NUMERICAL METHODS FOR PALINDROMIC EIGENVALUE PROBLEMS: COMPUTING THE ANTI-TRIANGULAR SCHUR FORM

D. STEVEN MACKEY *§¶, NILOUFER MACKEY *§¶, CHRISTIAN MEHL †¶, AND VOLKER MEHRMANN ‡¶

Abstract. We present structure-preserving numerical methods for complex palindromic polynomial eigenvalue problems via corresponding palindromic linearizations. A key ingredient is the development of an appropriate condensed form — the anti-triangular Schur form. Ill-conditioned problems which have eigenvalues near the unit circle, in particular near ± 1, are discussed. We show how a combination of unstructured methods followed by a structured refinement can be used to solve such problems very accurately.

Key words. nonlinear eigenvalue problem, palindromic matrix polynomial, Schur form, anti-triangular form, Jacobi algorithm, structured deflation method, palindromic QR-algorithm

AMS subject classification. 65F15, 15A18, 15A57

1. Introduction. The numerical solution of polynomial eigenvalue problems

$$P(\lambda)x = \left(\sum_{i=0}^{k} \lambda^i A_i\right) x = 0$$

with real or complex coefficient matrices A_i is an important task in many applications e.g., the vibration analysis for buildings, machines or vehicles [3, 9, 19]. In many cases the matrix polynomial $P(\lambda)$ has extra structure, such as symmetry or skew-symmetry of the coefficient matrices, leading to some kind of symmetry in the spectrum of $P(\lambda)$. One such type of structured polynomial arising in applications is the class of *palindromic matrix polynomials*, forming a generalization of symplectic matrices [13, 17]. Palindromic polynomials have the property that reversing the order of the coefficients (and possibly transposing) leads to the same matrix polynomial, analogous to the palindrome "never odd or even".

DEFINITION 1.1. *Let* $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ *be a matrix polynomial with* $A_i \in \mathbb{C}^{\ell \times \ell}$ *for* $i = 0, \ldots, k$ *and* $A_k \neq 0$. *Then the matrix polynomial*

$$\operatorname{rev} P(\lambda) := \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i$$

is called the reversal *of* $P(\lambda)$. *A matrix polynomial is said to be* **T-palindromic** *if it is the same as the transpose of its reversal, that is, if* $\operatorname{rev} P(\lambda)^T = P(\lambda)$.

Eigenvalue problems with underlying T-palindromic matrix polynomials arise, for example, in the mathematical modeling and numerical simulation of the behavior of

*Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA. (nil.mackey@wmich.edu,steve.mackey@wmich.edu).

†School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK. mehl@maths.bham.ac.uk.

‡Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG. mehrmann@math.tu-berlin.de.

§Partially supported by the National Science Foundation grant DMS-0713799.

¶Partially supported by the *Deutsche Forschungsgemeinschaft* through the DFG Research Center MATHEON *Mathematics for key technologies* in Berlin.

periodic surface acoustic wave (SAW) filters [8, 20], and in the vibration analysis of rail tracks under the excitation arising from high speed trains, see [6, 7].

A matrix polynomial is said to be *regular* if $\det P(\lambda) \not\equiv 0$. It was shown in [13] that eigenvalues of a regular T-palindromic matrix polynomial occur in pairs $(\lambda, 1/\lambda)$. Moreover, as long as certain exceptional points (e.g. -1 in the case of quadratic polynomials) are not in its spectrum, then $P(\lambda)$ admits a *T-palindromic linearization*, i.e., there exist matrix polynomials $E(\lambda), F(\lambda)$ with constant nonzero determinant such that

$$E(\lambda)\begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{\ell(k-1)} \end{bmatrix} F(\lambda) = \lambda Z + Z^T, \quad \text{where} \quad Z \in \mathbb{C}^{k\ell\times k\ell}.$$

It suffices, therefore, to determine methods for the numerical solution of eigenvalue problems for T-palindromic pencils $\lambda Z + Z^T$. A first step towards this end would be to derive a condensed form from which the eigenvalues, eigenvectors and/or deflating subspaces of the pencil can be easily read off. Clearly, one could just compute the (unstructured) generalized Schur form $\lambda T_1 + T_2 = U(\lambda Z + Z^T)V$ of the pencil $\lambda Z + Z^T$. But the $(\lambda, 1/\lambda)$ symmetry in the spectrum will likely be obscured by roundoff errors. Furthermore, for eigenvalues close to the unit circle, the number of eigenvalues inside the unit circle versus the number outside may be incorrectly computed. Since in applications it is typically important to compute the deflating subspace associated with eigenvalues inside the unit circle, it would be much better to obtain a structured condensed form under structure preserving transformations. In the case of T-palindromic matrix polynomials, a T-congruence transformation by any non-singular matrix S

$$(\lambda Z + Z^T) \mapsto S^T(\lambda Z + Z^T)S$$

preserves the palindromic structure. In the interest of numerical stability, however, we restrict ourselves to unitary matrices, and therefore look for a condensed form under *unitary T-congruence*

$$(\lambda Z + Z^T) \mapsto U^T(\lambda Z + Z^T)U\,.$$

Observe that since $U^T = \overline{U}^{-1}$, this transformation may also be viewed as a simultaneous unitary consimilarity transformation on Z and Z^T.

What might a useful structured condensed form look like? A triangular form will not be of any help in this context; if $U^T Z U$ is upper triangular, then $U^T Z^T U$ would be lower triangular, and the eigenvalues of the pencil cannot be easily read off. On the other hand if $U^T Z U$ is *anti-triangular*, that is,

$$U^T Z U = M = [m_{i,j}] = \left[\quad \diagup\!\!| \quad\right]_{n\times n}, \quad \text{with } m_{ij} = 0 \text{ whenever } i + j \le n,$$

then so is $M^T = U^T Z^T U$. The eigenvalues of the pencil $\lambda M + M^T$ can now be read off from the anti-diagonal as quotients $\lambda_j = -m_{n-j+1,j}/m_{j,n-j+1}$.

In section 2 we show that such an anti-triangular form always exists for any matrix $Z \in \mathbb{C}^{n\times n}$. Furthermore, we will also see how the existence of this anti-triangular form is intimately connected with the eigenproblem for palindromic pencils, rather than just an artifact of solving the palindromic eigenproblem in some particular way.

In subsequent sections we develop numerical methods for computing this anti-triangular form, which can then also be viewed as structure-preserving methods for the T-palindromic eigenproblem. The first is a structured deflation method taking its inspiration from an idea of Laub [12]. Known informally as the "Laub trick", this idea led to a method for computing the Hamiltonian (symplectic) Schur form of a Hamiltonian (symplectic) matrix using information from an unstructured Schur form [12], [16, p.105–6]. The common theme underlying both the "Ur"-Laub trick and our structured deflation procedure is that information from an *unstructured* condensed form can be used to build up a *structured* condensed form. This theme has recently been further developed by Byers and Kressner [1] to investigate how structured solutions to structured problems may be found from unstructured solutions by an appropriate projection onto a variety or manifold of structured objects.

Finally we discuss two other structure-preserving methods, the palindromic Jacobi and the palindromic-QR algorithms, and show how they can be combined with our structured deflation method to provide an effective and accurate means of solving the T-palindromic eigenvalue problem in a structure-preserving manner.

Note that throughout the rest of the paper we follow the convention that 0 and ∞ are considered to be *reciprocals* of each other. Also we use $\|v\|$ to denote the Euclidean norm of a vector v.

2. Anti-triangular forms for matrices and T-palindromic pencils.

To derive condensed forms for T-palindromic pencils, we first show that any matrix $Z \in \mathbb{C}^{n\times n}$ can be reduced to anti-triangular Schur-like form by a unitary T-congruence. The original motivation to investigate the possibility of such an anti-triangular form for complex matrices arose from the desire to solve the eigenvalue problem for the associated T-palindromic pencil $L_Z(\lambda) = \lambda Z + Z^T$ in a structure-preserving manner. From this point of view the connection between the eigenproblem for the pencil L_Z and the anti-triangular Schur form for the matrix Z may seem somewhat artificial. Surely the question whether such a form exists for a matrix Z is a problem just about matrices, solvable without reference to matrix pencils. Nevertheless, there is an intrinsic connection between these two problems, as we now demonstrate.

Suppose that $Z \in \mathbb{C}^{n\times n}$ is any matrix and that U is unitary with $M = U^T Z U$ in anti-triangular form. Then the first columns of both M and M^T are scalar multiples of e_n (the nth unit vector), so that for some constants α and β,

$$U^T Z U e_1 = U^T Z u_1 = \alpha e_n \quad \text{and} \quad (U^T Z U)^T e_1 = U^T Z^T u_1 = \beta e_n,$$

where u_1 denotes the first column of U. Hence $\beta U^T Z u_1 - \alpha U^T Z^T u_1 = 0$, or equivalently $(\beta Z - \alpha Z^T)u_1 = 0$, so that u_1 is an eigenvector of the pencil $L_Z(\lambda) = \lambda Z + Z^T$ with eigenvalue $\lambda = -\beta/\alpha$. (When $\alpha = \beta = 0$, then u_1 may still be viewed as an eigenvector of the *singular* pencil L_Z.) Thus any anti-triangular form for a matrix Z *necessarily* involves some eigenvector of the pencil $L_Z(\lambda)$. But not just any eigenvector of $L_Z(\lambda)$ will do. Observe that for $M = U^T Z U$ to be in anti-triangular form we must also have $m_{1,1} = u_1^T Z u_1 = 0$, so an eigenvector of $L_Z(\lambda)$ with this additional property is needed. The following technical lemma shows that such eigenvectors are not rare; indeed it turns out that "most" eigenvectors x of a regular T-palindromic pencil $\lambda Z + Z^T$ satisfy $x^T Z x = 0$.

Definition 2.1. *Let $x \in \mathbb{C}^n$ and $Z \in \mathbb{C}^{n\times n}$. If $x^T Z x = 0$ then the vector x is said to be* Z-isotropic. *More generally, suppose $\mathcal{S}$ is the subspace spanned by the* *columns of a matrix* $W \in \mathbb{C}^{n\times k}$*. Then the subspace* $\mathcal{S}$ *and the matrix* W *are said to be* Z-isotropic *if* $W^T Z W = 0$.

LEMMA 2.2. *Suppose* $Z \in \mathbb{C}^{n\times n}$ *is a matrix such that the associated* T*-palindromic pencil* $L_Z(\lambda) = \lambda Z + Z^T$ *is regular.*

(a) *Let* $x \in \mathbb{C}^n$ *be any eigenvector of* $L_Z(\lambda)$ *associated with either a finite eigenvalue* $\mu \in \mathbb{C} \setminus \{-1\}$ *or with the eigenvalue* $\mu = \infty$*. Then* x *is* Z*-isotropic.*

(b) *If* $\mu = -1$ *is an eigenvalue of* $L_Z(\lambda)$ *with algebraic multiplicity* $m > 1$*, then there exists an associated* Z*-isotropic eigenvector* $x \in \mathbb{C}^n$*.*

(c) *If* $L_Z(\lambda)$ *has no* Z*-isotropic eigenvector, then* $n = 1$*, i.e.,* Z *is scalar.*

Proof. (a) For a finite eigenvalue μ, the identity $(\mu Z + Z^T)x = 0$ implies that

$$0 = x^T(\mu Z + Z^T)x = \mu x^T Z x + x^T Z^T x = \mu x^T Z x + x^T Z x = (\mu + 1)x^T Z x\,,$$

and the desired conclusion follows for any finite $\mu \neq -1$. For $\mu = \infty$, an eigenvector is just a nonzero $x \in \ker Z$. But any such x is clearly Z-isotropic.

(b) Suppose first that there exist two linearly independent eigenvectors $w, y \in \mathbb{C}^n$ associated with the eigenvalue $\mu = -1$. If either w or y is Z-isotropic then we are done. If not, then for the eigenvectors $x(\beta) = w + \beta y$ consider

$$x(\beta)^T Z x(\beta) = w^T Z w + \beta(y^T Z w + w^T Z y) + \beta^2 y^T Z y\,.$$

Since y is not isotropic, $x(\beta)^T Z x(\beta) = 0$ is a quadratic equation in β with a solution $\widetilde{\beta}$ over $\mathbb{C}$, thus yielding a Z-isotropic eigenvector $x(\widetilde{\beta})$ associated with $\mu = -1$.

If, on the other hand, there is only one linearly independent eigenvector x for $\mu = -1$, then there exists a Jordan chain $(x_1, \ldots, x_m)$ associated with μ in which $x_1 = x$. Hence by definition [11]

$$(\mu Z + Z^T)x_1 = 0 \quad \text{and} \quad (\mu Z + Z^T)x_j = -Z x_{j-1} \quad \text{for } j = 2, \ldots, m.$$

In particular, we have $x_1^T(Z - Z^T) = 0$ and $(Z - Z^T)x_2 = Z x_1$. Thus we see that

$$x^T Z x = x_1^T(Z x_1) = \left(x_1^T(Z - Z^T)\right) x_2 = 0\,,$$

and so x is Z-isotropic.

(c) Suppose $L_Z(\lambda)$ has no Z-isotropic eigenvector. Then by (a) the only eigenvalue of L_Z is -1, and by (b) its algebraic multiplicity is one. Thus Z must be scalar. ◻

We are now ready to prove the main result of this section, the existence of an anti-triangular Schur-like form for any $n \times n$ complex matrix. It is instructive to compare the proof given here with the standard derivation of upper triangular Schur form. In both cases the argument proceeds inductively on the matrix size, using eigenvectors to reduce to a smaller problem. The key difference is the source of the eigenvectors. For the triangular Schur form they come from the matrix itself, whereas for anti-triangular form we will see that they come instead from the associated T-palindromic pencil.

THEOREM 2.3 (Anti-triangular Schur Form). *Let* $Z \in \mathbb{C}^{n\times n}$*. Then there exists a unitary matrix* $U \in \mathbb{C}^{n\times n}$ *such that*

$$M = U^T Z U = \begin{bmatrix} 0 & \cdots & 0 & m_{1,n} \\ \vdots & \iddots & \iddots & \vdots \\ 0 & \iddots & & \vdots \\ m_{n,1} & \cdots & \cdots & m_{n,n} \end{bmatrix} \tag{2.1}$$

is in anti-triangular form.

Proof. The proof proceeds by induction on n. For $n = 1$ there is nothing to prove, so let $n > 1$. We will show that there exists an $n \times n$ unitary matrix $Q = [q_1, \ldots, q_n]$ such that

$$\widetilde{Z} = Q^T Z Q = \begin{bmatrix} q_1^T Z q_1 & \cdots & q_1^T Z q_n \\ \vdots & \ddots & \vdots \\ q_n^T Z q_1 & \cdots & q_n^T Z q_n \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle n-2 & \scriptstyle 1 \end{array} \\ \begin{bmatrix} 0 & 0 & \widetilde{z}_{13} \\ 0 & \widetilde{Z}_{22} & * \\ \widetilde{z}_{31} & * & * \end{bmatrix} \end{array} \begin{array}{c} \scriptstyle 1 \\ \scriptstyle n-2 \\ \scriptstyle 1 \end{array} . \tag{2.2}$$

For $n = 2$ or $n = 3$ there is nothing more to do; $\widetilde{Z}$ is already in anti-triangular form. If $n > 3$, then the induction hypothesis applied to $\widetilde{Z}_{22}$ provides an $(n-2)\times(n-2)$ unitary matrix $\widetilde{U}$ such that $\widetilde{U}^T \widetilde{Z}_{22} \widetilde{U}$ is in anti-triangular form. Setting $U = Q \cdot \mathrm{diag}(1, \widetilde{U}, 1)$, then $U^T Z U$ is in anti-triangular form and the induction is complete.

To construct a unitary matrix Q such that (2.2) holds, we consider two cases:
<u>*Case 1*</u> (The pencil $L_Z(\lambda) = \lambda Z + Z^T$ is singular) : In this case the matrix Z must be singular, so choose u to be any unit vector in the left nullspace of Z, i.e. $u^T Z = 0$, and let P be any unitary matrix with u as its first column. Then

$$P^T Z P = \begin{bmatrix} 0 & 0 \\ x & Y \end{bmatrix} \tag{2.3}$$

where $x \in \mathbb{C}^{n-1}$. Let W be any $(n-1) \times (n-1)$ unitary matrix such that $W^T x = \beta\, e_{n-1}$ for some $\beta \in \mathbb{C}$. (For example, one could choose W^T to be an appropriate Householder reflector.) Setting $Q = P \cdot \mathrm{diag}(1, W)$, we see that

$$Q^T Z Q = \begin{bmatrix} 1 & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ x & Y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta\, e_{n-1} & W^T Y W \end{bmatrix}$$

has the desired form for (2.2).
<u>*Case 2*</u> (The pencil $L_Z(\lambda) = \lambda Z + Z^T$ is regular) : Since $n \geq 2$, by Lemma 2.2 the pencil $L_Z(\lambda)$ has a normalized Z-isotropic eigenvector u, i.e., $u^T Z u = 0$ and $u^* u = 1$. The vectors Zu and $Z^T u$ are linearly dependent, since u is an eigenvector of $L_Z(\lambda)$, and not both zero, since $L_Z(\lambda)$ is assumed to be a regular pencil. Let w be whichever of Zu and $Z^T u$ is nonzero, and let $q_2, \ldots, q_{n-1}$ be any orthonormal basis for the orthogonal complement of $\mathrm{Span}(u, \overline{w})$, so that the matrix

$$Q = \left[u, q_2, \ldots, q_{n-1}, \frac{\overline{w}}{\|\overline{w}\|} \right]$$

is unitary. The unitariness of Q together with the linear dependence of Zu and $Z^T u$ now imply that $0 = q_j^T w = q_j^T Z u = q_j^T Z^T u = u^T Z q_j$ for $j = 2, \ldots, n-1$. Thus, as desired, we obtain (2.2). $\square$

The anti-triangular Schur form of a matrix Z can now be used to read off basic information about the associated pencil $L_Z(\lambda)$: when is L_Z singular or regular, what is its spectrum, and in which order can the spectrum appear on the anti-diagonal? These issues are dealt with in Theorem 2.5.

DEFINITION 2.4. *A list of numbers* $(\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_i \in \mathbb{C} \cup \{\infty\}$ *is said to be* reciprocally ordered *if* λ_j *and* λ_{n+1-j} *are reciprocals for* $j = 1, \ldots, n$. (*Our convention that* 0 *and* ∞ *are reciprocals of each other is in effect here.*)

THEOREM 2.5 (Spectrum of T-palindromic pencils). *Let $Z \in \mathbb{C}^{n\times n}$ with associated T-palindromic pencil $L_Z(\lambda) = \lambda Z + Z^T$, and suppose $M = [m_{ij}] = U^T ZU$ is any anti-triangular form for Z.*

(1) *The pencil $L_Z(\lambda)$ is singular if and only if M has a symmetrically placed pair of zeroes on the anti-diagonal, i.e., $m_{j,n-j+1} = 0 = m_{n-j+1,j}$ for some j. On the other hand, if $L_Z(\lambda)$ is regular, then its spectrum is given by*

$$\sigma(\lambda Z + Z^T) = \left\{ -\frac{m_{n-j+1,j}}{m_{j,n-j+1}} \ : \ j = 1, \ldots, n \right\}. \tag{2.4}$$

(2) *Suppose $L_Z(\lambda)$ is regular, and $(\lambda_1, \ldots, \lambda_n)$ is the ordered list of eigenvalues of L_Z extracted from $\lambda M + M^T$ by reading from top to bottom, i.e.,*

$$\lambda_j = -\frac{m_{n-j+1,j}}{m_{j,n-j+1}}, \quad j = 1, \ldots, n \tag{2.5}$$

as in (2.4). Then the list $(\lambda_1, \ldots, \lambda_n)$ is reciprocally ordered. Indeed, for any *reciprocal ordering $(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_n)$ of the spectrum of $L_Z(\lambda)$ there exists a unitary matrix $\widetilde{U}$ so that the eigenvalues of $L_Z(\lambda)$ appear in this order, top-to-bottom, on the anti-diagonal of $\lambda\widetilde{M} + \widetilde{M}^T = \widetilde{U}^T L_Z(\lambda)\widetilde{U}$. (Note that if n is odd, then the middle eigenvalue $\lambda_{\frac{n+1}{2}}$ on any such list must be -1.)*

Proof. (1) Up to a sign, the determinant of the pencil $L_M(\lambda) = \lambda M + M^T$ is

$$\prod_{j=1}^{n} (m_{j,n-j+1}\lambda + m_{n-j+1,j}) \, .$$

Thus, $L_M(\lambda)$ is a singular pencil if and only if $m_{j,n-j+1} = 0 = m_{n-j+1,j}$ for some j. But the pencils $L_Z(\lambda)$ and $L_M(\lambda)$ are unitarily congruent, so they are either both singular or both regular with the same spectrum, and hence (2.4) follows.

(2) The reciprocal ordering of the list $(\lambda_1, \ldots, \lambda_n)$ follows immediately from (2.5). An induction on n shows that an arbitrary reciprocal ordering $(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_n)$ of the spectrum of L_Z can be realized by some anti-triangular form for Z. When $n = 1$ there is nothing to show. For $n \geq 2$ let u be a normalized Z-isotropic eigenvector of L_Z associated with $\widetilde{\lambda}_1$. The existence of such a u follows from Lemma 2.2(a) for any $\widetilde{\lambda}_1 \neq -1$, and for $\widetilde{\lambda}_1 = -1$ by Lemma 2.2(b), since reciprocal ordering implies that $\widetilde{\lambda}_n = -1$, and hence the multiplicity of $\widetilde{\lambda}_1 = -1$ is at least two. The procedure in Case 2 of the proof of Theorem 2.3 can now be applied with this eigenvector u, reducing the problem to a regular $(n-2)\times(n-2)$ pencil $\lambda\widetilde{Z}_{22} + \widetilde{Z}_{22}^T$ with reciprocally ordered spectrum $(\widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_{n-1})$. ∎

In this section, we have shown the existence of anti-triangular forms for arbitrary complex matrices and complex T-palindromic pencils under unitary T-congruence. The remaining sections are devoted to the numerical computation of these anti-triangular forms. For simplicity we restrict attention to the generic case of matrices Z such that the pencil $L_Z(\lambda) = \lambda Z + Z^T$ is regular.

3. Structure-preserving deflation methods. A first idea for a simple method to compute the anti-triangular form of a matrix Z comes directly from the constructive proof of Theorem 2.3. Suppose we have already computed the eigenvalues of the palindromic pencil $\lambda Z + Z^T$. Then we can proceed by computing one eigenvector at a time, and inductively reduce Z to anti-triangular form. We call this the *inductive reduction method* and summarize the procedure in the following algorithm.

ALGORITHM 3.1 (Inductive reduction method). *Given a matrix $Z \in \mathbb{C}^{n\times n}$ with $n \geq 2$ such that the pencil $L_Z(\lambda) = \lambda Z + Z^T$ is regular, and a reciprocally ordered list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the eigenvalues of $L_Z(\lambda)$, this algorithm computes a unitary matrix $U \in \mathbb{C}^{n\times n}$ and an anti-triangular matrix $M \in \mathbb{C}^{n\times n}$ such that $M = U^T ZU$ and $\lambda_j = -\frac{m_{n-j+1,j}}{m_{j,n-j+1}}$.*

- Let $M_1 := Z_1 := Z$.
- For $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ do:
  (1) With $n_k :=$ size of current subproblem $= n - 2k + 2$, compute a Z-isotropic eigenvector $v_k \in \mathbb{C}^{n_k}$ of the $n_k \times n_k$ pencil $\lambda Z_k + Z_k^T$ with eigenvalue λ_k. (Here we include eigenvectors associated with 0 or ∞.) Then $v_k^T Z_k v_k = 0$, so v_k and $w_k := \overline{Z_k v_k}$ are orthogonal. (If $Z_k v_k = 0$, i.e. if $\lambda_k = \infty$, use $w_k := \overline{Z_k^T v_k}$ instead.)
  (2) Set $q_1^{(k)} := \frac{v_k}{\|v_k\|}$ and $q_{n_k}^{(k)} := \frac{w_k}{\|w_k\|}$, and compute vectors $q_2^{(k)}, \ldots, q_{n_k-1}^{(k)}$ in $\mathbb{C}^{n_k}$ such that
  $$Q_k := [q_1^{(k)}, q_2^{(k)}, \ldots, q_{n_k-1}^{(k)}, q_{n_k}^{(k)}] \in \mathbb{C}^{n_k\times n_k}$$
  is unitary.
  (3) With $U_k := \operatorname{diag}(I_{k-1}, Q_k, I_{k-1})$, we now have
  $$M_{k+1} \;:=\; U_k^T M_k U_k \;=\; \begin{array}{c c} \begin{array}{ccccc} \scriptstyle k-1 & \scriptstyle 1 & \scriptstyle n_{k+1} & \scriptstyle 1 & \scriptstyle k-1 \end{array} & \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & Z_{k+1} & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{array}\right] & \begin{array}{c} \scriptstyle k-1 \\ \scriptstyle 1 \\ \scriptstyle n_{k+1} \\ \scriptstyle 1 \\ \scriptstyle k-1 \end{array} \end{array} .$$
- Then with $U := U_1 U_2 \cdots U_{\lfloor \frac{n}{2} \rfloor}$ we have $M := M_{\lfloor \frac{n}{2} \rfloor+1} = U^T ZU$.

Note that the final middle block $Z_{\lfloor \frac{n}{2} \rfloor+1}$ is 1×1 if n is odd; if n is even it is void.

Unfortunately, Algorithm 3.1 is only of theoretical value; it requires prior knowledge of all the eigenvalues of L_Z and has complexity at least $\mathcal{O}(n^4)$, because the cost of the computation of one eigenvector of a pencil is already of complexity $\mathcal{O}(n^3)$. Fortunately, though, the underlying idea of Algorithm 3.1 can be further developed to obtain an efficient numerical algorithm, by using deflating subspaces in place of eigenvectors.

Let us begin by seeing how deflating subspaces of the pencil L_Z fit naturally into the anti-triangular story for a matrix Z. Suppose $U \in \mathbb{C}^{n\times n}$ is a unitary matrix such that $U^T ZU$ is in *block-anti-triangular* form

$$U^T ZU \;=\; \begin{array}{c c} \begin{array}{ccc} \scriptstyle m & \scriptstyle n-2m & \scriptstyle m \end{array} & \\ \left[\begin{array}{ccc} 0 & 0 & Y^T \\ 0 & \widetilde{Z} & * \\ X & * & * \end{array}\right] & \begin{array}{c} \scriptstyle m \\ \scriptstyle n-2m \\ \scriptstyle m \end{array} \end{array} , \tag{3.1}$$

where $X, Y \in \mathbb{C}^{m\times m}$ with $m \leq n/2$. Let $E_m := [\, e_1 \; e_2 \; \ldots \; e_m \,] \in \mathbb{C}^{n\times m}$ denote the first m columns of I_n, and $\widetilde{E}_m := [\, e_{n-m+1} \; \ldots \; e_{n-1} \; e_n \,] \in \mathbb{C}^{n\times m}$ the last m columns of I_n. With U partitioned as $U = [\, W \;\; * \;\; V \,]$ where $W, V \in \mathbb{C}^{n\times m}$ we have $W = UE_m$ and $V = U\widetilde{E}_m$. Then from (3.1) we obtain

$$\widetilde{E}_m X \;=\; \begin{bmatrix} 0 \\ 0 \\ X \end{bmatrix} \;=\; (U^T ZU)E_m \;=\; U^T ZW$$

and

$$\widetilde{E}_m Y \;=\; \begin{bmatrix} 0 \\ 0 \\ Y \end{bmatrix} \;=\; (U^T Z^T U)E_m \;=\; U^T Z^T W\,.$$

Combining these yields $U^T(\lambda Z + Z^T)W = \widetilde{E}_m(\lambda X + Y)$, and hence

$$(\lambda Z + Z^T)W \;=\; (\overline{U}\widetilde{E}_m)(\lambda X + Y) \;=\; \overline{V}(\lambda X + Y)\,.$$

Thus the columns of W form an orthonormal basis for an m-dimensional deflating subspace of the pencil $L_Z(\lambda)$. From the $(1,1)$-block of (3.1) we see that

$$0 \;=\; (U^T ZU)_{1,1} \;=\; E_m^T U^T ZUE_m \;=\; W^T ZW\,,$$

so this deflating subspace must also necessarily be Z-isotropic.

Conversely, suppose we start out with a $W \in \mathbb{C}^{n\times m}$ whose columns form an orthonormal basis for a Z-isotropic deflating subspace for L_Z. Then from this W it is always possible to construct a unitary U such that $U^T ZU$ is in block-anti-triangular form, as follows. That the columns of W span an m-dimensional deflating subspace for L_Z means (by definition) that there exists $V \in \mathbb{C}^{n\times m}$ with rank m, and $X, Y \in \mathbb{C}^{m\times m}$ such that

$$(\lambda Z + Z^T)W \;=\; V(\lambda X + Y)\,. \tag{3.2}$$

Here we can assume without loss of generality that the columns of V are also orthonormal. (Using a QR decomposition $V = QR$, replace V by Q and $\lambda X + Y$ with $\lambda(RX) + (RY)$ in (3.2).) From the equality of rank W and rank V in (3.2) we see that $\lambda X + Y$ is nonsingular whenever $\lambda Z + Z^T$ is; hence $\lambda Z + Z^T$ being regular implies that $\lambda X + Y$ must also be regular. Since W is Z-isotropic, $0 = W^T ZW = W^T Z^T W$, so

$$W^T V(\lambda X + Y) \;=\; W^T(\lambda Z + Z^T)W \;=\; 0\,,$$

and the regularity of $\lambda X + Y$ now implies that $W^T V = 0$. Thus the columns of $\overline{V}$ are orthogonal to the columns of W, and we can extend W and $\overline{V}$ to a unitary matrix. Setting $U = \begin{bmatrix} W & \widetilde{U} & \overline{V} \end{bmatrix}$, where $\widetilde{U}$ is chosen in any way so that U is unitary, we obtain block-anti-triangular form

$$U^T ZU = \begin{bmatrix} W^T ZW & W^T Z\widetilde{U} & W^T Z\overline{V} \\ \widetilde{U}^T ZW & \widetilde{U}^T Z\widetilde{U} & \widetilde{U}^T Z\overline{V} \\ V^* ZW & V^* Z\widetilde{U} & V^* Z\overline{V} \end{bmatrix} = \begin{bmatrix} 0 & 0 & Y^T \\ 0 & \widetilde{Z} & \widetilde{U}^T Z\overline{V} \\ X & V^* Z\widetilde{U} & V^* Z\overline{V} \end{bmatrix}, \tag{3.3}$$

since from (3.2) we have $V^* ZW = V^* VX = X$, $W^T Z\overline{V} = Y^T V^T \overline{V} = Y^T$, $\widetilde{U}^T ZW = \widetilde{U}^T VX = 0$ and $W^T Z\widetilde{U} = Y^T V^T \widetilde{U} = 0$.

The spectrum of the pencil $L_Z(\lambda) = \lambda Z + Z^T$ can now be expressed as the union of the spectra of three subpencils of $\lambda(U^T Z U) + (U^T Z^T U)$:

$$\sigma\big(L_Z\big) \;=\; \sigma\big(\lambda X + Y\big) \,\cup\, \sigma\big(\lambda \widetilde{Z} + \widetilde{Z}^T\big) \,\cup\, \sigma\big(\lambda Y^T + X^T\big)\,.$$

Observe that the eigenvalues of $\lambda Y^T + X^T$ are just the reciprocals of the eigenvalues of $\lambda X + Y$. Thus once the spectrum of $\lambda X + Y$ is known, all that remains is to compute the eigenvalues of the *structured* subpencil $\lambda \widetilde{Z} + \widetilde{Z}^T$.

In order to effectively apply this idea, though, we need to be able to easily recognize when a deflating subspace is Z-isotropic. The next result establishes a simple sufficient condition for this property.

Definition 3.2 (Reciprocal-free sets). *A subset $\Lambda \subset \mathbb{C} \cup \{\infty\}$ is said to be* reciprocal-free *if $\mu \in \Lambda$ implies that $1/\mu \notin \Lambda$. (Our convention that 0 and ∞ are reciprocals of each other is in effect here.)*

Theorem 3.3 (Z-isotropic deflating subspaces). *Suppose $L_Z(\lambda) = \lambda Z + Z^T$ is a regular pencil, and the columns of $W \in \mathbb{C}^{n\times m}$ span an m-dimensional deflating subspace associated with the spectrum $\Lambda \subset \mathbb{C} \cup \{\infty\}$. If Λ is reciprocal-free, then W is Z-isotropic.*

Proof. By hypothesis there exists $V \in \mathbb{C}^{n\times m}$ of rank m, and $X, Y \in \mathbb{C}^{m\times m}$ such that

$$(\lambda Z + Z^T)W \;=\; V(\lambda X + Y) \tag{3.4}$$

with $\Lambda = \sigma\big(\lambda X + Y\big)$. Premultiplying (3.4) by W^T, we have $W^T Z W = W^T V X$. Thus it suffices to show that $W^T V = 0$ in order to conclude that W is Z-isotropic. We do this by constructing a Stein equation $ASB = S$ having $S = W^T V$ as a solution, and then proving that this equation can have only the trivial solution $S = 0$.

From (3.4) we immediately read off $ZW = VX$ and $Z^T W = VY$, and then $W^T Z = Y^T V^T$ by taking transpose. Thus

$$W^T V X \;=\; W^T Z W \;=\; Y^T V^T W \tag{3.5}$$

$$\text{and} \quad X^T V^T W \;=\; W^T V Y\,, \tag{3.6}$$

(3.6) coming from (3.5) by taking transpose. Because Λ is reciprocal-free, we know at least one of X and Y must be invertible, since the pencil $\lambda X + Y$ cannot have both 0 and ∞ as eigenvalues. Without loss of generality assume that X is invertible. Then

$$W^T V \;\overset{(3.5)}{=\!=}\; Y^T (V^T W) X^{-1} \;\overset{(3.6)}{=\!=}\; Y^T X^{-T} (W^T V) Y X^{-1}$$

shows that $S = W^T V$ is a solution of the Stein equation $ASB = S$, where

$$A \;:=\; -Y^T X^{-T} \quad \text{and} \quad B \;:=\; -Y X^{-1}\,.$$

It is well known that non-trivial solutions of $ASB = S$ exist only if some eigenvalue of A is the reciprocal of an eigenvalue of B [10, p.100, Thm 5.2.3]. Since the pencils $\lambda I - A$ and $\lambda I - B$ are equivalent to the pencils $\lambda X^T + Y^T$ and $\lambda X + Y$, respectively, we see that A and B have the *same* spectrum $\Lambda = \sigma(\lambda X + Y)$. Λ being reciprocal-free thus guarantees that $ASB = S$ has only the trivial solution $S = 0$, as desired. ☐

Suppose a block-anti-triangular form as in (3.3) has been obtained in which the blocks X and Y^T are themselves anti-triangular. Such a form will be referred to as a *partial anti-triangular form* for Z, since all that remains to achieve a "full" anti-triangular form is to solve the smaller subproblem for the middle block $\widetilde{Z}$. We formulate the discussion of this section as an algorithm for the reduction of a matrix Z to partial anti-triangular form. This algorithm may also be viewed as a structured deflation method for the T-palindromic eigenvalue problem $L_Z(\lambda) = \lambda Z + Z^T$, since it reduces this problem to a smaller structured pencil $L_{\widetilde{Z}}(\lambda) = \lambda\widetilde{Z} + \widetilde{Z}^T$. Note that F_m will be used here and in section 4.2 to denote the $m \times m$ "reverse identity", or "flip" matrix

$$F_m = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

ALGORITHM 3.4 (Structured deflation method). *Given $Z \in \mathbb{C}^{n\times n}$ such that $L_Z(\lambda)$ is regular and $m \le \frac{n}{2}$, the algorithm computes a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $M = U^T Z U$ is in partial anti-triangular form* (3.3).

(1) Compute unitary matrices $\widehat{V} = [V_1, V_2]$ and $\widehat{W} = [W_1, W_2]$ with $V_1, W_1 \in \mathbb{C}^{n\times m}$ and $V_2, W_2 \in \mathbb{C}^{n\times(n-m)}$ such that

$$\widehat{V}^*(\lambda Z + Z^T)\widehat{W} = \lambda \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix},$$

where $X_{11}, Y_{11} \in \mathbb{C}^{m\times m}$ are *upper* triangular and the eigenvalues are ordered in such a way that $\sigma(\lambda X_{11} + Y_{11})$ is reciprocal-free. (One way to achieve this is to apply the QZ algorithm with reordering of eigenvalues.) Then $(\lambda Z + Z^T)W_1 = V_1(\lambda X_{11} + Y_{11})$, so the columns of W_1 span a Z-isotropic deflating subspace for $L_Z(\lambda)$.

(2) Compute an isometric matrix $\widetilde{U} \in \mathbb{C}^{n\times(n-2m)}$ (i.e. $\widetilde{U}$ has orthonormal columns), such that the columns of $\widetilde{U}$ are orthogonal to the columns of W_1 and $\overline{V}_1$, and set $U = \begin{bmatrix} W_1 & \widetilde{U} & \overline{V}_1 F_m \end{bmatrix}$. Then U is unitary and $M := U^T Z U$ is in partial anti-triangular form:

$$M = \begin{bmatrix} 0 & 0 & Y_{11}^T F_m \\ 0 & \widetilde{U}^T Z \widetilde{U} & \widetilde{U}^T Z \overline{V}_1 F_m \\ F_m X_{11} & F_m V_1^* Z \widetilde{U} & F_m V_1^* Z \overline{V}_1 F_m \end{bmatrix}. \tag{3.7}$$

It is possible to use Algorithm 3.4 to reduce matrices all the way to anti-triangular form, but one needs a systematic procedure for identifying a reciprocal-free set of $m = \lfloor \frac{n}{2} \rfloor$ eigenvalues of $L_Z(\lambda)$. One way to do this is to preselect a reciprocal-free subset $\Delta \subset \mathbb{C}\cup\{\infty\}$ such that $\Delta\cup\Delta^{-1}$ covers all (or at least almost all) of $\mathbb{C}\cup\{\infty\}$, and then identify all the eigenvalues of L_Z that lie in Δ. For many applications, a natural choice for Δ is the set of all points outside the unit circle, i.e. $\Delta = \Lambda_1 := \{\lambda : |\lambda| > 1\}$.

There are several difficulties with this idea. The first stems from the fact that neither 1 nor -1 can ever be an element of a reciprocal-free set. Thus if L_Z has eigenvalues ± 1 with total multiplicity greater than one, then a reciprocal-free set of $m = \lfloor \frac{n}{2} \rfloor$ eigenvalues for L_Z won't even exist.

A second problem arises from the numerical difficulty of deciding, when Δ and Δ^{-1} have a common boundary, whether eigenvalues near this common boundary lie in Δ or in Δ^{-1}. For example, suppose L_Z has no eigenvalues on the unit circle and we take $\Delta = \Lambda_1$. If L_Z has eigenvalues *near* the unit circle, then the eigenvalues computed by the QZ-algorithm may not necessarily divide neatly into $m = \lfloor \frac{n}{2} \rfloor$ eigenvalues in

Δ and m eigenvalues in Δ^{-1}. Eigenvalues near ± 1 are especially problematic, since they tend to be more ill-conditioned than other eigenvalues of L_Z.

One way to address this issue is to set up a kind of "buffer zone" between Δ and Δ^{-1} that contains their common boundary. Then for any computed eigenvalue in this (small) buffer zone we simply avoid deciding whether this eigenvalue is in Δ or Δ^{-1}. Computed eigenvalues outside the buffer zone are deemed to be safely in Δ or Δ^{-1}, and with these "safe" eigenvalues we compute just a *partial* anti-triangular form using the non-structure-preserving QZ-algorithm together with the structured deflation method of Algorithm 3.4. The remaining middle block $\widetilde{Z}$ is then associated with the eigenvalues in the buffer zone. For this "bad part" we use a (possibly expensive) structure-preserving method to determine its anti-triangular form. In general, we can expect the size of the block $\widetilde{Z}$ to be small if the buffer zone is not too large, and so we are able to afford more expensive methods to compute its anti-triangular form. For the important case of $\Delta = \Lambda_1$, we use an annulus containing the unit circle in its interior as a buffer zone between Λ_1 and Λ_1^{-1}. In particular, for some choice of $\alpha > 1$ we take the annulus $\{\lambda \in \mathbb{C} : 1/\alpha \le |\lambda| \le \alpha\}$ as buffer zone, so that $\Lambda_\alpha := \{\lambda : |\lambda| > \alpha\}$ is the "safe part" of $\Delta = \Lambda_1$. The ideas discussed here are summarized in the following hybrid procedure for reducing a matrix to anti-triangular form.

ALGORITHM 3.5. *Given $Z \in \mathbb{C}^{n\times n}$ such that $L_Z(\lambda) = \lambda Z + Z^T$ is regular, the algorithm computes a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $M = U^T ZU$ is in anti-triangular form.*

(1) Select a value $\alpha > 1$, and let m denote the number of eigenvalues of $\lambda Z + Z^T$ in Λ_α.

(2) Use Algorithm 3.4 to compute a unitary matrix $\widehat{U} = \begin{bmatrix} W_1 & \widetilde{U} & V_1 \end{bmatrix}$ with $W_1, V_1 \in \mathbb{C}^{n\times m}$ such that the columns of W_1 span a deflating subspace associated with the spectrum of L_Z contained in Λ_α, and such that $\widehat{U}^T Z\widehat{U}$ is in partial anti-triangular form.

(3) Compute $\widetilde{Z} := \widetilde{U}^T Z\widetilde{U}$, then use a specific structure-preserving method to compute a unitary matrix $Q \in \mathbb{C}^{(n-2m)\times(n-2m)}$ such that $Q^T\widetilde{Z}Q$ is in anti-triangular form. Set $U = \widehat{U}\,\mathrm{diag}(I_m, Q, I_m)$. Then $U^T ZU$ is in anti-triangular form.

There are several possibilities for the structure-preserving method to be used in part (3) of Algorithm 3.5. We discuss two such methods in the next section.

4. Structure-preserving methods for small dense palindromic eigenvalue problems. In this section we describe two structure-preserving methods for computing the anti-triangular form of a small dense matrix. These are a palindromic version of the Jacobi method and a palindromic QR-algorithm.

4.1. A palindromic Jacobi method. The nonsymmetric Jacobi method for the computation of the Schur form of a complex matrix [2, 4, 5] was generalized in [15] for the computation of the anti-triangular form for Hermitian pencils. We now show how the algorithm in [15] can be readily adapted to the task of computing the anti-triangular form of any matrix $Z \in \mathbb{C}^{n\times n}$ for which the pencil $\lambda Z + Z^T$ is regular. We do not expect the method to be competitive for large dense matrices that are far from anti-triangular. But we will show later in section 5 how the method can be combined to advantage with a faster method to improve the accuracy of computed solutions.

As in all Jacobi algorithms, the basic idea is the repeated annihilation of suitably chosen entries in the matrix, usually referred to as "pivots". Our strategy here is to annihilate either *one diagonal* or *two symmetrically positioned off-diagonal elements* in the strict upper anti-triangular part of Z, in each iteration. These are depicted as bullets $\bullet$ in the sketch below.

diagonal pivot element

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

two off-diagonal pivot elements

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & * & * \\ \cdot & \bullet & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Let us first consider the task of annihilating a diagonal element z_{kk} with $k \leq \frac{n}{2}$. Our goal is to determine a unitary matrix $Q \in \mathbb{C}^{2\times 2}$ such that the target 2×2 subproblem

$$Z_{kk} = \begin{bmatrix} z_{k,k} & z_{k,n+1-k} \\ z_{n+1-k,k} & z_{n+1-k,n+1-k} \end{bmatrix}$$

is reduced to anti-triangular form by the T-congruence $Q^T Z_{kk} Q$. Let $\eta \in \mathbb{C}$ be either of the two solutions of

$$0 = \begin{bmatrix} 1 & \eta \end{bmatrix} Z_{kk} \begin{bmatrix} 1 \\ \eta \end{bmatrix} = z_{k,k} + \eta(z_{n+1-k,k} + z_{k,n+1-k}) + \eta^2 z_{n+1-k,n+1-k}. \tag{4.1}$$

Then the unitary matrix

$$Q = [q_{ij}] = \frac{1}{\sqrt{1+|\eta|^2}} \begin{bmatrix} 1 & -\overline{\eta} \\ \eta & 1 \end{bmatrix} \tag{4.2}$$

makes $Q^T Z_{kk} Q$ anti-triangular. Letting $U = [u_{ij}]$ be the $n \times n$ identity matrix except for the elements $u_{k,k} = q_{11}$, $u_{k,n+1-k} = q_{12}$, $u_{n+1-k,k} = q_{21}$, and $u_{n+1-k,n+1-k} = q_{22}$, we see that the (k,k)-element of $U^T Z U$ is 0. This procedure is depicted in the following sketch, where $\circ$ and $\bullet$ denote the elements of the 2×2 target subproblem Z_{kk}, with $\circ$ distinguishing the pivot element.

$$\begin{bmatrix} 1 &&&&&&& \\ & q_{11} &&&&& q_{21} & \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ & q_{12} &&&&& q_{22} & \\ &&&&&&& 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \circ & \cdot & \cdot & \cdot & \cdot & \bullet & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & \bullet & * & * & * & * & \bullet & * \\ * & * & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 &&&&&&& \\ & q_{11} &&&&& q_{12} & \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ & q_{21} &&&&& q_{22} & \\ &&&&&&& 1 \end{bmatrix}$$

The choice of η significantly influences the convergence behavior of this unsymmetric Jacobi algorithm; choosing the value that is smaller in magnitude produces the best results [6]. Thus among the two possible complex rotations in (4.2) that eliminate z_{kk}, we choose the one that is closer to the identity matrix.

Next, we show how to simultaneously eliminate two off-diagonal elements z_{kl} and z_{lk}, where $k < l$ and $k + l \leq n$. Focusing first on z_{kl}, consider the 2×2-submatrix

$$Z_{kl} = \begin{bmatrix} z_{k,l} & z_{k,n+1-k} \\ z_{n+1-l,l} & z_{n+1-l,n+1-k} \end{bmatrix},$$

and compute unitary matrices $V = [v_{ij}]$ and $W = [w_{ij}]$ so that z_{kl} is annihilated by the transformation $V^T Z_{kl} W$. Thus

$$V^T Z_{kl} W = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} z_{k,l} & z_{k,n+1-k} \\ z_{n+1-l,l} & z_{n+1-l,n+1-k} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}$$

is in anti-triangular form. Now obtain a unitary matrix $U = [u_{ij}]$ by embedding V and W into I_n as principal submatrices in the manner depicted in the following sketch. So we have $u_{kk} = v_{11}$, $u_{k,n+1-l} = v_{12}$, $u_{n+1-l,k} = v_{21}$, $u_{n+1-l,n+1-l} = v_{22}$; and $u_{ll} = w_{11}$, $u_{l,n+1-k} = w_{12}$, $u_{n+1-k,l} = w_{21}$, $u_{n+1-k,n+1-k} = w_{22}$. The symbols $\circ$ and $\bullet$ denote the submatrix Z_{kl}, while $\circ$ identifies the pivot element z_{kl}. Thus the (k,l)-element of $U^T Z U$ is made zero.

$$\begin{bmatrix} 1 &&&&&&& \\ & v_{11} &&&& v_{21} && \\ && w_{11} &&&& w_{21} & \\ &&& 1 &&&& \\ &&&& 1 &&& \\ & v_{12} &&&& v_{22} && \\ && w_{12} &&&& w_{22} & \\ &&&&&&& 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \circ & \cdot & \cdot & \cdot & \bullet & * \\ \cdot & + & \cdot & \cdot & \cdot & + & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & \bullet & * & * & * & \bullet & * \\ \cdot & + & * & * & * & + & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 &&&&&&& \\ & v_{11} &&&& v_{12} && \\ && w_{11} &&&& w_{12} & \\ &&& 1 &&&& \\ &&&& 1 &&& \\ & v_{21} &&&& v_{22} && \\ && w_{21} &&&& w_{22} & \\ &&&&&&& 1 \end{bmatrix}$$

It remains to construct matrices V and W that anti-triangularize Z_{kl}. Since we allow two different unitary matrices to act on Z_{kl}, there is a continuum of choices for V, W that annihilate z_{kl}. Now observe from the sketch that our transformation matrices work not just on the submatrix Z_{kl}, but also on the submatrix marked by the $+$ symbol. This submatrix, specified by

$$Z_{lk} = \begin{bmatrix} z_{lk} & z_{l,n+1-l} \\ z_{n+1-k,k} & z_{n+1-k,n+1-l} \end{bmatrix},$$

will be transformed into $W^T Z_{lk} V$. We can therefore exploit the freedom in V and W to anti-triangularize Z_{lk} as well. Indeed, if we choose V and W such that

$$V^T(\lambda Z_{kl} + Z_{lk}^T)W = \lambda \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} + \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}$$

is in anti-triangular form, then the two symmetrically positioned off-diagonal elements in the (k,l) and the (l,k) positions of Z will be annihilated in $U^T Z U$. The desired V and W can be found by computing the generalized Schur decomposition of $\lambda Z_{kl} + Z_{lk}^T$, and then premultiplying it by the 2×2 flip matrix F_2. Once again there are basically two choices for the matrices V and W, and we opt for the alternative that makes U closest to the identity.

Cyclic-by-row sweeps targeting elements in the strict upper anti-triangular part of Z were used in our numerical experiments. The number of iterations in a full sweep is $\approx n^2/4$, since two off-diagonal elements are annihilated by one iteration. In the following sequence of indices specifying a sweep, only index pairs (k, l) for which $k < l$ need to be listed, since z_{lk} is annihilated in the same iteration as z_{kl}. When n is even the sequence is specified by

$$(1, 1), (1, 2), \ldots, (1, n-1), (2, 2), \ldots, (2, n-2), \ldots, (\tfrac{n}{2}, \tfrac{n}{2}),$$

while for odd n, the sequence is

$$(1, 1), (1, 2), \ldots, (1, n-1), (2, 2), \ldots, (2, n-2), \ldots, (\tfrac{n-1}{2}, \tfrac{n-1}{2}), (\tfrac{n-1}{2}, \tfrac{n+1}{2}).$$

One cyclic-by-row sweep for the case $n = 6$ is displayed in the following sketch:

$$\begin{bmatrix} \circ & \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \cdot & * & * & * & * & * \\ \bullet & * & * & * & * & \bullet \end{bmatrix} \leadsto \begin{bmatrix} \cdot & \circ & \cdot & \cdot & \cdot & \bullet \\ \circ & \cdot & \cdot & \cdot & \bullet & * \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \cdot & \bullet & * & * & * & \bullet \\ \bullet & * & * & * & \bullet & * \end{bmatrix} \leadsto \begin{bmatrix} \cdot & \cdot & \circ & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \circ & \cdot & \cdot & \bullet & * & * \\ \cdot & \cdot & \bullet & * & * & \bullet \\ \cdot & * & * & * & * & * \\ \bullet & * & * & \bullet & * & * \end{bmatrix}$$
$$\leadsto \begin{bmatrix} \cdot & \cdot & \cdot & \circ & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \bullet & * & \bullet \\ \circ & \cdot & \bullet & * & * & * \\ \cdot & * & * & * & * & * \\ \bullet & * & \bullet & * & * & * \end{bmatrix} \leadsto \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \circ & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \circ & \bullet & * & * & * & * \\ \bullet & \bullet & * & * & * & * \end{bmatrix} \leadsto \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \circ & \cdot & \cdot & \bullet & * \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \cdot & \bullet & * & * & \bullet & * \\ * & * & * & * & * & * \end{bmatrix}$$
$$\leadsto \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \circ & \cdot & \bullet & * \\ \cdot & \circ & \cdot & \bullet & * & * \\ \cdot & \cdot & \bullet & * & \bullet & * \\ \cdot & \bullet & * & \bullet & * & * \\ * & * & * & * & * & * \end{bmatrix} \leadsto \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \circ & \bullet & * \\ \cdot & \cdot & \cdot & \bullet & \bullet & * \\ \cdot & \circ & \bullet & * & * & * \\ \cdot & \bullet & \bullet & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \leadsto \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \circ & \bullet & * & * \\ \cdot & \cdot & \bullet & \bullet & * & * \\ \cdot & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}.$$

A detailed investigation of the behavior of this method indicates global convergence to anti-triangular form at an asymptotically quadratic rate [6]. (See [14] for proofs of and comments on asymptotic convergence of nonsymmetric Jacobi algorithms.) The convergence is therefore quite fast for matrices that are already close to anti-triangular form, while for general matrices the algorithm is rather expensive — the cost of three sweeps is essentially equivalent to the cost of the QZ algorithm. In section 5 we show how the high accuracy with which this Jacobi method computes eigenvalues can be efficiently exploited.

4.2. The palindromic QR algorithm. In [18] Schröder proposed a QR-like algorithm for the computation of the anti-triangular form of a matrix $Z \in \mathbb{C}^{n \times n}$. This algorithm is called the *palindromic QR-algorithm*. The basic iteration (the so-called *palindromic QR-step*) for a matrix $Z_i \in \mathbb{C}^{n \times n}$ is given as follows:

1) compute a decomposition $Z_i = Q_i A_i$, where $Q_i \in \mathbb{C}^{n \times n}$ is unitary and $A_i \in \mathbb{C}^{n \times n}$ is anti-triangular; (this can be achieved by computing a QR decomposition $Z_i = QR$, and then choosing $Q_i = QF_n$ and $A_i = F_n R$);
2) compute $Z_{i+1} := A_i \overline{Q}_i$.

Starting with $Z_0 := Z$, this iteration produces a sequence of unitarily T-congruent matrices $(Z_i)_{i \in \mathbb{N}}$ (since $Q_i Z_{i+1} Q_i^T = Q_i A_i = Z_i$) that approach anti-triangular form. In particular, Schröder shows that two palindromic QR steps for Z are equivalent to one Francis QR step for the matrix $Z^{-T}Z$. Thus, the palindromic QR algorithm for Z shows similar convergence properties as the standard QR algorithm for $Z^{-T}Z$. Schröder also discusses the use of shifts in order to accelerate the speed of convergence, and introduces a Hessenberg-like form for which palindromic QR steps can be performed in $\mathcal{O}(n^2)$ floating point operations in order to improve the efficiency of the algorithm. However, unlike the Householder reduction to standard Hessenberg form used as a preliminary step of the Francis QR algorithm, a direct method for the computation of the Hessenberg-like form in [18] is only available in special situations. Therefore, a palindromic QR step requires $\mathcal{O}(n^3)$ floating point operations in general, resulting in a method whose overall complexity is $\mathcal{O}(n^4)$. Thus both the Jacobi-like method introduced in the previous section and the palindromic QR iteration are only appropriate for sufficiently small values of n.

5. Numerical experiments. Results of numerical experiments to test our algorithms for computing the anti-triangular form of complex matrices are now presented. As a measure for the performance of the algorithms, we compute the *distance from anti-triangularity*

$$\operatorname{dist}_{\triangle}(Z) := \sqrt{\sum_{i+j\leq n} |z_{ij}|^2}\,,$$

i.e., the Frobenius norm of the strict upper anti-triangular part of $Z \in \mathbb{C}^{n\times n}$, as well as the *distance from unitarity*

$$\operatorname{dist}_1(U) := \|U^*U - I_n\|_2$$

of the computed unitary transformations $U \in \mathbb{C}^{n\times n}$.

Two different types of random 100×100 complex matrices Z were used in our tests, corresponding to two different eigenvalue distributions of the corresponding palindromic pencil $L_Z(\lambda) := \lambda Z + Z^T$.

Type 1: Z is constructed so that $L_Z(\lambda)$ has at least 5 eigenvalues in an annulus in the complex plane with inner radius 1 and outer radius $\alpha := 1 + tol$. Since the eigenvalues are reciprocally paired, $L_Z(\lambda)$ has at least 10 eigenvalues close to the unit circle, with 5 of these lying outside, and 5 inside the unit circle. We generated these matrices in MATLAB by first selecting w_i, $i = 1, \ldots, 5$ of the form

```
(1+rand(1)*tol)*exp(i*2*pi*rand(1))
```

and the remaining w_i, $i = 6, \ldots, 50$ of the form

```
(1+abs(randn(1)))*exp(i*2*pi*rand(1)).
```

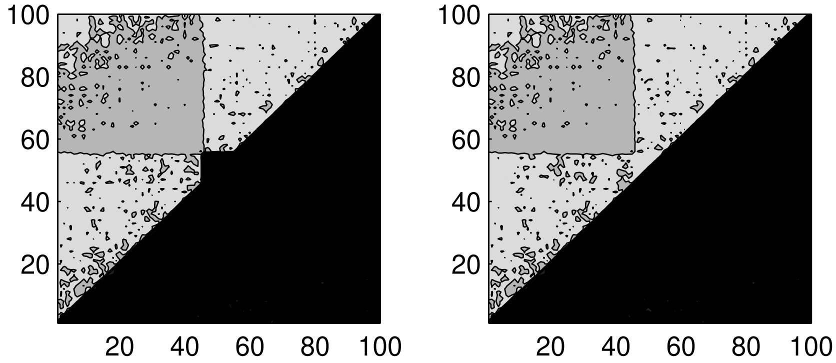
Setting $A = F_{100}\operatorname{diag}(w_1, \ldots, w_{50}, 1, \ldots, 1)$, we let $Z = P^TAP$, where the entries of P are normally distributed with mean zero and variance 1. Finally, Z was normalized so that $\|Z\|_2 = 1$. The palindromic pencil $L_Z(\lambda)$ now has the prescribed eigenvalues w_i, w_i^{-1}, $i = 1, \ldots, 50$.

Type 2: Z is constructed so that $L_Z(\lambda)$ has at least 10 random eigenvalues that are uniformly distributed in a disc around 1 with radius tol. We generated these matrices using the procedure for Type 1 matrices, except that w_i, $i = 1, \ldots, 5$ are determined by

```
1+sign(randn(1))*tol*rand(1)*exp(i*2*pi*rand(1)).
```

We first discuss a typical example of a Type 2 matrix with $tol = 10^{-10}$. After applying the structured deflation method (Algorithm 3.4) with $m = 45$, we obtain a matrix in partial anti-triangular form as depicted in Figure 5.1, on the left. (Here, a grey-scale is used to characterize the modulus of an element in the matrix. The lighter the color, the smaller the modulus of the corresponding element, ranging from moduli larger than one (black) to moduli smaller than the machine precision (white).) The partial anti-triangular form is clearly visible, with the small black triangle in the middle of the anti-diagonal depicting the 10×10 subproblem with eigenvalues close to the unit circle that remains to be solved. Applying the palindromic QR algorithm to this small subproblem yields the result depicted in Figure 5.1 on the right.

FIG. 5.1. *Structured deflation method for a* 100×100 *matrix with* 10 *eigenvalues close to* 1. *The matrix is shown before (left) and after (right) solving the remaining* 10×10 *subproblem.*



As Figure 5.1 suggests, one may not be satisfied with the results at this point: there seems to be a lot of "dirt" in the strict upper anti-triangular part and, indeed, the distance from anti-triangularity is about $4 \cdot 10^{-13}$ on average for matrices of this type.

To improve the result, one may consider applying one sweep of the palindromic Jacobi algorithm discussed in Section 4.1. However, if a Jacobi sweep is performed directly after the structured deflation method, i.e., before the remaining 10×10 subproblem is solved, then the corresponding unreduced block causes an increase in modulus of some of the elements in the strict upper part of the matrix in partial anti-triangular form, as shown on the left in Figure 5.2. This increase is not reduced when the remaining 10×10 subproblem is finally solved, as Figure 5.2 (right) reveals.

Therefore, it is advisable to first solve the remaining 10×10 subproblem and *then* apply a sweep of the Jacobi algorithm. Typically, this sweep will again blur the block corresponding to the eigenvalues close to 1, as seen in Figure 5.3 (left), because the problem is ill-conditioned so the palindromic Jacobi algorithm does not perform well in this case. This is remedied by solving the subproblem once again using the palindromic QR algorithm. The anti-triangular form emerges much better than before, as seen in Figure 5.3 (right). Indeed, after applying the algorithms in the prescribed sequence, the distance from anti-triangularity is about $3 \cdot 10^{-15}$ on average for matrices of this type.

With this typical performance of the structured deflation method in combination with different refinement methods in mind, we have performed several tests for matrices of Type 1 and Type 2 with different tolerances *tol*.

We have tested the structured deflation method (Algorithm 3.4) with $m = \frac{n}{2} =$

FIG. 5.2. *Structured deflation method for a* 100×100 *matrix with* 10 *eigenvalues close to* 1*, followed by one sweep of palindromic Jacobi. The matrix is shown before (left) and after (right) solving the remaining* 10×10 *subproblem.*

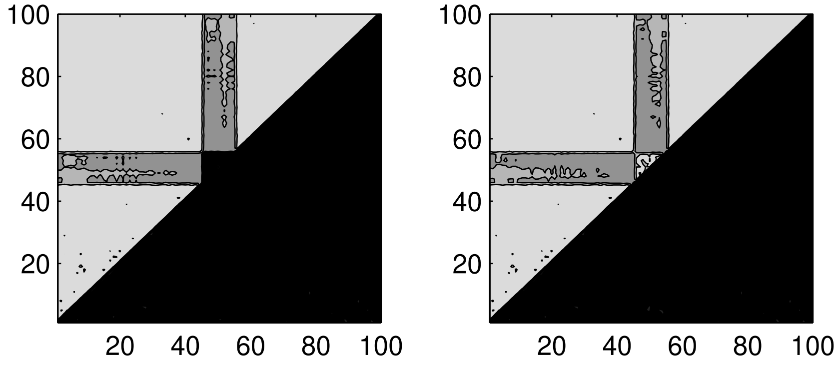
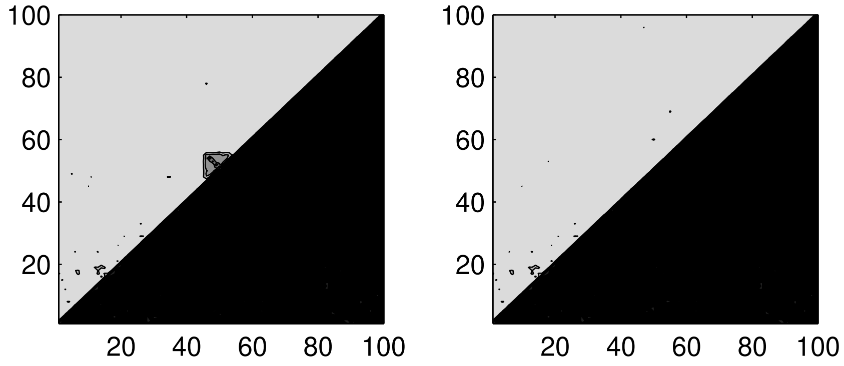


FIG. 5.3. *Structured deflation method for a* 100×100 *matrix with* 10 *eigenvalues close to* 1*, followed by one sweep of Jacobi performed* after *solving the remaining* 10×10 *subproblem. The matrix is shown before (left) and after (right) solving the remaining* 10×10 *problem once again.*



50 as well as Algorithm 3.5 with $\alpha = 1.01$ in combination with various algorithms for the solution of the remaining small palindromic subproblem associated with the eigenvalues close to the unit circle.

It should be noted that an increasing the outer radius of the annulus to, say, $\alpha = 1.5$ did not have a significant effect on the performance of the algorithms other than increasing the size of the remaining subproblem.

The following variations of the algorithms were tested:

(a) Algorithm 3.4 (the structured deflation method) with $m = \frac{n}{2} = 50$;

(b) Algorithm 3.5 with $\alpha = 1.01$ and the palindromic Jacobi algorithm (Section 4.1) for the solution the remaining subproblem;

(c) Algorithm 3.5 with $\alpha = 1.01$ and the inductive reduction method (Algorithm 3.1) for the solution the remaining subproblem;

(d) Algorithm 3.5 with $\alpha = 1.01$ and the palindromic QR algorithm (Section 4.2) for the solution the remaining subproblem;

(e) Algorithm 3.4 (the structured deflation method) with $m = \frac{n}{2} = 50$, followed by one sweep of the palindromic Jacobi method for the whole matrix;

(f) Algorithm 3.5 with $\alpha = 1.01$ and the palindromic Jacobi algorithm for the solution the remaining subproblem, followed by one sweep of palindromic Jacobi for the whole matrix;

(g) Algorithm 3.5 with $\alpha = 1.01$ and the inductive reduction method for the solution the remaining subproblem, followed by one sweep of the palindromic Jacobi algorithm for the whole matrix ;

(h) Algorithm 3.5 with $\alpha = 1.01$ and the palindromic QR algorithm for the solution the remaining subproblem, followed by one full sweep of the palindromic Jacobi algorithm for the whole matrix;

(i) Algorithm 3.5 with $\alpha = 1.01$ and the palindromic Jacobi algorithm for the solution the remaining subproblem, followed by one sweep of Jacobi for the whole matrix and one more application of the Jacobi algorithm to the small subproblem;

(j) Algorithm 3.5 with $\alpha = 1.01$ and the inductive reduction method for the solution the remaining subproblem, followed by one sweep of palindromic Jacobi for the whole matrix and one more application of the inductive reduction method to the small subproblem;

(k) Algorithm 3.5 with $\alpha = 1.01$ and the palindromic QR algorithm for the solution the remaining subproblem, followed by one sweep of the palindromic Jacobi method for the whole matrix and one more application of the palindromic QR algorithm to the small subproblem.

For the first series of tests, the algorithm variations (a) – (k) were tested on 100 random matrices of Type 1 with tolerances $tol = 10^{-5}$ and $tol = 10^{-12}$. The average distances from anti-triangularity $\mathrm{dist}_\triangle(\widetilde{Z})$ of the computed anti-triangular Schur forms $\widetilde{Z} = U^T Z U$ and the corresponding average distances from unitarity $\mathrm{dist}_1(U)$ are reported in Table 5.1.

TABLE 5.1

Average distance from anti-triangularity and distance from unitarity for computed anti-triangular Schur forms for matrices of Type 1 with different values for tol

$\mathrm{dist}_\triangle(\widetilde{Z})$

	$tol = 10^{-5}$	$tol = 10^{-12}$
(a)	1.39e-12	2.81e-13 (87%)
(b)	1.77e-13	1.71e-13
(c)	1.75e-13	1.73e-13
(d)	1.75e-13	1.73e-13
(e)	1.37e-15	1.37e-15 (87%)
(f)	1.37e-15	1.39e-15
(g)	1.38e-15	1.39e-15
(h)	1.37e-15	1.38e-15
(i)	2.73e-15	2.64e-15
(j)	2.74e-15	2.64e-15
(k)	2.72e-15	2.62e-15

$\mathrm{dist}_1(U)$

	$tol = 10^{-5}$	$tol = 10^{-15}$
(a)	1.01e-11	1.86e-12 (87%)
(b)	1.36e-12	1.32e-12
(c)	1.36e-12	1.32e-12
(d)	1.36e-12	1.32e-12

For $tol = 10^{-12}$, the structured deflation method used in (a) and (e) failed 13 times, when the spectrum computed by the QZ algorithm failed to contain 50 eigenvalues with modulus larger than 1. The averages were then taken over the remaining 87 test problems (indicated with the marker (87%)). As one can see from Table 5.1, all variations (a) – (k) yield satisfactory results. When the QZ algorithm is able to separate the eigenvalues inside from those outside the unit circle, the structured deflation method with $m = \frac{n}{2}$ works well. This can be explained by the fact that for eigenvalues close to the unit circle, the reciprocal-free condition in Algorithm 3.4 is always satisfied numerically, unless there are eigenvalues close to ± 1, because, in general, μ and $\frac{1}{\mu}$ are well separated. Therefore, we can only detect a slight improvement when passing from the structured deflation method to Algorithm 3.5 with $\alpha = 1.01$, regardless which algorithm is used for the solution of the remaining small subproblem (variations (b) – (d)). The distances from anti-triangularity decrease by a factor 100–1000 when a sweep of the Jacobi algorithm is applied in order to improve the results (variations (e) – (h)). Since small subproblems with eigenvalues close to the unit circle are generically well conditioned, the Jacobi algorithm performs well and does not blur the part of the anti-triangular form corresponding to the small subproblems. Therefore, an attempt at a subsequent refinement of the solution of the small subproblems yields no improvement in the distances from anti-triangularity. (In fact, a slight increase of $\mathrm{dist}_\triangle(\widetilde{Z})$ has been observed.) Concerning the distance from unitarity of the transformation matrices, we find that Algorithm 3.5 with $\alpha = 1.01$ produced slightly better results than the structured deflation method with $m = \frac{n}{2}$. Applying a sweep of the Jacobi algorithm and eventually solving the small subproblem once more had no significant impact on the distance from unitarity. Therefore, only the values for the variations (a) – (d) are reported in Table 5.1.

For the second series of tests, the variations (a) – (k) were tested on 100 random matrices of Type 2 with tolerances $tol = 10^{-5}, 10^{-8}, 10^{-10}, 10^{-12}$. The average distances from anti-triangularity $\mathrm{dist}_\triangle(\widetilde{Z})$ of the computed anti-triangular Schur forms $\widetilde{Z} = U^T Z U$ and the corresponding average distances from unitarity $\mathrm{dist}_1(U)$ are compiled in Table 5.2.

Table 5.2

Average distance from anti-triangularity and distance from unitarity for computed anti-triangular Schur forms for matrices of type 2 with different values for tol

$\mathrm{dist}_\triangle(\widetilde{Z})$

	$tol = 10^{-5}$	$tol = 10^{-8}$	$tol = 10^{-10}$	$tol = 10^{-12}$
(a)	1.77e-08	2.00e-05	1.45e-03	3.50e-02 (83%)
(b)	9.03e-11	1.63e-07	4.37e-06	4.77e-04
(c)	8.13e-11	5.34e-08	9.38e-06	6.04e-04
(d)	4.38e-13	4.32e-13	3.69e-13	2.74e-13
(e)	1.37e-10	1.41e-07	3.95e-05	5.07e-03 (83%)
(f)	1.29e-10	1.02e-07	2.77e-05	5.69e-03
(g)	1.01e-10	6.84e-08	2.53e-05	4.67e-03
(h)	3.53e-11	4.73e-08	1.83e-05	4.77e-03
(i)	7.57e-11	1.36e-07	5.34e-06	7.93e-04
(j)	1.04e-10	1.23e-07	1.04e-05	4.07e-04
(k)	2.83e-15	2.68e-15	2.65e-15	2.64e-15

$\mathrm{dist}_1(U)$

	$tol = 10^{-5}$	$tol = 10^{-8}$	$tol = 10^{-10}$	$tol = 10^{-12}$
(a)	1.53e-07	1.76e-04	1.25e-02	2.97e-01 (83%)
(b)	3.58e-12	3.29e-12	2.80e-12	2.02e-12
(c)	3.49e-12	3.24e-12	2.80e-12	2.02e-12
(d)	3.49e-12	3.24e-12	2.80e-12	2.02e-12

In contrast to the case of matrices of Type 1, here we observe a drastic increase in the distance from anti-triangularity when tol decreases. For $tol = 10^{-12}$, the structured deflation method for $m = \frac{n}{2}$ failed in 17 cases, because the QZ algorithm was not able to detect 50 eigenvalues outside the unit circle. The averages have then been taken over the remaining 83 cases (indicated by the marker (83%)). When passing from the structured deflation method to Algorithm 3.5 with $\alpha = 1.01$, a significant improvement can be observed (variations (b) – (d)). However, for matrices of Type 2 the choice of the algorithm used for solving the remaining subproblem with eigenvalues close to 1 is crucial. Both the Jacobi algorithm as well as the inductive reduction method had convergence difficulties due to the fact that the small subproblem is now very ill-conditioned. Only the palindromic QR algorithm was able to produce satisfactory results here. While the other methods showed worse performance as tol decreases, the distance from anti-triangularity remained approximately constant in variation (d). After applying a sweep of Jacobi (variations (e) – (h)), a slight increase in the distance from anti-triangularity could be observed. This is due to the fact that the Jacobi algorithm now blurs the part in the anti-triangular form that corresponds to the subproblem with eigenvalues close to 1 as it has been depicted in Figure 5.2. Solving the subproblem then once more (variations (i) – (k)) only has a significant effect on the distance from anti-triangularity when the palindromic QR algorithm is used. Concerning the distance from unitarity of the corresponding transformation matrix while tol decreases, one can observe a dramatic increase in the case the structured deflation method is used with $m = \frac{n}{2}$. This is due to the fact that the eigenvalues close to 1 are not well separated from their reciprocals. On the other hand, Algorithm 3.5 with $\alpha = 1.01$ produced results comparable to the case of matrices of Type 1, irrespective of the algorithm used for the solution of the small subproblems.

A final test was performed in connection with the T-palindromic eigenvalue problem arising in the vibration analysis of rail tracks [7]. This eigenvalue problem has the form

$$P(\lambda) = (\lambda^2 A + \lambda B + A^T)x = 0, \tag{5.1}$$

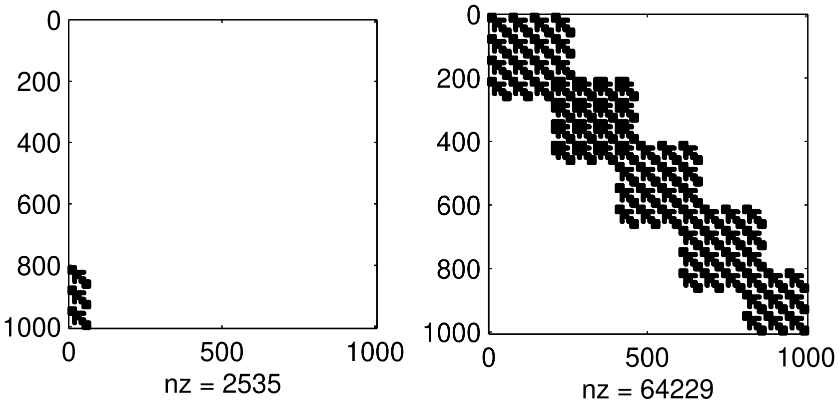
where $A, B \in \mathbb{C}^{1005\times 1005}$. Here A is highly singular with rank 67, and B is complex symmetric. The sparsity pattern of A and B is depicted in Figure 5.4.

Using the linearization theory from [13], we see that the 2010×2010 T-palindromic pencil

$$L_Z(\lambda) = \lambda \begin{bmatrix} A & B - A^T \\ A & A \end{bmatrix} + \begin{bmatrix} A^T & A^T \\ B - A & A^T \end{bmatrix}$$

is a linearization for $P(\lambda)$ provided that -1 is not an eigenvalue of $P(\lambda)$. Since A is rank deficient, ∞ is an eigenvalue of the pencil $L_Z(\lambda)$ with geometric multiplicity $1005-67 = 938$. We therefore applied the structured deflation method (Algorithm 3.4) with $m = 938$ as a first step, in order to directly deflate the eigenvalues ∞ and 0 of the pencil. This resulted in a matrix $\widetilde{Z} \in \mathbb{C}^{134\times 134}$ (normalized such that $\|\widetilde{Z}\|_2 = 1$) and a corresponding T-palindromic pencil $L_{\widetilde{Z}}(\lambda) = \lambda\widetilde{Z} + \widetilde{Z}^T$. For this matrix, the structured deflation method with $m = 67$ produced an anti-triangular Schur form $Z_1 = U_1^T \widetilde{Z} U_1$ with $\mathrm{dist}_\triangle(Z_1) = \texttt{2.8365e-15}$ and $\mathrm{dist}_1(U_1) = \texttt{2.4665e-10}$. Although this result was already satisfactory, we also applied Algorithm 3.5 with $\alpha = 1.5$ in combination with the palindromic QR algorithm for the remaining 8×8 subproblem. We

FIG. 5.4. *Sparsity pattern of the matrices A (left) and B (right) in* (5.1)



obtained an anti-triangular Schur form $Z_2 = U_2^T \widetilde{Z} U_2$ with $\mathrm{dist}_\triangle(Z_2) = \texttt{2.8189e-15}$ and $\mathrm{dist}_1(U_2) = \texttt{6.7083e-11}$. Thus there was no significant improvement on the results of the structured deflation method with $m = 67$.

6. Conclusions. We have discussed numerical methods for the solution of palindromic eigenvalue problems and have shown that a combination of structured deflation based on the unstructured QZ-Algorithm followed by a structure-preserving method for the solution of the typically small eigenvalue problem associated with the eigenvalues near ± 1 performs very well. Our observations indicate that one should preferably use Algorithm 3.4 with $m = \frac{n}{2}$ if the pencil has no eigenvalues close to ± 1; and if there are eigenvalues close to ± 1, then one should follow it by the palindromic QR algorithm for the solution of the remaining small subproblem. If the results are still not satisfactory, then improved accuracy is obtained by applying one sweep of the palindromic Jacobi algorithm to the whole problem and then solving the part corresponding to the eigenvalues close to ± 1 once again with the palindromic QR algorithm. If the number of eigenvalues close to ± 1 is small, then the major cost for this algorithm is that of the QZ algorithm.

REFERENCES

[1] R. Byers and D. Kressner. Exploiting bistructured computational problems. In *Sixth International Workshop on Accurate Solution of Eigenvalue Problems*, Pennsylvania State University, University Park , May 2006, 2006.

[2] P.J. Eberlein. On the Schur decomposition of a matrix for parallel computation. *IEEE Trans. Comp.*, 36:167–174, 1987.

[3] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Academic Press, New York, 1982.

[4] J. Greenstadt. A method for finding roots of arbitrary matrices. *Math. Tables Aids Comput.*, 9:47–52, 1955.

[5] J. Greenstadt. Some numerical experiments in triangularizing matrices. *Numer. Math.*, 4:187–195, 1962.

[6] A. Hilliges. Numerische Lösung von quadratischen Eigenwertproblemen mit Anwendung in der Schienendynamik. Diplomarbeit, TU Berlin, Institut für Mathematik, 2004.

[7] A. Hilliges, C. Mehl, and V. Mehrmann. On the solution of palindromic eigenvalue problems. In

Proceedings of the 4th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS). Jyväskylä, Finland, 2004. CD-ROM.

[8] M. Hofer, N. Finger, J. Schöberl, S. Zaglmayr, U. Langer, and R. Lerch. Finite element simulation of wave propagation in periodic piezoelectric saw structures. *IEEE Transactions on UFFC*, 53:1192–1201, 2006.

[9] P. Lancaster. *Lambda-matrices and Vibrating Systems*. Pergamon Press, Oxford, 1966.

[10] P. Lancaster and L. Rodman. *The Algebraic Riccati Equation*. Oxford University Press, Oxford, 1995.

[11] P. Lancaster and M. Tismenetsky. *The Theory of Matrices*. Academic Press, Orlando, 2nd edition, 1985.

[12] A. J. Laub. A Schur method for solving algebraic Riccati equations. *IEEE Trans. Automat. Control*, AC-24:913–921, 1979.

[13] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[14] C. Mehl. On asymptotic convergence of nonsymmetric Jacobi algorithms. To appear in SIAM J. Matrix Anal. Appl.

[15] C. Mehl. Jacobi-like algorithms for the indefinite generalized Hermitian eigenvalue problem. *SIAM J. Matrix Anal. Appl.*, 25:964–985, 2004.

[16] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution*. Number 163 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, July 1991.

[17] C. Schröder. A canonical form for palindromic pencils and palindromic factorizations. Technical Report 316, DFG Research Center MATHEON, *Mathematics for key technologies* in Berlin, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany, 2006. url: http://www.matheon.de/.

[18] C. Schröder. A QR-like algorithm for the palindromic eigenvalue problem. Technical Report 388, DFG Research Center MATHEON, *Mathematics for key technologies* in Berlin, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany, 2007. url: http://www.matheon.de/.

[19] F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem. *SIAM Rev.*, 43:235–286, 2001.

[20] S. Zaglmayr. Eigenvalue problems in SAW-filter simulations. Diplomarbeit, Inst. of Comp. Mathematics, J. Kepler Univ. Linz, Austria, 2002.