Spectral Intervals for Differential-Algebraic Equations and their Numerical Approximation *

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Abstract

Lyapunov and exponential dichotomy spectral theory is extended from ordinary differential equations (ODEs) to nonautonomous differential-algebraic equations (DAEs). By using orthogonal changes of variables, the original DAE system is transformed into appropriate condensed forms, for which concepts such as Lyapunov exponents, Bohl exponents, exponential dichotomy and spectral intervals of various kinds can be analyzed via the resulting underlying ODE. Some essential differences between the spectral theory for ODEs and that for DAEs are pointed out. Numerical methods for computing the spectral intervals associated with Lyapunov and Sacker-Sell (exponential dichotomy) spectra are derived by modifying and extending those methods proposed for ODEs [26]. Perturbation theory and error analysis are discussed, as well. Finally, some numerical examples are presented to illustrate the theoretical results and the properties of the numerical methods.

Keywords: differential-algebraic equations, strangeness index, Lyapunov exponent, Bohl exponent, Sacker-Sell spectrum, exponential dichotomy, spectral interval, smooth QR factorization, continuous QR algorithm, discrete QR algorithm, kinematic equivalence, Steklov function

AMS(MOS) subject classification: 65L07, 65L80, 34D08, 34D09

1 Introduction

More than a century ago, fundamental concepts and results for the stability theory of ordinary differential equations were presented in Lyapunov's famous thesis [58]. One of the most important notions, the so-called Lyapunov exponent (or Lyapunov characteristic number), has proved very useful in studying growth rates of solutions to linear ODEs. In the nonlinear case, by linearizing along a particular solution, Lyapunov exponents also give information about the convergence or divergence rates of nearby solutions. The spectral theory for ODEs was further developed throughout the 20th century, and concepts such as Bohl exponents, exponential dichotomy (also well-known as Sacker–Sell) spectra were introduced, see [1, 18, 19, 69]. Unlike the development of the analytic theory, the development of numerical methods to compute Lyapunov exponents and also other spectral intervals has only recently been studied. In a series of papers, see [22, 23, 25, 26, 28, 29, 30], Dieci and Van Vleck have developed algorithms for the computation of Lyapunov and Bohl exponents as well as Sacker-Sell spectral intervals. These methods have also been analyzed concerning their sensitivity under small perturbations (stability), the relationship between different spectra, the error analysis, and efficient implementation techniques.

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This paper is devoted to the generalization of some theoretical results as well as numerical methods from the spectral theory for ODEs to differential-algebraic equations (DAEs). In particular, we are interested in the characterization of the dynamical behavior of solutions to initial value problems for linear systems of DAEs

$$E(t)\dot{x} = A(t)x + f(t), \tag{1}$$

on the half-line $\mathbb{I} = [0, \infty)$, together with an initial condition

$$x(0) = x_0. \tag{2}$$

Here we assume that $E, A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$, and $f \in C(\mathbb{I}, \mathbb{R}^n)$ are sufficiently smooth. We use the notation $C(\mathbb{I}, \mathbb{R}^{n\times n})$ to denote the space of continuous functions from $\mathbb{I}$ to $\mathbb{R}^{n\times n}$.

Linear systems of the form (1) occur when one linearizes a general implicit nonlinear system of DAEs

$$F(t, x, \dot{x}) = 0, \quad t \geq 0, \tag{3}$$

along a particular solution [11]. In this paper for the discussion of spectral intervals, we restrict ourselves to *regular* DAEs, i. e., we require that (1) (or (3) locally) has a unique solution for sufficiently smooth right hand sides and appropriately chosen (consistent) initial conditions, see [51] for a discussion of existence and uniqueness of solution of more general nonregular DAEs.

DAEs like (1) and (3) arise in constrained multibody dynamics [36], electrical circuit simulation [39, 40], chemical engineering [31, 32] and many other applications, in particular when the dynamics of a system is constrained or when different physical models are coupled together in automatically generated models [63]. While DAEs provide a very convenient modeling concept, many numerical difficulties arise due to the fact that the dynamics is constrained to a manifold, which often is only given implicitly, see [8, 42, 66] or the recent textbook [51]. These difficulties are typically characterized by one of many index concepts that exist for DAEs, see [8, 38, 42, 51].

The fact that the dynamics of DAEs is constrained also requires a modification of most classical concepts of the qualitative theory that was developed for ODEs. Different stability concepts for DAEs have been discussed already in [2, 43, 44, 53, 59, 61, 67, 68, 70, 71, 72, 73]. Only very few papers, however, discuss the spectral theory for DAEs, see [16, 17] for results on Lyapunov exponents and Lyapunov regularity, [57] for the concept of exponential dichotomy used in numerical solution to boundary value problems, and [14, 35] for robustness results of exponential stability and Bohl exponents. All these papers use the tractability index approach as it was introduced in [38, 60] and consider linear systems of DAEs of tractability index 1, only. Here we allow general regular DAEs of arbitrary index and we use reformulations based on derivative arrays as well as the strangeness index concept [51]. As in the ODE case there is also a close relation of the spectral theory to the theory of adjoint equations which has recently been studied in the context of control problems in [3, 4, 5, 13, 52].

In this paper, we systematically extend the classical spectral concepts (Lyapunov, Bohl, Sacker-Sell) that were introduced for ODEs, to general linear DAEs with variable coefficients of the form (1). We show that substantial differences in the theory arise and that most statements in the classical ODE theory hold for DAEs only under further restrictions, here our results extend results on asymptotic stability given in [53]. After deriving the concepts and analyzing the relationship between the different concepts of spectral intervals, we then derive two alternative numerical approaches to compute the corresponding spectra.

The outline of the paper is as follows. In the following section, we recall some concepts from the theory of differential-algebraic equations. Then we give a short review on the spectral theory for ODEs and also on numerical methods for computing various spectral intervals. We discuss in detail the extension of spectral concepts to DAEs in Section 3. The relation between the spectral characteristics of DAE systems and those of their underlying ODE systems is investigated. Furthermore, the stability of the spectra with respect to perturbations arising in the system data is analyzed. In Section 4 we propose numerical methods for computing the Lyapunov and the Sacker-Sell (exponential dichotomy) spectral intervals and discuss implementation details as well as the associated error analysis. In Section 5 we present numerical examples to illustrate the theoretical results and the properties of the numerical methods. We finish the paper with a summary and a discussion of open problems.

2 Preliminaries

2.1 A review of DAE theory

In this section we briefly recall some concepts from the theory of differential-algebraic equations, see e.g. [8, 38, 51, 65]. We follow [51] in notation and style of presentation.

Definition 1 *Consider system (1) with sufficiently smooth coefficient functions E, A. A function $x : \mathbb{I} \to \mathbb{R}^n$ is called a* solution *of (1) if $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ and x satisfies (1) pointwise. It is called a* solution of the initial value problem *(1)–(2) if x is a solution of (1) and satisfies (2). An initial condition (2) is called* consistent *if the corresponding initial value problem has at least one solution.*

For the analysis as in [10, 12, 49, 51], we use *derivative arrays*

$$M_\ell(t)\dot{z}_\ell = N_\ell(t)z_\ell + g_\ell(t), \tag{4}$$

where

$$\begin{aligned}
(M_\ell)_{i,j} &= \tbinom{i}{j}E^{(i-j)} - \tbinom{i}{j+1}A^{(i-j-1)},\ i,j = 0,\dots,\ell,\\
(N_\ell)_{i,j} &= \begin{cases} A^{(i)} & \text{for } i = 0,\dots,\ell,\ j = 0,\\ 0 & \text{otherwise,}\end{cases}\\
(z_\ell)_j &= x^{(j)},\ j = 0,\dots,\ell,\\
(g_\ell)_i &= f^{(i)},\ i = 0,\dots,\ell,
\end{aligned} \tag{5}$$

using the convention that $\binom{i}{j} = 0$ for $i < 0$, $j < 0$ or $j > i$. In more detail, we have

$$M_\ell = \begin{bmatrix} E & & & & \\ \dot{E} - A & E & & & \\ \ddot{E} - 2\dot{A} & 2\dot{E} - A & E & & \\ \vdots & & \ddots & \ddots & \\ E^{(\ell)} - \ell A^{(\ell-1)} & \cdots & \cdots & \ell\dot{E} - A & E \end{bmatrix},\ N_\ell = \begin{bmatrix} A & 0 & \cdots & 0 \\ \dot{A} & 0 & \cdots & 0 \\ \ddot{A} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A^{(\ell)} & 0 & \cdots & 0 \end{bmatrix}. \tag{6}$$

To guarantee existence and uniqueness of solutions, we make the following hypothesis, see [51].

Hypothesis 2 *There exist integers μ, a, and d such that the inflated pair (M_μ, N_μ) associated with the given pair of matrix functions (E, A) has the following properties:*

1. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} M_\mu(t) = (\mu + 1)n - a$ such that there exists a smooth matrix function Z_2 of size $(\mu + 1)n \times a$ and pointwise maximal rank satisfying $Z_2^T M_\mu = 0$.*

2. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} \hat{A}_2(t) = a$, where $\hat{A}_2 = Z_2^T N_\mu [I_n\ 0 \cdots 0]^T$ such that there exists a smooth matrix function T_2 of size $n \times d$, $d = n - a$, and pointwise maximal rank satisfying $\hat{A}_2 T_2 = 0$.*

3. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} E(t)T_2(t) = d$ such that there exists a smooth matrix function Z_1 of size $n \times d$ and pointwise maximal rank satisfying $\operatorname{rank} \hat{E}_1 T_2 = d$ with $\hat{E}_1 = Z_1^T E$.*

Since Gram-Schmidt orthonormalization is a continuous process, we may assume without loss of generality that the columns of the matrix functions Z_1, Z_2, and T_2 in Hypothesis 2 are pointwise orthonormal.

Definition 3 *The smallest possible μ for which Hypothesis 2 holds is called the* strangeness index *of (1). Systems with vanishing strangeness index are called* strangeness-free.

The strangeness index can be considered as a generalization of the *differentiation index* as introduced in [7], see [51] for a detailed analysis of the relationship between different index concepts. It has been shown in [48], see also [51], that under some constant rank conditions, every uniquely solvable (regular) linear DAE of the form (1) with sufficiently smooth E, A satisfies Hypothesis 2 and that there exists a *reduced system*

$$\hat{E}(t)\dot{x} = \hat{A}(t)x + \hat{f}(t), \tag{7}$$

that is strangeness-free and has the same solution as (1), where

$$\hat{E}(t) = \begin{bmatrix} \hat{E}_1(t) \\ 0 \end{bmatrix}, \ \hat{A} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix},$$

with block entries

$$\hat{E}_1 = Z_1^T E, \quad \hat{A}_1 = Z_1^T A, \quad \hat{A}_2 = Z_2^T N_{\hat{\mu}} [\, I_n \ 0 \ \cdots \ 0 \,]^T. \tag{8}$$

System (7) can be viewed as a different representation (remodeling) of system (1), where all necessary differentiations of (1) that are needed to describe the solution are already represented in the model. This representation avoids many of the numerical difficulties that are associated with DAEs that have a non-vanishing strangeness-index (differentiation index larger than 1), see [8, 51]. The reduction to the form (7) can be carried out in a numerically stable way at any time instance t, see [55, 51] and this idea can also be extended to over- and underdetermined systems as well as locally to general nonlinear systems, [54, 50, 51]. For this reason, in the following, we assume that the DAE is given in the form (7) and for ease of notation we leave off the hats.

2.2 Spectral intervals for ODEs and their approximation

In this section, we give a brief review of important qualitative and numerical results on the spectral theory for linear ODEs. We refer to [25, 26, 29, 45] for more details. Consider the linear homogeneous ODE

$$\dot{x} = A(t)x, \quad t \in \mathbb{I}, \tag{9}$$

where $A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ is a bounded matrix function. For a nontrivial solution x of (9), we define

$$\lambda^u(x) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\|, \quad \lambda^l(x) = \liminf_{t\to\infty} \frac{1}{t} \ln \|x(t)\|. \tag{10}$$

These quantities, which characterize the asymptotic behavior of the solution x, are called the *upper and lower Lyapunov exponents*, respectively.

Since A is bounded, the Lyapunov exponents are finite. Let $X : \mathbb{I} \to \mathbb{R}^{n\times n}$ be a *fundamental solution matrix* of (9), i.e. X satisfies,

$$\dot{X} = A(t)X, \tag{11}$$

then, the Lyapunov exponents for the i-th column of X are well-defined as

$$\lambda^u(Xe_i), \quad \text{and} \quad \lambda^\ell(Xe_i), \quad i = 1, 2, ..., n, \tag{12}$$

where e_i denotes the i-th unit vector. Without loss of generality, we may assume that the columns of X are ordered such that the upper Lyapunov exponents are ordered decreasingly as

$$\lambda^u(Xe_1) \geq \lambda^u(Xe_2) \geq ... \geq \lambda^u(Xe_n).$$

When $\sum_{i=1}^n \lambda^u(Xe_i)$ is minimized with respect to all possible fundamental solution matrices, then the $\lambda_i^u = \lambda^u(Xe_i)$ are called *(upper) Lyapunov exponents* of (9) and the columns of the corresponding fundamental solution matrix are said to form a *normal basis* of (9).

In general, the Lyapunov exponents satisfy the *Lyapunov inequality*

$$\sum_{i=1}^{n} \lambda_i^u \geq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr}(A(s))ds, \tag{13}$$

where $\operatorname{tr}(A(s))$ is the trace of the matrix $A(s)$. Furthermore, it is always possible to construct a normal basis from an arbitrary fundamental solution matrix. Hence, we may assume that the columns of X form a normal basis with decreasingly ordered Lyapunov exponents. We may proceed analogously for the lower Lyapunov exponents. Alternatively, we may consider the *adjoint equation* associated with (9) given by

$$\dot{y} = -A^T(t)y, \tag{14}$$

and let $\{-\mu_i^u\}_{i=1}^n$ be the upper Lyapunov exponents (ordered increasingly) of (14) with associated fundamental solution matrices $Y(t)$. Then, see [15], the fundamental solution matrices satisfy the *Lagrange identity*

$$Y^T(t)X(t) = Y^T(0)X(0) \quad \text{for all } t \in \mathbb{I}.$$

If the columns of X form a normal basis for (9), then the columns of Y form a normal basis for (14). Hence, we have the following relationship between the Lyapunov exponents of the ODE and its adjoint,

$$\lambda_i^\ell = -\mu_i^u, \quad i = 1, 2, ..., n.$$

Definition 4 *The* Lyapunov spectrum Σ_L *of (9) is the union of intervals*

$$\Sigma_L := \bigcup_{i=1}^{n} [\lambda_i^\ell, \lambda_i^u],$$

and the intervals $[\lambda_i^\ell, \lambda_i^u], i = 1, 2, ..., n$, *are called* Lyapunov spectral intervals. *If each of the Lyapunov spectral intervals shrinks to a point, i.e., if* $\lambda_i^\ell = \lambda_i^u$ *for all* $i = 1, 2, ..., n$, *then the system is called* Lyapunov-regular. *(Note that this property is often called just* regularity *and it is not related to the regularity notion for DAEs.) If a system is Lyapunov-regular, then we simply write* λ_i *for the Lyapunov exponents.*

Remark 5 The definition of (Lyapunov-)regularity in Definition 4 is due to Perron [64] and it is different from, but equivalent to, the original definition by Lyapunov [58], which defines the system (9) to be regular if the time average of the trace has a finite limit and equality holds in (13). In [19], only the set of all upper Lyapunov exponents for (9) is called the Lyapunov spectrum. Clearly, these two different definitions of Lyapunov spectra are equivalent if and only if the system is Lyapunov-regular.

A fundamental property of Lyapunov exponents and Lyapunov spectra is that they are preserved under specific similarity transformations.

Definition 6 *A change of variables* $z = V^{-1}x$ *with an invertible matrix function* $V \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ *is called a* kinematic similarity transformation *if* V *and* V^{-1} *are bounded. If* $\dot{V}$ *is bounded as well, then it is called a* Lyapunov transformation.

The key idea for the numerical computation of Lyapunov exponents for Lyapunov-regular systems comes from the following two results.

Theorem 7 *[58] Let* $B = [b_{ij}] \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ *be bounded and upper-triangular. Then the system*

$$\dot{R} = B(t)R, \tag{15}$$

is Lyapunov-regular if and only if all the limits

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t b_{ii}(s)ds, \quad i = 1, 2, ..., n, \tag{16}$$

exist and these limits coincide with the Lyapunov exponents λ_i *for* $i = 1, 2, ..., n$.

Theorem 8 *[33, 64] Consider the linear system (9). Then there exists a Lyapunov transformation to upper triangular form, and moreover, this transformation can be chosen to be pointwise orthogonal.*

Proof. A constructive proof for the existence of an orthogonal Lyapunov transformation to triangular form is as follows. We want to determine an orthogonal matrix function $Q \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that the change of variables $X(t) = Q(t)R(t)$ transforms the system (11) to the triangular form (15). Inserting this change of variables we see that B has to satisfy

$$B(t) = Q^T(t)A(t)Q(t) - Q^T(t)\dot{Q}(t).$$

We introduce the skew-symmetric matrix function $S(Q) := Q^T\dot{Q} = Q^TAQ - B$. Then, since Q is orthogonal and B is upper triangular, we obtain the strict lower triangular part of the skew-symmetric matrix function S by the corresponding part of Q^TAQ and the remaining parts by skew-symmetry. Thus, Q is well-defined and (numerically) computable as the unique solution of the initial value problem

$$\dot{Q} = QS(Q), \quad Q(0) = Q_0$$

where Q_0 is the Q factor in a QR factorization of $X(0)$. □

It should be noted that the norm of a fundamental solution matrix X of (11) is preserved under orthogonal changes of variable. Hence, if the columns of X form a normal basis, then the columns of $R = Q^TX$ form a normal (and in addition, triangular) basis as well.

The constructive proof of Theorem 8 gives the basic idea for computing Lyapunov exponents as well as Lyapunov spectra. Unfortunately, one faces a (numerical) difficulty arising from the sensitivity of Lyapunov exponents under small perturbations. Even if the system is Lyapunov-regular, then it is not guaranteed that this property is preserved under small perturbations, i.e. the Lyapunov spectrum of a slightly perturbed system does not necessarily stay close to that of the unperturbed system. To treat this problem, we introduce the following concept of stability.

Definition 9 *Consider a homogeneous linear system of the form (9) and a perturbed system $\dot{x} = [A(t) + \Delta A(t)]x$. The upper Lyapunov exponents, $\lambda_1^u \geq ... \geq \lambda_n^u$, are called* stable *if for any $\varepsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\sup_{t\geq 0} \|\Delta A(t)\| < \delta$ implies*

$$|\lambda_i^u - \nu_i^u| < \epsilon, \quad i = 1, \dots, n,$$

where the quantities ν_i^u, $i = 1, \dots, n$ are the (decreasingly ordered) upper Lyapunov exponents of $\dot{x} = [A(t) + \Delta A(t)]x$.

To give a characterization when Lyapunov exponents are stable, we need the following concept, see [25].

Definition 10 *A fundamental solution matrix X of (11) (as well as the columns of X) are said to be* integrally separated *if for $i = 1, 2, ..., n-1$, there exist constants $\beta > 0$ and $\gamma > 0$ such that*

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq \gamma e^{\beta(t-s)},$$

for all t, s with $t \geq s$. If the system (9) has an integrally separated fundamental solution matrix, then it is called an integrally separated system.

We will see later that integral separation plays a key role in the computation of spectral intervals. The properties of an integrally separated system can be summarized as follows, see [25] and the references therein.

Proposition 11

i) Integral separation is invariant under Lyapunov transformations (or kinematic similarity transformations).

ii) An integrally separated system has pairwise distinct upper (and pairwise distinct lower) Lyapunov exponents.

iii) Distinct upper Lyapunov exponents are stable if and only if there exists an integrally separated fundamental solution matrix.

iv) If a system is integrally separated, then so is its adjoint system and thus the lower Lyapunov exponents are stable as well.

v) Let $\mathcal{B}$ be the Banach space of continuous bounded matrix valued functions A, supplied with the norm $\|A\| = \sup_{t\geq 0} \|A(t)\|$. Systems with integral separation property form an open and dense subset of $\mathcal{B}$, i.e., integral separation is a generic property in $\mathcal{B}$.

For the triangular system (15), one can define the integral separation by using an alternative concept.

Definition 12 *Consider the triangular system (15). The diagonal functions $b_{ii}(t)$, $i = 1, 2, ..., n$, of B are said to be* integrally separated *if for $i = 1, 2, ..., n-1$ there exist constants $\beta > 0$ and $\gamma \in \mathbb{R}$ such that*

$$\int_s^t [b_{ii}(\tau) - b_{i+1,i+1}(\tau)]d\tau \geq \beta(t-s) - \gamma, \quad \text{for all } t \geq s. \tag{17}$$

Thus, if the diagonal elements of B are integrally separated, then for t, s such that $t-s$ is sufficiently large, the integral in the left hand side of (17) is strictly positive.

Definition 13 *Consider a scalar continuous function f and suppose that $H > 0$. The* Steklov function *f^H is defined by*

$$f^H(t) := \frac{1}{H}\int_t^{t+H} f(\tau)d\tau.$$

The following two results provide the basis for numerical tests of integral separability.

Theorem 14 *[1] Two scalar continuous functions f_1, f_2 are integrally separated if and only if there exists scalar $H > 0$ such that their Steklov difference is positive, i.e., for H sufficiently large, there exists $\beta > 0$ such that*

$$f_1^H(t) - f_2^H(t) \geq \beta > 0, \quad \text{for all } t \geq 0.$$

Theorem 15 *[26, 29] For the system (15) with B bounded, continuous, and triangular, a necessary and sufficient condition for the existence of an integrally separated fundamental solution matrix is that the diagonal elements of B are integrally separated.*

An important consequence of the existence of an integrally separated fundamental solution matrix is that in this case we can numerically approximate the Lyapunov exponents using the following result.

Theorem 16 *[25, 26] Consider (15) with B bounded, continuous, and triangular. If the diagonal of B is integrally separated, then $\Sigma_L = \Sigma_{CL}$, where*

$$\Sigma_{CL} := \bigcup_{i=1}^n [\lambda_{i,i}^\ell, \lambda_{i,i}^u],$$

with

$$\lambda_{i,i}^\ell := \liminf_{t\to\infty} \frac{1}{t}\int_0^t b_{ii}(s)ds, \ \lambda_{i,i}^u := \limsup_{t\to\infty} \frac{1}{t}\int_0^t b_{ii}(s)ds, \ i = 1, 2, ..., n.$$

Thus, under the integral separability condition, the Lyapunov spectrum of (9) can be computed by first transforming the system into the triangularized form (15) and then approximating the Lyapunov spectrum for the scalar equations associated with the diagonal elements. In general, if the system is not integrally separated, then we have $\Sigma_{CL} \subseteq \Sigma_L$. However, in [29] it has been shown that equality holds under an assumption that is a weaker condition than the integral separation.

Another concept that can be used to describe the behavior of solutions to (9) is that of Bohl exponents [6], see also [19].

Definition 17 *Let x be a nontrivial solution of (9). The* (upper) Bohl exponent $\kappa_B^u(x)$ *of this solution is the greatest lower bound of all those values ρ for which there exist constants $N_\rho > 0$ such that*

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\| \tag{18}$$

for any $t \geq s \geq 0$. If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$.

Similarly, the lower Bohl exponent $\kappa_B^\ell(x)$ is the least upper bound of all those values ρ' for which there exist constants $N'_\rho > 0$ such that

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t. \tag{19}$$

The interval $[\kappa_B^\ell(x), \kappa_B^u(x)]$ is called the Bohl interval *of the solution x.*

It follows directly from the definition, that Lyapunov exponents and Bohl exponents are related via

$$\kappa_B^\ell(x) \leq \lambda^\ell(x) \leq \lambda^u(x) \leq \kappa_B^u(x).$$

Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$. If the supremum of the upper Lyapunov exponents for all solutions to (9) is negative, then the system is *asymptotically stable*, a property which for constant coefficient systems is characterized by the property that all eigenvalues have negative real part. If the same holds for the supremum of the upper Bohl exponents of (9), then the system is *(uniformly) exponentially stable.*

One then has the following formulas characterizing the Bohl exponents, see e.g. [19],

$$\kappa_B^u(x) = \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \quad \kappa_B^\ell(x) = \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}.$$

It is also well-known [19] that if the coefficient matrix function $A(t)$ is *integrally bounded*, i.e., if

$$\sup_{t\geq 0} \int_t^{t+1} \|A(s)\|\, ds < \infty,$$

then the Bohl exponents for (9) are finite. For a continuous bounded function A, this condition trivially holds. Moreover, unlike the Lyapunov exponents, the Bohl exponents are stable without any extra assumption.

A third alternative concept to Lyapunov spectra is the theory of *exponential dichotomy spectra* developed by Sacker and Sell [69].

Definition 18 *The fundamental matrix solution X is said to admit an* exponential dichotomy *if there exist a constant projection matrix $P : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ (i.e. $P^2 = P$) and constants $\alpha, \beta > 0$, as well as $K, L \geq 1$, such that*

$$\begin{array}{rcl} \left\|X(t)PX^{-1}(s)\right\| & \leq & Ke^{-\alpha(t-s)}, \quad t \geq s, \\ \left\|X(t)(I-P)X^{-1}(s)\right\| & \leq & Le^{\beta(t-s)}, \quad t \leq s. \end{array} \tag{20}$$

The Sacker-Sell (or exponential-dichotomy) spectrum $\Sigma_S \subset \mathbb{R}$ *for (9) is given by those values λ such that the* shifted system

$$\dot{x}_\lambda = [A(t) - \lambda I]x_\lambda$$

does not have exponential dichotomy. The complement of Σ_S is called the resolvent set.

It is worth noting that the property that a system possesses an exponential dichotomy, as well as the exponential dichotomy spectrum are preserved under Lyapunov transformations (or kinematic similarity transformations). In [69] it has been shown that Σ_S is the union of at most n disjoint closed intervals, and furthermore that it is stable. It is also not difficult to see that

$$\Sigma_L \subseteq \Sigma_S.$$

Theorem 19 *[30] Consider the system (15) with $B = [b_{ij}]$ bounded, continuous, and upper triangular. The Sacker-Sell spectrum of the triangular system (15) and that of the corresponding diagonal system*

$$\dot{x} = D(t)x, \qquad \textit{with } D(t) = \mathrm{diag}(b_{11}(t), \ldots, b_{nn}(t)), \quad t \in \mathbb{I}, \tag{21}$$

coincide.

Thus, one can retrieve Σ_S of the original system (9) from the diagonal elements of the triangularized system (15). Furthermore, the Sacker-Sell spectrum of the diagonal system (21) can be approximated as follows. For $i = 1, 2, ..., n$, and for $\lambda \in \mathbb{R}$, one introduces the two diagonal systems

$$\dot{y}_i = \begin{bmatrix} \lambda & 0 \\ 0 & b_{ii}(t) \end{bmatrix} y_i \quad \text{and} \quad \dot{y}_i = \begin{bmatrix} b_{ii}(t) & 0 \\ 0 & \lambda \end{bmatrix} y_i. \tag{22}$$

Considering the sets $\Lambda_i := \{\lambda \in \mathbb{R} : \text{ the systems in (22) are not integrally separated}\}$, $i = 1, 2, ..., n$ one defines the *integral separation spectrum* Σ_I for (21) as

$$\Sigma_I := \bigcup_{i=1}^{n} \Lambda_i. \tag{23}$$

Then it has been shown in [25, 26] that for a diagonal system $\Sigma_I = \Sigma_S$. For $H > 0$ and $i = 1, 2, ..., n$ one then may compute

$$\alpha_i^H := \inf_{t \geq 0} \frac{1}{H} \int_t^{t+H} b_{ii}(s)ds \quad \text{and} \quad \beta_i^H := \sup_{t \geq 0} \frac{1}{H} \int_t^{t+H} b_{ii}(s)ds, \tag{24}$$

and these quantities present practically (but expensively) numerically computable quantities to approximate the integral separation spectrum for (21), and thus, the Sacker-Sell spectrum for (15).

Theorem 20 *[26] Consider the diagonal system (21) and let Λ_i, $i = 1, 2, ..., n$, be the i-th interval in Σ_S for this system. Then, for $H > 0$ sufficiently large,*

$$\Lambda_i = [\alpha_i^H, \beta_i^H], \quad \text{for all } i = 1, 2, ..., n,$$

where α_i^H, β_i^H are defined in (24).

In order to discuss the relationship between the Sacker-Sell spectrum and the Bohl intervals, we first consider the scalar case.

Lemma 21 *For a scalar equation*

$$\dot{x} = a(t)x, \quad t \geq 0, \tag{25}$$

where a is a bounded and continuous function, the Sacker-Sell spectrum and the Bohl interval coincide, i.e., $\Sigma_S = [\kappa_B^l, \kappa_B^u]$.

Proof. Let $\lambda \in \mathbb{R}$ such that $\lambda \notin \Sigma_S$. Then there exist constants $\alpha, \beta > 0$ and $K, L \geq 1$ such that for $t \geq s$, either $e^{-\lambda(t-s)} e^{\int_s^t a(\tau)d\tau} \leq K e^{-\alpha(t-s)}$ or $e^{\lambda(t-s)} e^{-\int_s^t a(\tau)d\tau} \leq L e^{-\beta(t-s)}$, which holds if and only if either $\lambda - \alpha \geq \kappa_B^u$ or $\lambda + \beta \leq \kappa_B^l$, respectively, i.e. $\lambda \notin [\kappa_B^l, \kappa_B^u]$. Consequently, the Bohl interval is contained in the Sacker-Sell spectrum. The proof of the converse direction is similar. The proof is complete. □

For the scalar case we thus have the following analytic representation of the Bohl exponents.

Proposition 22 *Consider the scalar equation (25), where a is a bounded and continuous function. Then there exists a (sufficiently large but finite) $H > 0$ such that*

$$\inf_{s\geq 0} \frac{1}{H}\int_s^{s+H} a(\tau)d\tau = \liminf_{s,t-s\to\infty} \frac{1}{t-s}\int_s^t a(\tau)d\tau = \kappa_B^l,$$

and

$$\sup_{s\geq 0} \frac{1}{H}\int_s^{s+H} a(\tau)d\tau = \limsup_{s,t-s\to\infty} \frac{1}{t-s}\int_s^t a(\tau)d\tau = \kappa_B^u.$$

Proof. To verify the first identity in each formula, it suffices to repeat the arguments in the proof of [26, Theorem 8.4]. The second identity follows directly by the definition of Bohl exponents. □

The presented summary of results for ODEs shows that triangularization allows the numerical computation of the Sacker-Sell spectra and, therefore, also of inclusion regions for the Lyapunov spectra. Essentially there are two ways to realize this triangularization, i.e., the transformation of the system (9) to the form (15). Using the (discrete) QR-algorithm [37] one may compute the triangular form for discrete time points. With this, one may compute the fundamental solution matrix X directly by an appropriate numerical integration. As an alternative way, one can use the continuous QR-algorithm to determine the triangular matrix coefficient B in (15) by solving the matrix differential equation for the orthogonal matrix function $Q(t)$ [22, 25, 26]. For further details on numerical methods and their implementation, see [21, 24, 25, 26, 27].

3 Spectral theory for DAEs

In this section we generalize the classical spectral results for ODEs that we have reviewed in Subsection 2.2 to DAEs. An essential step in the computation of spectral intervals for linear DAEs of the form (1) is to first transform the system to a reduced strangeness-free form (7), which has the same solution set as (1), see [51], and then to consider the spectral results in this framework. This transformation will not alter the spectral sets which will be defined in terms of the fundamental solution matrices that have not changed. Under Hypothesis 2 this transformation can always be done and this reduced form can even be computed numerically at every time instance t. For this reason, we may assume in the following that the system is given in the reduced form (7), i.e. we assume that our homogeneous DAE is already strangeness-free and has the form

$$E(t)\dot{x} = A(t)x, \quad t \in \mathbb{I}, \tag{26}$$

where

$$E(t) = \begin{bmatrix} E_1(t) \\ 0 \end{bmatrix}, \; A(t) = \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix},$$

and $E_1 \in C(\mathbb{I}, \mathbb{R}^{d\times n})$ and $A_2 \in C(\mathbb{I}, \mathbb{R}^{(n-d)\times n})$ are of full column rank.

Note that for convenience of notation, we have left off the hats in the coefficients.

3.1 Lyapunov exponents and Lyapunov spectral intervals

We first generalize the concepts of Lyapunov and Bohl exponents.

Definition 23 *A matrix function $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times k})$, $d \leq k \leq n$, is called* fundamental solution matrix of (26) *if each of its columns is a solution to (26) and* rank $X(t) = d$*, for all $t \geq 0$.*

A fundamental solution matrix is said to be maximal *if $k = n$ and* minimal *if $k = d$, respectively. A maximal fundamental matrix solution, denoted by $X(t,s)$, is called* principal *if it satisfies the* projected initial condition $E(t_0)(X(t_0,t_0) - I) = 0$*, for some $t_0 \geq 0$.*

We see that a major difference between ODEs and DAEs is that fundamental solution matrices for DAEs are not necessarily square and of full-rank. Every fundamental solution matrix has exactly d linearly independent columns and a minimal fundamental matrix solution can be easily made maximal by adding $n-d$ zero columns.

Definition 24 *For a given fundamental solution matrix X of a strangeness-free DAE system of the form (26), and for $d \le k \le n$, we introduce*

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\| \ \text{ and } \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\|, \quad i = 1, 2, ..., k.$$

The columns of a minimal fundamental solution matrix form a normal basis *if $\Sigma_{i=1}^d \lambda_i^u$ is minimal. The $\lambda_i^u, i = 1, 2, ..., n$, belonging to a normal basis are called* (upper) Lyapunov exponents *and the intervals $[\lambda_i^\ell, \lambda_i^u], i = 1, 2, ..., d$, are called* Lyapunov spectral intervals. *The set of the Lyapunov spectral intervals is called the* Lyapunov spectrum *of (26).*

Definition 25 *Suppose that $U \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ are pointwise nonsingular matrix functions such that V and V^{-1} are bounded. Then the transformed DAE system*

$$\tilde{E}(t)\dot{\tilde{x}} = \tilde{A}(t)\tilde{x}, \tag{27}$$

with $\tilde{E} = UEV$, $\tilde{A} = UAV - UE\dot{V}$ and $x = V\tilde{x}$ is called globally kinematically equivalent *to (26) and the transformation is called a* global kinematical equivalence transformation. *If, furthermore, also U and U^{-1} are bounded then we call this a* strong global kinematical equivalence transformation.

It is clear that the Lyapunov exponents of a DAE system as well as the normality of a basis formed by the columns of a fundamental solution matrix are preserved under global kinematic equivalence transformations.

Lemma 26 *Consider a strangeness-free DAE system of the form (26) with continuous coefficients and a minimal fundamental solution matrix X. Then there exist pointwise orthogonal matrix functions $U \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that in the fundamental matrix equation $E\dot{X} = AX$ associated with (26), the change of variables $X = VR$, with $R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ and $R_1 \in C^1(\mathbb{I}, \mathbb{R}^{d\times d})$, and the multiplication of both sides of the system from the left with U^T leads to the system*

$$\mathcal{E}_1\dot{R}_1 = \mathcal{A}_1 R_1, \tag{28}$$

where $\mathcal{E}_1 := U_1^T E V_1$ is pointwise nonsingular and $\mathcal{A}_1 := U_1^T A V_1 - U_1^T E \dot{V}_1$. Here, U_1, V_1 are the matrix functions consisting of the first d columns of U, V, respectively.

Proof. Since a smooth and full column rank matrix function has a smooth QR-decomposition, see [24, Prop. 2.3], there exists a pointwise orthogonal matrix function V such that $X = VR = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$, where R_1 is pointwise nonsingular. By substituting $X = VR$ into the fundamental matrix equation $E\dot{X} = AX$, we obtain

$$EV \begin{bmatrix} \dot{R}_1 \\ 0 \end{bmatrix} = (AV - E\dot{V}) \begin{bmatrix} R_1 \\ 0 \end{bmatrix}.$$

Since, by assumption, the first d rows of E are of full row rank, we have that the first d columns of EV, given by EV_1, have full column rank. Thus, there exists a smooth QR-decomposition

$$EV_1 = U \begin{bmatrix} \mathcal{E}_1 \\ 0 \end{bmatrix},$$

where U is pointwise orthogonal and $\mathcal{E}_1$ is pointwise nonsingular. Looking at the leading $d \times d$ block in the transformed equation, we arrive at

$$\mathcal{E}_1 \dot{R}_1 = [U_1^T A V_1 - U_1^T E \dot{V}_1] R_1,$$

which proves the assertion. □

The system (28) is an implicitly given ODE, since $\mathcal{E}_1$ is nonsingular. It is called *essentially underlying implicit ODE system* of (26). Since orthonormal changes of basis keep the Euclidean norm invariant, the Lyapunov exponents of the columns of the matrices X and R, and therefore those of the two systems are the same.

Theorem 27 *Let Z be a minimal fundamental solution matrix for (26) such that the upper Lyapunov exponents of its columns are ordered decreasingly. Then there exists a nonsingular upper triangular matrix $C \in \mathbb{R}^{d\times d}$ such that the columns of $X(\cdot) = Z(\cdot)C$ form a normal basis.*

Proof. By Lemma 26, there exists a pointwise orthogonal matrix function V such that $V^T Z = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ with R_1 satisfying the implicit system

$$\mathcal{E}_1 \dot{R}_1 = \mathcal{A}_1 R_1,$$

or equivalently, satisfying the explicit ODE system

$$\dot{R}_1 = \mathcal{E}_1^{-1} \mathcal{A}_1 R_1.$$

Here $\mathcal{E}_1, \mathcal{A}_1$ are defined as in Lemma 26. Note that the Lyapunov exponents of Z are exactly the Lyapunov exponents of R_1. Due to Lyapunov's theorem on the construction of a normal basis for ODEs (see [58, p. 233]), there exists an upper triangular nonsingular matrix $C \in \mathbb{R}^{d\times d}$ such that the columns of $R_1 C$ form a normal basis of (28). This implies that the columns of $RC = V^T ZC$ form a normal basis as well. Because the normality is preserved under global kinematical equivalence transformations, the proof is complete. □

As in the case of ODEs it is useful to introduce the adjoint equation to (26), see e.g. [4, 52].

Definition 28 *The DAE system*

$$\frac{d}{dt}(E^T y) = -A^T y, \qquad or \qquad E^T(t)\dot{y} = -[A^T(t) + \dot{E}^T(t)]y, \quad t \in \mathbb{I}, \tag{29}$$

is called the adjoint system *associated with (26).*

Lemma 29 *Fundamental solution matrices X, Y of (26) and its adjoint equation (29) satisfy the Lagrange identity*

$$Y^T(t)E(t)X(t) = Y^T(0)E(0)X(0),\ t \in \mathbb{I}.$$

Let $U, V \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ define a strong global kinematic equivalence for system (26). Then the adjoint of the transformed DAE system (27) is strongly globally kinematically equivalent to the adjoint of (26).

Proof. Differentiating the product $Y(t)^T E(t)X(t)$ and using the definition of the adjoint equation, we obtain (leaving off the arguments) that

$$\frac{d}{dt}(Y^T E)X + Y^T E\dot{X} = -Y^T AX + Y^T AX = 0$$

and hence the Lagrange identity follows. By assumption, the matrices V^T, U^T define a strong global kinematic equivalence transformation for the adjoint equation leading to the adjoint of (27). □

Despite these results, the relationship between the dynamics of a DAE system and its adjoint is more complicated than in the ODE case, except if some extra assumptions are added. In order to see this and to better understand the dynamical behavior of DAEs, we apply an orthogonal change of basis to transform the system (26) into appropriate condensed forms.

Theorem 30 *Consider the strangeness-free DAE system (26). If the pair of coefficient matrices is sufficiently smooth, then there exists an orthogonal matrix function $\hat{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that by the change of variables $\hat{x} = \hat{Q}^T x$, the submatrix E_1 is compressed, i.e., the transformed system has the form*

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \in \mathbb{I}. \tag{30}$$

Furthermore, the system (30) is still strangeness-free and thus $\hat{E}_{11}$ and $\hat{A}_{22}$ are pointwise nonsingular.

Proof. In order to show the existence of appropriate transformations, we use again the theorem on the existence of smooth QR decompositions, see [20, Prop. 2.3] and [51, Thm. 3.9]. If E is continuously differentiable, then there exist a matrix function $\hat{Q}_1 \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$ with pointwise orthonormal columns and a pointwise nonsingular $\hat{E}_{11} \in C^1(\mathbb{I}, \mathbb{R}^{d\times d})$ such that

$$E_1 = \hat{E}_{11}\hat{Q}_1^T.$$

Since d rows of $\hat{Q}_1^T$ pointwise form orthonormal basis in $\mathbb{R}^n$ and since the Gram-Schmidt process is continuous, we can complete this basis by adding a smooth (and pointwise orthonormal) matrix $\hat{Q}_2 \in C^1(\mathbb{I}, \mathbb{R}^{n\times(n-d)})$ so that

$$\hat{Q} := \begin{bmatrix} \hat{Q}_1^T \\ \hat{Q}_2^T \end{bmatrix}$$

is pointwise orthogonal. Then, we have

$$E_1 = \begin{bmatrix} \hat{E}_{11} & 0 \end{bmatrix} \hat{Q}.$$

Since we have started with a strangeness-free system, it follows that the corresponding transformed matrix $\hat{A}$ partitioned as in (30) has a nonsingular block $\hat{A}_{22}$. □

Remark 31 Alternatively we could have used a transformation in Theorem 30 that compresses the block A_2, thus obtaining a transformed system

$$\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \tilde{x}, \quad t \in \mathbb{I}. \tag{31}$$

with $\tilde{E}_{11}$ and $\tilde{A}_{22}$ nonsingular. The proof for the condensed from (31) follows analogously to that of Theorem 30 by compressing the second block row of A. Most of the results that we present below carry over directly to this system. For the analysis we prefer to use (30), while for the construction of numerical algorithms the form (31).

System (30) is a strangeness-free DAE in *semi-implicit form.* Since $\hat{Q}$ is orthogonal and since the Euclidean norm is used, it follows that $\|\hat{x}\| = \|x\|$. Performing this transformations allows to separate the differential and the algebraic components of the solutions. Partitioning $\hat{x} = [\hat{x}_1^T, \hat{x}_2^T]^T$ appropriately, solving for the second component and substituting it into the first block equation one gets the *associated underlying (implicit) ODEs,*

$$\hat{E}_{11}\dot{\hat{x}}_1 = \hat{A}_s\hat{x}_1, \tag{32}$$

where $\hat{A}_s := \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}$ denotes the Schur complement. For (31), the associated underlying implicit ODE system is

$$\tilde{E}_{11}\dot{\tilde{x}}_1 = \tilde{A}_{11}\tilde{x}_1, \tag{33}$$

respectively. The following result extends the asymptotic stability results of [53] in terms of Lyapunov exponents.

Theorem 32 *Let $\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21})$ be the upper Lyapunov exponent of the matrix function $\hat{A}_{22}^{-1}\hat{A}_{21}$. If*

$$\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21}) \leq 0, \tag{34}$$

then the (upper and lower) Lyapunov exponents of (30) and those of (32) coincide if they are both ordered decreasingly.

An analogous result holds for the Lyapunov exponents of (31) and those of (33).

Proof. It is clear that each minimal fundamental solution matrix $\hat{X}$ of (30) has the form

$$\hat{X} = \begin{bmatrix} \hat{X}_1 \\ \hat{A}_{22}^{-1}\hat{A}_{21}\hat{X}_1 \end{bmatrix},$$

where $\hat{X}_1$ is a fundamental solution of (32). Let $\hat{x}$ be a column of $\hat{X}$. Then

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{A}_{22}^{-1}\hat{A}_{21}\hat{x}_1 \end{bmatrix},$$

where $\hat{x}_1$ is the corresponding column of $\hat{X}_1$. Using the triangle inequality, we then have

$$\|\hat{x}_1\| \leq \|\hat{x}\| \leq \sqrt{2}(1 + \left\|\hat{A}_{22}^{-1}\hat{A}_{21}\right\|)\,\|\hat{x}_1\|, \tag{35}$$

from which it follows that

$$\lambda^u(\hat{x}_1) \leq \lambda^u(\hat{x}) \leq \lambda^u(1 + \left\|\hat{A}_{22}^{-1}\hat{A}_{21}\right\|) + \lambda^u(\hat{x}_1) = \lambda^u(\hat{x}_1)$$

and thus, $\lambda^u(\hat{x}) = \lambda^u(\hat{x}_1)$. Analogously we prove that $\lambda^l(\hat{x}_1) \leq \lambda^l(\hat{x})$.

Since $\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21}) \leq 0$, for any $\varepsilon > 0$, there exists $T \geq 0$ such that

$$\frac{1}{t}\ln(1 + \left\|\hat{A}_{22}^{-1}\hat{A}_{21}\right\|) \leq \varepsilon \quad \text{for all } t \geq T,$$

which implies that $1 + \left\|\hat{A}_{22}^{-1}\hat{A}_{21}\right\| \leq e^{\varepsilon t}$, for all $t \geq T$. As in the case of upper exponents, we have

$$\|\hat{x}(t)\| \leq \sqrt{2}e^{\varepsilon t}\,\|\hat{x}_1(t)\|, \quad t \geq T.$$

Hence, we obtain that $\lambda^l(\hat{x}) \leq \varepsilon + \lambda^l(\hat{x}_1)$. Since ε can be chosen arbitrarily, it follows that $\lambda^l(\hat{x}) \leq \lambda^l(\hat{x}_1)$. Thus, it follows that $\lambda^l(\hat{x}_1) = \lambda^l(\hat{x})$. As a consequence of this construction, the columns of the fundamental solution matrix $\hat{X}$ of (30) form a normal basis if and only if the corresponding columns of X_1 form a normal basis of (32).

The proof for the Lyapunov exponents of (31) and (33) is trivial. □

Remark 33 A sufficient condition for (34) is that $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded or has a less than exponential growth rate. This is for example the case if there exist constants $\gamma > 0$ and $k \in \mathbb{N}$ such that $\left\|\hat{A}_{22}^{-1}\hat{A}_{21}(t)\right\| \leq \gamma t^k$ for all $t \in \mathbb{I}$.

Alternatively, we could use (31) and the corresponding underlying ODE (33). In this case such a boundedness or restriction in the growth rate is not required. However, a similar boundedness condition on $\tilde{A}_{22}^{-1}$ in (31) will be needed, if one considers perturbed or inhomogeneous DAE systems.

The next step of our analysis is the extension of the concept of Lyapunov-regularity to DAEs.

Definition 34 *The DAE system (26) is said to be* Lyapunov-regular *if each of its Lyapunov spectral intervals reduces to a point, i.e., $\lambda_i^l = \lambda_i^u, i = 1, 2, ..., d$.*

To analyze the Lyapunov-regularity of the DAE system (26), we again study the transformed semi-implicit system (30) and the underlying ODE system. Since the Lyapunov exponents for a DAE system are preserved under global kinematic equivalence transformations, also the Lyapunov-regularity is preserved, i. e. the DAE system (26) is Lyapunov-regular if and only if the semi-implicit DAE system (30) is Lyapunov-regular. Thus, we immediately have the following equivalence result.

Proposition 35 *Consider the DAE system (30) and suppose that the boundedness condition (34) holds. Then, the DAE system (30) is Lyapunov-regular if and only if the underlying ODE system (32) is Lyapunov-regular.*

Unlike for ODEs, to obtain the equivalence between the Lyapunov-regularity of (26) and that of its adjoint system we need some extra conditions.

Theorem 36 *Consider the DAE system (30) and suppose that the boundedness condition (34) holds. Assume further, that for the transformed system (30) the conditions*

$$\lambda^u(\hat{A}_{12}\hat{A}_{22}^{-1}) \leq 0, \quad \lambda^u(\hat{E}_{11}) \leq 0, \quad \lambda^u(\hat{E}_{11}^{-1}) \leq 0. \tag{36}$$

hold. If λ_i^l are the lower Lyapunov exponents order of (26) and $-\mu_i^u$ are the upper Lyapunov exponents of the adjoint system (29), both in increasing order, then

$$\lambda_i^l = \mu_i^u, \quad i = 1, 2, ..., d,$$

Furthermore, system (26) is regular if and only if (29) is regular, and in this case we have the Perron identity

$$\lambda_i = \mu_i, \quad i = 1, 2, ..., d, \tag{37}$$

where $\{-\mu_i\}_{i=1}^d$ are the Lyapunov exponents of (29) in increasing order.

Proof. Without loss of generality, we may consider the adjoint system (29) for the semi-implicit form (30)

$$\begin{bmatrix} \hat{E}_{11}^T & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{y}} = -\left(\begin{bmatrix} \hat{A}_{11}^T + \dot{\hat{E}}_{11}^T & \hat{A}_{21}^T \\ \hat{A}_{12}^T & \hat{A}_{22}^T \end{bmatrix} \right) \hat{y}. \tag{38}$$

The underlying ODE of the adjoint system is then given by

$$\hat{E}_{11}^T \dot{\hat{y}}_1 = -(\hat{A}_{11}^T - \hat{A}_{21}^T \hat{A}_{22}^{-T} \hat{A}_{12}^T + \dot{\hat{E}}_{11}^T)\hat{y}_1, \tag{39}$$

which is exactly the adjoint of the underlying ODE system (32). It also follows immediately that the Lagrange identity $\hat{Y}_1^T \hat{E}_{11} \hat{X}_1 \equiv$ const holds for fundamental matrix solutions $\hat{X}_1, \hat{Y}_1$ of (32) and its adjoint, respectively. Note that if the columns of X_1 form a normal basis for (32) then those of $\hat{Y}_1 := \hat{E}_{11}^{-T} \hat{X}_1^{-T}$ form a normal basis for the adjoint. Hence, (37) holds for the Lyapunov spectra of the underlying ODE systems and due to the preservation of Lyapunov spectra under global kinematic equivalence, the proof is complete. $\square$

Note that the last two inequalities in (36) just imply $\lambda^u(\hat{E}_{11}) = \lambda^u(\hat{E}_{11}^{-1}) = 0$.

Corollary 37 *Suppose that (26) is Lyapunov-regular and the assumptions of Theorem 36 hold. Then, with A_s defined as in (30), the limit*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \operatorname{tr}(\hat{E}_{11}^{-1} \hat{A}_s(s)) ds,$$

exists and is equal to $\sum_{i=1}^d \lambda_i$.

If one of the conditions (34), (36) fails to hold, then in general the spectral symmetry between the DAE system (30), the underlying ODE system (32), and their adjoint systems does not hold.

The following examples show that the DAE and its adjoint may be Lyapunov-regular, but the Perron identity (37) does not hold and also the converse implication may not hold. It is also possible that the Lyapunov-regularity of a DAE system does not imply the Lyapunov-regularity of its adjoint and vice versa.

Example 38 Consider the DAE system

$$\begin{aligned} e^{at}\dot{x}_1 &= e^{at}\lambda x_1 + x_2, \\ 0 &= -e^{bt}x_2, \end{aligned} \tag{40}$$

for $t \in \mathbb{I}$, with constants $a \leq 0, b \leq 0$ and $\lambda \in \mathbb{R}$. The adjoint system is

$$\begin{aligned} e^{at}\dot{y}_1 &= -(e^{at}\lambda + ae^{at})y_1, \\ 0 &= -y_1 + e^{bt}y_2. \end{aligned} \tag{41}$$

It is easy to see that both (40) and (41) are Lyapunov-regular. The only Lyapunov exponent for (40) is λ, while the only Lyapunov exponent for (41) is $-\lambda - a - b$. So, the Perron identity (37) between the Lyapunov exponents does not hold if $a + b \neq 0$. In addition, the Lyapunov exponent of (41) is not necessarily equal to that of its underlying ODE. Note that in this example all the coefficient matrices are bounded.

Example 39 Consider the systems (40) and (41) as in Example 38 but assume that a is positive, i. e. the leading coefficient matrix is unbounded and assume that λ is given by the time-varying function $\lambda(t) = \sin(\ln(t+1)) + \cos(\ln(t+1))$. Then, the Lyapunov spectrum of (40) is $[-1, 1]$ and that of the adjoint (41) is $[-1 - a - b, 1 - a - b]$. Neither the DAEs nor their underlying ODEs are Lyapunov-regular. However, if $a + b = 2$, then (37) holds for the upper Lyapunov exponents but the spectra of the DAE and its adjoint are not symmetric at all.

As we have defined it, Lyapunov-regularity is an asymptotic property of solutions to a DAE system. Hence, the Lyapunov-regularity definition presented here seems to be more natural than that based on the Perron identity (37) given in [17]. Clearly, if the conditions (34) and (36) hold, then the different definitions of Lyapunov-regularity are equivalent.

The following two examples demonstrate the effect of the algebraic constraint on the dynamical behavior of solutions. We stress that again in these examples the coefficient matrices are bounded.

Example 40 Consider the DAE system

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ 0 &= x_1 - e^{-t+t\sin(t)}x_2. \end{aligned}$$

Here the underlying ODE is Lyapunov-regular, but the DAE itself is not. Conversely, consider the DAE

$$\begin{aligned} \dot{x}_1 &= [\sin(\ln(t+1) + \cos(\ln(t+1))]x_1, \\ 0 &= -x_1 + e^{t\sin(\ln(t+1))-t}x_2. \end{aligned}$$

Here the DAE is Lyapunov-regular but the underlying ODE is not.

Example 41 The DAE system

$$\begin{aligned} \dot{x}_1 &= -3x_1 + e^{t\sin(t)-t}x_2, \\ 0 &= x_2, \end{aligned}$$

is Lyapunov-regular. However, its adjoint system

$$\begin{aligned} \dot{y}_1 &= 3y_1, \\ 0 &= e^{t\sin(t)-t}y_1 + y_2, \end{aligned}$$

is not.

We see from these examples that not the boundedness of the original coefficient matrices, but the boundedness of the coefficients in (34), (36) is relevant.

In this section we have introduced the concepts of Lyapunov spectra and Lyapunov-regularity for strangeness-free DAEs of the form (26). Since these concepts only depend on the solution of the DAE and not on the representation of the system of DAEs, whether it is in the form (26) or in the general form (1), we immediately have all the results also for DAEs in the general form.

3.2 Stability of Lyapunov exponents

The analysis performed in the last subsection changes substantially if the DAE (26) is subject to perturbations, i. e. if one studies perturbed DAEs

$$(E(t) + \Delta E(t))\dot{x} = (A(t) + \Delta A(t))x, \quad t \in \mathbb{I}, \tag{42}$$

with perturbation functions $\Delta E(t),\ \Delta A(t)$. If we allow general perturbations, then it is very difficult to analyze the behavior of the system due to the fact that the strangeness-index may change or the solvability of the system may be destroyed, see [9, 34, 62]. The complete perturbation analysis for this case is still an **open problem** even for constant coefficient systems. For this reason we require that the pair of perturbation functions $(\Delta E, \Delta A),\ \Delta E,\ \Delta A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ are sufficiently smooth such that by applying a similar orthogonal transformation as from (26) to (30) (but not the same), we obtain

$$\begin{bmatrix} \hat{E}_{11} + \Delta\hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} + \Delta\hat{A}_{11} & \hat{A}_{12} + \Delta\hat{A}_{12} \\ \hat{A}_{21} + \Delta\hat{A}_{21} & \hat{A}_{22} + \Delta\hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \in \mathbb{I}. \tag{43}$$

If this is the case then we say that the perturbations are *admissible*.

Lemma 42 *Consider a strangeness-free DAE of the form (26) and the set $\mathcal{P}$ of all pairs of admissible perturbation functions $(\Delta E, \Delta A)$ such that in the transformed systems (43) the blocks $\hat{E}_{11}$ and $\hat{A}_{22}$ are still invertible and have bounded inverses. If $(\Delta E, \Delta A) \in \mathcal{P}$ is sufficiently small, then (43) remains strangeness-free.*

Proof. The assertion follows, since for sufficiently small admissible pairs of perturbations $(\Delta E, \Delta A)$ the functions $I - \hat{E}_{11}^{-1}\Delta\hat{E}_{11}$ and $I - \hat{A}_{22}^{-1}\Delta\hat{A}_{22}$ remain pointwise nonsingular. □

If the unperturbed DAE systems corresponding to the transformed system (43) has boundedly invertible blocks $\hat{E}_{11}$ and $\hat{A}_{22}$, then we call these DAEs *robustly strangeness-free*. In the following we restrict ourselves to robustly strangeness-free DAE systems under admissible perturbations.

Definition 43 *The upper Lyapunov exponents $\lambda_1^u \geq ... \geq \lambda_d^u$ of (30) are said to be* stable *if for any $\epsilon > 0$, there exists $\delta > 0$ such that the conditions $\sup_t \|\Delta E(t)\| < \delta$, $\sup_t \|\Delta A(t)\| < \delta$ on the perturbations imply that the perturbed DAE system (43) is strangeness-free and*

$$|\lambda_i^u - \gamma_i^u| < \epsilon, \quad \text{for all } i = 1, 2, ..., d,$$

where the γ_i^u are the ordered upper Lyapunov exponents of the perturbed system (43).

The DAE system (30) and the perturbed system (43) are called asymptotically equivalent *if they are strangeness-free and*

$$\lim_{t\to\infty} \|\Delta E(t)\| = \lim_{t\to\infty} \|\Delta A(t)\| = 0.$$

It is clear that the stability of upper Lyapunov exponents and the asymptotic equivalence of DAE systems are invariant under strong global kinematic equivalence transformations. Since the Lyapunov exponents do not depend on the behavior of the coefficient matrices on a finite interval, we have the following result (see also [1, Theorem 5.2.1] or [26, Theorem 3.1]).

Theorem 44 *Suppose that the DAE system (30) and the perturbed system (43) are asymptotically equivalent. Then the stability of the Lyapunov exponents of (30) implies $\lambda_i^u = \gamma_i^u$, for all $i = 1, 2, ..., d$, where again the γ_i^u are the ordered upper Lyapunov exponents of the perturbed system (43).*

Proof. Due to the asymptotic equivalence of the two systems, given an arbitrary $\epsilon > 0$, there exists $T > 0$ (sufficiently large) such that

$$\sup_{t\geq T} \|\Delta E(t)\| < \delta, \quad \sup_{t\geq T} \|\Delta A(t)\| < \delta, \tag{44}$$

where δ (depending on ϵ) is as in Definition 43. By definition, the Lyapunov exponents are invariant with respect to changes occurring in the coefficient matrices on a finite interval $[0, T]$. On the other hand, due to the stability (of the Lyapunov exponents), the inequalities (44) imply that

$$|\lambda_i^u - \gamma_i^u| < \epsilon, \quad \text{for all } i = 1, 2, ..., d.$$

Since ϵ can be chosen arbitrarily small, the proof is complete. □

As our next step we extend the concept of integral separation to DAEs.

Definition 45 *A minimal fundamental solution matrix X for (26) (or (30)) is called* integrally separated *if for $i = 1, 2, ..., d-1$ there exist constants $\beta > 0$ and $\gamma > 0$ such that*

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq \gamma e^{\beta(t-s)},$$

for all t, s with $t \geq s \geq 0$. If a DAE system has an integrally separated minimal fundamental solution matrix, then we say it has the integral separation property.

Analogous to the result for ODEs in Proposition 11, we then have the following facts.

Proposition 46 *Consider a strangeness-free DAE system of the form (30).*

1. *If (30) is integrally separated then the same holds for any globally kinematically equivalent system, i. e. also for (26).*
2. *If (30) is integrally separated, then it has pairwise distinct upper and pairwise distinct lower Lyapunov exponents.*
3. *Suppose that $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded. Then, the DAE system (30) is integrally separated if and only if and the underlying ODE (32) is integrally separated.*

Proof.

1. Let $\hat{X}$ be a fundamental solution matrix of (30). Suppose that, under a global kinematic equivalence transformation, the transformed fundamental solution matrix $\tilde{X}$ is given as $\tilde{X} = V\hat{X}$, where V is smooth, and V as well as V^{-1} are bounded. Then,

$$\frac{\left\|V(t)\hat{X}(t)e_i\right\|}{\left\|V(s)\hat{X}(s)e_i\right\|} \geq \frac{1}{\mathrm{cond}(V)} \frac{\left\|\hat{X}(t)e_i\right\|}{\left\|\hat{X}(s)e_i\right\|},$$

for all t, s and $i = 1, 2, ..., d$, where

$$\mathrm{cond}(V) := \sup_{t,s} \|V(t)\| \left\|V^{-1}(s)\right\|.$$

A similar estimate for $i+1$ with the change of variables $t \leftrightarrow s$, yields

$$\frac{\left\|V(s)\hat{X}(s)e_{i+1}\right\|}{\left\|V(t)\hat{X}(t)e_{i+1}\right\|} \geq \frac{1}{\mathrm{cond}(V)} \frac{\left\|\hat{X}(s)e_{i+1}\right\|}{\left\|\hat{X}(t)e_{i+1}\right\|}.$$

Multiplying the left and right side of these two inequalities for i and $i+1$, we obtain

$$\frac{\left\|V(t)\hat{X}(t)e_i\right\|}{\left\|V(s)\hat{X}(s)e_i\right\|}\cdot\frac{\left\|V(s)\hat{X}(s)e_{i+1}\right\|}{\left\|V(t)\hat{X}(t)e_{i+1}\right\|} \geq \frac{1}{[\mathrm{cond}(V)]^2}\frac{\left\|\hat{X}(t)e_i\right\|}{\left\|\hat{X}(s)e_i\right\|}\cdot\frac{\left\|\hat{X}(s)e_{i+1}\right\|}{\left\|\hat{X}(t)e_{i+1}\right\|} \geq \frac{1}{[\mathrm{cond}(V)]^2}\gamma e^{\beta(t-s)},$$

for all t, s and $i = 1, 2, ..., d-1$, which immediately yields the assertion.

2. This part is immediate.

3. The proof is similar to that of Part 1. We use again the estimates (35) between the columns of $\hat{X}$ and the corresponding columns of the fundamental solution for (32).

□

Remark 47 Note that the integral separation property is of a rather uniform nature. It is stronger than the property that

$$\frac{\|X(t)e_i\|}{\|X(t)e_{i+1}\|} \geq \gamma e^{\beta t}, \quad t \in \mathbb{I},$$

which is sufficient for the Lyapunov exponents to be pairwise distinct. This is the reason, why in Part 3. of Proposition 46 we require a stronger assumption than the condition (34) that we have used before.

Theorem 48 *Suppose that the coefficient matrices in (30) are such that*

$$\hat{A}_{22}^{-1}\hat{A}_{21},\ \hat{A}_{12}\hat{A}_{22}^{-1},\ \hat{E}_{11},\ \hat{E}_{11}^{-1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}) \text{ are bounded.} \tag{45}$$

If the system (30) has d pairwise distinct upper and pairwise distinct lower Lyapunov exponents and they are stable, then the system admits integral separation. Conversely, if there exists an integrally separated fundamental solution matrix to (30), then the system has d stable pairwise distinct upper and stable pairwise distinct lower Lyapunov exponents.

Proof. Under the boundedness assumption of $\hat{A}_{22}^{-1}\hat{A}_{21}$, the DAE system (30) possesses the same Lyapunov exponents as its underlying ODE (32). The boundedness conditions (45) imply that if the perturbations ΔE and ΔA are small enough, then the underlying explicit ODE

$$\dot{\hat{x}}_1 = \hat{A}\hat{x}_1 = \hat{E}_{11}^{-1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})\hat{x}_1$$

is only affected by a small perturbation in the coefficient matrix $\hat{A}$. By invoking Part 3 of Proposition 46 and the well-known result for ODEs [1], see also [25], the proof for system (30) follows. □

Remark 49 Theorem 46 is stated for both the upper and the lower Lyapunov exponents. But one should note that although both upper and lower Lyapunov exponents are pairwise distinct, the Lyapunov spectral intervals may intersect each other, see Example 53 below.

Unlike the case of ODEs, the integral separation of a DAE system does not automatically imply that of its adjoint system.

Theorem 50 *Consider a strangeness-free DAE system of the form (30) and suppose that $\hat{A}_{22}^{-1}\hat{A}_{21}$, $\hat{A}_{12}\hat{A}_{22}^{-1}$, $\hat{E}_{11}$, and $\hat{E}_{11}^{-1}$ are bounded. Then, the system has an integrally separated fundamental solution matrix if and only if its adjoint (29) has an integrally separated fundamental solution matrix.*

Proof. The proof follows immediately by using the structure of the fundamental solution matrices of (30), the Lagrange identity, the first statement of Theorem 36, and Part 3. of Proposition 46. □

Recall that if the columns of $\hat{X}_1$ form a normal basis for (32) then the columns of $Y_1 := \hat{E}_{11}^{-T}\hat{X}_1^{-T}$ form a normal basis for the adjoint. But, as the following example shows, without the boundedness conditions in Theorem 50, the integral separation of the DAE (30) does not imply the integral separation of the underlying ODE (32).

Example 51 *Consider the DAE system*

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2, \\ \dot{x}_2 &= x_2, \\ 0 &= x_1 - x_3, \\ 0 &= x_2 - e^{-t}x_4. \end{aligned}$$

Clearly, the underlying ODE

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is not integrally separated, but the DAE system has a minimal fundamental solution

$$X(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \\ e^t & te^t \\ 0 & e^{2t} \end{bmatrix},$$

which is integrally separated. This illustrates the need for the boundedness assumptions in (45).

3.3 Bohl exponents and Sacker-Sell spectrum

The extension of Bohl exponents and Bohl intervals to strangeness-free DAE systems of the form (26) is straightforward. Definition 17 can be applied directly to DAEs and it is not difficult to verify the following statements.

Proposition 52 *Consider the DAE system (26) and the transformed system (30). Then we have the following properties of Bohl exponents.*

1. *Bohl exponents are invariant under global kinematical equivalence transformations.*
2. *Consider a minimal fundamental solution matrix $\hat{X}$ for (30). If $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded, then the Bohl intervals for the columns of $\hat{X}$ are exactly the Bohl intervals for the corresponding fundamental solution matrix $\hat{X}_1$ of the underlying ODE (32).*
3. *If the Bohl intervals of the columns of a minimal fundamental solution matrix X of (26) are d disjoint closed intervals, then X has integrally separated columns.*
4. *If the columns of a fundamental solution matrix X of (26) are integrally separated, then the upper (or the lower) Bohl exponents of the columns of X columns are distinct, but the Bohl intervals may intersect each other.*

Proof.

1. Suppose that $V \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that V and V^{-1} are bounded. Let x be an arbitrary solution to (26). Since
$$\frac{\|V(t)x(t)\|}{\|V(s)x(s)\|} \geq \frac{1}{\mathrm{cond}(V)} \frac{\|x(t)\|}{\|x(s)\|},$$

we have
$$\frac{\ln\|V(t)x(t)\| - \ln\|V(s)x(s)\|}{t-s} \geq \frac{\ln\|x(t)\| - \ln\|x(s)\| - \ln \operatorname{cond}(V)}{t-s}.$$
Taking the lim sup on both sides as $s, t-s$ tend to ∞, we obtain
$$\kappa_B^u(Vx) \geq \kappa_B^u(x).$$
Conversely, we have $x = V^{-1}Vx$, hence similarly we obtain
$$\kappa_B^u(x) \geq \kappa_B^u(Vx).$$
As a consequence, we have $\kappa_B^u(Vx) = \kappa_B^u(x)$. The proof for the lower Bohl exponents is obtained analogously.

2. The proof is similar to that of Theorem 32.

3. Without loss of generality, we assume that the Bohl intervals of X are ordered decreasingly. For the sake of simplicity, it suffices to consider the first two columns x_1 and x_2. By the definition of the Bohl exponents, there exist positive constants $N_i, M_i,\ i = 1, 2$ such that
$$N_i e^{\kappa_i^l(t-s)}\|x_i(s)\| \leq \|x_i(t)\| \leq M_i e^{\kappa_i^u(t-s)}\|x_i(s)\|, \quad \text{for all} \ \ t \geq s,$$
for $i = 1, 2$, where κ_i^l and κ_i^u are lower and upper Bohl exponents for x_i, respectively. Then, it is easy to see that
$$\frac{\|x_1(t)\|}{\|x_1(s)\|}\frac{\|x_2(s)\|}{\|x_2(t)\|} \geq \frac{N_1 e^{\kappa_1^l(t-s)}}{M_2 e^{\kappa_2^u(t-s)}} = \frac{N_1}{M_2} e^{(\kappa_1^l - \kappa_2^u)(t-s)}$$
holds for all $t \geq s$. Since the intervals $[\kappa_1^l, \kappa_1^u]$ and $[\kappa_2^l, \kappa_2^u]$ are disjoint, the positivity of $(\kappa_1^l - \kappa_2^u)$ is obvious, which verifies the integral separation of x_1 and x_2.

4. For the converse statement, we consider again the first two columns x_1 and x_2. By the integral separation property, there exist positive constants β, γ such that
$$\frac{\|x_1(t)\|}{\|x_1(s)\|}\frac{\|x_2(s)\|}{\|x_2(t)\|} \geq \gamma e^{\beta(t-s)} \quad t \geq s,$$
or equivalently
$$\frac{\|x_1(t)\|}{\|x_1(s)\|} \geq \gamma e^{\beta(t-s)}\frac{\|x_2(t)\|}{\|x_2(s)\|},$$
which implies
$$\frac{\ln\|x_1(t)\| - \ln\|x_1(s)\|}{t-s} \geq \frac{\ln\gamma + \beta(t-s) + \ln\|x_2(t)\| - \ln\|x_2(s)\|}{t-s}.$$
Taking the lim sup on both sides of the above inequality as $s, t-s$ tend to infinity, we obtain
$$\limsup_{s,t-s\to\infty} \frac{\ln\|x_1(t)\| - \ln\|x_1(s)\|}{t-s} \geq \limsup_{s,t-s\to\infty} \frac{\ln\|x_1(t)\| - \ln\|x_1(s)\|}{t-s},$$
which implies that $\kappa_1^u > \kappa_2^u$. By taking the lim inf instead of the lim sup, we obtain the result for the lower Bohl exponents.

□

Even though we have shown that integral separation implies disjoint upper and lower Bohl exponents, the Bohl intervals may still overlap as the following example demonstrates.

Example 53 Consider the system

$$\begin{aligned}\dot{x}_1(t) &= [\sin(\ln t) + \cos(\ln t)]x_1(t),\\ \dot{x}_2(t) &= [\sin(\ln t) + \cos(\ln t) - 1]x_2(t),\end{aligned}$$

with $t \geq 1$that is an extension to DAEs of an example by Perron [64]. It is easy to see that this system is integrally separated. However, the Bohl intervals $[-\sqrt{2}, \sqrt{2}]$ and $[-\sqrt{2}-1, \sqrt{2}-1]$ are clearly not disjoint. Similarly, also the Lyapunov spectral intervals of this system $[-1, 1]$ and $[-2, 0]$ overlap.

In order to extend the concept of exponential dichotomy to DAEs, we first introduce shifted DAE systems.

Definition 54 *Consider a strangeness-free DAE of the form (26). For $\lambda \in \mathbb{R}$, the DAE system*

$$E(t)\dot{x} = [A(t) - \lambda E(t)]x, \quad t \in \mathbb{I}, \tag{46}$$

is called a shifted DAE system.

By the transformation of Theorem 30, the shifted DAE transforms as

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} - \lambda\hat{E}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \in \mathbb{I}, \tag{47}$$

and clearly, the shifted DAE system inherits the strangeness-free property from the original DAE.

In the previous subsection, we have seen that minimal fundamental solution matrices are useful in the analysis of Lyapunov exponents. Unfortunately, they do not have the semi-group property as fundamental solutions in the ODE case have. However, for strangeness-free systems it is easy to introduce a (unique) maximal fundamental solution matrix X which possesses a semi-group property by demanding that it satisfies the projected initial condition

$$E(t_0)\,(X(t_0) - I_n) = 0. \tag{48}$$

Example 55 Condition (48) is, however, not the right condition, if the system is not strangeness-free. Consider the

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{I},$$

with

$$E(t) = \begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix},$$

see [47, 51]. The strangeness index of this system is 1. It is easy to see that the trivial solution $X \equiv 0$ is the only maximal fundamental matrix solution, but it does not satisfy (48).

In the following, when it is necessary to emphasize the dependence of the fundamental solution matrix on the initial time t_0, we write $X(t, t_0)$.

In the case of the transformed strangeness-free system (30), a maximal fundamental solution matrix that satisfies the corresponding projected initial condition (48) is easily obtained as

$$\hat{X}(t, t_0) = \begin{bmatrix} \hat{X}_1(t, t_0) & 0 \\ \hat{A}_{22}^{-1}\hat{A}_{21}(t)\hat{X}_1(t, t_0) & 0 \end{bmatrix}, \tag{49}$$

where $\hat{X}_1$ is a fundamental solution matrix for (32) satisfying $\hat{X}_1(t_0, t_0) = I_d$. For this maximal fundamental solution matrix $\hat{X}(t, t_0)$, we introduce the generalized inverse matrix function $\hat{X}^-(t, t_0)$ defined by

$$\hat{X}^-(t, t_0) := \begin{bmatrix} \hat{X}_1^{-1}(t, t_0) & 0 \\ \hat{A}_{22}^{-1}\hat{A}_{21}(t_0)\hat{X}_1^{-1}(t, t_0) & 0 \end{bmatrix}, \tag{50}$$

for $t \geq t_0$. The matrix functions $\hat{X}$ and $\hat{X}^-$ satisfy the following relations.

Proposition 56 *Let $\hat{X}$ as in (49) be the maximal fundamental solution matrix of (30) and let $\hat{X}^-$ be as in (50). Then for $t_1 \geq t_0$, the following identities hold.*

1. $\hat{X}(t_1,t_0)\hat{X}^-(t_1,t_0)\hat{X}(t_1,t_0) = \hat{X}(t_1,t_0)$,
2. $\hat{X}^-(t_1,t_0)\hat{X}(t_1,t_0)\hat{X}^-(t_1,t_0) = \hat{X}^-(t_1,t_0)$,
3. $\hat{X}(t_1,t_0)\hat{X}^-(t_1,t_0) = \begin{bmatrix} I & 0 \\ \hat{A}_{22}^{-1}(t_1)\hat{A}_{21}(t_1) & 0 \end{bmatrix}$,
4. $\hat{X}^-(t_1,t_0)\hat{X}(t_1,t_0) = \begin{bmatrix} I & 0 \\ \hat{A}_{22}^{-1}(t_0)\hat{A}_{21}(t_0) & 0 \end{bmatrix}$,

and these four properties define the matrix $\hat{X}^-$ uniquely.

Furthermore, $\hat{X}$ satisfies the semigroup property, i.e. for $t_2 \geq t_1 \geq t_0$, we have

$$\hat{X}(t_2,t_1)\hat{X}(t_1,t_0) = \hat{X}(t_2,t_0).$$

Proof. By using the formulae (49) and (50), the identities 1.-4. as well as the semigroup property are easily verified by elementary matrix calculations. □

The identities 1.-2. in Proposition 56 mean that $X^-(t_1,t_0)$ is a reflexive generalized inverse of $X(t_1,t_0)$, while the identities 3.-4. guarantee that this generalized inverse is unique, see [4, 5].

In the following, for ease of notation, we use $t_0 = 0$ and the abbreviation $\hat{X}(t) := \hat{X}(t,0)$. We then introduce the concept of exponential dichotomy for DAEs as in [57].

Definition 57 *The semi-implicit DAE system (30) is said to have an* exponential dichotomy *if for a maximal fundamental solution matrix$\hat{X}(t)$, there exists a projection matrix $P \in \mathbb{R}^{d\times d}$ and constants $\alpha, \beta > 0$, and $K, L \geq 1$ such that*

$$\begin{aligned} \left\| \hat{X}(t) \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^-(s) \right\| &\leq K e^{-\alpha(t-s)}, \quad t \geq s, \\ \left\| \hat{X}(t) \begin{bmatrix} I_d - P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^-(s) \right\| &\leq L e^{\beta(t-s)}, \quad t \leq s. \end{aligned} \tag{51}$$

Furthermore, we say that a general DAE system (26) has an exponential dichotomy *if there exists a global kinematical equivalence transformation that reduces (26) to the semi-implicit form and the reduced system has an exponential dichotomy.*

For a strangeness-free DAE in the form (30), exponential dichotomy can again be characterized via the underlying ODE.

Theorem 58 *The DAE system (30) has an exponential dichotomy if and only if $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded and the corresponding underlying ODE (32) has an exponential dichotomy.*

Proof. Suppose that (51) holds. Using the structure of $\hat{X}$ and $\hat{X}^-$, we can rewrite (51) as

$$\begin{aligned} \left\| \begin{bmatrix} \hat{X}_1(t)P\hat{X}_1^{-1}(s) \\ \hat{A}_{22}^{-1}(t)\hat{A}_{21}(t)\hat{X}_1(t)P\hat{X}_1^{-1}(s) \end{bmatrix} \right\| &\leq K e^{-\alpha(t-s)}, \quad t \geq s, \\ \left\| \begin{bmatrix} \hat{X}_1(t)(I_d - P)\hat{X}_1^{-1}(s) \\ \hat{A}_{22}^{-1}(t)\hat{A}_{21}(t)\hat{X}_1(t)(I_d - P)\hat{X}_1^{-1}(s) \end{bmatrix} \right\| &\leq L e^{\beta(t-s)}, \quad t \leq s, \end{aligned} \tag{52}$$

which implies that the underlying ODE (32) has an exponential dichotomy. Setting $t = s$, we obtain

$$\left\| \begin{bmatrix} I_d \\ \hat{A}_{22}^{-1}(t)\hat{A}_{21}(t) \end{bmatrix} \right\| \leq \left\| \hat{X}(t) \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^-(t) \right\| + \left\| \hat{X}(t) \begin{bmatrix} I_d - P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^-(t) \right\| \leq K + L, \quad t \in \mathbb{I},$$

from which the boundedness of $\hat{A}_{22}^{-1}(t)\hat{A}_{21}(t)$ is clear.

If $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded and (32) has an exponential dichotomy, then clearly the inequalities in (52) hold. □

The following facts associated with exponential dichotomy of DAEs follow easily.

Proposition 59 *Consider a strangeness-free DAE of the form (30) that has an exponential dichotomy.*

1. *Every globally kinematically equivalent system has an exponential dichotomy, i. e. in particular the exponential dichotomic property is invariant under global kinematical equivalence transformations.*
2. *If a fundamental solution matrix $\hat{X}$ of (30) has an exponential dichotomy, then so does the fundamental solution matrix that fulfills the projected initial condition (48) at $t_0 = 0$. Furthermore, the projection P can be chosen to be orthogonal.*

Proof. The first part simply follows by the definition of exponential dichotomic DAE systems. By Theorem 58, to verify the second statement, it suffices to consider the underlying ODE system (32) and analyze its exponential dichotomy. Invoking [26, Lemma 6.1], the underlying ODE system (32) also admits an exponential dichotomy for its (principal) matrix solution that satisfies $\hat{X}_1(0) = I_d$ and the projection P can be chosen to be orthogonal. Finally, note that the fundamental matrix solution for (30) constructed with this $\hat{X}_1$, see (49), is exactly the unique fundamental solution matrix that fulfills the projected initial condition. □

After these preparations we can define Sacker-Sell spectra for DAEs.

Definition 60 *The* Sacker-Sell (or exponential dichotomy) spectrum *of the DAE system (30) is defined by*

$$\Sigma_S := \{\lambda \in \mathbb{R},\ \text{the shifted DAE (47) does not have an exponential dichotomy}\}\,. \tag{53}$$

The complement of Σ_S is called the resolvent set *for the DAE system (30).*

The Sacker-Sell spectrum of the DAE system (26) is defined as the Sacker-Sell spectrum of its transformed DAE system (30).

With these definitions we have the following properties of Sacker-Sell spectra for DAEs.

Lemma 61 *Consider the DAE system (30) and suppose that $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded. Then,*

1. *the Sacker-Sell spectrum of the DAE system (30) is exactly the Sacker-Sell spectrum of the underlying ODE (32). Thus, it consists of at most d closed intervals.*
2. *the Sacker-Sell spectrum of the DAE system (26) does not depend on the choice of an orthogonal change of basis that transforms it into the form (30).*

Proof.

1. Consider an arbitrary $\lambda \in \mathbb{R}$. By Theorem 58, the shifted DAE system (47) with this λ has an exponential dichotomy if and only if the corresponding shifted underlying ODE system has an exponential dichotomy. This implies that the resolvent set of (30) and that of (32) are exactly the same which proves the assertion. Since the dimension (the size) of the underlying ODE system (32) is d, it has been shown in [69] that the Sacker-Sell spectrum of (32) consists of at most d closed intervals.
2. As a consequence of Proposition 59, Part 1, two globally kinematically equivalent semi-implicit DAE systems must possess the same Sacker-Sell spectrum. Therefore, the Sacker-Sell spectrum of the DAE system (26) does not depend on the choice of a global kinematic equivalence transformation that transforms it into the form (30). □

We also obtain the relationship of Sacker-Sell spectra and the integral separation property.

Theorem 62 *Suppose that the Sacker-Sell spectrum of (26) is given by d disjoint closed intervals. Then there exists a minimal fundamental solution matrix of (26) with integrally separated columns.*

Proof. Without loss of generality, it is sufficient to consider the transformed system (30). By Theorem 58, the assumption implies that $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded. Then, by Lemma 61, the Sacker-Sell spectrum of the underlying ODE (32) consists of the same d disjoint intervals. By invoking [26, Theorem 6.3], then (32) has an integrally separated fundamental solution matrix denoted by $\hat{X}_1$. Hence, by construction (see Part 3. of Proposition 46), there exists a corresponding fundamental solution for the DAE system (30) whose columns are integrally separated. □

For the converse of this result we again need a boundedness condition.

Theorem 63 *Suppose that for the DAE system (30), $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded and there exists a minimal and integrally separated fundamental solution matrix $\hat{X}$. Then the Sacker-Sell spectrum for (30) is given exactly by the d (not necessarily disjoint) Bohl intervals associated with the columns of $\hat{X}$.*

Proof. Because of the relations between (30) and its underlying ODE (32), the verification of the statement reduces to that for the underlying ODE system (32). Due to a theorem of Bylov (see [1, Corollary 5.3.2] or [25, Theorem 2.31]), there exists a kinematic equivalence transformation that transforms (32) into diagonal form. The diagonalized system obtained in this way is integrally separated as well. By Lemmas 21 and 22 we have that the Sacker-Sell spectrum for the diagonal ODE system is exactly the set of Bohl intervals for all scalar equations corresponding to the diagonal elements. Since Bohl intervals are invariant under global kinematic equivalence transformations, the proof is complete. □

Remark 64 As we have seen already for ODEs, if we take an arbitrary minimal fundamental solution matrix of (30), then the set of Bohl intervals associated with its columns is only a subset of the Sacker-Sell spectrum of (30). The integral separation assumption then ensures that the two sets coincide.

Corollary 65 *Consider the strangeness-free DAE system (26). Then the Lyapunov spectrum is contained in the Sacker-Sell spectrum, i.e. we have $\Sigma_L \subseteq \Sigma_S$.*

Proof. Suppose that the columns of a fundamental solution matrix X of (26) form a normal basis. Then, by definition, the Lyapunov spectrum is exactly the set of Lyapunov intervals for the columns of X. Since for an arbitrary solution x of (26), the Lyapunov interval is contained in the Bohl interval and since the Bohl intervals are contained in the Sacker-Sell spectrum (see Remark 64), the proof is complete. □

The following well-known example example of Perron [64] shows that Lyapunov spectral intervals can be strict subsets of Sacker-Sell spectral intervals.

Example 66 Consider the ODE

$$\dot{x} = [\sin(\ln(t+1)) + \cos(\ln(t+1))]x,$$

with solution $x(t) = x(0)e^{(t+1)\sin(\ln(t+1))}$. The Lyapunov spectrum is given by $[-1, 1]$, while the Bohl interval (and the Sacker-Sell spectrum) is given by $[-\sqrt{2}, \sqrt{2}]$, see [19] and [25].

Finally we analyze the relation between the Sacker-Sell spectra of the DAE system (30) and its adjoint (38).

Lemma 67 *Consider the implicit scalar differential equation*

$$e(t)\dot{x} = a(t)x, \quad t \in \mathbb{I},$$

where e, e^{-1}, and $e^{-1}a$ are continuous and bounded. Let the Sacker-Sell spectrum associated with this system be given by the interval $[\alpha, \beta]$. Then the Sacker-Sell spectrum of the adjoint equation

$$e(t)\dot{y} = -(a(t) + \dot{e}(t))y$$

is given by $[-\beta, -\alpha]$.

Proof. Without loss of generality, we assume that $e(t) > 0$ for all $t \in \mathbb{I}$. Let the Sacker-Sell spectrum of the adjoint equation be denoted by $[\bar{\alpha}, \bar{\beta}]$. Due to Proposition 22, we have

$$\alpha = \liminf_{s,t-s\to\infty} \frac{1}{t-s} \int_s^t \frac{a(\tau)}{e(\tau)} d\tau$$

and

$$\bar{\beta} = \limsup_{s,t-s\to\infty} \frac{-1}{t-s} \int_s^t \left[\frac{a(\tau)}{e(\tau)} + \frac{\dot{e}(\tau)}{e(\tau)} \right] d\tau.$$

By some elementary manipulations, we obtain

$$\bar{\beta} = \limsup_{s,t-s\to\infty} \frac{-1}{t-s} \left[\int_s^t \frac{a(\tau)}{e(\tau)} d\tau + \ln e(t) - \ln e(s) \right].$$

Since e and e^{-1} are bounded, we get

$$\lim_{s,t-s\to\infty} \frac{1}{t-s} (\ln e(t) - \ln e(s)) = 0,$$

and, therefore, $\bar{\beta} = -\alpha$. The proof that $\bar{\alpha} = -\beta$ follows analogously. □

Before we can prove the symmetry property for the Sacker-Sell spectra of the DAE and its adjoint, we need the following lemma.

Lemma 68 *Consider an implicit ODE of the form*

$$E(t)\dot{x} = A(t)x \tag{54}$$

with $E(t)$ pointwise nonsingular. Suppose that both $E(t)$ and $A(t)$ are continuous and that $E^{-1}A$ is bounded. Then there exist pointwise orthogonal matrix functions $U \in C(\mathbb{I}, \mathbb{R}^{n,n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that the transformed matrix functions

$$\mathcal{E} = [e_{ij}] = U^T E V \quad \text{and} \quad \mathcal{A} = [a_{ij}] = U^T A V - U^T E \dot{V}$$

are both in upper triangular form.

Proof. We give a constructive proof. We want to determine triangular matrix functions $\mathcal{E}$ and $\mathcal{A}$ with

$$\mathcal{A} = U^T A V - U^T E \dot{V} = U^T A V - U^T E V V^T \dot{V} = U^T A V - \mathcal{E} V^T \dot{V},$$

and

$$\mathcal{E}^{-1}\mathcal{A} = \mathcal{E}^{-1} U^T A V - V^T \dot{V}$$

being both triangular. Due to the orthogonality of V we must have that $S(V) = [s_{i,j}] := V^T \dot{V}$ is skew-symmetric. Hence, the strictly lower triangular part of $S(V)$ can be determined first by the corresponding part of $W = [w_{ij}] = \mathcal{E}^{-1}U^TAV$ and the remaining part then is determined by the skew-symmetry of $S(V)$, i.e.,

$$s_{ij} = \begin{cases} w_{ij}, & i > j, \\ 0, & i = j, \\ -w_{ji}, & i < j, \end{cases} \quad 1 \le i, j \le d. \tag{55}$$

The orthogonal matrix function V then solves the differential equation

$$\dot{V} = VS(V). \tag{56}$$

At $t = 0$, we can obtain $V(0)$ by using a QR-factorization of an initial value for the fundamental solution matrix $X(0)$ associated with (54) say, the identity matrix. Since $W = \mathcal{E}^{-1}U^TAV = V^TE^{-1}AV$, the boundedness of $E^{-1}A$ implies the boundedness of W, thus the boundedness of $S(V)$, as well. Thus, V can be determined as the (unique) solution to the initial value problem (56). With this, a unique matrix function U is then determined via

$$EV = U\mathcal{E} \tag{57}$$

and this then uniquely determines $\mathcal{A}$. □

It should be noted that the computation of the triangularization in (68) can actually be implemented numerically by using smooth QR-factorizations, see [20, Subsection 2.1].

Using (68), we can now prove the following theorem.

Theorem 69 *Consider the DAE system (30). Suppose that $\hat{A}_{22}^{-1}\hat{A}_{21}$, $\hat{A}_{12}\hat{A}_{22}^{-1}$, $\hat{E}_{11}$, and $\hat{E}_{11}^{-1}$ are bounded. Then, the Sacker-Sell spectrum of (30) and that of the adjoint system are symmetric with respect to the origin, i. e., if $[\alpha_i, \beta_i], (1 \le i \le d)$ is an arbitrary Sacker-Sell spectral interval for (30), then $[-\beta_i, -\alpha_i]$ is a Sacker-Sell spectral interval for the adjoint system and vice versa.*

Proof. By Lemma 61, it suffices to consider the Sacker-Sell spectrum associated with the underlying ODEs (32) and (39). By Lemma 68, there exist pointwise orthogonal matrix functions $U_1 \in C(\mathbb{I}, \mathbb{R}^{d,d})$ and $V_1 \in C^1(\mathbb{I}, \mathbb{R}^{d,d})$ such that (32) is equivalently transformed to

$$\mathcal{E}_1\dot{z}_1 = \mathcal{A}_1 z_1,$$

where $\mathcal{E}_1 = U_1^T\hat{E}_{11}V_1, \mathcal{A}_1 = U_1^T\hat{A}_sV_1 - U_1^T\hat{E}_{11}\dot{V}_1$ are both upper triangular. Similarly, the equivalence transformation associated with U, V transforms the adjoint equation (39) to

$$\mathcal{E}^T\dot{\zeta}_1 = -(A_1^T + \dot{\mathcal{E}})\zeta_1,$$

with lower triangular coefficients. By Theorem 19, the Sacker-Sell spectrum of systems in triangular form is exactly the union of the Sacker-Sell spectra associated with each scalar equations corresponding to the diagonal elements. By invoking Lemma 67, the proof is complete. □

In this section we have introduced the concepts of Sacker-Sell (exponential dichotomy) spectra for strangeness-free DAEs of the form (26). Again, since these concepts only depend on the solution of the DAE and not on the representation of the system of DAEs, whether it is in the form (26) or in the general form (1), we immediately have all the results also for DAEs in the general form.

3.4 Stability of the Sacker-Sell spectrum

In this section we analyze the stability properties of the Sacker-Sell spectrum of a DAE system of the form (26) under admissible perturbations. This means we restrict ourselves to consider the perturbed DAE system (43). We assume that the boundedness condition (45) holds and that there exist positive constants $M_i, 1 \le i \le 4$, $N_j, j = 1,2$ and δ such that for $t \in \mathbb{I}$ the following bounds are given,

$$\begin{array}{lll}
\sup_t \left\| \hat{E}_{11}^{-1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}) \right\| \le M_1, & \sup_t \left\| \hat{A}_{22}^{-1} \right\| \le M_2, & \sup_t \left\| \hat{A}_{22}^{-1}\hat{A}_{21} \right\| \le M_3, \\
\sup_t \left\| \hat{A}_{12}\hat{A}_{22}^{-1} \right\| \le M_4, & \sup_t \left\| \hat{E}_{11} \right\| \le N_1, & \sup_t \left\| \hat{E}_{11}^{-1} \right\| \le N_2, \\
\sup_t \left\| \Delta\hat{E} \right\| \le \delta, & \sup_t \left\| \Delta\hat{A} \right\| \le \delta. &
\end{array} \tag{58}$$

Lemma 70 *Consider a strangeness-free DAE in the form (30) and a perturbed system of the form (43) with the bounds (58). If*

$$\delta < \min\{M_2^{-1}, N_2^{-1}\}, \tag{59}$$

then the perturbed system (43) remains strangeness-free and we have the estimates

$$\left\| (\hat{A}_{22} + \Delta\hat{A}_{22})^{-1} \right\| \le \frac{M_2}{1 - M_2\delta}, \quad \left\| (\hat{E}_{11} + \Delta\hat{E}_{11})^{-1} \right\| \le \frac{N_2}{1 - N_2\delta}, \quad t \in \mathbb{I}.$$

Proof. We have

$$\hat{A}_{22} + \Delta\hat{A}_{22} = \hat{A}_{22}(I + \hat{A}_{22}^{-1}\Delta\hat{A}_{22}).$$

Since $\left\| \hat{A}_{22}^{-1}\Delta\hat{A}_{22} \right\| \le \delta M_2 < 1$, we conclude the matrix $(I + \hat{A}_{22}^{-1}\Delta\hat{A}_{22})$ is invertible for all $t \in \mathbb{I}$ and the estimate

$$\left\| (I + \hat{A}_{22}^{-1}\Delta\hat{A}_{22})^{-1} \right\| \le \frac{1}{1 - M_2\delta}$$

holds. Hence, the matrix $\hat{A}_{22} + \Delta\hat{A}_{22}$ is invertible for all $t \in \mathbb{I}$ as well, and

$$\left\| (\hat{A}_{22} + \Delta\hat{A}_{22})^{-1} \right\| \le \frac{M_2}{1 - M_2\delta}.$$

The claim for $\hat{E}_{11} + \Delta\hat{E}_{11}$ is proved similarly. Note that the invertibility of these matrices implies the strangeness-free property of the perturbed DAE system (43). $\square$

It is not difficult to verify that (43) satisfies the boundedness condition (45). Using the estimates in Lemma 70, we immediately see that

$$\left\| (\hat{A}_{22} + \Delta\hat{A}_{22})^{-1}(\hat{A}_{21} + \Delta\hat{A}_{21}) \right\| \le \frac{M_3 + M_2\delta}{1 - M_2\delta},$$

and we obtain an estimate for the perturbation arising in the coefficient of the underlying ODE associated with the perturbed system (43) given by

$$\left\| (\hat{A}_{12} + \Delta\hat{A}_{12})(\hat{A}_{22} + \Delta\hat{A}_{22})^{-1}(\hat{A}_{21} + \Delta\hat{A}_{21}) - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} \right\| \le \frac{M_3 + M_2\delta}{1 - M_2\delta}(1 + M_4)\delta =: M_5(\delta)\delta,$$

where $M_5(\delta) = M_3(1 + M_4) + O(\delta)$ for sufficiently small δ. Let us reformulate the underlying unperturbed and perturbed ODE into explicit form as

$$\dot{\hat{x}}_1 = \hat{E}_{11}^{-1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})\hat{x}_1,$$

and

$$\dot{\bar{x}}_1 = (\bar{E}_{11})^{-1}(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21})\bar{x}_1,$$

respectively, where $\bar{E}_{11} = \hat{E}_{11} + \Delta\hat{E}_{11}$ and $\bar{A}_{ij} = \hat{A}_{ij} + \Delta\hat{A}_{ij}$, $i,j = 1,2$. Setting $\hat{A}_s = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}$ and $\bar{A}_s = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$, we then obtain

$$\left\| \bar{E}_{11}^{-1}\bar{A}_s - \hat{E}_{11}^{-1}\hat{A}_s \right\| = \left\| (1 + \hat{E}_{11}^{-1}\Delta\hat{E}_{11})^{-1}\hat{E}_{11}^{-1}\bar{A}_s - \hat{E}_{11}^{-1}\hat{A}_s \right\| \le N_2(M_1 + M_5)\delta + N_2^2 M_5\delta^2 =: M_6(\delta)\delta,$$

where $M_6(\delta) = N_2(M_1 + M_3 + M_3 M_4) + O(\delta)$.

These elementary calculations show that the explicit underlying ODE is only affected by a small perturbation if the perturbations ΔE, ΔA affecting the DAE system are sufficiently small. Furthermore, since the boundedness condition (45) is fulfilled, the Sacker-Sell spectrum of the perturbed DAE system is exactly that of the associated underlying ODE system. This then implies the stability result for the Sacker-Sell spectrum (and the resolvent set).

Theorem 71 *Consider the DAE system (30) and the perturbed system (43). Assume that (45) holds and that (30) has a Sacker-Sell spectrum that consists of k, $k \le d$, disjoint increasingly ordered closed finite intervals, i. e.,*

$$\Sigma_S = \bigcup_{i=1}^{k} [\alpha_i, \beta_i].$$

Let $\varepsilon > 0$ be sufficiently small such that $\beta_i + \varepsilon < \alpha_{i+1} - \varepsilon$ for some i, $0 \le i \le k$. For $i = 0$ and $i = k$, set $\beta_0 = -\infty$ and $\alpha_{k+1} = \infty$, respectively. Then there exists $\delta > 0$ so that if the perturbations satisfy the inequality

$$\max\{\sup_t \left\|\Delta\hat{E}(t)\right\|, \sup_t \left\|\Delta\hat{A}(t)\right\|\} \le \delta,$$

then the interval $(\beta_i + \varepsilon, \alpha_{i+1} - \varepsilon)$ is contained in the resolvent set of the perturbed DAE system (43).

Proof. Consider an arbitrary $\lambda \in (\beta_i + \varepsilon, \alpha_{i+1} - \varepsilon)$. Let the fundamental solution of the shifted system (47) be denoted by $\hat{X}_\lambda$ and the corresponding part for the underlying ODE by $\hat{X}_{1,\lambda}$. Since λ belongs to the resolvent set of (30) and since $\lambda > \beta_i + \varepsilon$ and $\lambda < \alpha_{i+1} - \varepsilon$, it is clear that there exist a projection matrix P, and constants $\alpha, \beta > \varepsilon$ as well as $K, L \ge 1$ such that

$$\begin{aligned} \left\|\hat{X}_{1,\lambda}(t) P \hat{X}_{1,\lambda}^{-1}(s)\right\| &\le K e^{-\alpha(t-s)}, \quad t \ge s, \\ \left\|\hat{X}_{1,\lambda}(t)(I-P)\hat{X}_{1,\lambda}^{-1}(s)\right\| &\le L e^{\beta(t-s)}, \quad t \le s. \end{aligned}$$

By invoking the Roughness Theorem (see [18, p. 34]) and using the estimate for the perturbation appearing in the associated explicit underlying ODE, if δ is sufficiently small such that in addition to the condition (59), the inequality

$$\Delta = M_6(\delta)\delta < \min\{\frac{\alpha}{4K^2}, \frac{\beta}{4L^2}\}, \tag{60}$$

holds, then there exists a projection $\bar{P}$ with the same null-space as P such that for the fundamental solution matrices $\bar{X}_\lambda$ and $\bar{X}_{1,\lambda}$ associated with the corresponding shifted system for (43) and its underlying shifted ODE, respectively, we have

$$\begin{aligned} \left\|\bar{X}_{1,\lambda}(t) \bar{P} \bar{X}_{1,\lambda}^{-1}(s)\right\| &\le \tfrac{5}{2} K^2 e^{-(\alpha - 2K\Delta)(t-s)}, \quad t \ge s, \\ \left\|\bar{X}_{1,\lambda}(t)(I-\bar{P})\bar{X}_{1,\lambda}^{-1}(s)\right\| &\le \tfrac{5}{2} L^2 e^{(\beta - 2L\Delta)(t-s)}, \quad t \le s. \end{aligned}$$

It is easy to see that by (60) it follows that $(\alpha - 2K\Delta)$ and $(\beta - 2L\Delta)$ are strictly positive. Thus, the perturbed and shifted underlying ODE has an exponential dichotomy. Applying Theorem 58, we obtain that the shifted and perturbed DAE system has an exponential dichotomy as well. But this means that λ belongs to the resolvent set of the perturbed DAE system (43). $\square$

Corollary 72 *Let the assumptions of Theorem 71 hold and let $\varepsilon > 0$ be sufficiently small such that $\beta_{i-1} + \varepsilon < \alpha_i - \varepsilon < \alpha_i \le \beta_i < \beta_i + \varepsilon < \alpha_{i+1} - \varepsilon$, for $0 \le i \le k$. For $i = 0$ and $i = k$, set $\beta_0 = -\infty$ and $\alpha_{k+1} = \infty$, respectively. Then, there exists $\delta > 0$ so that if*

$$\max\{\sup_t \left\|\Delta\hat{E}(t)\right\|, \sup_t \left\|\Delta\hat{A}(t)\right\|\} \le \delta,$$

then the Sacker-Sell interval $[\alpha_i, \beta_i]$ either remains a Sacker-Sell interval associated with the perturbed system or it is possibly split into several new Sacker-Sell intervals, but the smallest left end-point and the largest right end-point stay in the interval $[\alpha_i - \varepsilon, \beta_i + \varepsilon]$.

We remark that by using the Roughness Theorem (see [18, p. 34]) and the estimates of the perturbation for the explicit underlying ODE, the bound δ in Theorem 71 and Corollary 72 can be calculated explicitly in terms of ε and the constants M_i, N_i. Furthermore, the exponential dichotomy spectrum does not depend on the finite-time behavior of the coefficients, i. e. we have the following corollary for asymptotically equivalent DAE systems.

Corollary 73 *Let the assumptions of the Theorem 71 hold. If the perturbations $\Delta\hat{E}, \Delta\hat{A}$ are such that the DAE system (30) and the perturbed system (43) are asymptotically equivalent, then their Sacker-Sell spectra are the same.*

In this section we have analyzed the Lyapunov, and Sacker-Sell spectra for DAEs and their stability under perturbations. We have shown that the classical results for ODEs can be extended to DAEs. These results then form the basis for the computational methods that we consider in the following section.

4 Numerical Computation of Spectral Intervals for DAEs

In this section we extend the approaches that were derived for the computation of spectral intervals for ODEs in [25, 26, 29] to DAEs. We derive numerical methods for computing Lyapunov and Sacker-Sell spectra for DAEs of the form (26) based on smooth QR factorizations. We discuss both continuous time and discrete time versions of these numerical methods.

4.1 Continuous QR-algorithm

The basic idea for the numerical computation of spectral intervals for DAEs is to first transform the DAE system into an appropriate semi-implicit form, and then to apply a triangularization process to the coefficient matrices of the underlying implicit ODE. Throughout this section we assume that the DAE system is given in the strangeness-free form (26), i. e. whenever the value of $E(t), A(t)$ is needed, this has to be computed from the derivative array as described in Section 2.1. This can be done for example with the `FORTRAN` code `GELDA` [55] or the corresponding `MATLAB` version [56].

Although for the analysis we have preferred the semi-explicit form (30), in the numerical realization we use the transformation from the strangeness-free system (26) to the form (31), whose existence has been shown in Theorem 30. Note that the two forms (30) and (31) are globally kinematically equivalent so they have the same spectral intervals.

Suppose that the lower row-block A_2 in (26) is continuously differentiable. By assumption it has full-row rank. Therefore, see [20], there exist a pointwise nonsingular (and upper triangular) matrix function $\tilde{A}_{22} \in C^1(\mathbb{I}, \mathbb{R}^{(n-d)\times(n-d)})$ and a pointwise orthogonal matrix function $\tilde{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that

$$A_2 = \begin{bmatrix} 0 & \tilde{A}_{22} \end{bmatrix} \tilde{Q}. \tag{61}$$

A numerical implementation of this smooth factorization can be obtained by using a sequence of Householder transformations applied to the augmented matrix

$$\begin{bmatrix} I_n \\ A_2 \end{bmatrix}.$$

The triangularization process should be carried out pointwise from the bottom and the explicit multiplication of the elementary Householder transformations can be avoided. To make the factorization unique and to obtain the smoothness, we require the diagonal elements of $\tilde{A}_{22}$ to be positive, see [20]. Another possibility would be to derive differential equations for $\tilde{Q}$ (or its Householder factors) and to solve the corresponding initial value problems, see [20, 46].

The transformation $\tilde{x} = \tilde{Q}^T x$ leads to a DAE of the form (31), where

$$\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tilde{Q}, \quad \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tilde{Q} - \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{\tilde{Q}}. \tag{62}$$

In order to evaluate $\dot{\tilde{Q}}$ at any time instance, we use either an appropriate finite difference formula or the method derived in [46].

Since in the form (31) the solution component $\tilde{x}_2$ associated with the algebraic equations vanishes, i. e., $\tilde{x}_2 = 0$, we only have to deal with the underlying implicit ODE (33) for the dynamic component $\tilde{x}_1$.

By the construction given in the proof of Lemma 68, there exist pointwise orthogonal matrix functions $U_1 \in C(\mathbb{I}, \mathbb{R}^{n,n})$ and $V_1 \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that the transformed matrix functions

$$\mathcal{E}_1 = [e_{ij}] = U_1^T \tilde{E}_{11} V_1 \quad \text{and} \quad \mathcal{A}_1 = [a_{ij}] = U_1^T \tilde{A}_{11} V_1 - U_1^T \tilde{E}_{11} \dot{V}_1$$

are both in upper triangular form. Combining this transformation with the preliminary change of variables $\tilde{x} = \tilde{Q}^T x$, we obtain that there exist pointwise orthogonal matrix functions $U = \operatorname{diag}(U_1, I_a) \in C(\mathbb{I}, \mathbb{R}^{n \times n})$, $V = \operatorname{diag}(V_1, I_a)\tilde{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ such that by the new change of variables $z = V^T x$ and by multiplying both sides of (26) with U^T from the left, we arrive at a special upper triangular DAE system

$$\begin{bmatrix} \mathcal{E}_1 & U_1^T \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \mathcal{A}_1 & U_1^T \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} z,$$

where $\mathcal{E}_1$, $\mathcal{A}_1$, and $\tilde{A}_{22}$ are upper triangular matrix functions of appropriate sizes.

In the case of explicit ODEs, i. e., if $E = I_n$, it is easy to see that $\tilde{Q} = I_n$, $U = V = U_1 = V_1$ and the presented triangularization procedure reduces to that for ODEs, see Subsection 2.2 and [25, 26].

As a consequence of the formula (55), in practice, we evaluate $\mathcal{A}_1$ by setting $K = [k_{ij}] = U_1^T \tilde{A}_{11} V_1$, and obtain

$$a_{ij} = \begin{cases} (k_{ij} - k_{ji}), & i < j, \\ k_{ij}, & i = j, \\ 0, & i > j, \end{cases} \quad 1 \le i, j \le d. \tag{63}$$

Note that it is not necessary to invert $\mathcal{E}_1$ in order to compute $S(Q)$ in (55). Indeed, let $L = [l_{ij}]$ be the strictly lower triangular part of $W_1 = \mathcal{E}_1^{-1} U_1^T \tilde{A}_{11} V_1$, then (55) implies the linear system of equations

$$\begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,d} \\ 0 & e_{2,2} & \cdots & e_{2,d} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_{d,d} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ l_{2,1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & 0 \\ l_{d,1} & l_{d,2} & \cdots & 0 \end{bmatrix} = \begin{bmatrix} * & * & \cdots & * \\ k_{2,1} & * & \cdots & * \\ \cdot & \cdot & \cdots & * \\ k_{d,1} & k_{d,2} & \cdots & * \end{bmatrix}. \tag{64}$$

Solving for the entries l_{ij} from the bottom row up to the top row we obtain

$$\begin{aligned} l_{d,j} &= \tfrac{k_{d,j}}{e_{d,d}}, && j = 1, 2, ..., d-1, \\ l_{d-1,j} &= \tfrac{k_{d-1,j} - e_{d-1,d} l_{d,j}}{e_{d-1,d-1}}, && j = 1, 2, ..., d-2, \\ l_{i,j} &= \tfrac{k_{i,j} - \sum_{k=i+1}^{d} e_{i,k} l_{k,j}}{e_{i,i}}, && i = d-2, ..., 2;\ j = 1, ..., i-1. \end{aligned}$$

In this way the computational cost to determine L and $S(V_1)$ is $d^3/3 + \mathcal{O}(d^2)$, only. The skew-symmetric matrix $S(V_1)$ is then given by

$$S(V_1) = L - L^T. \tag{65}$$

Another important issue in the numerical implementation is to preserve the orthogonality of V_1 during the integration. There are different choices of methods to achieve this. The first is to use orthogonal integrators, e.g., Runge-Kutta-Gauß schemes, see [41]. The second is to apply first an arbitrary integration scheme, and then to reorthogonalize the obtained numerical solution at every grid point by a standard method, e. g., the Gram-Schmidt orthogonalization process. This is called *projected integration*. In the ODE case, in [24, 27], the authors have suggested and analyzed a third possibility, which is based on solving initial value problems for the elementary Householder or Givens transformations. An extension of the latter approach to implicit ODEs would give an efficient solution as well. We refer to [22, 26] for more details on the methods and numerical experiments in the case of explicit ODEs.

Finally, we extend the procedures for computing the Lyapunov and Sacker-Sell spectral intervals to the implicit ODE of the form

$$\mathcal{E}_1(t)\dot{R}_1 = \mathcal{A}_1(t)R_1, \quad t \in \mathbb{I}, \tag{66}$$

with upper triangular matrix functions $\mathcal{E}_1, \mathcal{A}_1$. Here R_1 is a fundamental solution matrix of the triangularized underlying implicit ODE and it is exactly the R-part of a QR-factorization of the fundamental solution to the underlying implicit ODE (33). By multiplying both sides of (66) by $\mathcal{E}_1^{-1}$, one arrives at an explicit ODE system of upper triangular form as in the ODE case. From the boundedness of $\tilde{E}_{11}^{-1}\tilde{A}_{11}$ and the proof of Lemma 68, the boundedness of $\mathcal{E}_1^{-1}\mathcal{A}_1$ is obvious. However, the computation of $\mathcal{E}_1^{-1}$ should be avoided, because only the information lying in the diagonal elements is relevant for the computation of the spectral intervals.

In the following, we proceed as in the case of explicit ODEs in [26]. Due to Theorem 16, if the functions a_{ii}/e_{ii}, $i = 1, \ldots, d$ are integrally separated, then the Lyapunov spectrum of the implicit ODE (66), which coincides with the Lyapunov spectrum of the DAE (26), can be determined as follows

$$\Sigma_L = \bigcup_{i=1}^{d} [\lambda_i^l, \lambda_i^u], \tag{67}$$

with

$$\lambda_i^l := \liminf_{t\to\infty} \frac{1}{t}\int_0^t \frac{a_{ii}(s)}{e_{ii}(s)}ds, \quad \lambda_i^u := \limsup_{t\to\infty} \frac{1}{t}\int_0^t \frac{a_{ii}(s)}{e_{ii}(s)}ds, \ i = 1, 2, ..., d.$$

Let

$$\lambda_i(t) := \frac{1}{t}\int_0^t \frac{a_{ii}(s)}{e_{ii}(s)}ds, \quad i = 1, 2, ..., d, \tag{68}$$

which can be approximated by solving auxiliary initial value problems

$$\left\{ \begin{array}{lcl} \dot{\phi}_i & = & \frac{a_{ii}(t)}{e_{ii}(t)}\phi_i, \quad t \in \mathbb{I}, \\ \phi_i(0) & = & 0; \end{array} \right. \qquad i = 1, 2, ..., d, \tag{69}$$

and then setting

$$\lambda_i(t) = \frac{1}{t}\phi_i(t), \quad i = 1, 2..., d. \tag{70}$$

Since

$$\lambda_i^l = \lim_{\tau\to\infty} \inf_{t\geq\tau} \lambda_i(t) \quad \text{ and } \lambda_i^u = \lim_{\tau\to\infty} \sup_{t\geq\tau} \lambda_i(t),$$

for given t_0 and T, $0 < t_0 < T$, $i = 1, 2, ..., d$, the quantities

$$\lambda_i^l(t_0, T) := \inf_{t_0\leq t\leq T} \lambda_i(t) \quad \text{ and } \lambda_i^u(t_0, T) := \sup_{t_0\leq t\leq T} \lambda_i(t)$$

give approximate values for λ_i^l and λ_i^u, respectively.

To test the integral separation of the functions $\{a_{ii}/e_{ii}\}_{i=1}^{d}$ in practice, we use Theorem 14. Choosing a sufficiently large H, then the Steklov difference of a_{ii}/e_{ii} and $a_{i+1,i+1}/e_{i+1,i+1}$ is given by

$$S_i(t,H) := \frac{1}{H}\left\{[\phi_i(t+H)-\phi_i(t)]-[\phi_{i+1}(t+H)-\phi_{i+1}(t)]\right\}, t\in\mathbb{I},\ i=1,...,d-1. \tag{71}$$

Analogously, by Theorem 19 and Proposition 22, the functions a_{ii}/e_{ii}, $i=1,\ldots,d$ also present information about the Sacker-Sell intervals of the implicit ODE (66) even without the integral separability. Concretely, for given $T\geq H>0$ and $0<t_0<T-H$, we set $b_{ii}(t)=a_{ii}(t)/e_{ii}(t)$, $i=1,2,...,d$, defined on $[0,T]$. We compute the Steklov averages of b_{ii} with respect to the given H as

$$\psi_{i,H}(t) := \frac{1}{H}\int_t^{t+H} b_{ii}(s)ds, \quad T-H\geq t\geq t_0.$$

This computation can be realized by solving auxiliary initial value problems as in the case of testing integral separation. Then, we use the quantities

$$\kappa_i^l(t_0,T,H) := \inf_{t_0\leq t\leq T-H}\psi_{H,i}(t) \quad \text{and } \kappa_i^u(t_0,T,H) := \sup_{t_0\leq t\leq T-H}\psi_{H,i}(t)$$

as approximations to the endpoints of the Sacker-Sell spectral intervals. Due to the property that the Sacker-Sell intervals include the Lyapunov intervals we then have obtain also bounds for the Lyapunov intervals.

We summarize the procedure for computing approximations to Lyapunov and Sacker-Sell spectral intervals in the following algorithm.

Algorithm 1 (Continuous QR algorithm for computing Lyapunov and Sacker-Sell spectra)

- **Input:** `A pair of sufficiently matrix functions` (E,A) `in the form of the strangeness-free DAE (26) (if they are not available directly they must be obtained pointwise as output of a routine such as GELDA); the values` T,H,τ `such that` $H\in(0,T)$ `and` $\tau\in(0,T)$; $V_1(t_0)$ `as initial value for (56). Here we may use` $V_1(t_0)=I_d$.
- **Output:** `Bounds for spectral intervals` $\{\lambda_i^l,\lambda_i^u\}_{i=1}^d$.
- **Initialization:**
  1. `Set` $j=0, t_0:=0$. `Compute` $\tilde{Q}(t_0)$, $\tilde{E}_{11}(t_0)$, `and` $\tilde{A}_{11}(t_0)$ `as in (31).`
  2. `Compute` $U_1(t_0),\mathcal{E}_1(t_0),\mathcal{A}_1(t_0)$.
  3. `Set` $\lambda_i(t_0)=0,\phi_i(t_0)=0,\ i=1,...,d$.

  `While` $t_j<T$

  1. $j:=j+1$.
  2. `Choose a stepsize` h_j `and set` $t_j=t_{j-1}+h_j$.
  3. `Compute` $\tilde{Q}(t_j)$, `then` $\tilde{E}_{11}(t_j),\tilde{A}_{11}(t_j)$, `see (61) and (62).`
  4. `Evaluate` $V_1(t_j)$ `by solving (56).`
  5. `Compute` $U_1(t_j),\mathcal{E}_1(t_j),\mathcal{A}_1(t_j)$ `as in (57), (63), respectively.`
  6. `Compute` $\phi_i(t_j)$, $\lambda_i(t_j)$, $i=1,...,d$ `as in (69), (70).`
  7. `Compute` $S_i(t,H)$, $i=1,2,...,d-1$, `by (71).`
  8. `If desired, test integral separation via Theorem 14.`
  9. `Update` $\min_{\tau\leq t\leq t_j}\lambda_i(t)$ `and` $\max_{\tau\leq t\leq t_j}\lambda_i(t)$.

The corresponding algorithm for computing Sacker-Sell spectra is similar. A slight difference is that instead of computing $\lambda_i(t)$ at each meshpoint (see Step 6.), we evaluate the Steklov averages $\psi_{H,i}(t)$ by the formula

$$\psi_{H,i}(t) = \frac{1}{H}(\phi_i(t+H) - \phi_i(t)), \quad i = 1, 2, ..., d.$$

Finally, we use the last step for computing $\inf_{\tau \le t \le T-H} \psi_{H,i}(t)$ and $\sup_{\tau \le t \le T-H} \psi_{H,i}(t)$.

4.2 Discrete QR-algorithm

While in the continuous QR-algorithm, the fundamental solution matrix R_1 of the triangularized implicit ODE system (66) is not evaluated directly, in the discrete QR-algorithm, R_1 is indirectly evaluated by a reorthogonalized integration of DAE system (26), an implicitly determined transformation to the semi-implicit form (31), and an appropriate QR-factorization. Note that R_1 is upper triangular as well and the diagonal elements of the normalized R_1 are given by $e^{\phi_i(t)},\ i = 1, 2, ..., d$, with the auxiliary functions ϕ_i defined in (69).

To apply the discrete QR-algorithm, we first choose a mesh $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T$. (This mesh may be different from that in Algorithm 1). At t_0, we set

$$Z_0 = Q_0 := I_d.$$

For $j = 1, 2, \ldots, N$, let $X^{[j]}$ be the solution to the matrix initial value problem

$$\begin{cases} E\dot{X}^{[j]} = AX^{[j]}, & t_{j-1} \le t \le t_j, \\ X^{[j]}(t_{j-1}) = \chi_{j-1}, & \end{cases} \tag{72}$$

with the initial condition

$$\chi_{j-1} := \tilde{Q}(t_{j-1}) \begin{bmatrix} Q_{j-1} \\ 0 \end{bmatrix}. \tag{73}$$

Here, $\tilde{Q}$ is defined and computed as in (61). We stress that χ_{j-1} defined in this way is a consistent initial value assigned at t_{j-1} for DAE system (26). Then, we have that

$$\tilde{Q}(t_j)^T X^{[j]}(t_j) = \begin{bmatrix} Z_j \\ 0 \end{bmatrix}, \tag{74}$$

where Z_j is the value of the rescaled fundamental solution matrix for the underlying ODE (33).

Then, we compute QR-factorizations

$$Z_j = Q_j \Theta_j, \quad j = 1, 2, \ldots, N$$

where all the diagonal elements of the triangular matrices Θ_j are chosen to be positive. Now, letting $\tilde{X}_1$ be the normalized fundamental solution matrix of (33), then it follows that

$$\begin{aligned} \tilde{X}_1(t_j) &= Z_j Q_{j-1}^T \tilde{X}_1(t_{j-1}) \\ &= Q_j \Theta_j Q_{j-1}^T Q_{j-1} \Theta_{j-1} \cdots \Theta_1 Q_0 \\ &= Q_j \Theta_j \Theta_{j-1} \cdots \Theta_1. \end{aligned}$$

Hence

$$R_1(t_j) = \Theta_j \Theta_{j-1} \cdots \Theta_1.$$

Note that the quantities Θ_j give information about the local growth rates of the fundamental solution matrix $\tilde{X}_1$ on $[t_{j-1}, t_j]$. Furthermore, we obtain

$$\lambda_i(t_j) = \frac{1}{t_j} \ln[R_1(t_j)]_{i,i} = \frac{1}{t_j} \ln \prod_{\ell=1}^{j} [\Theta_\ell]_{i,i} = \frac{1}{t_j} \sum_{\ell=1}^{j} \ln[\Theta_\ell]_{i,i}, \quad i = 1, 2, \ldots, d,$$

where the functions λ_i are defined as in (68). For computing the Lyapunov spectrum, we solve the associated optimization problems $\inf_{\tau\leq t\leq T}\lambda_i(t)$ and $\sup_{\tau\leq t\leq T}\lambda_i(t)$, $i=1,2,\ldots,d$, respectively, with a given $\tau\in(0,T)$.

The approximation of the Sacker-Sell spectrum is obtained analogously. We summarize the procedure in the following algorithm.

Algorithm 2 (Discrete QR-algorithm for computing Lyapunov and Sacker-Sell spectra)

- **Input:** `A pair of sufficiently smooth matrix functions` (E,A) `in the form of the strangeness-free DAE (26) (if they are not available directly they must be obtained pointwise as output of a routine such as GELDA), the time interval` $[0,T]$, $\tau\in(0,T)$, `and a mesh` $0=t_0<t_1<...<t_{N-1}<t_N=T$
- **Output:** `Bounds for spectral intervals` $\{\lambda_i^l,\lambda_i^u\}_{i=1}^d$.
- **Initialization:**

  1. `Set` $t_0:=0, Z_0=Q_0=I_d$ `and compute` $\tilde{Q}(t_0)$ `as in (31).`
  2. `Set` $\lambda_i(t_0):=0$ `and` $s_i:=0$ `for` $i=1,\ldots,d$ `(for computing the sum` s_i `of the logarithms).`

  `While` $j\leq N$

  1. $j:=j+1$.
  2. `Compute the initial values` χ_{j-1} `via (73).`
  3. `Solve the initial value problem (72) for` $X^{[j]}$ `on` $[t_{j-1},t_j]$.
  4. `Compute` $\tilde{Q}(t_j)$ `by (61) and then` Z_j `by (74).`
  5. `Compute the` QR `factorization` $Z_j=Q_j\Theta_j$.
  6. `Update` $s_i:=s_i+\ln[\Theta_j]_{i,i}$ `and` $\lambda_i(t_j)=\frac{1}{t_j}s_j,\quad i=1,2,\ldots,d$.
  7. `If desired, test the integral separation property by using` $\{s_i\}_{i=1}^d$, `if necessary.`
  8. `Update` $\min_{\tau\leq t\leq t_j}\lambda_i(t)$ `and` $\max_{\tau\leq t\leq t_j}\lambda_i(t)$, $i=1,2,\ldots,d$.

Remark 74 If the same mesh is used in Algorithms 1 and 2 and all calculations are done in exact arithmetic and without discretization errors, then the quantities s_i at the end of the j-th step of Algorithm 2 are exactly the values $\phi_i(t_j)$ defined in Algorithm 1.

Advantages of the discrete Algorithm are a simpler implementation and that existing DAE solvers for strangeness-free problems like BDF or Runge-Kutta methods, see [8, 42, 51] can be used. However, a disadvantage of the discrete method is that it creates numerical integration errors on each of the local intervals and these may grow very fast, in particular if the DAE system is very unstable and the subintervals are very long.

In Algorithms 1 and 2 we face the following error sources in the course of the numerical approximation of the spectral intervals.

In the continuous QR method, Algorithm 1 the following sources of errors arise.

1. The error in computing the QR factorization to compute $\tilde{Q}$ has to be considered together with the additional error in the numerical differentiation of $\tilde{Q}$. While the QR factorization is numerically backward stable and so the errors can be neglected, the error that arises in the numerical differentiation can be avoided by using the method of [46].
2. In the computation of $V_1(t_j)$ the integration error has to be considered, while the error in the QR factorizations to determine $U_1(t_j),\mathcal{E}_1(t_j)$, and the error arising in the solution of the linear system for L can be neglected again if backward stable methods are used.

3. We definitely have to consider the integration error for ϕ_i as well as
4. the error from truncating to a finite interval and
5. the error in the solution of the optimization problems.

In the discrete QR method, Algorithm 2 the following sources of errors arise.

1. The error in computing the QR-factorization of $\tilde{Q}$ causes an error in the initial value χ_{j-1} which then influences the
2. integration error for $X^{[j]}$ arising from discretization.
3. The error in computing the QR-factorizations that determine Q_j, Θ_j can again be neglected compared to these errors and
4. the error from truncating to a finite interval and
5. the error in solving the optimization problems.

For each method, the errors arising from the first 3 sources can be estimated and kept under control as well. In the literature, there have been some attempts to give error estimates for truncated time exponents and for the accumulated integration error, see [22]. Special and careful attention was paid to the efficient orthogonal integration in [24, 27]. The first systematical error analysis for QR methods in computing Lyapunov exponents was given in [28, 30]. We refer to these papers and references therein for more details.

Unfortunately, all existing results were given for regular ODE systems, only. For non-regular systems, choosing sufficiently large bounds for T, t_0, and H and giving error estimates for the approximate values of $\liminf$ and $\limsup$ are difficult tasks and this is work in progress. For the illustration of some of the difficulties, see the numerical examples given in the next section.

5 Numerical examples

We have implemented both the continuous and the discrete variants of the QR methods described in Section 4 in MATLAB. The following results are obtained with Version 7.0 on an IBM computer with Intel CPU T2300 1.66 GHz. For the orthogonal integration, we have used the projected integration technique, see [21].

To illustrate the properties of the procedures we consider two examples, one of a Lyapunov regular DAE system and another DAE system which is not Lyapunov regular. In the second case, we calculated not only the Lyapunov spectral intervals, but also the Sacker-Sell intervals.

Example 75 Our first example is a Lyapunov-regular DAE system which is constructed similar to the ODE examples in [22, 26]. We derived a DAE system of the form (26) as follows. We began with an upper triangular implicit ODE system, applied appropriate transformations and then added additional algebraic variables. In this way we obtained a semi-implicit DAE system of the form (31) which was then transformed again to obtain a DAE system of the form (26) whose spectral information is the same as that of original implicit ODE system.

The original triangular implicit ODE system had the form $D(t)\dot{\bar{x}}_1 = B(t)\bar{x}_1$, where

$$D(t) = \begin{bmatrix} 1+\frac{1}{t+1} & 0 \\ 0 & 1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} \lambda_1 - \frac{1}{t+1} & 1 \\ 0 & \lambda_2 + \cos(t+1) \end{bmatrix}, \quad t \in \mathbb{I},\ \lambda_i \in \mathbb{R} \quad (i=1,2).$$

Here $\lambda_i, i=1,2,\ (\lambda_1 < \lambda_2)$ are given real parameters. We then transformed and obtained the implicit ODE system $\tilde{E}_{11}(t)\dot{\tilde{x}}_1 = \tilde{A}_{11}(t)\tilde{x}_1$ given by

$$\tilde{E}_{11} = U_1 D V_1^T, \quad \tilde{A}_{11} = U_1 B V_1^T + U_1 D V_1^T \dot{V}_1 V_1^T,$$

T	h	λ_1	λ_2	$CPU - time$
500	0.1	4.9341	-0.0043	2.55
500	0.05	4.9337	-0.0038	5.01
500	0.01	4.9337	-0.0037	24.89
1000	0.1	4.9632	-0.0006	5.01
1000	0.05	4.9628	-0.0001	10.01
1000	0.01	4.9627	-0.0001	49.84
2000	0.1	4.9799	-0.0010	10.17
2000	0.05	4.9794	-0.0005	20.02
10000	0.1	4.9956	-0.0009	49.91
10000	0.05	4.9951	-0.0003	99.71

Table 1: Lyapunov exponents for Example 78 computed via the continuous QR-Euler method

with $U_1(t) = G_{\gamma_1}(t),\ V_1(t) = G_{\gamma_2}(t)$ with the Givens rotation

$$G_\gamma(t) = \begin{bmatrix} \cos\gamma t & \sin\gamma t \\ -\sin\gamma t & \cos\gamma t \end{bmatrix}$$

and some real parameters γ_1, γ_2. We chose additional blocks $\tilde{E}_{12} = U_1, \tilde{A}_{12} = V_1, \tilde{A}_{22} = U_1V_1$ and finally

$$\tilde{E} = \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}.$$

Using a 4×4 orthogonal matrix

$$G(t) = \begin{bmatrix} \cos\gamma_3 t & 0 & 0 & \sin\gamma_3 t \\ 0 & \cos\gamma_4 t & \sin\gamma_4 t & 0 \\ 0 & -\sin\gamma_4 t & \cos\gamma_4 t & 0 \\ -\sin\gamma_3 t & 0 & 0 & \cos\gamma_3 t \end{bmatrix},$$

with real values γ_3, γ_4 we obtained $E = \tilde{E}G^T,\ A = AG^T + \tilde{E}G^T\dot{G}G^T$ and applied the methods to the DAE system $E(t)\dot{x} = A(t)x$ which is a strangeness-free DAE system of the form (26). Furthermore, because Lyapunov-regularity together with Lyapunov exponents are invariant with respect to orthogonal change of variables, this system is Lyapunov-regular with the Lyapunov exponents λ_1, λ_2.

For our numerical tests we have used the values

$$\lambda_1 = 5, \lambda_2 = 0, \gamma_1 = \gamma_4 = 2, \gamma_2 = \gamma_3 = 1.$$

As numerical integration method in the continuous QR algorithm, we used the (projected) first order explicit Euler method and the (projected) second order explicit trapezoidal rule, both with constant stepsize h. The approximate values of the Lyapunov exponents are then calculated with different stepsizes h and for different time intervals $[0, T]$. The results are displayed in Tables 1 and 2, respectively. We display the CPU time measured in seconds.

The graph of the functions $\lambda_1(t), \lambda_2(t)$ is depicted in Figure 1. The monotonic, respectively oscillatory behavior of the two Lyapunov exponents is well approximated and the (admittedly slow) convergence of the computed Lyapunov exponents towards the exact Lyapunov exponents can be observed.

The numerical results of the discrete QR algorithm are displayed in Table 3. We have used the same meshes as in the computations with the continuous QR algorithm and we have used the implicit Euler method with a constant stepsize $h/10$ for the numerical integration of the DAE in the subintervals. Without this refinement, the approximate values are substantially less

T	h	λ_1	λ_2	$CPU - time$
500	0.1	4.9333	-0.0033	4.95
500	0.05	4.9336	-0.0036	9.83
500	0.01	4.9337	-0.0037	48.81
1000	0.1	4.9624	-0.0004	9.88
1000	0.05	4.9626	-0.0002	19.61
1000	0.01	4.9951	-0.0003	100.28
2000	0.1	4.9789	0.0000	19.63
2000	0.05	4.9951	-0.0003	101.02
10000	0.1	4.9946	0.0002	97.55
10000	0.05	4.9948	-0.0001	195.51

Table 2: Lyapunov exponents for Example 78 computed via the continuous QR-Trapezoid method

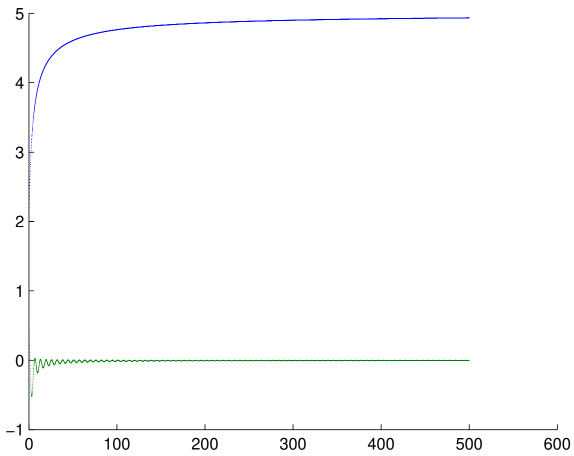

Figure 1: Graph of the functions $\lambda_i(t), i = 1, 2$ in Example 75.

T	h	λ_1	λ_2	$CPU - time$
500	0.1	5.0324	-0.0137	9.87
500	0.05	4.9818	-0.0087	19.59
500	0.01	4.9431	-0.0047	97.31
1000	0.1	5.0625	-0.0100	19.63
1000	0.05	5.0114	-0.0050	38.87
2000	0.1	5.0799	-0.0104	39.20
2000	0.05	5.0284	-0.0053	78.15
10000	0.1	5.0963	-0.0102	194.89
10000	0.05	5.0443	-0.0052	389.64

Table 3: Lyapunov exponents computed by the discrete QR method with the implicit Euler method as integrator

T	t_0	h	$[\lambda_1^l, \lambda_1^u]$	$[\lambda_2^l, \lambda_2^u]$	$CPU - time$
1000	100	0.1	$[-1.0018, 0.5865]$	$[-6.0006, -4.8928]$	5.28
5000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.3846]$	26.02
10000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.0235]$	51.52
10000	500	0.1	$[-0.0647, 1.0004]$	$[-6.0006, -4.0235]$	51.63
10000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0229]$	103.42
20000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.0007]$	103.50
20000	500	0.1	$[-0.4598, 1.0004]$	$[-6.0006, -4.0007]$	103.32
20000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	210.95
50000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	519.45
50000	500	0.05	$[-0.9844, 1.0000]$	$[-6.0001, -4.0001]$	518.15
100000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	1044.94
100000	500	0.05	$[-0.9998, 1.0000]$	$[-6.0001, -4.0001]$	1050.36

Table 4: Lyapunov spectral intervals computed by the continuous QR-Euler method

accurate than the corresponding values computed by the continuous QR method. By comparing the numerical results for the continuous and discrete QR algorithm we see that the continuous QR method is more efficient and accurate than the discrete QR method. It is also interesting to observe that the discrete QR method oscillates when the stepsize is decreased.

Example 76 (A DAE system which is not Lyapunov regular) With the same transformations as in Example 75 we also constructed a DAE that is not Lyapunov regular by changing the matrix $B(t)$ in Example 75 to

$$B(t) = \begin{bmatrix} \sin(\ln(t+1)) + \cos(\ln(t+1)) + \lambda_1 & 1 \\ 0 & \sin(\ln(t+1)) - \cos(\ln(t+1)) + \lambda_2 \end{bmatrix}, \quad t \in \mathbb{I}.$$

Here we chose $\lambda_1 = 0, \lambda_2 = -5$. Since Lyapunov and Sacker-Sell spectra are invariant with respect to global kinematical equivalence transformation, it is easy to compute the Lyapunov spectral intervals as $[-1, 1]$ and $[-6, -4]$ and the Sacker-Sell spectral intervals as $[-\sqrt{2}, \sqrt{2}]$ and $[-5-\sqrt{2}, -5+\sqrt{2}]$.

We computed first the approximate Lyapunov spectral intervals with different initial and end points t_0, T, and stepsizes h via the continuous QR-Euler method. The results are displayed in Table 4 and we observe that the method computes reasonably good approximations to the Lyapunov spectral intervals but that the method is sensitive to the choice of the values of T and t_0. This is already a well-known difficulty in the case of ODEs, see [26]. The graphs of the functions $\lambda_1(t), \lambda_2(t)$ are shown in Figure 2.

Finally, we used the continuous QR algorithm for approximating the Sacker-Sell intervals with different T, H and h. The numerical results displayed in Table 5 illustrate well the success of the QR algorithm but also the difficulty in choosing appropriately large values of T and H.

A plot of the graph of the Steklov averages $\psi_{H,i}(t), i = 1, 2$ with $H = 500$ is given in Figure 3.

In order to improve the described numerical methods it is important to carry out a careful error analysis of the different components of the method as well as a detailed analysis of the convergence behavior with respect to the choice of initial and end point t_0, T. This is current work in progress.

6 Conclusion

In this paper we have extended the classical spectral concepts and numerical methods for approximating (Lyapunov, Bohl and Sacker-Sell) spectral intervals that are well-known for ordinary differential equations to linear differential-algebraic equations with variable coefficients.

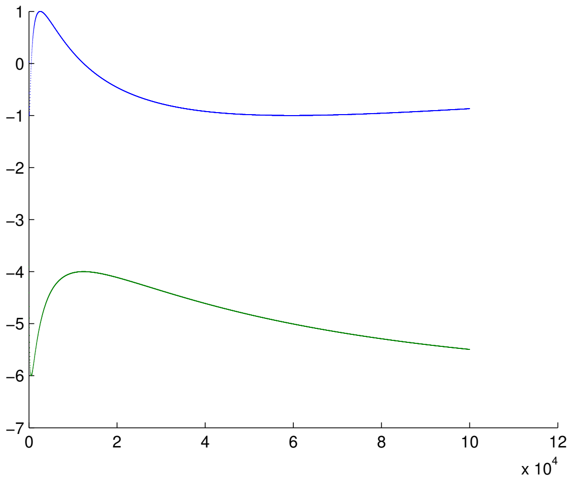

Figure 2: The graph of functions $\lambda_i(t), i = 1, 2$ in Example 76.

T	H	h	$[\kappa_1^l, \kappa_1^u]$	$[\kappa_2^l, \kappa_2^u]$	$CPU - time$
1000	100	0.1	$[-1.2042, 1.3811]$	$[-6.4049, -4.8927]$	6.20
5000	100	0.1	$[-1.2042, 1.4131]$	$[-6.4049, -3.5990]$	30.79
10000	100	0.1	$[-1.2042, 1.4131]$	$[-6.4049, -3.5867]$	61.94
10000	500	0.1	$[-0.7327, 1.4030]$	$[-6.2142, -3.5872]$	94.80
10000	100	0.05	$[-1.2049, 1.4127]$	$[-6.4046, -3.5860]$	147.19
20000	100	0.1	$[-1.3461, 1.4131]$	$[-6.4049, -3.5867]$	123.57
20000	500	0.1	$[-1.3416, 1.4030]$	$[-6.2142, -3.5872]$	201.26
20000	100	0.05	$[-1.3468, 1.4127]$	$[-6.4046, -3.5860]$	283.10
50000	100	0.1	$[-1.4132, 1.4131]$	$[-6.4049, -3.5867]$	310.36
50000	500	0.1	$[-1.4132, 1.4030]$	$[-6.2142, -3.5872]$	506.65
100000	100	0.1	$[-1.4132, 1.4131]$	$[-6.4049, -3.5867]$	646.15
100000	500	0.1	$[-1.4132, 1.4030]$	$[-6.3633, -3.5872]$	976.30
200000	500	0.1	$[-1.4132, 1.4030]$	$[-6.4147, -3.5872]$	1973.43

Table 5: Sacker-Sell spectral intervals computed by the continuous QR-Euler method

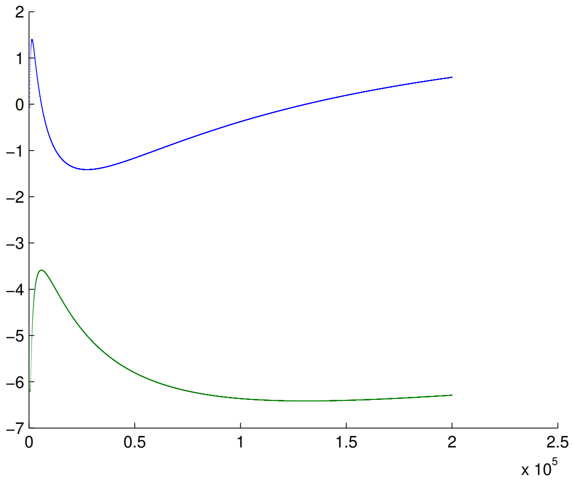

Figure 3: Graph of Steklov averages $\psi_{H,i}(t), i = 1, 2,\ H = 500$.

In the theoretical analysis of the spectral theory we have used appropriate orthogonal changes of variables to transform the original DAE system to a particular strangeness-free form for which the underlying ODE systems are easily obtained. The relationship between different spectra of the DAE systems and those of their corresponding underlying ODE system has been analyzed. We have proven that under some boundedness conditions, the Lyapunov and the Sacker-Sell (exponential dichotomy) spectrum of a DAE system and those of its underlying ODE system coincide. Several significant differences between the spectral theory for ODEs and that for DAEs have been discussed as well and the stability of these spectra has been investigated. In particular, we have shown that the Sacker-Sell spectrum of a robustly strangeness-free DAE system is stable with respect to admissible structured perturbations. In general, if either the DAE system in consideration is not robustly strangeness-free or it is subject to an unstructured perturbation, then the spectral stability cannot be expected. We have proposed two numerical methods based on QR factorization for calculating Lyapunov and Sacker-Sell spectra. The algorithms as well as related implementation techniques have been discussed. Finally, two DAE examples have been presented for illustration.

Experimental numerical results have not only illustrated the efficiency and the reliability of the computational methods, but the numerical results also indicate the difficulties that may arise in the implementation and the use of these methods. In particular a detailed error and perturbation analysis is necessary. Similarly to the ODE case, an extension of such algorithms to nonlinear DAEs should also be carried out.

Further work is also necessary in developing more efficient implementation techniques and a complete error analysis for the overall numerical methods proposed in this paper.

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