The perturbation and ADAE index of a degenerated hyperbolic system modelling a heat exchanger

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Abstract

The heat exchanger in a heat pump can be modelled by the zero Mach-number limit of the Euler equations of compressible fluid flow. This system turns out to be a coupled hyperbolic/parabolic equation with coupled, time-dependent boundary conditions. Using the theory of abstract differential-algebraic equations it is shown that the frozen coefficient system has ADAE index 1. Moreover, the much stronger result is proven that the system has time-perturbation index one and space-perturbation index two even in the case of time-dependent boundary conditions. The results are stated in terms of the original physical variables. The estimates agree well with numerical experiments.

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AMS:
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1 Introduction

A heat pump is basically an air conditioner which can switch between acting as cooler and heater. Heat pumps have been around for a long time and traditionally the green house gas *freon* has been used as the refrigerant sunstance. Lately, concern for the environment has led engineers to experiment with other refrigerant substances, one of them being *carbon dioxide*. The mathematical and numerical modelling of heat pumps using *carbon dioxide* is a rather complex problem. In principle, the state of each component making up the complete device must be described by the full system of the Euler equations of compressible flow. For the routine task of modelling a heat pump in a system simulation environment, this description is much too hard to be feasible. Simplified descriptions are necessary.

In a recent paper [11], a reduced model for the heat exchangers appearing in a heat pump has been developed. It consists essentially of the zero-Mach number limit of the compressible Euler equations. This degenerated hyperbolic system turns out to be a coupled hyperbolic/parabolic equation. In the cited paper, energy estimates for a (simplified) frozen coefficient system transformed to a certain normal form are derived. In fact, the system is weakly ill-posed. In terms of the perturbation index (cf. [4]), the time index is 1, while the space index is 2. This statement depends essentially on the fact that time-independent boundary conditions are used exclusively. This is an unrealistic assumption for practical simulations. In case of time-dependent boundary conditions, the time index is known not to exceed 3 [21]. Numerical experiments in [21] indicate that this is an overestimate of the time index.

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Here, we will consider the frozen coefficient system, too. The aim of this paper is however two-fold:

- We will derive all estimates immediately in terms of the original, that is physical, variables without any simplifications, e.g., by neglecting terms.
- We will show that the perturbation index is 1 with respect to time and 2 with respect to space even for time-dependent boundary condtions.

The main tool is the theory of *abstract differential-algebraic equations* (ADAEs) [17, 18, 19]. The ADAE decoupling procedure leads to a boundary control system for a coupled hyperbolic/parabolic sytem of partial differential equations subject to Dirichlet and Robin boundary controls.
This work is organized as follows: In the remaining part of this section, we arrange the notation and introduce some basic facts concerning function spaces. In section 2 we provide the mathematical model of the heat exchanger. Section 3 contains a short review of the tools needed for the ADAE analysis. This theory is applied to the heat exchanger model in section 4. Theorem 4.1 and its conclusions are the main results of the paper. In section 5 we prove stability estimates for coupled hyperbolic/parabolic systems. Their application completes the proof of Theorem 4.1. Some numerial experiments with the complete nonlinear heat exchanger model supporting our theory are given in section 6.
We will use the following notation in this work. $\mathbb{R}^+$ is the set of non-negative real numbers while for $\omega \in \mathbb{R}$, the set $\mathbb{C}_\omega^+$ consists of the complex numbers whose real part exceeds ω.
For an Hilbert space X we denote the associated inner product by $\langle \cdot, \cdot \rangle_X$ and the norm by $\|\cdot\|_X$ or simply by $\langle \cdot, \cdot \rangle$ and respectively $\|\cdot\|$ if it is clear from the context. For a further Hilbert space Y, the *product space* $X \times Y$ is a Hilbert space with the inner product $\langle (x_1, y_1), (x_2, y_2) \rangle_{X\times Y} = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y$. The space of *bounded linear operators* mapping X to Y by $\mathcal{B}(X, Y)$ and we moreover set $\mathcal{B}(X) = \mathcal{B}(X, X)$. An operator $Q \in \mathcal{B}(X)$ with the property $Q^2 = Q$ is called *projector*. For a closed operator $A : D(A) \subset X \to Y$, the space $D(A)$ is called *domain of* A and it is provided with the *graph norm* $\|\cdot\|_{D(A)}$ defined by the relation $\|x\|^2_{D(A)} = \|x\|^2_X + \|Ax\|^2_Y$. For a linear operator A, $\ker A$ and $\operatorname{im} A$ denote its nullspace and range, respectively.
Now we arrange the required facts concerning spaces of functions taking values in some Hilberts space X. Note that if we neglect the argument X, it is just meant $X = \mathbb{R}$. If $I \subset \mathbb{R}$ is an interval, the symbols $L_2(I, X)$ and $C(I, X)$ stand for the *Lebesgue space* of classes of square integrable functions and, respectively, the space of continuous functions $f : I \to X$. We define $\langle f, g \rangle_{L_2(I,X)} := \int_I \langle f(y), g(y) \rangle_X \, dy$ and $\|f\|_{C(I,X)} := \max_{y\in I} \|f(y)\|_X$.
For a further interval $\tilde{I} \subset I$, we abbreviate the restriction of $f : I \to X$ to $\tilde{I}$ by $f|_{\tilde{I}}$. For $k \in \mathbb{N}$, the k-th distributional derivative is denoted by $f^{(k)}$. Further, the *Sobolev space* $H^k(I, X)$ is given by

$$H^k(I, X) := \{f : I \to X,\ f^{(j)} \in L_2(I, X) \text{ for } j = 0, \dots, k\}.$$

In the case where $0 \in I$, we provide this space with the inner product

$$\langle f, g \rangle_{H^k(I,X)} := \sum_{j=0}^{k-1} \left\langle f^{(j)}(0), g^{(j)}(0) \right\rangle_X + \left\langle f^{(k)}, g^{(k)} \right\rangle_{L_2(I,X)}.$$

The spatial indeterminate will usually denoted by ξ while t and τ are time variables. Further, for $L, T > 0$ we use the abbreviations $\Omega_L = [0, L]$ for a spatial and $S_T = [0, T]$ for a time interval.
The derivative of a function f with respect to space is denoted by $\partial f := \frac{\partial}{\partial \xi} f$ and for derivatives with respect to time, we use $\dot{f} := \frac{\partial}{\partial t} f$. Furthermore, for $\alpha \in \Omega_L$ we introduce the *evaluation operator* $C_\alpha : H^1(\Omega_L, X) \to X$ by $C_\alpha f := f(\alpha)$.
The *Laplace transform* of $f \in L_{2,\omega}(X)$ is denoted by $\mathcal{L}f$ or simply by $\widehat{f}$. We introduce the *Hardy space* $\mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$ consisting of all bounded analytical functions $G : \mathbb{C}_\omega^+ \to \mathcal{B}(X, Y)$ provided with the norm $\|G\|_{\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)} := \sup_{s\in\mathbb{C}_\omega^+} \|G(s)\|_{\mathcal{B}(X,Y)} < \infty$. As a result from [24], for all $T \in \mathbb{R}^+$, a $G \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$ well-defines an operator $\mathcal{G} \in \mathcal{B}(L_2(S_T, X), L_2(S_T, Y))$ via

$\mathcal{G}f := \mathcal{L}^{-1}\left(G(s)\widehat{f}(s)\right)$ with

$$\|\mathcal{G}\|_{(L_2(S_T,X),L_2(S_T,Y)} \leq \max\{1, e^{\omega T}\} \cdot \|G\|_{\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)}.$$

Another Hardy space $\mathcal{H}_2(\mathbb{C}_\omega^+, X)$ consists of analytical functions $g : \mathbb{C}_\omega^+ \to X$ with

$$\|g\|_{\mathcal{H}_2(\mathbb{C}_\omega^+,X)} = \sup_{\gamma>\omega} \|g(\gamma + i\cdot)\|_{L_2(\mathbb{R},X)} < \infty.$$

For some function $f : \mathbb{R}^+ \to X$ with $e^{-\omega\cdot} f(\cdot) \in L_2(\mathbb{R}^+, X)$, Parseval's identity yields that $\|e^{-\omega\cdot} f(\cdot)\|_{L_2(\mathbb{R}\cdot X)} = \|\widehat{f}\|_{\mathcal{H}_2(\mathbb{C}_\omega^+,X)}$.

2 The heat exchanger model

The heat exchanger is considered to be a long and narrow pipe with length L and cross section A. The state of the refrigerant substance is described by the system of incompressible Euler equations,

$$\begin{aligned}
\tfrac{\partial}{\partial t}(A\rho) + \tfrac{\partial}{\partial \xi}(A\rho u) &= 0, \\
\tfrac{\partial}{\partial t}(A\rho u) + \tfrac{\partial}{\partial \xi}(A(\rho u^2 + p)) &= R, \\
\tfrac{\partial}{\partial t}(AE_{\text{tot}}) + \tfrac{\partial}{\partial \xi}(A(E_{\text{tot}} + p)u) &= Q.
\end{aligned}$$

Here, ρ is the density, u the velocity, p the pressure, and E_{tot} the total energy per unit volume. R models the momentum loss due to friction in the pipe, while Q models the energy exchange with the surrounding air. Let $\xi = 0$ be the inflow boundary and $\xi = L$ the outflow boundary (hence, $u > 0$). The Euler equations are accompanied by constitutive relationships which relate pressure p, mass specific enthalpy h, density ρ, and temperature T,

$$\rho = f(p, h), \quad T = g(p, h).$$

Since carbon dioxide is a mono-component medium, the choice of (p, h) as primary thermodynamic variables is advantageous. The more common choice of (p, T) is not suitable since p depends on T alone in the two-phase region of the phase space. As a third variable it is convenient to use the mass flow rate $F = A\rho u$.

Assuming that the flow in the heat exchanger is characterized by very low Mach numbers a reduced model, the so-called *zero-Mach number limit* [12], can be used. Using the constitutive relations this leaves us with the system

$$\begin{aligned}
A\tfrac{\partial f}{\partial p}\tfrac{\partial p}{\partial t} + A\tfrac{\partial f}{\partial h}\tfrac{\partial h}{\partial t} + \tfrac{\partial F}{\partial \xi} &= 0, \\
A\tfrac{\partial p}{\partial \xi} &= R, \\
-A\tfrac{\partial p}{\partial t} + Af\tfrac{\partial h}{\partial t} + F\tfrac{\partial h}{\partial \xi} &= Q.
\end{aligned}$$

The friction and energy exchange models are given by

$$\begin{aligned}
R &= -L_f F|u| = -L_f \tfrac{F^2}{Af(p,h)} \operatorname{sign}(F), \\
Q &= A_{\text{exch}}\alpha(T_{\text{air}} - T) = A_{\text{exch}}\alpha(T_{\text{air}} - g(p, h)).
\end{aligned}$$

In this paper, we will consider the linearized version of the zero-Mach number limit with frozen coefficients,

$$A\tfrac{\partial u}{\partial t} + B\tfrac{\partial u}{\partial \xi} + Cu = g \quad \text{for all } (t, \xi) \in S_T \times \Omega_L \tag{1}$$

and

$$u(t,\xi) = \begin{bmatrix} F(t,\xi) \\ p(t,\xi) \\ h(t,\xi) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$$

We can assume that all indicated coefficients are non-zero and

$$d = \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} \neq 0. \tag{2}$$

So what about the boundary conditions? The energy estimates (and numerical experiments) in [11] suggest that the following two sets of boundary conditions are admissible:

(a) specific enthalpy h and mass flow rate F at inflow, pressure p at outflow;

(b) specific enthalpy h and pressure p at inflow, mass flow rate F at outflow.

According to the results of [11], the choice of the set of conditions depends on the magnitude of c_{33} and c_{22}: If $c_{33} > c_{22}$, then (a) gives a weakly ill-posed problem while, for $c_{33} < c_{22}$, (b) yields weak ill-posedness. In our numerical tests, both combinations were admissible independently of the relation between c_{33} and c_{22} with a slightly more stable behavior of (a). In fact, the distinction between these two cases is an artifact of the method of proof in the paper cited. The results of the present paper will show that the weak ill-posedness is independent of that relation even for the linearized frozen coefficient case.

In what follows, we will exclusively consider case (a). The boundary conditions can be formulated as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t,0) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} u(t,L) = f_3(t) \tag{3}$$

3 Abstract Differential-Algebraic Models

The heat exchanger model is now rewritten as an abstract differential-algebraic equation (ADAE). First, we will briefly revisit the solvability, decoupling and index theory of ADAEs from [19, 18, 17]). Let Hilbert spaces X, Z and an ADAE

$$\begin{aligned} \mathcal{E}\dot{x}(t) &= \mathcal{A}x(t) + q(t) \\ x(0) &= x_0 \end{aligned} \tag{4}$$

with bounded $\mathcal{E} : X \to Z$ and closed, denslely defined $\mathcal{A} : D(\mathcal{A}) \subset X \to Z$. As in the finite dimensional case [3], solvability questions lead to additional smoothness claims on the inhomogeneity $q(\cdot)$ and the initial value has to be *consistent* meaning that it has to fulfill some further constraints coming from algebraic relations which are contained in the differential-algebraic equation. A further question arising is the so-called *perturbation analysis* where one is interested in the continuous dependence of the solution trajectory $x(\cdot)$ upon certain norms of the inhomogeneity and the initial value. The above mentioned notion will be concretized in the following.

Definition 3.1 *A function $x(\cdot) : J \to \mathbb{R}$ is called* weak solution *of (4), if it is continuous and for all $z^* \in D(A^*)$, the scalar function $\langle x(\cdot), E^*z^* \rangle$ is weakly differentiable and for almost all $t \in J$ holds*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \langle x(t), E^*z^* \rangle_X = \langle x(t), A^*z^* \rangle_X + \langle q(t), z^* \rangle_Z \,, \tag{5}$$

where the adjoint operator $A^* : D(A^*) \subset Z \to X$ *defined on*

$$D(A^*) := \{y^* \in Y : \exists\, x^* \in X : \textit{such that } \langle y^*, Ax \rangle = \langle x^*, x \rangle \textit{ for all } x \in D(A)\}$$

is well-defined via the relation $\langle A^*z, x \rangle = \langle z, Ax \rangle$ for all $x \in D(A)$.
We call the initial value $x_0 \in X$ consistent *with q if (4) possesses a weak solution.*
The perturbation index *of (4) is the smallest $\nu \in \mathbb{N}$ for which a dense subspace $X_\nu \subset X$ with norm $\|\cdot\|_{X_\nu}$ and a constant $c \in \mathbb{R}^+$ exists such that for all initial values $x_0 \in X_\nu$ being consistent with $q \in H^\nu(S_T, Z)$ it holds for all $t \in S_T$ that*

$$\|x(t)\|_X \leq c \left(\|x_0\|_{X_\nu} + \|q(\cdot)\|_{H^\nu(S_T,Z)} \right). \tag{6}$$

In the finite-dimensional linear and constant-coefficient case, the questions of solvability, consistent initialization and perturbation index can be answered by making use of the Kronecker normal form, i.e. the decoupling of differential and algebraic parts. Indeed, the perturbation index coincides with the Kronecker index of the associated matrix pencil and further, the spaces X and X_ν in the above definition coincide since there is no proper and dense subspace of $\mathbb{R}^n$. However, the situation becomes more difficult in infinite dimensions, since a possibly stronger norm of the initial values has to be chosen if one wants to have an estimate of the present state upon inhomogeneity and initial value. For instance, in [18], electrical circuits with transmission lines have been considered and some Sobolev instead of Lebesgue norms of the initial voltages and currents on the transmission lines have to be taken. This leads to the other concept of *spatial perturbation index* that - roughly speaking - measures the smallest integer ν_p such that $\|\cdot\|_{X_\nu}$ involves the norm H^{ν_p}. For general ADAEs, such a concept is not formulable and one has to restrict the consideration to systems whose state space is some cartesian product of $\mathbb{R}^n$ and some function spaces. However, we will see in the present case study that such a concept makes sense.
A useful tool for the above motivated tasks is the generalization of the Kronecker normal form considered in [19] with applications to consistent initialization and perturbation analysis presented in [17]. The *decoupling form* is constructed with projectors $\mathcal{Q}_i \in \mathcal{B}(X) \cap \mathcal{B}(D(A))$ for $i = 0, 1, 2, \dots, \nu - 1$ fulfilling

$$\begin{aligned}
&\mathcal{E}_0 := \mathcal{E},\ \mathcal{A}_0 := \mathcal{A},\\
&\operatorname{im}\mathcal{Q}_i = \ker\mathcal{E}_i,\ \textstyle\sum_{j=0}^{i-1}\ker\mathcal{E}_j \subset \ker\mathcal{Q}_i,\\
&\mathcal{E}_{i+1} = \mathcal{E}_i - \mathcal{A}_i\mathcal{Q}_i,\ D(\mathcal{E}_{i+1}) = D(\mathcal{E}_i) \cap (D(\mathcal{A}_i) + \ker\mathcal{Q}_i),\\
&\mathcal{A}_{i+1} = \mathcal{A}_i(I - \mathcal{Q}_i),\ D(\mathcal{A}_{i+1}) = D(\mathcal{A}_i) + \operatorname{im}\mathcal{Q}_i.
\end{aligned}$$

such that $\mathcal{E}_\nu$ is injective and has a bounded left inverse $\mathcal{E}_\nu^-$. We assume that ν is chosen to be minimal with this property, i.e., $Q_{\nu-1} \neq 0$. The number ν with this property is called *ADAE index.* For $i = 0, \dots, \nu - 1$, $i = 0, \dots, \nu - 2$, $k = j + 1, \dots, \nu - 1$, we denote $\mathcal{P}_i = I - \mathcal{Q}_i$ and

$$\begin{aligned}
x_i(t) &= \mathcal{Q}_i\mathcal{P}_{i+1}\cdots\mathcal{P}_{\nu-1}x(t), & q_i(t) &= \mathcal{Q}_i\mathcal{P}_{i+1}\cdots\mathcal{P}_{\nu-1}\mathcal{E}_\nu^- q(t),\\
x_\nu(t) &= \mathcal{P}_0\cdots\mathcal{P}_{\nu-1}x(t), & q_\nu(t) &= \mathcal{P}_i\cdots\mathcal{P}_{\nu-1}\mathcal{E}_\nu^- q(t),\\
q_{\nu+1}(t) &= (I - \mathcal{E}_\nu\mathcal{E}_\nu^-)q(t), & \mathcal{N}_{jk} &= \mathcal{Q}_j\mathcal{P}_{j+1}\cdots\mathcal{P}_{k-1}\mathcal{Q}_k,\\
\mathcal{K}_i &= -\mathcal{Q}_i\mathcal{P}_{i+1}\cdots\mathcal{P}_{\nu-1}\mathcal{E}_\nu^-\mathcal{A}_\nu, & \mathfrak{U} &= \mathcal{P}_0\cdots\mathcal{P}_{\nu-1}\mathcal{E}_\nu^-\mathcal{A}_\nu,\\
\mathfrak{P} &= (I - \mathcal{E}_\nu\mathcal{E}_\nu^-)\mathcal{A}_\nu,
\end{aligned}$$

then (4) is equivalent to an ADAE in *decoupling form*

$$\left[\begin{array}{cccc|c}
0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0\,\nu-1} & 0\\
\vdots & \ddots & \ddots & \vdots & \vdots\\
\vdots & & \ddots & \mathcal{N}_{\nu-2\,\nu-1} & \vdots\\
0 & \cdots & \cdots & 0 & 0\\ \hline
0 & \cdots & \cdots & 0 & I\\ \hline
0 & \cdots & \cdots & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\dot{x}_0(t)\\ \vdots\\ \vdots\\ \dot{x}_{\nu-1}(t)\\ \hline \dot{x}_\nu(t)
\end{array}\right]
=
\left[\begin{array}{cccc|c}
I & 0 & \cdots & 0 & \mathcal{K}_0\\
0 & \ddots & \ddots & \vdots & \vdots\\
\vdots & & \ddots & 0 & \vdots\\
0 & \cdots & 0 & I & \mathcal{K}_{\nu-1}\\ \hline
0 & \cdots & \cdots & 0 & \mathfrak{U}\\ \hline
0 & \cdots & \cdots & 0 & \mathfrak{P}
\end{array}\right]
\left[\begin{array}{c}
x_0(t)\\ \vdots\\ \vdots\\ x_{\nu-1}(t)\\ \hline x_\nu(t)
\end{array}\right]
+
\left[\begin{array}{c}
q_0(t)\\ \vdots\\ \vdots\\ q_{\nu-1}(t)\\ \hline q_\nu(t)\\ \hline q_{\nu+1}(t)
\end{array}\right]. \tag{7}$$

The variable x_ν fulfills the *abstract boundary control system*

$$\begin{aligned}
\dot{x}_\nu(t) &= \mathfrak{U}x_\nu(t) + q_\nu(t),\\
0 &= \mathfrak{P}x_\nu(t) + q_{\nu+1}(t),\\
x_\nu(0) &= x_{\nu,0}.
\end{aligned} \tag{8}$$

The other components are then recursively determined by the algebraic and hidden algebraic constraints

$$\begin{aligned} x_{\nu-1}(t) &= -\mathcal{K}_{\nu-1}x_\nu(t) - q_{\nu-1}(t), \\ x_{\nu-2}(t) &= \mathcal{N}_{\nu-2\,\nu-1}\dot{x}_{\nu-1}(t) - \mathcal{K}_{\nu-2}x_\nu(t) - q_{\nu-1}(t), \\ &\ \ \vdots \\ x_0(t) &= \textstyle\sum_{j=1}^{\nu-1}\mathcal{N}_{0\,j}\dot{x}_j(t) - \mathcal{K}_0 x_\nu(t) - q_0(t). \end{aligned}$$

By solving these expressions for $x_0(t), \dots, x_{\nu-1}(t)$, we obtain that $x_i(t)$ depends on the first j derivatives of $\mathcal{K}_{\nu-1-j}x_\nu(t)$ and $q_{\nu-1-j}(t)$ for $j = 0, \dots, i$. Thus, for the existence of a continuous solution, both $q_i(\cdot)$ and $\mathcal{K}_i x_\nu(\cdot)$ have to be i-times continuously differentiable. The smoothness claims on $\mathcal{K}_i x_\nu(\cdot)$ in general leads to further conditions on the initial value $x_\nu(0)$, $q_\nu(\cdot)$ and $q_{\nu+1}(\cdot)$, the so-called *(hidden) boundary constraints.*
Assuming that there exists $\bar{c} > 0$ such that an estimate

$$\left\| \sum_{j=0}^{\nu-1} \frac{\mathrm{d}^{\mathrm{i}}}{\mathrm{d}t^i}\mathcal{K}_i x_\nu(t) \right\|_{X_1} \leq \bar{c}\left(\|x_\nu(0)\|_{X_\nu} + \|q_\nu(\cdot)\|_{Z_\nu} + \|q_{\nu+1}(\cdot)\|_{Z_{\nu+1}} \right) \tag{9}$$

holds for all $q_\nu(\cdot)$, $q_{\nu+1}(\cdot)$ and some function spaces $Z_\nu \subset L_2(S_T, \operatorname{im}\mathcal{P}_0\cdots\mathcal{P}_{\nu-1})$ and $Z_{\nu+1} \subset L_2(S_T, \operatorname{im}\mathcal{P}_0\cdots\mathcal{P}_{\nu-1})$, respectively, and consistent initial values $x_\nu(0)$ in some subspace $X_\nu \subset \operatorname{im}\mathcal{P}_0\cdots\mathcal{P}_{\nu-1}$, we get the existence of $c \in \mathbb{R}^+$ such that a uniform perturbation result holds

$$\left\| \begin{bmatrix} x_0(T) \\ \vdots \\ \vdots \\ x_{\nu-1}(T) \\ \hline x_\nu(T) \end{bmatrix} \right\|_{X^{\nu+1}} \leq c\left(\|x_\nu(0)\|_{X_\nu} + \|q_\nu(\cdot)\|_{Z_\nu} + \|q_{\nu+1}(\cdot)\|_{Z_{\nu+1}} + \sum_{i=0}^{\nu-1}\|q_i(\cdot)\|_{C^i(S_T, \operatorname{im}\mathcal{Q}_i)} \right). \tag{10}$$

For the class of ADAEs for which the system (8) is well-posed - meaning that $x(t)$ continuously depends on the L_2-norms of $q_\nu(\cdot)$ and $q_{\nu+1}(\cdot)$, an explicit characterization of the consistency of an initial value is given in [17] by the existence of the chain $x_{\nu,j} := q_\nu^{(j-1)}(0) + \mathfrak{U}x_{\nu,j-1}$ which satisfies $x_{\nu,j} \in D(\mathfrak{U})$ und $\mathfrak{P}x_{\nu,j} = -q_{\nu+1}^{(j)}(0)$ for $j = 1, \dots, l-1$ for an l fulfilling $\mathcal{N}^l\mathcal{K} = 0$ where $\mathcal{N} = [\mathcal{N}_{jk}]$ and $\mathcal{K} = [\mathcal{K}_j]$. As we will see later, the system that is analyzed in this work does not satisfy this well-posedness condition. Nevertheless, the ADAE approach will lead to a certain estimate of the state upon some norms of the inhomogeneity $q(\cdot)$ and the initial value x_0.

4 The heat exchanger as an ADAE

We now consider the model (1) with boundary conditions (3). Defining the spaces $X = L_2(\Omega_L)^3$, $Z = L_2(\Omega_L) \times H^1(\Omega_L) \times L_2(\Omega_L) \times \mathbb{R}^3$ and

$$x(t) = u(t,\cdot) = \begin{bmatrix} F(t,\cdot) \\ p(t,\cdot) \\ h(t,\cdot) \end{bmatrix} \in X,\ q(t) = \begin{bmatrix} g_1(t,\cdot) \\ g_2(t,\cdot) \\ g_3(t,\cdot) \\ f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix} \in Z \tag{11}$$

the system is equivalent to the ADAE (4) with

$$\mathcal{E} = \begin{bmatrix} 0 & a_{12}I & a_{13}I \\ 0 & 0 & 0 \\ 0 & a_{32}I & a_{33}I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{A} = -\begin{bmatrix} \partial & 0 & 0 \\ 0 & b_{22}\partial & 0 \\ 0 & 0 & b_{33}\partial \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & c_{12}I & c_{13}I \\ c_{21}I & c_{22}I & c_{23}I \\ c_{31}I & c_{32}I & c_{33}I \\ C_0 & 0 & 0 \\ 0 & 0 & C_0 \\ 0 & C_L & 0 \end{bmatrix}. \tag{12}$$

Recall that $\partial = \partial/\partial\xi$ and $C_\alpha u = u(\alpha)$. The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = H^1(\Omega_L) \times H^2(\Omega_L) \times H^1(\Omega_L)$$

Now we compute the projectors for the decoupling of the ADAE. The operator chain leads to $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{A}_0 = \mathcal{A}$, $\ker \mathcal{E}_0 = H^1(\Omega_L) \times \{0\} \times \{0\}$ and projectors $\mathcal{Q}_0, \mathcal{P}_0 \in \mathcal{B}(X) \cap \mathcal{B}(D(\mathcal{A}))$ with $\mathcal{Q}_0 = \operatorname{diag}(I, 0, 0)$, $\mathcal{P}_0 = \operatorname{diag}(0, I, I)$ and

$$\mathcal{E}_1 = \mathcal{E}_0 - \mathcal{A}_0\mathcal{Q}_0 = \begin{bmatrix} \partial & a_{12}I & a_{13}I \\ c_{21}I & 0 & 0 \\ c_{31}I & a_{32}I & a_{33}I \\ C_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $D(\mathcal{E}_1) = H^1(\Omega_L) \times L_2(\Omega_L) \times L_2(\Omega_L)$. Due to (2), we now have have that $\mathcal{E}_1$ is injective, i.e. the system is of ADAE index 1. A left inverse of $\mathcal{E}_1$ is given by

$$\mathcal{E}_1^- = \begin{bmatrix} 0 & \frac{1}{c_{21}}I & 0 & 0 & 0 & 0 \\ \frac{a_{33}}{d}I & -\frac{a_{33}}{c_{21}d}\partial + \frac{a_{13}c_{31}}{c_{21}d}I & -\frac{a_{13}}{d}I & 0 & 0 & 0 \\ -\frac{a_{32}}{d}I & \frac{a_{32}}{c_{21}d}\partial - \frac{a_{12}c_{31}}{c_{21}d}I & \frac{a_{12}}{d}I & 0 & 0 & 0 \end{bmatrix} \in \mathcal{B}(L_2(\Omega_L) \times H^1(\Omega_L) \times L_2(\Omega_L) \times \mathbb{R}^3, L_2(\Omega_L)^3).$$

A decoupling form is now given by (7) with

$$\mathcal{K}_0 = \begin{bmatrix} 0 & \frac{b_{22}}{c_{21}}\partial + \frac{c_{22}}{c_{21}}I & \frac{c_{23}}{c_{21}}I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{b_{22}}{c_{21}}C_0\partial - \frac{c_{22}}{c_{21}}C_0 & -\frac{c_{23}}{c_{21}}C_0 \\ 0 & 0 & -C_0 \\ 0 & -C_L & 0 \end{bmatrix}, \quad \mathfrak{U} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathfrak{u}_{22} & \mathfrak{u}_{23} \\ 0 & \mathfrak{u}_{32} & \mathfrak{u}_{33} \end{bmatrix}, \tag{13}$$

where

$$\begin{aligned}
\mathfrak{u}_{22} &= \tfrac{-a_{33}c_{12}^2 - a_{13}c_{31}c_{22} + a_{13}c_{32}c_{21}}{c_{21}d}I + \tfrac{c_{22}a_{33} - a_{13}c_{31}b_{22}}{c_{21}d}\partial + \tfrac{a_{33}b_{22}}{c_{21}d}\partial^2, \\
\mathfrak{u}_{23} &= \tfrac{-a_{33}c_{13}c_{21} - c_{23}a_{13}c_{31} + a_{13}c_{21}c_{33}}{dc_{21}}I + \tfrac{c_{23}a_{33} + b_{33}a_{13}c_{21}}{dc_{21}}\partial, \\
\mathfrak{u}_{32} &= \tfrac{a_{32}c_{12}^2 - a_{12}c_{31}c_{22} - a_{12}c_{32}c_{21}}{c_{21}d}I + \tfrac{-c_{22}a_{32} + a_{12}c_{31}b_{22}}{c_{21}d}\partial - \tfrac{a_{32}b_{22}}{c_{21}d}\partial^2, \\
\mathfrak{u}_{33} &= \tfrac{a_{32}c_{13}c_{21} + c_{23}a_{12}c_{31} - a_{12}c_{21}c_{33}}{dc_{21}}I - \tfrac{c_{23}a_{32} + a_{12}b_{33}c_{21}}{dc_{21}}\partial.
\end{aligned}$$

Specific components of state and inhomogeneity are given by

$$x_0(t) = \begin{bmatrix} F(t,\cdot) \\ 0 \\ 0 \end{bmatrix}, \quad x_1(t) = \begin{bmatrix} 0 \\ p(t,\cdot) \\ h(t,\cdot) \end{bmatrix}, \quad q_0(t) = \begin{bmatrix} \frac{1}{c_{21}}g_2(t,\cdot) \\ 0 \\ 0 \end{bmatrix},$$

$$q_1(t) = \begin{bmatrix} 0 \\ \frac{a_{33}}{d}g_1(t,\cdot) - \frac{a_{33}}{c_{21}d}\partial g_2(t,\cdot) + \frac{a_{13}c_{31}}{c_{21}d}g_2(t,\cdot) - \frac{a_{13}}{d}g_3(t,\cdot) \\ -\frac{a_{32}}{d}g_1(t,\cdot) + \frac{a_{32}}{c_{21}d}\partial g_2(t,\cdot) - \frac{a_{12}c_{31}}{c_{21}d}g_2(t,\cdot) - \frac{a_{12}}{d}g_3(t,\cdot) \end{bmatrix}, \quad q_2(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_1(t) - \frac{1}{c_{21}}g_2(t,0) \\ f_2(t) \\ f_3(t) \end{bmatrix}. \tag{14}$$

Now taking a look at the operators $\mathfrak{P}$ and $\mathfrak{U}$, we can see that the inherent boundary control system is in general not well-posed. This is due to the appearance of the Dirichlet boundary condition for p that satisfies an heat equation with some additional inhomogeneity. It is shown in [8] that this class of systems does not have the well-posedness property if one chooses a Lebesgue space as state space. However, strengthening the smoothness claims on the boundary leads to an estimate. The main result about the consistent initialization and perturbation analysis is now presented.

Theorem 4.1 *The ADAE (4) with operators, state and inhomogeneity as in (12) possesses a continuous solution $x(\cdot) : S_T \to X$ for all $q(\cdot)$, $x(0)$ in the spaces*

$$\begin{aligned} g_1, g_3 \in H^1(S_T, L_2(\Omega_L)), \qquad & g_2 \in H^1(S_T, H^1(\Omega_L)), \\ f_1, f_2, f_3 \in H^1(S_T), \qquad & F(0,\cdot), p(0,\cdot), h(0,\cdot) \in H^1(\Omega_L) \end{aligned} \tag{15}$$

that satisfy the conditions

$$p(0,L) = f_3(0), \quad h(0,0) = f_2(0), \quad 0 = c_{21}F(0,\cdot) + b_{22}\partial p(0,\cdot) + c_{22}p(0,\cdot) + c_{23}h(0,\cdot) + g_2(0,\cdot). \tag{16}$$

Moreover, there exists a constant $c > 0$ such that for all $t \in S_T$ and all initial states and inhomogeneities satisfying (15) and (16), the solution of (4) satisfies the estimate

$$\begin{aligned} &\|F(t,\cdot)\|_{L_2(\Omega_L)} + \|p(t,\cdot)\|_{H^1(\Omega_L)} + \|h(t,\cdot)\|_{H^1(\Omega_L)} \\ \le & c \left(\|p(0,\cdot)\|_{H^1(\Omega_L)} + \|h(0,\cdot)\|_{H^1(\Omega_L)} + \|g_1\|_{H^1(S_T,L_2(\Omega_L))} + \|g_2\|_{H^1(S_T,H^1(\Omega_L))} \right. \\ & \left. + \|g_3\|_{H^1(S_T,L_2(\Omega_L))} + \|f_1\|_{H^1(S_T)} + \|f_2\|_{H^1(S_T)} + \|f_3\|_{H^1(S_T)} \right). \end{aligned} \tag{17}$$

A weaker version of the above result is that with $x(\cdot)$ and $q(\cdot)$ defined in (11) and for $X_1 := L_2(\Omega_L) \times H^1(\Omega_L)^2$ we have an estimate

$$\|x(t)\|_X \le c \left(\|x(0)\|_{X_1} + \|q\|_{H^1(S_T,Z)} \right).$$

This means that - according to Definition 3.1 - we have perturbation index one. Furthermore, the first spatial derivatives of the initial value and some components of the inhomogenity are involved. This leads to the fact that it is spoken about spatial perturbation index 2.

Having a look at (17) we see that the mass flow rate plays the rôle of an index two variable which depends on spatial derivatives of the pressure.

5 Stability estimates for a mixed parabolic/hyperbolic boundary control system

The boundary control system (8) corresponding to the heat exchanger can be read off from the decoupled system (14). Setting $x_1 = p$ and $x_2 = p + \frac{a_{32}}{a_{33}}h$ we obtain by using suitable abbreviations,

$$\begin{bmatrix} \dot{x}_1(t,\xi) \\ \dot{x}_2(t,\xi) \end{bmatrix} = \begin{bmatrix} \kappa_1\partial^2 + \kappa_2\partial + \kappa_3 & \kappa_4\partial + \kappa_5 \\ \kappa_6\partial + \kappa_7 & \kappa_8\partial + \kappa_9 \end{bmatrix} \begin{bmatrix} x_1(t,\xi) \\ x_2(t,\xi) \end{bmatrix} + \begin{bmatrix} \bar{g}_1(t,\xi) \\ \bar{g}_2(t,\xi) \end{bmatrix} \tag{18}$$

subject to the boundary conditions

$$\tfrac{\partial}{\partial\xi}x_1(t,0) + \kappa_{10}x_1(t,0) = \bar{f}_1(t), \tag{19a}$$

$$\kappa_{11}x_1(t,0) + x_2(t,0) = \bar{f}_2(t), \tag{19b}$$

$$x_1(t,L) = \bar{f}_3(t) \tag{19c}$$

and the initial condition

$$\begin{bmatrix} x_1(0,\xi) \\ x_2(0,\xi) \end{bmatrix} = \begin{bmatrix} x_{10}(\xi) \\ x_{20}(\xi) \end{bmatrix}. \tag{20}$$

Note that the first boundary condition (19a) has been slighly reformulated by using (19b). The critical point in this system is the presence of the Dirichlet boundary control (19c) for the parabolic component. Correspondingly, we will split the proof into two steps: In the first step we assume that $\bar{f}_3(t) \equiv 0$, while in the second part, all inhomogenities in the partial differential equations and the boundary conditions are set to 0 with the exception of $\bar{f}_3(t)$.

Assumption 5.1 *It holds $\kappa_1 > 0$ and $\kappa_8 < 0$.*

In the present application, these conditions are always fulfilled. It holds $\kappa_8 = -u$ where u denotes the speed of the fluid. The value of κ_1 depends only on the constitutive relations. $\kappa_1 > 0$ holds for ideal gases, and it has been numerically verified also for carbon dioxide.

First we present an auxiliary result concerning the existence of solutions.

Lemma 5.2 *Let Assumption 5.1be valid. If $\bar{f}_i \in C^2(S_T)$ for $i = 1, 2, 3$ and $\bar{g}_i \in C^1(S_T, L_2(\Omega_L))$ for $i = 1, 2$, and initial values $x_{10} \in H^2(\Omega_L)$, $x_{20} \in H^1(\Omega_L)$ with $\frac{\partial}{\partial \xi} x_{10}(0) + \kappa_{10} x_{10}(0) = \bar{f}_1(0)$, $\kappa_{11} x_{10}(0) + x_{20}(0) = \bar{f}_2(0)$, $x_{10}(L) = \bar{f}_3(0)$, the system (18) with boundary condition (19) and initial data (20) possesses a unique solution $[x_1, x_2]^T \in C(S_T, H^2(\Omega_T) \times H^1(\Omega_T))$.*

Proof:
For

$$\begin{bmatrix} \bar{x}_1(t,\xi) \\ \bar{x}_2(t,\xi) \end{bmatrix} := \begin{bmatrix} x_1(t,\xi) - \frac{\xi(L-\xi)}{L} \bar{f}_1(t) - \frac{\xi^2}{L_2} \bar{f}_3(t) \\ x_2(t,\xi) - \bar{f}_2(t) \end{bmatrix},$$

the given equation is equivalent to

$$\begin{aligned} \begin{bmatrix} \dot{\bar{x}}_1(t,\xi) \\ \dot{\bar{x}}_2(t,\xi) \end{bmatrix} = &\begin{bmatrix} \kappa_1 \partial^2 + \kappa_2 \partial + \kappa_3 & \kappa_4 \partial + \kappa_5 \\ \kappa_6 \partial + \kappa_7 & \kappa_8 \partial + \kappa_9 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t,\xi) \\ \bar{x}_2(t,\xi) \end{bmatrix} + \begin{bmatrix} \bar{g}_1(t,\xi) \\ \bar{g}_2(t,\xi) \end{bmatrix} + \begin{bmatrix} -\frac{\xi(L-\xi)}{L} \dot{\bar{f}}_1(t) + \frac{\xi^2}{L_2} \dot{\bar{f}}_3(t) \\ -\dot{\bar{f}}_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} \kappa_1 \partial^2 + \kappa_2 \partial + \kappa_3 & \kappa_4 \partial + \kappa_5 \\ \kappa_6 \partial + \kappa_7 & \kappa_8 \partial + \kappa_9 \end{bmatrix} \begin{bmatrix} \frac{\xi(L-\xi)}{L} \bar{f}_1(t) + \frac{\xi^2}{L_2} \bar{f}_3(t) \\ \bar{f}_2(t) \end{bmatrix}. \end{aligned} \tag{21}$$

with homogeneous boundary conditions, i.e.,

$$\tfrac{\partial}{\partial \xi} \bar{x}_1(t,0) + \kappa_{10} \bar{x}_1(t,0) = \kappa_{11} \bar{x}_1(t,0) + \bar{x}_2(t,0) = \bar{x}_1(t,L) = 0$$

and the initial condition

$$\begin{bmatrix} \bar{x}_1(0,\xi) \\ \bar{x}_2(0,\xi) \end{bmatrix} = \begin{bmatrix} x_{10} - \frac{\xi(L-\xi)}{L} \bar{f}_1(0) - \frac{\xi^2}{L_2} \bar{f}_3(0) \\ x_{20} - \bar{f}_2(0) \end{bmatrix}.$$

Now consider the operator

$$A := \begin{bmatrix} \kappa_1 \partial^2 + \kappa_2 \partial + \kappa_3 & \kappa_4 \partial + \kappa_5 \\ \kappa_6 \partial + \kappa_7 & \kappa_8 \partial + \kappa_9 \end{bmatrix}$$

with domain $D(A)$ given by the space

$$\left\{ (x_1, x_2) \in H^2(\Omega_L) \times H^1(\Omega_L) : \tfrac{\partial}{\partial \xi} x_{10}(0) + \kappa_{10} x_{10}(0) = \kappa_{11} x_{10}(0) + x_{20}(0) = x_{10}(L) = 0 \right\}.$$

If we now show that A generates a strongly continuous semigroup $T(\cdot)$ on $X = L_2(\Omega_L)^2$, Theorem 3.1.3 of [6] implies that there exists a unique solution

$$\begin{bmatrix} \bar{x}_1(\cdot,\cdot) \\ \bar{x}_2(\cdot,\cdot) \end{bmatrix} \in C(S_T, H^2(\Omega_L) \times H^2(\Omega_L)).$$

In the following, the semigroup property is shown with the Lumer-Philips Theorem [16] stating that A generates a strongly continuous semigroup $T(\cdot)$ with $\|T(t)\| \leq e^{\gamma t}$ for some $\gamma \in \mathbb{R}$ if there exists a $\lambda > \gamma$ such that $A - \lambda I : D(A) \to X$ is onto and there for all $x \in D(A)$ holds $\langle Ax, x\rangle_X \le \gamma\|x\|_X^2$. Let us assume that $\kappa_3 = \kappa_5 = \kappa_7 = \kappa_9 = 0$, since bounded perturbations of an operator do not destroy its property of being the generator of a semigroup [9]. The surjectivity of $A-\lambda I$ holds since the boundary value problem $\kappa_1\partial^2 x_1(\xi)+\kappa_2\partial x_1(\xi)-\lambda x_1(\xi)+\kappa_4\partial x_2(\xi) = y_1(\xi)$, $\kappa_6\partial x_1(\xi) + \kappa_8\partial x_2(\xi) - \lambda x_2(\xi) = y_2(\xi)$ with $\frac{\partial}{\partial\xi}x_1(0) + \kappa_{10}x_1(0) = \kappa_{11}x_1(0) + x_2(0) = x_1(L) = 0$ is uniquely solvable for all $y_1, y_2 \in L_2(\Omega_L)$, if $\lambda \in \mathbb{R}$ is large enough. Further, by making use of integration by parts and the boundary conditions, we compute for $x = [x_1, x_2] \in D(A)$

$$\begin{aligned}
\langle Ax, x\rangle_X =&\kappa_1\int_0^L \partial^2 x_1(\xi)x_1(\xi)d\xi + \kappa_2\int_0^L \partial x_1(\xi)x_1(\xi)d\xi + \kappa_4\int_0^L \partial x_2(\xi)x_1(\xi)d\xi \\
&+ \kappa_6\int_0^L \partial x_1(\xi)x_2(\xi)d\xi + \kappa_8\int_0^L \partial x_2(\xi)x_2(\xi)d\xi \\
=& -\kappa_1\int_0^L \partial x_1(\xi)\partial x_1(\xi)d\xi + \kappa_2\int_0^L \partial x_1(\xi)x_1(\xi)d\xi + (\kappa_6-\kappa_4)\int_0^L \partial x_1(\xi)x_2(\xi)d\xi \\
&+ \left(\kappa_1\kappa_{10} + \kappa_4\kappa_{11} - \kappa_8\kappa_{10}^2\right)x_1(0)^2 + \tfrac{\kappa_8}{2}x_2(L)^2 \\
=& -\kappa_1\int_0^L \partial x_1(\xi)\partial x_1(\xi)d\xi + \left(\kappa_2 + 2\kappa_1\kappa_{10} + 2\kappa_4\kappa_{11} - 2\kappa_8\kappa_{10}^2\right)\int_0^L \partial x_1(\xi)x_1(\xi)d\xi \\
&+ (\kappa_6-\kappa_4)\int_0^L \partial x_1(\xi)x_2(\xi)d\xi + \tfrac{\kappa_8}{2}x_2(L)^2 \\
\le& -\kappa_1\|\partial x_1\|^2_{L_2(\Omega_L)} + |\kappa_2 + 2\kappa_1\kappa_{10} + 2\kappa_4\kappa_{11} - 2\kappa_8\kappa_{10}^2|\cdot\varepsilon_1\cdot\|\partial x_1\|^2_{L_2(\Omega_L)} \\
&+ \tfrac{|\kappa_2+2\kappa_1\kappa_{10}+2\kappa_4\kappa_{11}-2\kappa_8\kappa_{10}^2|}{\varepsilon_1}\|x_1\|^2_{L_2(\Omega_L)} + |\kappa_6-\kappa_4|\varepsilon_2\|\partial x_1\|^2_{L_2(\Omega_L)} + \tfrac{|\kappa_6-\kappa_4|}{\varepsilon_2}\|x_2\|^2_{L_2(\Omega_L)}
\end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^+$. By choosing $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^+$ small enough, we obtain that $\langle Ax, x\rangle_X \le \gamma\|x\|_X^2$ for some $\gamma \in \mathbb{R}$. □

The above result yields existence of a solution in the case where initial, boundary and inhomogeneous data are sufficiently smooth. In the following, we present several results that lead to less smoothness claims. First, we treat the case where $\bar f_3 \equiv 0$. Later on, we generalize this result to arbitrary $\bar f_3 \in H^1(S_T)$.

Subsequently, we need the space $V_2 := \{v \in H^1(\Omega_L) | v(L) = 0\}$ that is normed by $\|x\|_{V_2} := \|\partial x\|_{L_2(\Omega_L)}$. Further, we introduce V_2^* as its dual space with respect to $L_2(\Omega_L)$. The spaces $V_2 \subset L_2(\Omega_L) \subset V_2^*$ are a so-called Gelfand tripel. We will use the notation $\langle\cdot,\cdot\rangle$ in order to denote the dual pairing between elements from V_2^* and V_2. We further introduce the space $W(S_T) = L_2(S_T, V_2) \cap H^1(S_T, V_2^*)\}$ that is continuously imbedded into $C(S_T, L_2(\Omega_L))$, see [25, pp. 390ff].

Lemma 5.3 *Let Assumption 5.1 be valid and $[x_1, x_2]^T$ be a solution of the initial-boundary value problem (18) subject to the boundary conditions (19) with $\bar f_1, \bar f_2 \in L_2(S_T)$, $\bar f_3 \equiv 0$ and the initial condition (20) with $\bar x_{10}, \bar x_{20} \in L_2(\Omega_L)$. Then for all $t \in S_T$ it holds*

$$\begin{aligned}
\|x_1(t)\|^2_{L_2(\Omega_L)} + \|x_2(t)\|^2_{L_2(\Omega_L)} \le& C\Big(\|x_{10}\|^2_{L_2(\Omega_L)} + \|x_{20}\|^2_{L_2(\Omega_L)} + \|\bar f_1\|^2_{L_2(S_T)} + \|\bar f_2\|^2_{L_2(S_T)} \\
&+ \|\bar g_1\|^2_{L_2(S_T, L_2(\Omega_L))} + \|\bar g_2\|^2_{L_2(S_T, L_2(\Omega_L))}\Big).
\end{aligned}$$

for some constant $C \in \mathbb{R}^+$ independent of the inhomogenities but which may depend on T, L, and κ_i.

Proof:

The proof will be given in several steps. Let C denote a generic constant in th following. Note that at different places, this constant may take different values.

1. For the moment, let us assume that the solution x_2 of the hyperbolic equation belongs to the space $x_2 \in L_2(S_T, H^1(\Omega_L))$. Note that the smoothness with respect to t can be reduced.

Consider the following parabolic problem: Find $x_1 \in W(S_T)$ such that

$$\langle \dot{x}_1, v\rangle + a(x_1, v) = l(v) \text{ for all } v \in V_2.$$

Here, $a(\cdot,\cdot) : V_2 \times V_2 \to \mathbb{R}$ is a continous and coercive bilinear form, that is the existence of $M, m \in \mathbb{R}^+$ such that for all $w, v \in V_2$ holds

$$|a(w, v)| \leq M\|w\|_{V_2}\|v\|_{V_2}, \qquad a(v, v) \geq m\|v\|^2_{V_2} - k_0\|v\|_{L_2(\Omega_L)}.$$

Here, $k_0 \geq 0$ and $m, M > 0$. Then, the parabolic problem subject to the initial condition $x_1(0,\cdot) = x_{10}(\cdot)$ has a unique solution for all $x_{10} \in L_2(\Omega_L)$ and all $l \in L_2(S_T, V_2^*)$. Moreover, we have the estimates

$$\begin{aligned} \|x_1\|^2_{L_2(S_T,V_2)} &\leq e^{k_0T}\left(\tfrac{1}{m}\|x_{10}\|^2_{L_2(\Omega_L)} + \tfrac{1}{m^2}\|l\|^2_{L_2(S_T,V_2^*)}\right), \\ \|\dot{x}_1\|_{L_2(S_T,V_2^*)} &\leq \|l\|_{L_2(S_T,V_2^*)} + M\|u\|_{L_2(S_T,V_2)}. \end{aligned} \tag{22}$$

For a proof of this theorem see [25].

This theorem should be applied to our problem. Let us define

$$\begin{aligned} a(u, v) &= \int_0^L (\kappa_1\partial u(\xi)\partial v(\xi) - \kappa_2\partial u(\xi)v(\xi) - \kappa_3 u(\xi)v(\xi))d\xi - (\kappa_1\kappa_{10} + \kappa_4\kappa_{11})\, u(0)v(0), \\ \langle l(t), v\rangle &= \int_0^L (\bar{g}_1(t,\xi)v + \kappa_5 x_2(t,\xi)v - \kappa_4 x_2(t,\xi)\partial v(\xi))d\xi - \kappa_1\bar{f}_1(t) - \kappa_4\bar{f}_2(t). \end{aligned}$$

In order to show that this is the correct energetic prolongation of the parabolic differential equation, assume $u \in V_2$ to hold and integrate by parts,

$$\begin{aligned} 0 =& - a(u, v) + \langle l, v\rangle \\ =& - \int_0^L (\kappa_1\partial u(\xi)\partial v(\xi) - \kappa_2\partial u(\xi)v(\xi) - \kappa_3 u(\xi)v(\xi))d\xi + (\kappa_1\kappa_{10} + \kappa_4\kappa_{11})\, u(0)v(0) \\ & + \int_0^L (\bar{g}_1(t)v + \kappa_5 x_2(t)v(\xi) - \kappa_4\bar{x}_2(t,\xi)\partial v(\xi))d\xi - \kappa_1\bar{f}_1(t) - \kappa_4\bar{f}_2(t) \\ =& \int_0^L (\kappa_1\partial^2 u(\xi) + \kappa_2\partial u(\xi)v(\xi) + \kappa_3 u(\xi))v(\xi)d\xi - \kappa_1\partial u(L)v(L) + \kappa_1\partial u(0)v(0) \\ & + (\kappa_1\kappa_{10} + \kappa_4\kappa_{11})\, u(0)v(0) + \int_0^L (\bar{g}_1(t,\xi)v(\xi) + \kappa_5\bar{x}_2(t,\xi)v(\xi) + \kappa_4\partial\bar{x}_2(t,\xi)v(\xi))d\xi \\ & - \kappa_4\bar{x}_2(t,L)v(L) + \kappa_4\bar{x}_2(t,0)v(0) - \kappa_1\bar{f}_1(t) - \kappa_4\bar{f}_2(t) \\ =& \int_0^L (\kappa_1\partial^2 u(\xi) + \kappa_2\partial u(\xi)v(\xi) + \kappa_3 u(\xi))v(\xi)d\xi + \kappa_1(\kappa_{10}u(0) + \partial u(0) - \bar{f}_1(t))v(0) \\ & + \int_0^L (\bar{g}_1(t,\xi)v(\xi) + \kappa_5\bar{x}_2(t,\xi)v(\xi) + \kappa_4\partial x_2(t,\xi)v(\xi))d\xi \\ & + \kappa_4[(x_2(t,0) + \kappa_{11}u(0) - \bar{f}_2(t))]v(0). \end{aligned}$$

The integral part yields immediately the right hand side of the differential equation. In the solution, the term in brackets disappears thus providing the Robin boundary condition for x_1.

It remains to show the boundedness of $a(\cdot,\cdot)$ and l and to show Gårding's inequality for $a(\cdot,\cdot)$. Obviously, the estimate $|a(w, v)| \leq M\|w\|_{V_2}\|v\|_{V_2}$ is true for all $u, v \in V_2$ for some constant M because of the continous imbedding of $C(\Omega_L)$ in V_2. By using integration by parts one can estimate

$$
\begin{aligned}
a(v,v) &= \int_0^L (\kappa_1 \partial v(\xi)\partial v(\xi) - \kappa_2 \partial v(\xi) v(\xi) - \kappa_3 v(\xi)v(\xi))d\xi - (\kappa_1\kappa_{10} + \kappa_4\kappa_{11})\, v(0)v(0) \\
&\geq \kappa_1 \|v\|_{V_2}^2 - |\kappa_2| \|v\|_{V_2} \|v\|_{L_2(\Omega_L)} - |\kappa_3| \|v\|_{L_2(\Omega_L)}^2 - \tfrac{|\kappa_1\kappa_{10}+\kappa_4\kappa_{11}|}{2} \|v\|_{V_2}\|v\|_{L_2(\Omega_L)} \\
&\geq \kappa_1 \|v\|_{V_2}^2 - \left(|\kappa_2| + \tfrac{|\kappa_1\kappa_{10}+\kappa_4\kappa_{11}|}{4}\right)(\varepsilon \|v\|_{V_2}^2 + \tfrac{1}{\varepsilon}\|v\|_{L_2(\Omega_L)}^2) - |\kappa_3| \|v\|_{L_2(\Omega_L)}^2 \\
&= \left(\kappa_1 - \varepsilon \tfrac{|\kappa_1\kappa_{10}+\kappa_4\kappa_{11}|}{4}\right)\|v\|_{V_2}^2 - \left(|\kappa_3| + |\kappa_2| + \tfrac{|\kappa_1\kappa_{10}+\kappa_4\kappa_{11}|}{4\varepsilon}\right)\|v\|_{L_2(\Omega_L)}^2
\end{aligned}
$$

for all $\varepsilon \in \mathbb{R}^+$. Choosing ε sufficiently small, the desired estimate arises.

The functional $l(t)$ can be estimated

$$
|\langle l(t), v\rangle| \leq C\big(\|x_2(t)\|_{L_2(\Omega_L)} + \|\bar{g}_1(t)\|_{L_2(\Omega_L)} + |\bar{f}_1(t)| + |\bar{f}_2(t)|\big)\|v\|_{V_2}.
$$

This provides us with

$$
\|l\|_{L_2(S_T,V_2^*)} \leq C\big(\|x_2\|_{L_2(S_T,L_2(\Omega_L))} + \|\bar{g}_1\|_{L_2(S_T,L_2(\Omega_L))} + \|\bar{f}_1\|_{L_2(S_T)} + \|\bar{f}_2\|_{L_2(S_T)}\big).
$$

Using the estimate for the solution of a parabolic problem (22), we obtain the relations

$$
\begin{aligned}
\|x_1\|_{L_2(S_T,V_2)} \leq C\,\big(&\|x_{10}\|_{L_2(S_T)} + \|x_2\|_{L_2(S_T,L_2(\Omega_L))} + \|\bar{g}_1\|_{L_2(S_T,L_2(\Omega_L))} \\
&+ \|\bar{f}_1\|_{L_2(S_T)} + \|\bar{f}_2\|_{L_2(S_T)}\big), \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\|x_1(t)\|_{L_2(\Omega_L)} \leq C\,\big(&\|x_{10}\|_{L_2(S_T)} + \|x_2\|_{L_2(S_T,L_2(\Omega_L))} + \|\bar{g}_1\|_{L_2(S_T,L_2(\Omega_L))} \\
&+ \|\bar{f}_1\|_{L_2(S_T)} + \|\bar{f}_2\|_{L_2(S_T)}\big). \qquad (24)
\end{aligned}
$$

For the last inequality, we used the continuous imbedding of $W(S_T)$ into $C(S_T, L_2(\Omega_L))$.

2. The hyperbolic equation reads $\dot{x}_2(t,\xi) = \kappa_8 \partial x_2(t,\xi) + \kappa_6 \partial x_1(t,\xi) + \bar{g}_2(t,\xi)$ subject to the boundary and initial conditions $x_2(0,\xi) = x_{20}(\xi), \quad x_2(t,0)) = \bar{f}_2 - \kappa_{11}x_1(t,0)$. For the following derivations it becomes important to note that $\kappa_8 < 0$. Let $H^1(\Omega_L)^*$ be the dual space of $H^1(\Omega_L)$ with respect to $L_2(\Omega_L)$.

   We require only $x_2 \in \{u \in L_2(S_T, H^1(\Omega_L)) | u' \in L_2(S_T, H^1(\Omega_L)^*)\}$. Since $x_1 \in W(S_T)$ and $\bar{g}_2 \in L_2(S_T, L_2(\Omega_L))$, it holds even $x_2 \in H^1(S_T, H^1(\Omega_L))$. Then we have

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|x_2(t)\|_{L_2(\Omega_L)}^2 = \left\langle \tfrac{d}{dt}x_2(t), x_2(t)\right\rangle_{L_2(\Omega_L)} \\
&= \int_0^L (\kappa_8 \partial x_2(t,\xi) + \kappa_9 x_2(t,\xi) + \kappa_6 \partial x_1(t,\xi) + \kappa_7 x_1(t,\xi) + \bar{g}_2) x_2(t,\xi) d\xi \\
&= \tfrac{\kappa_8}{2} x_2^2(t,L) - \tfrac{\kappa_8}{2} x_2^2(t,0) + \int_0^L (\bar{g}_2(t,\xi) + \kappa_9 x_2(t,\xi) + \kappa_6 \partial x_1(t,\xi) + \kappa_7 x_1(t,\xi))\, x_2(t,\xi) d\xi \\
&\leq \tfrac{-\kappa_8}{2}\|C_0 x_2\|_{L_2(S_T)}^2 + \big(\|\bar{g}_2(t)\|_{L_2(S_T)} + |\kappa_9| \|x_2(t)\|_{L_2(\Omega_L)} + C\|x_1(t)\|_{V_2}\big)\,\|x_2(t)\|_{L_2(\Omega_L)} \\
&\leq \tfrac{-\kappa_8}{2}\|\bar{f}_2 - \kappa_{11}C_0 x_1\|_{L_2(S_T)}^2 + |\kappa_9| \|x_2(t)\|_{L_2(\Omega_L)}^2 + (\|\bar{g}_2(t)\|_{L_2(\Omega_L)} + C\|x_1(t)|_{V_2})\|x_2(t)\|_{L_2(\Omega_L)} \\
&\leq C\left(\|\bar{f}_2\|_{L_2(S_T)}^2 + \|C_0 x_1\|_{L_2(S_T)}^2 + \left(\|\bar{g}_2(t)\|_{L_2(\Omega_L)}^2 + \|x_1(t)\|_{V_2}^2\right) + (|\kappa_9|+1)\|x_2(t)\|_{L_2(\Omega_L)}^2\right).
\end{aligned}
$$

   Sorting all expressions,

$$
\tfrac{d}{dt}\|x_2(t)\|_{L_2(\Omega_L)}^2 \leq k_1(1+|\kappa_9|)\|x_2(t)\|_{L_2(\Omega_L)}^2 + k_2\left(2\|x_1(t)\|_{V_2}^2 + \|\bar{f}_2\|_{L_2(S_T)}^2 + \|\bar{g}_2(t)\|_{L_2(\Omega_L)}^2\right).
$$

   Using Gronwall's inequality we obtain the estimate

$$
\begin{aligned}
\|x_2(t)\|_{L_2(\Omega_L)}^2 &\leq e^{k_1 t}\left(\|x_{20}\|_{L_2(\Omega_L)}^2 + k_2 \int_0^t \left(2\|x_1(\tau)\|_{V_2}^2 + \|\bar{f}_2\|_{L_2(S_T)}^2 + \|\bar{g}_2(\tau)\|_{L_2(\Omega_L)}^2\right) d\tau\right) \\
&\leq e^{k_1 t}\Big(\|x_{20}\|_{L_2(\Omega_L)}^2 + k_2\big(2\|x_1\|_{L_2(S_T,V_2)}^2 + \|\bar{f}_2\|_{L_2(S_T)}^2 + \|\bar{g}_2\|_{L_2(S_T,L_2(\Omega_L))}^2\big)\Big). \qquad (25)
\end{aligned}
$$

Combining this estimate with (23), we obtain that there exists a constant $C > 0$ with

$$\begin{aligned}\|x_2(t)\|^2_{L_2(\Omega_L)} \le & e^{k_1 t} C \Big(\|x_{20}\|^2_{L_2(\Omega_L)} + k_2 \Big(2\|x_2\|^2_{L_2(S_T, L_2(\Omega_L))} + \|\bar{f}_1\|^2_{L_2(S_T)} \\ & + \|\bar{f}_2\|^2_{L_2(S_T)} + \|\bar{g}_1\|^2_{L_2(S_T, L_2(\Omega_L))} + \|\bar{g}_2\|^2_{L_2(S_T, L_2(\Omega_L))} \Big) \Big) .\end{aligned} \tag{26}$$

Furthermore using Gronwall's inequality then leads to an estimate

$$\begin{aligned}\|x_2(t)\|^2_{L_2(\Omega_L)} \le & C \Big(\|x_{20}\|^2_{L_2(\Omega_L)} + \|\bar{f}_1\|^2_{L_2(S_T)} + \|\bar{f}_2\|^2_{L_2(S_T)} \\ & + \|\bar{g}_1\|^2_{L_2(S_T, L_2(\Omega_L))} + \|\bar{g}_2\|^2_{L_2(S_T, L_2(\Omega_L))} \Big) .\end{aligned} \tag{27}$$

3. The estimates (24) and (27) lead to

$$\begin{aligned} & \|x_1(t)\|^2_{L_2(\Omega_L)} + \|x_2(t)\|^2_{L_2(\Omega_L)} \\ \le & C \Big(\|x_{10}\|^2_{L_2(\Omega_L)} + \|x_{20}\|^2_{L_2(\Omega_L)} + \|\bar{f}_1\|^2_{L_2(S_T)} + \|\bar{f}_2\|^2_{L_2(S_T)} \\ & + \|\bar{g}_1\|^2_{L_2(S_T, L_2(\Omega_L))} + \|\bar{g}_2\|^2_{L_2(S_T, L_2(\Omega_L))} + \|x_2\|_{L_2(S_T, L_2(\Omega_L))} \Big) .\end{aligned} \tag{28}$$

for some constant independent of $t \in S_T$, and $\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$, $\bar{g}_1$, $\bar{g}_2$, x_{10} and x_{20}. Thus we have for $x(t) = [x_1(t), x_2(t)]$ that

$$\begin{aligned}\|x(t)\|^2_{L_2(\Omega_L)^2} \le & C \Big(\|x_{10}\|^2_{L_2(\Omega_L)} + \|x_{20}\|^2_{L_2(\Omega_L)} + \|\bar{f}_1\|^2_{L_2(S_T)} + \|\bar{f}_2\|^2_{L_2(S_T)} \\ & + \|\bar{g}_1\|^2_{L_2(S_T, L_2(\Omega_L))} + \|\bar{g}_2\|^2_{L_2(S_T, L_2(\Omega_L))} + \|x\|_{L_2(S_T, L_2(\Omega_L))^2} \Big) .\end{aligned} \tag{29}$$

Then, Gronwall's inequality leads to a new constant $C > 0$ such that

$$\begin{aligned}\|x(t)\|^2_{L_2(\Omega_L)^2} \le & C \Big(\|x_{10}\|^2_{L_2(\Omega_L)} + \|x_{20}\|^2_{L_2(\Omega_L)} + \|\bar{f}_1\|^2_{L_2(S_T)} + \|\bar{f}_2\|^2_{L_2(S_T)} \\ & + \|\bar{g}_1\|^2_{L_2(S_T, L_2(\Omega_L))} + \|\bar{g}_2\|^2_{L_2(S_T, L_2(\Omega_L))} \Big) .\end{aligned} \tag{30}$$

which is the desired result.

□

Corollary 5.4 *Let $[x_1, x_2]^T$ be a solution of the initial-boundary value problem (18) subject to the boundary conditions (19) with $\bar{f}_1, \bar{f}_2 \in H_1(S_T)$, $\bar{f}_3 \equiv 0$ and inhomogeneity $\bar{g}_1 \in H^1(S_T, L_2(\Omega_L))$, $\bar{g}_2 \in H^1(S_T, L_2(\Omega_L))$ and the initial condition (20) with $x_{10} \in H^2(\Omega_L)$, $x_{20} \in H^1(\Omega_L)$ with $\partial x_{10}(0) + \kappa_{10} x_{10}(0) = \bar{f}_1(0)$, $\kappa_{11} x_{10}(0) + x_{20}(0) = \bar{f}_2(0)$ and $x_{20}(L) = 0$. Then it holds*

$$\begin{aligned}\|x_1(t)\|^2_{V_2} + \|x_2(t)\|^2_{H^1(\Omega_L)} \le & C \Big(\|x_{10}\|^2_{H^2(\Omega_L)} + \|x_{20}\|^2_{H^1(\Omega_L)} + \|\bar{g}_1\|^2_{H^1(S_T, L_2(\Omega_L))} \\ & + \|\bar{g}_2\|^2_{H^1(S_T, L_2(\Omega_L))} + \|\bar{f}_1\|^2_{L_2(S_T)} + \|\bar{f}_2\|^2_{L_2(S_T)} \Big) .\end{aligned}$$

for some constant $C \in \mathbb{R}^+$ independent of the inhomogenities but which may depend on T, L and κ_i.

Proof:
By differentiation of the partial differential equations and the boundary conditions it is obvious that $\bar{x}_1 = \dot{x}_1$, $\bar{x}_2 = \dot{x}_2$ solve the equation

$$\begin{bmatrix} \dot{\bar{x}}_1(t,\xi) \\ \dot{\bar{x}}_2(t,\xi) \end{bmatrix} = \begin{bmatrix} \kappa_1 \partial^2 + \kappa_2 \partial + \kappa_3 & \kappa_4 \partial + \kappa_5 \\ \kappa_6 \partial + \kappa_7 & \kappa_8 \partial + \kappa_9 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t,\xi) \\ \bar{x}_2(t,\xi) \end{bmatrix} + \begin{bmatrix} \dot{\bar{g}}_1(t,\xi) \\ \dot{\bar{g}}_2(t,\xi) \end{bmatrix},$$

subject to the boundary conditions $\frac{\partial}{\partial\xi}\bar{x}_1(t,0)+\kappa_{10}\bar{x}_1(t,0)=\dot{\bar{f}}_1(t)$, $\kappa_{11}\bar{x}_1(t,0)+\bar{x}_2(t,0)=\dot{\bar{f}}_2(t)$, $\bar{x}_1(t,L)=0$ and initial conditions

$$\begin{bmatrix}\bar{x}_1(0,\xi)\\ \bar{x}_2(0,\xi)\end{bmatrix}=\begin{bmatrix}\kappa_1\partial^2+\kappa_2\partial+\kappa_3 & \kappa_4\partial+\kappa_5\\ \kappa_6\partial+\kappa_7 & \kappa_8\partial+\kappa_9\end{bmatrix}\begin{bmatrix}x_{10}(\xi)\\ x_{20}(\xi)\end{bmatrix}+\begin{bmatrix}\bar{g}_1(0,\xi)\\ \bar{g}_2(0,\xi)\end{bmatrix}.$$

Then Lemma 5.3 leads to

$$\begin{aligned}\|\dot{x}_1(t)\|^2_{L_2(\Omega_L)}+\|\dot{x}_2(t)\|^2_{L_2(\Omega_L)}\leq & C\Big(\|\bar{x}_1(0,\cdot)\|^2_{L_2(\Omega_L)}+\|\bar{x}_2(0,\cdot)\|^2_{L_2(\Omega_L)}+\|\bar{f}_1\|^2_{H^1(S_T)}+\|\bar{f}_2\|^2_{H^1(S_T)}\\ &+\|\bar{g}_1\|^2_{H_1(S_T,L_2(\Omega_L))}+\|\bar{g}_2\|^2_{H_1(S_T,L_2(\Omega_L))}\Big).\end{aligned}\tag{31}$$

However, from the given system of differential equations, we obtain that there exists a constant $\bar{C}>0$ such that

$$\begin{aligned}\|x_1(t)\|^2_{H^2(\Omega_L)}+\|x_2(t)\|^2_{H^1(\Omega_L)}\leq & C\cdot\Big(\|\dot{x}_1(t)\|^2_{L_2(\Omega_L)}+\|\dot{x}_2(t)\|^2_{L_2(\Omega_L)}+\|\bar{g}_1(t)\|^2_{L_2(\Omega_L)}\\ &+\|\bar{g}_2(t)\|^2_{L_2(\Omega_L)}+|\bar{f}_1(t)|+|\bar{f}_2(t)|^2\Big).\end{aligned}\tag{32}$$

Now a combination of (31) and (32) together with the continuity of evaluation in $H^1(S_T)$ leads to the desired result. □

It is now tempting to use a homogenization technique in order to handle time-dependent Dirichlet boundary conditions, $x_1(t,L)=\bar{f}_3(t)$. Define a new function $\hat{x}_1$ by $\hat{x}_1(t,\xi)=x_1(t,\xi)-\bar{f}_3(t)$. If one replaces x_1 by $\hat{x}_1$ one gets a system of the type considered in Lemma 5.3 with a homogeneous Dirichlet boundary condition for $\hat{x}_1$. However, the right-hand side $\bar{g}_1$ is now replaced by $\bar{g}_1-\dot{\bar{f}}_3$. The estimate of Lemma 5.3 indicates a perturbation index of 2 with respect to time. The next lemma shows that this is clearly an overestimate.

Lemma 5.5 *Let $[x_1,x_2]^T$ be the solution of the initial-boundary value problem (18) with $\bar{g}_1=\bar{g}_2=0$, boundary conditions (19) with $\bar{f}_1=\bar{f}_2=0$, $\bar{f}_3\in H^1(S_T)$ and initial value (20) with $\bar{x}_{10},\bar{x}_{20}\in H^1(\Omega_L)$, $\bar{x}_{10}(L)=\bar{f}_3(0)$ and $\bar{x}_{10}(0)=0$. Then there exists a $T>0$ such that*

$$\|x_1(t)\|^2_{H^1(\Omega_L)}+\|x_2(t)\|^2_{H^1(\Omega_L)}\leq C\big(\|\bar{x}_{10}\|^2_{H^1(\Omega_L)}+\|\bar{x}_{20}\|^2_{H^1(\Omega_L)}+\|\bar{f}_3\|_{H^1(S_T)}\big)\tag{33}$$

for some constant independent of $\bar{f}_3$ and the initial value.

Proof:
Without loss of generality, we assume that $\kappa_2=\kappa_3=0$. This can be reached by a transformation $\bar{x}_i(t,\xi)=e^{\zeta t+\eta\xi}x_i(t,\xi)$ for $i=1,2$ and some suitable $\zeta,\eta\in\mathbb{R}$.

1. First we show an estimate

$$\|C_0x_1\|^2_{H^1(S_T)}\leq c\left(\|\bar{f}_3\|^2_{H^1(S_T)}+\|x_{10}\|^2_{H^1(\Omega_L)}+\|x_1\|^2_{L_2(S_T,H^1(\Omega_L))}+\|x_2\|^2_{L_2(S_T,H^1(\Omega_L))}\right).\tag{34}$$

   Consider the first equation in frequency domain, that is

$$s\widehat{x}_1(s,\xi)-x_{10}(\xi)=\kappa_1\partial^2\widehat{x}_1(s,\xi)+\kappa_4\partial\widehat{x}_2(s,\xi)+\kappa_5\widehat{x}_2(s,\xi).\tag{35}$$

   with boundary condition $\widehat{x}_1(s,L)=\widehat{\bar{f}}_3(s)$. Solving (35) for $\widehat{x}_1(s,\xi)$ and evaluating at $\xi=0$, we obtain for $\gamma(s,\xi):=\frac{\sqrt{s}(L-\xi)}{\sqrt{\kappa_1}}$ that

$$\begin{aligned}\widehat{x}_1(s,0)=&\frac{\widehat{\bar{f}}_3(s)}{\cosh(\gamma(s,0))}-\frac{\tanh(\gamma(s,0))}{\sqrt{\kappa_1 s}}\partial\widehat{x}_1(s,0)\\ &+\frac{\int_0^L\sinh\gamma(s,\xi)(\kappa_4\partial\widehat{x}_2(s,\xi)+\kappa_5\widehat{x}_2(s,\xi)+x_{10}(\xi))d\xi}{\sqrt{s\kappa_1}\cosh\left(\gamma(s,0)\right)}.\end{aligned}$$

Integrating by parts and making use of the further boundary relations $\partial\widehat{x}_1(s,0) = -\kappa_{10}\widehat{x}_1(s,0)$, $\widehat{x}_2(s,0) = -\kappa_{11}\widehat{x}_1(s,0)$ as well as $x_{10}(L) = \bar{f}_3(0)$, $x_{10}(0) = 0$ yields

$$\begin{aligned}&\left(1 - \tfrac{\sqrt{s}\kappa_4\kappa_{11} - \sqrt{s}\kappa_1\kappa_{10} + \sqrt{\kappa_1}\kappa_5\kappa_{11}}{\sqrt{\kappa_1}s}\tanh(\gamma(s,0))\right) s\widehat{x}_1(s,0)\\ =&\frac{\widehat{s\bar{f}_3}(s) - \bar{f}_3(0)}{\cosh(\gamma(s,0))} + \frac{\int_0^L \cosh(\gamma(s,\xi))(\kappa_4 s\widehat{x}_2(s,\xi) + \kappa_1\kappa_5\partial\widehat{x}_2(s,\xi) + \kappa_1\partial x_{10}(\xi))d\xi}{\kappa_1\cosh(\gamma(s,0))}\end{aligned}$$

and thus

$$\begin{aligned}&\left(1 - \tfrac{\sqrt{s}\kappa_4\kappa_{11} - \sqrt{s}\kappa_1\kappa_{10} + \sqrt{\kappa_1}\kappa_5\kappa_{11}}{\sqrt{\kappa_1}s}\tanh(\gamma(s,0))\right) s\widehat{x}_1(s,0)\\ =&\frac{\widehat{s\bar{f}_3}(s) - \bar{f}_3(0)}{\cosh(\gamma(s,0))} + \frac{\int_0^L \cosh(\gamma(s,\xi))(\sqrt{\kappa_1}\partial x_{10}(\xi) - \frac{\kappa_4}{\kappa_1}x_{20}(\xi))d\xi}{\cosh(\gamma(s,0))}\\ &+ \frac{\int_0^L \cosh(\gamma(s,\xi))(\kappa_4(s\widehat{x}_2(s,\xi) - x_{20}(\xi)) + \kappa_5\kappa_1\partial\widehat{x}_2(s,\xi))d\xi}{\kappa_1\cosh(\gamma(s,0))}.\end{aligned}$$

We now introduce the function $g(s) := \frac{\sqrt{s}\kappa_4\kappa_{11} - \sqrt{s}\kappa_1\kappa_{10} + \sqrt{\kappa_1}\kappa_5\kappa_{11}}{\sqrt{\kappa_1}s}\tanh(\gamma(s,\xi))$. Then we have $g \in \mathcal{H}_\infty(\mathbb{C}^+_{\omega_1}, \mathbb{R}, \mathbb{R})$ for some $\omega \in \mathbb{R}$ and furthermore

$$\lim_{\substack{\sigma\to\infty\\ \sigma\in\mathbb{R}}} \sup_{\omega\in\mathbb{R}} |g(\sigma + i\omega)| = 0,$$

there exists a ω_2 such that $(1 + g(s))^{-1} \in \mathcal{H}_\infty(\mathbb{C}^+_{\omega_2}, \mathbb{R}, \mathbb{R})$. Furthermore, we introduce the functions $G_1 : \mathbb{C}^+_0 \to \mathbb{R}$, $G_2 : \mathbb{C}^+_0 \to \mathcal{B}(L_2(\Omega_L), \mathbb{R})$ and $G_3 : L_2(\Omega_L) \to \mathcal{H}_2(\mathbb{C}^+_0)$ with

$$\begin{aligned}G_1(s) :=& \frac{1}{\cosh(\gamma(s,0))},\\ G_2(s)x :=& \frac{\int_0^L \cosh(\gamma(s,\xi))x(\xi)d\xi}{\kappa_1\cosh(\gamma(s,0))},\\ (G_3x)(s) :=& \frac{\int_0^L \cosh(\gamma(s,\xi))x(\xi)d\xi}{\cosh(\gamma(s,0))}.\end{aligned}$$

It can be seen that $G_1 \in \mathcal{H}_\infty(\mathbb{C}^+_0, \mathbb{R}, \mathbb{R})$, $G_2 \in \mathcal{H}_\infty(\mathbb{C}^+_0, L_2(\Omega_L), \mathbb{R})$. Moreover, the boundedness and well-definedness of G_2 follows from the fact that $y := \mathcal{L}^{-1}G_2x$ is the function satisfying $\dot{x}(t,\xi) = \kappa_1\partial^2 x(t,\xi)$, $x(t,L) = \partial x(t,0) = 0$, $x(0,\xi) = \partial x_{110}$, $y(t) = x(t,0)$ and the well-posedness property of a heat equation with Dirichlet-observation [5]. Altogether, we have

$$\begin{aligned}\mathcal{L}(\dot{x}_1)(s) = \tfrac{1}{1-g(s)}\Big(&G_1(s)\mathcal{L}(\dot{\bar{f}}_3)(s) + G_2(s)\left(\kappa_4\mathcal{L}(\dot{x}_2)(s) + \kappa_5\mathcal{L}(\partial x_2)(s)\right)\\ &+ G_3\left(\left(\sqrt{\kappa_1}\partial x_{10} - \tfrac{\kappa_4}{\kappa_1}x_{20}\right)(s)\right)\Big).\end{aligned}$$

Parseval's identity then leads to an estimate

$$\|C_0x_1\|^2_{H^1(S_T)} \le c\left(\|\bar{f}_3\|^2_{H^1(S_T)} + \|x_{10}\|^2_{H^1(\Omega_L)} + \|x_2\|^2_{L_2(S_T,H^1(\Omega_L))} + \|x_2\|^2_{H^1(S_T,L_2(\Omega_L))}\right). \tag{36}$$

From the equation $\dot{x}_2(t,\xi) = \kappa_6\partial x_1(t,\xi) + \kappa_7 x_1(t,\xi) + \kappa_8\partial x_2(t,\xi) + \kappa_9 x_2(t,\xi)$, we can derive an estimate

$$\|x_2\|^2_{H^1(S_T,L_2(\Omega_L))} \le C\left(\|x_1\|^2_{L_2(S_T,H^1(\Omega_L))} + \|x_2\|^2_{L_2(S_T,H^1(\Omega_L))}\right).$$

A combination with (36) then leads to (34).

2. Now we show the inequality (33). We assume to have a sufficiently smooth solution $x = [x_1, x_2]$. Let $t \in S_T$ such that
$$\max_{\tau \in S_T} \|x(\tau)\|_{H^1(\Omega_L)^2} = \|x(\tau)\|_{H^1(\Omega_L)^2}. \tag{37}$$
Then for
$$\mathfrak{U} := \begin{bmatrix} \kappa_1 \partial^2 + & \kappa_4 \partial + \kappa_5 \\ \kappa_6 \partial + \kappa_7 & \kappa_8 \partial + \kappa_9 \end{bmatrix}$$
we obtain
$$\begin{aligned} \|\partial x(t)\|^2_{L_2(\Omega_L)^2} =& \|\partial x(0)\|^2_{L_2(\Omega_L)^2} + 2 \int_0^t \langle \partial x(\tau), \partial \dot{x}(\tau) \rangle_{L_2(\Omega_L)^2} \, d\tau \\ =& \|\partial x(0)\|^2_{L_2(\Omega_L)^2} + 2 \int_0^t \langle \partial x(\tau), \partial \mathfrak{U} x(\tau) \rangle_{L_2(\Omega_L)^2} \, d\tau. \end{aligned} \tag{38}$$
Especially, we have
$$\begin{aligned} &\langle \partial x(\tau), \partial \mathfrak{U} x(\tau) \rangle_{L_2(\Omega_L)^2} \\ =& \kappa_1 \left\langle \partial x_1(\tau), \partial^3 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_4 \left\langle \partial x_1(\tau), \partial^2 x_2(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_5 \langle \partial x_2(\tau), \partial x_1(\tau) \rangle_{L_2(\Omega_L)} \\ &+ \kappa_6 \left\langle \partial x_2(\tau), \partial^2 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_8 \left\langle \partial x_2(\tau), \partial^2 x_2(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_9 \langle \partial x_2(\tau), \partial x_2(\tau) \rangle_{L_2(\Omega_L)} \\ =& -\kappa_1 \left\langle \partial^2 x_1(\tau), \partial^2 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_1 \partial^2 x_1(\tau, L) \partial x_1(\tau, L) - \kappa_1 \partial^2 x_1(\tau, 0) \partial x_1(\tau, 0) \\ &- \kappa_4 \left\langle \partial^2 x_1(\tau), \partial x_2(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_4 \partial x_1(\tau, L) \partial x_2(\tau, L) - \kappa_4 \partial x_1(\tau, 0) \partial x_2(\tau, 0) \\ &+ \kappa_6 \left\langle \partial x_2(\tau), \partial^2 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \tfrac{\kappa_8}{2} \partial x_2(\tau, L)^2 - \tfrac{\kappa_8}{2} \partial x_2(\tau, L)^2 \\ &+ \kappa_5 \langle \partial x_2(\tau), \partial x_1(\tau) \rangle_{L_2(\Omega_L)} + \kappa_9 \langle \partial x_2(\tau), \partial x_2(\tau) \rangle_{L_2(\Omega_L)}. \end{aligned}$$
Now using $\kappa_1 \partial^2 x_1(\tau, 0) = \dot{x}_1(\tau, 0) - \kappa_4 \partial x_2(\tau, 0)$, $\kappa_1 \partial^2 x_1(\tau, L) = \dot{\bar{f}}_3(\tau) - \kappa_4 \partial x_2(\tau, L)$, and $\partial x_1(\tau, 0) = -\kappa_{10} x_1(\tau, 0)$ we get
$$\begin{aligned} &\langle \partial x(\tau), \partial \mathfrak{U} x(\tau) \rangle_{L_2(\Omega_L)^2} \\ =& -\kappa_1 \left\langle \partial^2 x_1(\tau), \partial^2 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \dot{\bar{f}}_3(\tau) \partial x_1(\tau, L) + \kappa_{10} \dot{x}_1(\tau, 0) x_1(\tau, 0) \\ &- \kappa_4 \left\langle \partial^2 x_1(\tau), \partial x_2(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_6 \left\langle \partial x_2(\tau), \partial^2 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \tfrac{\kappa_8}{2} \partial x_2(\tau, L)^2 - \tfrac{\kappa_8}{2} \partial x_2(\tau, 0)^2 \\ &+ \kappa_5 \langle \partial x_2(\tau), \partial x_1(\tau) \rangle_{L_2(\Omega_L)} + \kappa_9 \langle \partial x_2(\tau), \partial x_2(\tau) \rangle_{L_2(\Omega_L)}. \end{aligned}$$
The boundary relation $\kappa_8 \partial x_2(\tau, 0) = -\kappa_{11} \dot{x}_1(\tau, 0) - \kappa_6 x_1(\tau, 0)$ then leads to
$$\begin{aligned} &\langle \partial x(\tau), \partial \mathfrak{U} x(\tau) \rangle_{L_2(\Omega_L)^2} \\ \leq& -\kappa_1 \left\langle \partial^2 x_1(\tau), \partial^2 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \dot{\bar{f}}_3(\tau) \partial x_1(\tau, L) + \kappa_{10} \dot{x}_1(\tau, 0) x_1(\tau, 0) \\ &- \kappa_4 \left\langle \partial^2 x_1(\tau), \partial x_2(\tau) \right\rangle_{L_2(\Omega_L)} + \kappa_6 \left\langle \partial x_2(\tau), \partial^2 x_1(\tau) \right\rangle_{L_2(\Omega_L)} + \tfrac{\kappa_8}{2} \partial x_2(\tau, L)^2 \\ &- \tfrac{\kappa_8}{2} \left(\kappa_{11} \dot{x}_1(\tau, 0) + \kappa_6 x_1(\tau, 0) \right)^2. \end{aligned}$$
By using $|\partial x_1(\tau, L)| \leq C \|x_2(\tau)\|_{H^2(\Omega_L)}$, $|x_1(\tau, 0)| \leq C \|x_2(\tau)\|_{H^1(\Omega_L)}$ and taking into account that $\kappa_8 < 0$, we obtain
$$\langle \partial x(\tau), \partial \mathfrak{U} x(\tau) \rangle_{L_2(\Omega_L)^2} \leq C \left(|\dot{\bar{f}}_3(\tau)|^2 + |C_0 \dot{x}_1(\tau)|^2 + \|x_1(\tau)\|^2_{H^1(\Omega_L)} + \|x_2(\tau)\|^2_{H^1(\Omega_L)} \right).$$
This can be plugged into (38), which leads to
$$\begin{aligned} \|x_1(t)\|^2_{H^1(\Omega_L)} + \|x_2(t)\|^2_{H^1(\Omega_L)} \leq& C \Big(\|x_{10}\|^2_{H^1(\Omega_L)} + \|x_{20}\|^2_{H^1(\Omega_L)} + \|\bar{f}_3\|^2_{H^1(S_T)} \\ &+ \|C_0 x_1\|^2_{H^1(S_T)} + \|x_1\|^2_{L_2(S_T, H^1(\Omega_L))} + \|x_2\|^2_{L_2(S_T, H^1(\Omega_L))} \Big). \end{aligned}$$

Moreover, the expression $\|C_0x_1\|_{H^1(S_T)}$ can be estimated with (34), and thus we have

$$\begin{aligned}&\|x_1(t)\|^2_{H^1(\Omega_L)}+\|x_2(t)\|^2_{H^1(\Omega_L)}\\ \le& C\left(\|x_{10}\|^2_{H^1(\Omega_L)}+\|x_{20}\|^2_{H^1(\Omega_L)}+\|\bar f_3\|^2_{H^1(S_T)}+\|x_2\|^2_{L_2(S_T,H^1(\Omega_L))}+\|x_2\|^2_{L_2(S_T,H^1(\Omega_L))}\right).\end{aligned}$$

Due to (37), we have

$$\|x_1\|^2_{L_2(S_T,H^1(\Omega_L))}+\|x_1\|^2_{L_2(S_T,H^1(\Omega_L))}\le Ct\left(\|x_1(t)\|^2_{H^1(\Omega_L)}+\|x_1(t)\|^2_{H^1(\Omega_L)}\right).$$

Thus we obtain for small enough T that

$$\|x_1(t)\|^2_{H^1(\Omega_L)}+\|x_2(t)\|^2_{H^1(\Omega_L)}\le \tfrac{C}{1-CT}\left(\|x_{10}\|^2_{H^1(\Omega_L)}+\|x_{20}\|^2_{H^1(\Omega_L)}+\|\bar f_3\|^2_{H^1(S_T)}\right).$$

□

Summarizing, we can now formulate the following result.

Theorem 5.6 *Let Assumption 5.1be valid and let $\bar f_i\in H^1(S_T)$, $\bar g_j\in C^1(S_T,L_2(\Omega_L))$ for $i=1,2,3$ and $j=1,2$, and initial values $x_{10}\in H^1(\Omega_L)$, $x_{20}\in H^1(\Omega_L)$ with $\kappa_{11}x_{10}(0)+x_{20}(0)=\bar f_2(0)$, $x_{10}(L)=\bar f_3(0)$, the system (18) with boundary condition (19) and initial data (20) possesses a unique solution $[x_1,x_2]^T\in C(S_T,H^1(\Omega_T)^2)$. Moreover, there exists a constant $C\in\mathbb{R}^+$ such that for all data with the above properties holds*

$$\begin{aligned}\|x_1(t)\|^2_{H^1(\Omega_L)}+\|x_2(t)\|^2_{H^1(\Omega_L)}\le& C\Big(\|x_{10}\|^2_{H^1(\Omega_L)}+\|x_{20}\|^2_{H^1(\Omega_L)}+\|\bar g_1\|^2_{H^1(S_T,L_2(\Omega_L))}\\ &+\|\bar g_2\|^2_{H^1(S_T,L_2(\Omega_L))}+\|\bar f_1\|^2_{H^1(S_T)}+\|\bar f_2\|^2_{H^1(S_T)}+\|\bar f_3\|^2_{H^1(S_T)}\Big).\end{aligned}\tag{39}$$

Proof:
We split the problem into the following two parts

(i)

$$\begin{bmatrix}\dot x_{11}(t,\xi)\\ \dot x_{21}(t,\xi)\end{bmatrix}=\begin{bmatrix}\kappa_1\partial^2+\kappa_2\partial+\kappa_3 & \kappa_4\partial+\kappa_5\\ \kappa_6\partial+\kappa_7 & \kappa_8\partial+\kappa_9\end{bmatrix}\begin{bmatrix}x_{11}(t,\xi)\\ x_{21}(t,\xi)\end{bmatrix},\tag{40}$$

with boundary conditions

$$\tfrac{\partial}{\partial\xi}x_1(t,0)+\kappa_{10}x_1(t,0)=\kappa_{11}x_{11}(t,0)+x_{21}(t,0)=0,\quad x_{11}(t,L)=\bar f_3(t)$$

and initial value

$$\begin{bmatrix}x_{11}(0,\xi)\\ x_{21}(0,\xi)\end{bmatrix}=\begin{bmatrix}x_{10}(\xi)-\frac{L-\xi}{L}\frac{(L\kappa_{10}+1)\xi+L}{L}x_{10}(0)-\bar f_1(0)\xi\frac{L-\xi}{L}\\ x_{20}(\xi)+\kappa_{11}x_{10}(0)-\bar f_2(0)\end{bmatrix}.$$

(ii)

$$\begin{bmatrix}\dot x_{12}(t,\xi)\\ \dot x_{22}(t,\xi)\end{bmatrix}=\begin{bmatrix}\kappa_1\partial^2+\kappa_2\partial+\kappa_3 & \kappa_4\partial+\kappa_5\\ \kappa_6\partial+\kappa_7 & \kappa_8\partial+\kappa_9\end{bmatrix}\begin{bmatrix}x_{12}(t,\xi)\\ x_{22}(t,\xi)\end{bmatrix}+\begin{bmatrix}\bar g_1(t,\xi)\\ \bar g_2(t,\xi)\end{bmatrix},\tag{41}$$

with boundary conditions

$$\tfrac{\partial}{\partial\xi}x_{12}(t,0)+\kappa_{10}x_{22}(t,0)=\bar f_1,\quad \kappa_{11}x_{12}(t,0)+x_{22}(t,0)=\bar f_2,\quad x_{12}(t,L)=0$$

and initial value

$$\begin{bmatrix}x_{12}(0,\xi)\\ x_{22}(0,\xi)\end{bmatrix}=\begin{bmatrix}\frac{L-\xi}{L}\frac{(L\kappa_{10}+1)\xi+L}{L}x_{10}(0)+\bar f_1(0)\xi\frac{L-\xi}{L}\\ -\kappa_{11}x_{10}(0)+\bar f_2(0)\end{bmatrix}.$$

By the superposition principle, the solution is given by $x_1 = x_{11} + x_{12}$ and $x_2 = x_{21} + x_{22}$. For the system (i), holds $x_{11}(0,0) = 0$ and $\kappa_{11}x_{11}(0,0) + x_{21}(0,0) = 0$. Hence we can apply Lemma 5.5 in order to obtain the existence of $T, C_1 > 0$ such that for all $t \in S_T$ holds

$$\begin{aligned} &\|x_{11}(t)\|^2_{H^1(\Omega_L)} + \|x_{21}(t)\|^2_{H^1(\Omega_L)} \\ \le &C_1 \left(\|\bar{f}_3\|^2_{H^1(S_T)} + \left\| x_{10} - \tfrac{L-\xi}{L} \tfrac{(L\kappa_{10}+1)\xi+L}{L} x_{10}(0) - \bar{f}_1(0)\xi\tfrac{L-\xi}{L} \right\|^2_{H^1(\Omega_L)} \right. \\ &\quad \left. + \|x_{20} + \kappa_{11}x_{10}(0) - \bar{f}_2(0)\|^2_{H^1(\Omega_L)} \right) \\ \le &\bar{C}_1 \left(\|\bar{f}_1\|^2_{H^1} + \|\bar{f}_2\|^2_{H^1} + \|\bar{f}_3\|^2_{H^1} + \|x_{10}\|^2_{H^1(\Omega_L)} + \|x_{20}\|^2_{H^1(\Omega_L)} \right), \end{aligned} \tag{42}$$

where the existence of $\bar{C}_1$ is guaranteed by the triangular inequality and the continuity of evaluation.
Considering System (ii), we have $x_{12}(0,\xi) \in H^2(\Omega_L)$, $x_{22}(0,\xi) \in H^1(\Omega_L)$ with $\partial x_{12}(0,0) + \kappa_{10}x_{12}(0,0) = \bar{f}_1(0)$ and $\kappa_{11}x_{12}(0,0) + x_{22}(0,0) = 0$ and $x_{12}(0,L) = 0$. Corollary 5.4 then implies the existence of C_2 with

$$\begin{aligned} &\|x_{12}(t)\|^2_{H^1(\Omega_L)} + \|x_{22}(t)\|^2_{H^1(\Omega_L)} \\ \le &C_2 \left(\|\bar{f}_1\|^2_{H^1} + \|\bar{f}_2\|^2_{H^1} + \| \tfrac{L-\xi}{L} \tfrac{(L\kappa_{10}+1)\xi+L}{L} x_{10}(0) + \bar{f}_1(0)\xi\tfrac{L-\xi}{L} \|^2_{H^2(\Omega_L)} \right. \\ &\quad \left. + \| - \kappa_{11}x_{10}(0) + \bar{f}_2(0)\|^2_{H^1(\Omega_L)} \right) \\ \le &\bar{C}_2 \left(\|\bar{f}_1\|^2_{H^1} + \|\bar{f}_2\|^2_{H^1} + \|x_{10}\|^2_{H^1(\Omega_L)} \right), \end{aligned} \tag{43}$$

Now the combination of (42) with (43) leads to (39) for all $t < T$. In order to prove that for all $t > 0$ there exists a $C > 0$ such that (39) holds, we can concatenate the corresponding inequalities shown for limited time intervals. □
Now we are in the position to prove the main result in Theorem 4.1.
Proof:
By defining $x_1(t,\xi) = p(t,\xi)$, $x_2(t,\xi) = p(t,\xi) + \frac{a_{32}}{a_{33}}h(t,\xi)$, an application of Theorem 5.6 with suitable choice of the constants κ_i leads to

$$\begin{aligned} \left\| \begin{bmatrix} p(t,\cdot) \\ h(t,\cdot) \end{bmatrix} \right\|_{H^1(\Omega_L)} \le &C \big(\|p(0,\cdot)\|_{H^1(\Omega_L)} + \|h(0,\cdot)\|_{H^1(\Omega_L)} \\ &+ \left\| \tfrac{a_{33}}{d} g_1(t,\cdot) - \tfrac{a_{33}}{c_{21}d}\partial g_2(t,\cdot) + \tfrac{a_{13}c_{31}}{c_{21}d} g_2(t,\cdot) - \tfrac{a_{13}}{d} g_3(t,\cdot) \right\|_{H^1(S_T,L_2(\Omega_L))} \\ &+ \left\| -\tfrac{a_{32}}{d} g_1(t,\cdot) + \tfrac{a_{32}}{c_{21}d}\partial g_2(t,\cdot) - \tfrac{a_{12}c_{31}}{c_{21}d} g_2(t,\cdot) - \tfrac{a_{12}}{d} g_3(t,\cdot) \right\|_{H^1(S_T,L_2(\Omega_L))} \\ &+ \left\| f_1 + \tfrac{1}{c_{21}} g_2(\cdot,0) \right\|_{H^1(S_T)} + \|f_2\|_{H^1(S_T)} + \|f_3\|_{H^1(S_T)} \Big) \end{aligned}$$

and then the triangular inequality together with continuity of evaluation in H^1 leads to

$$\begin{aligned} \left\| \begin{bmatrix} p(t,\cdot) \\ h(t,\cdot) \end{bmatrix} \right\|_{H^1(\Omega_L)^2} \le &C \big(\|p(0,\cdot)\|_{H^1(\Omega_L)} + \|h(0,\cdot)\|_{H^1(\Omega_L)} \\ &+ \|g_1\|_{H^1(S_T,L_2(\Omega_L))} + \|g_2\|_{H^1(S_T,H^1(\Omega_L))} + \|g_3\|_{H^1(S_T,L_2(\Omega_L))} \\ &+ \|f_1\|_{H^1(S_T)} + \|f_2\|_{H^1(S_T)} + \|f_3\|_{H^1(S_T)} \big). \end{aligned}$$

By furthermore using $F(t,\xi) = \frac{c_{22}}{c_{21}}p(t,\xi) + \frac{b_{22}}{c_{21}}\partial p(t,\xi) + \frac{c_{23}}{c_{21}}h(t,\xi) + \frac{1}{c_{21}}g_2(t,\xi)$, we obtain

$$\|F(t)\|_{L_2(\Omega_L)} \le C \left(\|p(t)\|_{H^1(\Omega_L)} + \|h(t)\|_{H^1(\Omega_L)} + \|g_2\|_{H^1(S_T,H^1(\Omega_L))} \right).$$

Combining this with the estimate for the components $F(t)$ and $p(t)$, the inequality (17) follows immediately. □

6 Conclusions

In this work, we continued the index analysis of the frozen coefficient version of a reduced model of a heat exchanger appearing in a heat pump system. The model consists essentially of the zero Mach-number limit of the compressible Euler equations. This degenerated hyperbolic system turns out to be a coupled hyperbolic/parabolic equation subject to coupled boundary conditions. We have shown that the system has ADAE and (time-) perturbation index 1 while the space pertubation index is two. This is in accordance with numerical experiments.

The perturbation analysis has been done for the practical relevant case of time-dependent boundary conditions. An approach via homogenization of boundary conditions as used for example in [21] would have led to a higher time index. The theory of abstract differential-algebraic equations provided immediately the form of consistent initial values. Even if the underlying boundary control system is not well-posed, perturbation estimates have been derived.

By using the ADAE theory we obtained the estimates immediately in the original physical variables (mass flow rate, pressure, mass speciphic enthalpy).

Numerical experiments indicate that the perturbation results hold for the nonlinear system, too. It would be interested to know if such results can be proved strictly in the nonlinear case. Because of the ill-posedness of the underlying systems linearization arguments will not do.

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