. Let $S_\pm = \frac{1}{2}(M \pm M^\top)$. Then $S_\pm$ can be written in the form $S_\pm = A B_\pm^*$, where

$$A = \begin{bmatrix} a_1 & a_2 & \bar{b}_1 & \bar{b}_2 \end{bmatrix}, \qquad B_\pm = \frac{1}{2} \begin{bmatrix} b_1 & b_2 & \pm\bar{a}_1 & \pm\bar{a}_2 \end{bmatrix}.$$

We have

$$A^* A = \begin{bmatrix} \|a_1\|^2 & a_1^* a_2 & \overline{a_1^\top b_1} & \overline{a_1^\top b_2} \\ a_2^* a_1 & \|a_2\|^2 & \overline{a_2^\top b_1} & \overline{a_2^\top b_2} \\ b_1^\top a_1 & b_1^\top a_2 & \|b_1\|^2 & \overline{b_1^* b_2} \\ b_2^\top a_1 & b_2^\top a_2 & \overline{b_2^* b_1} & \|b_2\|^2 \end{bmatrix},$$

$$B_\pm^* B_\pm = \frac{1}{4} \begin{bmatrix} \|b_1\|^2 & b_1^* b_2 & \pm\overline{b_1^\top a_1} & \pm\overline{b_1^\top a_2} \\ b_2^* b_1 & \|b_2\|^2 & \pm\overline{b_2^\top a_1} & \pm\overline{b_2^\top a_2} \\ \pm a_1^\top b_1 & \pm a_1^\top b_2 & \|a_1\|^2 & \overline{a_1^* a_2} \\ \pm a_2^\top b_1 & \pm a_2^\top b_2 & \overline{a_2^* a_1} & \|a_2\|^2 \end{bmatrix}.$$

Using Proposition 7.2 we obtain for the Frobenius norm of $S_\pm$,

$$\begin{aligned}
\|S_\pm\|_{(2)}^2 &= \operatorname{tr}((A^* A)(B_\pm^* B_\pm)) \\
&= \frac{1}{2}\left(\|a_1\|^2 \|b_1\|^2 + \|a_2\|^2 \|b_2\|^2 \pm |a_1^\top b_1|^2 \pm |a_2^\top b_2|^2 \right) \\
&\qquad + \Re\left((a_1^* a_2)\,(\overline{b_1^* b_2}) \;\pm\; (a_1^\top b_2)\,(\overline{a_2^\top b_1}) \right).
\end{aligned}$$

We now determine the Schatten-1-norm of S_-. Since $\operatorname{rank}(S_-) \leq 4$, at most 4 singular values of S_- are nonzero. Let $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4$ denote these singular values. Since

S_- is skew-symmetric, its singular values have even multiplicity [10, Sect. 4.4, Exercise 26]. Thus $\sigma_1 = \sigma_2$ and $\sigma_3 = \sigma_4$. This yields

$$\begin{aligned}\|S_-\|_{(1)}^2 &= (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)^2 \\ &= (2\sigma_1 + 2\sigma_3)^2 \\ &= 2(2\sigma_1^2 + 2\sigma_3^2) + 8\sigma_1\sigma_3 \\ &= 2\|S_-\|_{(2)}^2 + 8\left(\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\right)^{1/4} \\ &= 2\|S_-\|_{(2)}^2 + 8\det((A^*A)(B_-^*B_-))^{1/4}.\end{aligned}$$

Since $4\,\overline{B_-^*\,B_-}$ is unitarily similar to A^*A, we have $\det(B_-^*B_-) = \frac{1}{64}\det(A^*A)$. Hence,

$$\|S_-\|_{(1)}^2 = 2\left(\|S_-\|_{(2)}^2 + \sqrt{\det(A^*A)}\right).$$

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