

STABILITY ANALYSIS OF POSITIVE DESCRIPTOR SYSTEMS*

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Dedicated to Richard Varga on the occasion of his 80th birthday

Abstract. In this paper, we discuss stability properties of positive descriptor systems in the continuous-time as well as in the discrete-time case. We present different characterisations of positivity and establish generalised stability criteria for the case of positive descriptor systems. We show that if the spectral projector onto the right finite deflating subspace of the matrix pair (E, A) is non-negative, then all stability criteria for standard positive systems take a comparably simple form in the positive descriptor case. Furthermore, we provide sufficient conditions that guarantee entry-wise non-negativity along with positive semi-definiteness of solutions of generalised projected Lyapunov equations. As an application of the framework established throughout this paper, we exemplarily generalise two criteria for the stability of two switched standard positive systems under arbitrary switching to the descriptor case.

Key words. Positive descriptor systems; stability; generalised projected Lyapunov equations; switched positive systems.

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1. Introduction. We consider linear time-invariant positive descriptor systems in continuous-time

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (1.1a)$$

$$y(t) = Cx(t), \quad (1.1b)$$

and in discrete-time

$$Ex(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (1.2a)$$

$$y(t) = Cx(t), \quad (1.2b)$$

where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ are real constant coefficient matrices. In the continuous-time case the state x , input u and output y are real-valued vector functions. In the discrete-time case x, u and y are real-valued vector sequences. Positive systems arise naturally in many applications such as pollutant transport, chemotaxis, pharmacokinetics, Leontief input-output models, population models and compartmental systems, [2], [4], [5], [7], [14], [20], [24]. In these models, the variables represent concentrations, population numbers of bacteria or cells or, in general, measures that are per se non-negative. Positive standard systems, i.e., where $E = I$, are subject to ongoing research by many authors, [1], [15], [16], [20], [24], [34], [35], [36], [38], [39]. Recent advances on control theoretical issues have been made especially in the positive discrete-time case. Yet, there are still many open problems, especially for standard positive systems in continuous-time. Control theory of descriptor systems without the non-negativity restriction is to a large extent well understood, see, e.g., [17]. Very little is known about positive descriptor systems up to now, however, some properties mainly in the discrete-time case were studied in [8], [9], [10], [24].

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It is well known that stability properties of standard systems, where $E = I$, are closely related to the spectral properties of the system matrix A . If the dynamics of the system, however, is described by an implicit differential or difference equation, then stability properties are determined by the eigenvalues and eigenvectors associated with the matrix pencil $\lambda E - A$, or just the matrix pair (E, A) .

In the case of standard positive systems, classical stability criteria take a simpler form, [20], [24]. In this paper we present generalisations of these stability criteria for the case of positive descriptor systems. It turns out, that if the spectral projector onto the right finite deflating subspace of the matrix pair (E, A) is non-negative, then all stability criteria for standard positive systems take a comparably simple form in the positive descriptor case.

Stability properties and also many other control theoretical issues such as model reduction methods or the quadratic optimal control problem are, furthermore, closely related to the solution of Lyapunov equations, see. e.g., [3], [21], [22], [27], [32]. For descriptor systems, generalised projected Lyapunov equations were presented in [37]. Entry-wise non-negative solutions to standard Lyapunov equations were discussed, e.g., in [18]. In this paper, we provide sufficient conditions that guarantee entry-wise non-negativity of solutions of generalised projected Lyapunov equations.

This paper is organised as follows. In Section 2 we recall fundamental properties of matrix pencils, descriptor systems, projectors and non-negative matrices. In particular, we recall the generalised Perron-Frobenius Theorem for matrix pairs given in [33] that forms the basis for many results in this paper. In Section 3 we give characterisations of positive continuous-time and discrete-time descriptor systems going from the most general one to the case where the spectral projector onto the right finite deflating subspace is non-negative. In the latter case the given characterisation shows a greater correspondence to the standard case and for discrete-time descriptor systems the characterisation was already given in [9]. In Section 4 we generalise the special stability conditions for positive systems from the standard case, see [20], to the descriptor case. In Section 5 we establish conditions for the solutions of the continuous-time and discrete-time generalised projected Lyapunov equations, as introduced in [37], to be entry-wise non-negative. Finally, in Section 6 we exemplarily show how we can use the framework established throughout this paper in order to generalise the results on stability of two standard switched positive systems, see [30], [31], to positive descriptor systems.

2. Preliminaries.

2.1. Matrix pairs. Let $E, A \in \mathbb{R}^{n \times m}$. A matrix pair (E, A) , or matrix pencil $\lambda E - A$, is called *regular* if E and A are square ($n = m$) and $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{R}$. It is called *singular* otherwise. In this paper we only consider square and regular pencils.

A scalar $\lambda \in \mathbb{C}$ is said to be a (*generalised*) *eigenvalue* of the matrix pair (E, A) if $\det(\lambda E - A) = 0$. A vector $x \in \mathbb{C}^n \setminus \{0\}$ such that $(\lambda E - A)x = 0$ is called (*generalised*) *eigenvector* of (E, A) corresponding to λ . If E is singular and $v \in \mathbb{C}^n \setminus \{0\}$, such that $Ev = 0$ holds, then v is called eigenvector of (E, A) corresponding to the eigenvalue ∞ .

The set of all finite eigenvalues is called *finite spectrum* of (E, A) and is denoted by $\sigma_f(E, A)$. In the case that E is invertible, we denote by $\rho(E, A)$ the spectral radius of (E, A) defined by $\rho(E, A) = \max_{\lambda \in \sigma(E, A)} |\lambda|$. If E is singular, then we denote by

$\rho_f(E, A) = \max_{\lambda \in \sigma_f(E, A)} |\lambda|$ the *finite spectral radius* of (E, A) .

A k -dimensional subspace $S_\lambda^{def} \subset \mathbb{C}^n$ is called *right deflating subspace* of (E, A) corresponding to λ , if there exists a k -dimensional subspace $\mathcal{W} \subset \mathbb{C}^n$ such that $ES_\lambda^{def} \subset \mathcal{W}$ and $AS_\lambda^{def} \subset \mathcal{W}$. Let $\lambda_1, \dots, \lambda_p$ be the pairwise distinct finite eigenvalues of (E, A) and let $S_{\lambda_i}^{def}$, $i = 1, \dots, p$ be the corresponding deflating subspaces associated with these eigenvalues. We call the subspace defined by

$$S_f^{def} := S_{\lambda_1}^{def} \oplus \dots \oplus S_{\lambda_p}^{def} \quad (2.1)$$

the *right finite deflating subspace* of (E, A) .

For an eigenvalue λ we denote by $\Re(\lambda)$ its real part. For a vector x we denote by $X := \text{diag}(x)$ a diagonal matrix with diagonal entries $X_{ii} = x_i$.

LEMMA 2.1. *Let (E, A) be a regular matrix pair. Let $\hat{\lambda}$ be chosen such that $\hat{\lambda}E - A$ is non-singular. Then, the matrices*

$$\hat{E} = (\hat{\lambda}E - A)^{-1}E \text{ and } \hat{A} = (\hat{\lambda}E - A)^{-1}A$$

commute.

Proof. See, e.g., [13], [26]. \square

Throughout the paper, we refer to \hat{E}, \hat{A} as defined in Lemma 2.1 independent of the special choice of $\hat{\lambda}$. Furthermore, for a matrix B from system 1.1 or (1.2) we define $\hat{B} := (\hat{\lambda}E - A)^{-1}B$.

We denote by A^D the Drazin inverse of a matrix A , see, e.g., [13], [19]. The following Theorem gives an explicit solution representation in terms of the Drazin inverse.

THEOREM 2.2. *Let (E, A) be a regular matrix pair with $E, A \in \mathbb{C}^{n \times n}$ and let $\text{ind}(E, A) = \nu$. Furthermore, for the continuous-time case, let $u \in \mathcal{C}^\nu$. Then, every solution $x \in C^1$ to Equation (1.1a) has the form:*

$$\begin{aligned} x(t) = & e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v + \int_0^t e^{\hat{E}^D \hat{A} (t-\tau)} \hat{E}^D \hat{B} u(\tau) d\tau - \\ & -(I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} u^{(i)}(t). \end{aligned} \quad (2.2)$$

for some $v \in \mathbb{C}^n$. In the discrete-time case, every solution sequence $x(t)$ to Equation (1.2a) has the form:

$$\begin{aligned} x(t) = & (\hat{E}^D \hat{A})^t \hat{E}^D \hat{E} v + \sum_{\tau=0}^{t-1} (\hat{E}^D \hat{A})^{t-1-\tau} \hat{E}^D \hat{B} u(\tau) - \\ & -(I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} u(t+i). \end{aligned} \quad (2.3)$$

for some $v \in \mathbb{C}^n$.

Proof. See, e.g., [11], [26]. \square

COROLLARY 2.3. *With the same assumptions as in Theorem 2.2, the continuous-time initial value problem (1.1) has a (unique) solution if and only if there exists a*

vector $v \in \mathbb{C}^n$ such that

$$x_0 = \hat{E}^D \hat{E}v - (I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B}u^{(i)}(0). \quad (2.4)$$

and the discrete-time initial value problem (1.2) has a (unique) solution if and only if

$$x_0 = \hat{E}^D \hat{E}v - (I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B}u(i). \quad (2.5)$$

for some $v \in \mathbb{C}$.

Proof. See, e.g., [11], [26]. \square

2.2. Projectors and index of (E, A) . A matrix Q is called *projector* if $Q^2 = Q$. A projector Q is called projector *onto* a subspace $S \subseteq \mathbb{R}^n$ if $\text{im } Q = S$. It is called projector *along* a subspace $S \subseteq \mathbb{R}^n$ if $\ker Q = S$.

Let (E, A) be a regular matrix pair. As introduced in [23] we define a matrix chain by setting

$$E_0 := E, \quad A_0 := A \quad \text{and} \quad (2.6a)$$

$$E_{i+1} := E_i - A_i \tilde{Q}_i, \quad A_{i+1} := A_i \tilde{P}_i, \quad \text{for } i \geq 0, \quad (2.6b)$$

where \tilde{Q}_i are projectors onto $\ker E_i$ and $\tilde{P}_i = I - \tilde{Q}_i$. Since we have assumed (E, A) to be regular, there exists an index ν such that E_ν is nonsingular and all E_i are singular for $i < \nu$, [28]. Note, that ν is independent of a special choice of the projectors Q_i . Then, we say that the matrix pair (E, A) has (*tractability*) *index* ν and denote it by $\text{ind}(E, A) = \nu$. It is well known that for regular pairs (E, A) the tractability index is equal to the differentiation index, see, e.g., [12], and it can be determined as the size of the largest Jordan block associated with the eigenvalue infinity in the Weierstraß canonical form of the pair (E, A) , see [26, 28]. In the following we, therefore, only speak of the *index* of the pair (E, A) .

It is possible to construct the matrix chain in (2.6) with special, so called *canonical projectors*, see [29], [33]. For such projectors Q_i , it holds that for all $v \in S_f^{def}$ and for all $i = 0, \dots, \nu - 1$ we have $Q_i v = 0$. In the following, whenever we refer to the matrix chain in (2.6), we assume that it is constructed with canonical projectors.

2.3. Nonnegative matrices and matrix pairs. A matrix $A \in \mathbb{R}^{n \times n}$, $A = [a_{ij}]_{i,j=1}^n$ is called *nonnegative* and we write $A \geq 0$ if all entries a_{ij} are nonnegative. Matrices A for which $e^{At} \geq 0$ are called *exponentially non-negative*.

The matrix A is called *Z-matrix* if its off-diagonal entries are non-positive. In the literature, a matrix for which $-A$ is a Z-matrix sometimes is called *L-matrix*, Metzler matrix or essentially positive matrix, see, e.g., [6], [20], [24], [40]. Throughout this paper we will use the term *-Z-matrix*. For a matrix A we have that $e^{At} \geq 0$ if and only if A is a *-Z-matrix*, see, e.g., [40]. Let $B \geq 0$ with spectral radius $\rho(B)$. A matrix A of the form $A = sI - B$, with $s > 0$, and $s \geq \rho(B)$ is called *M-matrix*. If $s > \rho(B)$ then A is a *nonsingular M-matrix*, if $s = \rho(B)$ then A is a *singular M-matrix*. The class of *M-matrices* is a subclass of the *Z-matrices*.

A matrix A is called *strictly diagonally dominant* if for all $i = 1, \dots, n$ we have $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$. A matrix A is called *positive (semi-)definite* if for all $x \neq 0$ we have

($x^T Ax \geq 0$) $x^T Ax > 0$. If this holds for $-A$ then A is called *negative (semi-)definite*. A strictly diagonally dominant matrix A with positive diagonal entries is positive definite, see, e.g., [40].

The following generalised Perron-Frobenius-type condition for matrix pairs is presented in [33].

THEOREM 2.4. *Let (E, A) , with $E, A \in \mathbb{R}^{n \times n}$, be a regular matrix pair of $\text{ind}(E, A) = \nu$. Let a matrix chain as in (2.6) be constructed with canonical projectors Q_i, P_i . If*

$$E_\nu^{-1} A_\nu \geq 0, \quad (2.7)$$

holds, then the finite spectral radius $\rho_f(E, A)$ is an eigenvalue and there exists a corresponding nonnegative eigenvector $v \geq 0$. If $E_\nu^{-1} A_\nu$ is, in addition, irreducible, then we have that $\rho_f(E, A)$ is simple and $v > 0$ is unique up to a scalar multiple.

Proof. See [33]. \square

COROLLARY 2.5. *Let P_r be a projector onto the right finite deflating subspace S_f^{def} , let \hat{E}, \hat{A} be defined as in Lemma 2.1. Under the assumptions of Theorem 2.4 the conditions*

$$P_r E_\nu^{-1} A \geq 0, \quad (2.8)$$

$$E_\nu^{-1} A \hat{E}^D \hat{E} \geq 0, \quad (2.9)$$

$$\hat{E}^D \hat{A} \geq 0, \quad (2.10)$$

are equivalent to condition (2.7).

Proof. See [33]. \square

In particular, Corollary 2.5 implies the following identities that we will frequently use throughout the paper:

- $P_r = \hat{E}^D \hat{E}$;
- $S_f^{def} = \text{im}(P_r) = \text{im}(\hat{E}^D \hat{E})$;
- $\hat{E}^D \hat{A} = E_\nu^{-1} A_\nu = E_\nu^{-1} A P_r = P_r E_\nu^{-1} A$.

3. Positive descriptor systems. For standard systems internal positivity implies that for any initial condition $x_0 \geq 0$ and any input function $u(t) \geq 0$ we have $x(t) \geq 0$ and $y(t) \geq 0$ for all $t \geq 0$, see [20], [24]. In the case of descriptor systems, an initial value is constrained in order to be consistent, see Corollary 2.3. Hence, for the positivity of the initial condition only the part that can be chosen freely, i.e., the part that corresponds to the finite deflating subspace, has to be non-negative.

DEFINITION 3.1 (Consistent non-negative initial condition). *Let the matrices E, A, B, C be the system matrices in (1.1) or (1.2), respectively with $\text{ind}(E, A) = \nu$. We call the initial value x_0 in (1.1) or (1.2) consistent non-negative if there exists a vector $v \in \mathbb{C}^n$ such that (2.4) or (2.5) holds, respectively, and furthermore $\hat{E}^D \hat{E} v \geq 0$.*

Note, that for standard systems, this non-negative consistency condition reduces to $x_0 \geq 0$. In contrast to standard systems, here it is only required that the initial condition is non-negative on the finite deflating subspace S_f^{def} . On the subspace corresponding to the infinite eigenvalues, the solution does not depend on the initial condition and its positivity. Hence, the positivity of the initial condition on the subspace corresponding to infinite eigenvalues should be a property of the system.

DEFINITION 3.2 (Positivity). *We call the continuous-time system (1.1) with $\text{ind}(E, A) = \nu$ internally positive if for every consistent non-negative initial condition x_0 and every non-negative input function u such that $u^{(i)}(t) \geq 0$ for $i = 0, \dots, \nu - 1$, the state vector $x(t)$ and the output $y(t)$ are non-negative for all $t \geq 0$.*

The discrete-time system (1.2) is called internally positive if for any consistent non-negative initial condition x_0 and every non-negative input sequence $u(t)$, $t \geq 0$, the output $y(t)$ and the state $x(t)$ are non-negative for all $t > 0$.

In the following theorem we formulate an extension of the characterisation of positivity in the standard case in, e.g., [20], [24].

THEOREM 3.3 (Continuous-time case). *Let E, A, B, C be the matrices in system (1.1) with $\text{ind}(E, A) = \nu$ and $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu - 1$. Then, the continuous-time system (1.1) is internally positive if and only if the matrix $\hat{E}^D \hat{A}$ is exponentially non-negative on the right finite deflating subspace S_f^{def} and $\hat{E}^D \hat{B}, C \geq 0$.*

Proof. “ \Rightarrow ” Let the system (1.1) be internally positive. If the initial condition x_0 is consistent non-negative, then by Corollary 2.3 there exists a $v \in \mathbb{C}^n$, such that (2.4) holds. For internally positive systems, we have $x(t) \geq 0$ and $y(t) \geq 0$ for every $u(t) \geq 0$ and for every consistent non-negative x_0 . Hence, in particular for $u \equiv 0$, from (2.2) we get that

$$x(t) = e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v \geq 0, \quad (3.1)$$

holds for any $x_0 = \hat{E}^D \hat{E} v \geq 0$. Hence, $e^{\hat{E}^D \hat{A} t}$ is non-negative on $S_f^{def} = \text{im}(\hat{E}^D \hat{E})$, i.e., $\hat{E}^D \hat{A}$ is exponentially non-negative on the right finite deflating subspace S_f^{def} .

For $x_0 = 0$ and $u \equiv \xi \geq 0$, where $\xi \in \mathbb{R}_+^m$ is a constant vector and hence, all derivatives of u are zero, we get from (2.4) that

$$x_0 = \hat{E}^D \hat{E} v - (I - \hat{E}^D \hat{E}) \hat{A}^D \hat{B} \xi = 0. \quad (3.2)$$

Since, $\text{im}(\hat{E}^D \hat{E}) \oplus \text{im}(I - \hat{E}^D \hat{E}) = \mathbb{C}^n$, we conclude that $\hat{E}^D \hat{E} v = 0$ and $(I - \hat{E}^D \hat{E}) \hat{A}^D \hat{B} \xi = 0$. Thus, from (2.2) we obtain that

$$x(t) = \int_0^t e^{\hat{E}^D \hat{A} (t-\tau)} \hat{E}^D \hat{B} \xi d\tau \geq 0, \quad (3.3)$$

for any $\xi \in \mathbb{R}_+^m$. Since $\hat{E}^D \hat{A}$ is exponentially non-negative on $\text{im}(\hat{E}^D \hat{E})$ and since integration is monotone and (3.3) must hold for all $\xi \in \mathbb{R}_+^m$, we conclude that $\hat{E}^D \hat{B} \geq 0$. Additionally, since the system is internally positive, we get $y(t) = Cx(t) \geq 0$ for any $x \geq 0$ and therefore, $C \geq 0$.

“ \Leftarrow ” Let $\hat{E}^D \hat{A}$ be exponentially non-negative on the right finite deflating subspace S_f^{def} , let $\hat{E}^D \hat{B}, C \geq 0$ and $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu - 1$. We have to show that for any consistent non-negative x_0 and for $u(t)$ with $u^{(i)}(t) \geq 0$ for all $t \geq 0$ and $i = 0, \dots, \nu - 1$ we get $x(t) \geq 0$ and $y(t) \geq 0$ for all t . The solution at time t is given by

$$\begin{aligned} x(t) = & e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v + \int_0^t e^{\hat{E}^D \hat{A} (t-\tau)} \hat{E}^D \hat{B} u(\tau) d\tau - \\ & -(I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} u^{(i)}(t), \end{aligned} \quad (3.4)$$

and $x_0 = \hat{E}^D \hat{E} v - (I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A}^D)^i \hat{A}^D \hat{B} u^{(i)}(0)$. Since x_0 is consistent non-negative, we get $\hat{E}^D \hat{E} v \geq 0$, see Definition 3.1. Since, additionally, $\hat{E}^D \hat{A}$ is exponentially non-negative on $S_f^{def} = \text{im}(\hat{E}^D \hat{E})$, the first term is non-negative. For the second term we have that $\hat{E}^D \hat{B} \geq 0$ and therefore $e^{\hat{E}^D \hat{A}(t-\tau)} \hat{E}^D \hat{B} u(\tau) \geq 0$ for any $u \geq 0$, since $\hat{E}^D \hat{B} u(\tau) \in \text{im} \hat{E}^D \hat{E}$. Hence, since integration is monotone, the second term is non-negative. Furthermore, we have $-(I - \hat{E}^D \hat{E})(\hat{E}^D \hat{A}^D)^i \hat{A}^D \hat{B} \geq 0$ for $i = 0, \dots, \nu - 1$ and therefore the third term is also non-negative for any $u(t)$ with $u^{(i)}(t) \geq 0$ for all $t \geq 0$ and $i = 0, \dots, \nu - 1$. Thus, $x(t) \geq 0$. From $y(t) = Cx(t)$ and $C \geq 0$, we also conclude that $y(t) \geq 0$. \square

LEMMA 3.4. Consider the matrix pair (E, A) in the positive continuous-time system (1.1). If the spectral projector $P_r = \hat{E}^D \hat{E}$ is non-negative, where \hat{E} is defined as in Lemma 2.1, then there exists $\alpha \geq 0$ such that

$$\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E} \geq 0.$$

Proof. For continuous-time positive systems (1.1), from the proof of Theorem 3.3 we have that equation (3.1) holds, i.e.,

$$e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v \geq 0 \quad \text{for all } \hat{E}^D \hat{E} v \geq 0 \quad \text{and all } t \geq 0.$$

If $\hat{E}^D \hat{E} \geq 0$, then we obtain that

$$e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v \geq 0 \quad \text{for all } v \geq 0 \quad \text{and all } t \geq 0,$$

and hence,

$$e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} \geq 0. \tag{3.5}$$

From this, we have that $(\hat{E}^D \hat{A})_{ij} \geq 0$ for all pairs (i, j) such that $(\hat{E}^D \hat{E})_{ij} = 0$, since otherwise, for $t > 0$ small enough, we obtain

$$e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} = (\hat{E}^D \hat{E})_{ij} + (\hat{E}^D \hat{A})_{ij} t + O(t^2) < 0,$$

which contradicts equation (3.5). Setting $\alpha \geq |\min_{(i,j)} (\hat{E}^D \hat{A})_{ij}|$, we obtain $\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E} \geq 0$. \square

REMARK 3.5. The important implication of Lemma 3.4 is that we can shift the finite spectrum of the matrix pair (E, A) as in the standard case, see, e.g., [20, p.38], such that the shifted matrix pair $(E, A + \alpha E)$ fulfils the assumptions of Theorem 2.4 and its finite spectral radius is an eigenvalue. Furthermore, for any finite eigenvalue μ of $(E, A + \alpha E)$ we have that $\lambda = \mu - \alpha$ is a finite eigenvalue of (E, A) . The eigenvectors and eigenspaces of (E, A) and $(E, A + \alpha E)$ are the same. In particular, the eigenspace that corresponds to the eigenvalue ∞ remains unchanged.

Proof. Let (λ, v) be a finite eigenpair of (E, A) , i.e.,

$$\begin{aligned} (\lambda E - A)v &= 0 \\ \Leftrightarrow ((\lambda + \alpha)E - A - \alpha E)v &= 0. \end{aligned}$$

Hence, (μ, v) with $\mu := \lambda + \alpha$ is a finite eigenpair of $(E, A + \alpha E)$. \square

The following theorem characterises positivity in the case that the spectral projector is non-negative.

THEOREM 3.6. *Let E, A, B, C be the matrices in system (1.1) with $\text{ind}(E, A) = \nu$ and $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D B \leq 0$ for $i = 0, \dots, \nu - 1$. If in addition $\hat{E}^D \hat{E} \geq 0$, then we have the following characterisations of internal positivity.*

The continuous-time system (1.1) is internally positive if and only if there exists a scalar $\alpha > 0$ such that the matrix

$$M := -\alpha I + (\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E})$$

is a $-Z$ -matrix and $\hat{E}^D B, C \geq 0$. Note, that on the right finite deflating subspace S_f^{def} the matrix M is identical to $\hat{E}^D \hat{A}$, i.e., for any $v \in S_f^{def}$ we have

$$Mv = (-\alpha I + \hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E})v = \hat{E}^D \hat{A}v.$$

Proof. By Lemma 3.4 we have that for the continuous-time internally positive system (1.1) if $\hat{E}^D \hat{E} \geq 0$, then there exists a scalar $\alpha > 0$ such that $\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E} \geq 0$. Hence, the matrix M has non-negative off-diagonal entries and we have

$$e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} = e^{Mt} \hat{E}^D \hat{E} \geq 0,$$

since M is a $-Z$ -matrix. The rest of the proof follows from Theorem 3.3. \square

COROLLARY 3.7. *Let E, A, B, C be the matrices in system (1.1) with $\text{ind}(E, A) = \nu$ and $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu - 1$. If the matrix $\hat{E}^D \hat{A}$ is a $-Z$ -matrix and $\hat{E}^D \hat{B}, C \geq 0$, then the continuous-time system (1.1) is internally positive.*

Proof. If $\hat{E}^D \hat{A}$ is a $-Z$ -matrix, this implies that $\hat{E}^D \hat{A}$ is exponentially non-negative on S_f^{def} . Internal positivity follows from Theorem 3.3. \square

The first of the following two examples demonstrates that the property that $\hat{E}^D \hat{A}$ is a $-Z$ -matrix is not necessary for the system (1.1) to be internally positive. The second example is a system that is not (internally) positive.

EXAMPLE 3.8. *Consider the system*

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Since the matrices E and A commute, we can directly compute

$$E^D A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is not a $-Z$ -matrix and

$$E^D E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E^D B = 0.$$

For the state vector, we then obtain

$$\begin{aligned} x(t) &= e^{E^D A t} E^D E v - (I - E^D E) A^D B u(t) = \\ &= \begin{bmatrix} e^{-t} & e^{-t} - 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 + v_2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u(t). \end{aligned}$$

Hence, the system is internally positive, although $E^D A$ is not a $-Z$ -matrix.

EXAMPLE 3.9. Consider the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Here, the matrices E and A also commute and we can compute

$$E^D A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is also not a $-Z$ -matrix and

$$E^D E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E^D B = 0.$$

For the solution, we obtain

$$x(t) = e^{E^D A t} E^D E v - (I - E^D E) A^D B u(t) = \begin{bmatrix} e^t & -te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u(t).$$

The system is not internally positive, since the first component of x may become negative.

The following Theorem 3.10 gives a characterisation of positivity in the case of discrete-time systems.

THEOREM 3.10 (Discrete-time case). Let E, A, B, C be the matrices in system (1.2) with $\text{ind}(E, A) = \nu$. Then, the discrete-time system (1.2) is internally positive if and only if $\hat{E}^D \hat{A}$ is non-negative on the right finite deflating subspace S_f^{def} , $\hat{E}^D \hat{B}, C \geq 0$ and $(I - \hat{E}^D \hat{E})(\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu - 1$.

Proof. “ \Rightarrow ” Let the system (1.2) be internally positive. If the initial condition x_0 is consistent non-negative, then by Corollary 2.3 there exists a $v \in \mathbb{C}^n$, such that (2.5) holds. For internally positive systems, we have $x(t) \geq 0$ and $y(t) \geq 0$ for every sequence $u(t) \geq 0$ and for every consistent non-negative x_0 . Then, in particular, for $u \equiv 0$, we have

$$x(1) = \hat{E}^D \hat{A}(\hat{E}^D \hat{E}v) \geq 0, \quad (3.6)$$

for any $x_0 = \hat{E}^D \hat{E}v \geq 0$. Hence, $\hat{E}^D \hat{A}$ is non-negative on $S_f^{def} = \text{im}(\hat{E}^D \hat{E})$. Furthermore, for $u \equiv 0$ we have that $y(1) = Cx_0 \geq 0$ for any $x_0 \geq 0$ and therefore $C \geq 0$. Now, choose a time $t > 0$ and set x_0 such that $\hat{E}^D \hat{E}v = 0$ and

$$u(\tau) = \begin{cases} e_i, & \tau = t - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where e_i is the i -th unit vector and obtain

$$x(1) = \hat{E}^D \hat{B}e_i \geq 0,$$

for $i = 1, \dots, m$. Hence, we have $\hat{E}^D \hat{B} \geq 0$. Finally, choose x_0 such that $\hat{E}^D \hat{E}v = 0$ and

$$u(\tau) = \begin{cases} e_i, & \tau = t, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain

$$x(1) = -(I - \hat{E}^D \hat{E}) \hat{A}^D \hat{B} e_i \geq 0,$$

for $i = 1, \dots, m$. Hence, we have $-(I - \hat{E}^D \hat{E}) \hat{A}^D \hat{B} \geq 0$. By repeating the procedure for $\tau = t+1, \dots, t+\nu-1$, we obtain $-(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \geq 0$ for $i = 1, \dots, \nu-1$.

“ \Leftarrow ” Let $\hat{E}^D \hat{A}$ be non-negative on the right finite deflating subspace, $\hat{E}^D \hat{B}, C \geq 0$ and $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu-1$. We have to show that for any consistent non-negative x_0 and any input sequence $u \geq 0$ we get $x(t) \geq 0$ and $y(t) \geq 0$ for all t . The solution at time t is given by

$$\begin{aligned} x(t) = & (\hat{E}^D \hat{A})^t \hat{E}^D \hat{E}v + \sum_{\tau=0}^{t-1} (\hat{E}^D \hat{A})^{(t-1-\tau)} \hat{E}^D \hat{B} u(\tau) - \\ & -(I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} u(t+i), \end{aligned}$$

and $x_0 = \hat{E}^D \hat{E}v - (I - \hat{E}^D \hat{E}) \sum_{i=0}^{\nu-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} u(i)$. Since x_0 is consistent non-negative, we have $\hat{E}^D \hat{E}v \geq 0$. Since, additionally, $\hat{E}^D \hat{A}$ is non-negative on $\text{im } \hat{E}^D \hat{E}$ the first term is non-negative. Since $\hat{E}^D \hat{B} \geq 0$, we have that $(\hat{E}^D \hat{A})^{t-1-\tau} \hat{E}^D \hat{B} u(\tau) \geq 0$ for any $u \geq 0$ and the second term is also non-negative. Additionally, we have $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu-1$ and therefore, the third term is non-negative for any $u \geq 0$. Hence, $x(t) \geq 0$. From $y(t) = Cx(t)$ and $C \geq 0$, we also conclude that $y(t) \geq 0$. \square

The discrete-time analogon to Theorem 3.6 was given In [9]. Under the assumption that $\hat{E}^D \hat{E} \geq 0$ it states that internal positivity of the system (1.2) is equivalent to $\hat{E}^D \hat{A}, \hat{E}^D \hat{B}, C \geq 0$ and $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu-1$. Furthermore, the following holds.

COROLLARY 3.11. *Let E, A, B, C be the matrices in system (1.2) with $\text{ind}(E, A) = \nu$. If $\hat{E}^D \hat{A}, \hat{E}^D \hat{B}, C \geq 0$ and $(I - \hat{E}^D \hat{E})(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0$ for $i = 0, \dots, \nu-1$, then the discrete-time system (1.2) is internally positive.*

Proof. If $\hat{E}^D \hat{A} \geq 0$, this implies that $\hat{E}^D \hat{A}$ is non-negative on S_f^{def} . Internal positivity then follows from Theorem 3.10. \square

4. Stability conditions for positive descriptor systems. In the course of this section, we consider linear homogeneous positive time-invariant systems:

- in continuous-time:

$$E\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{4.1}$$

- or in discrete-time:

$$Ex(t+1) = Ax(t), \quad x(0) = x_0. \tag{4.2}$$

DEFINITION 4.1 (c-/d-positive matrix pair). *We call a matrix pair (E, A) c-positive if system (4.1) is internally positive. We call a matrix pair (E, A) d-positive if system (4.2) is internally positive.*

DEFINITION 4.2 (c-/d-stable matrix pair). *A matrix pair (E, A) is called c-stable if all finite eigenvalues of (E, A) have negative real part. A matrix pair (E, A) is called d-stable if $\rho_f(E, A) < 1$.*

In this subsection we generalise the special stability conditions in the case of positive systems from the standard case, see [20], to the descriptor case under the assumption that the spectral projector P_r onto the right finite deflating subspace S_f^{def} is non-negative. This is a reasonable assumption, since in this case for any vector $v \geq 0$ a consistent initial value x_0 is also consistent non-negative. In [9], this condition is imposed to characterise positivity of discrete-time positive descriptor systems.

Stability conditions for positive systems are closely related to and can be characterised by the so called dominant eigenvalue(s) of the system.

DEFINITION 4.3 (c-/d-dominant eigenvalue). *For linear continuous-time systems (1.1), we call a finite eigenvalue λ of the matrix pair (E, A) c-dominant if its real part is larger or equal to the real part of any other eigenvalue of the matrix pair (E, A) , i.e. $\Re(\lambda) \geq \Re(\lambda_i)$ for all $\lambda_i \in \sigma_f(E, A)$. For linear discrete-time systems (1.2), we call a finite eigenvalue of the matrix pair (E, A) d-dominant if it is larger or equal in modulus than any other eigenvalue of the matrix pair (E, A) , i.e. $|\lambda| \geq |\lambda_i|$ for all $\lambda_i \in \sigma_f(E, A)$.*

Note, that for standard continuous-time systems, where $E = I$, a c-dominant eigenvalue is an eigenvalue of the matrix A with the largest real part and for standard discrete-time systems a d-dominant eigenvalue is an eigenvalue of A that is largest in modulus.

By using Lemma 3.4 and Remark 3.5, we can generalise the result on dominant eigenvalues in [20, Theorem 11] to descriptor systems.

THEOREM 4.4. *Consider the positive continuous-time system (1.1). If the spectral projector $P_r = \hat{E}^D \hat{E}$ onto the right finite deflating subspace is non-negative, then the c-dominant eigenvalue v of the system is real and unique. Furthermore, there exists a non-negative eigenvector corresponding to this c-dominant eigenvalue. Consider the positive discrete-time system (1.2). If the spectral projector $P_r = \hat{E}^D \hat{E}$ onto the right finite deflating subspace is non-negative, then $\rho_f(E, A)$ is a d-dominant eigenvalue and there exists a corresponding non-negative eigenvector.*

Proof. In the continuous-time case, since $\hat{E}^D \hat{E} \geq 0$, by Lemma 3.4 and Remark 3.5 we have that there exists a scalar $\alpha > 0$ such that for the shifted matrix pair $(E, A + \alpha E)$ the finite spectral radius $\rho_f(E, A + \alpha E)$ is an eigenvalue. Hence, $\lambda = \mu - \alpha$ is an eigenvalue of (E, A) and it is the eigenvalue with the largest real part, i.e., the c-dominant eigenvalue of the positive system (1.1). Hence, the c-dominant eigenvalue λ is real and unique. Since by Remark 3.5, the eigenvectors remain unchanged by the shift, there exists a corresponding non-negative eigenvector.

In the discrete-time case, if $\hat{E}^D \hat{E} \geq 0$, then for a positive system (1.2) we have that $\hat{E}^D \hat{A} \geq 0$, see [9]. Hence, by Theorem 2.4, the finite spectral radius of (E, A) is an eigenvalue and there exists a corresponding non-negative eigenvector. \square

EXAMPLE 4.5. Let $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. We have $\hat{E}^D \hat{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\hat{E}^D \hat{A} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, the system (4.1) is positive, since

$$e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v = \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} e^{-t} v_1 \\ 0 \end{bmatrix} \geq 0,$$

for all $v_1 \geq 0$. Choose $\alpha = |\min_{i,j} (\hat{E}^D \hat{A})_{i,j}| = 1$. We obtain

$$\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

Hence, $\mu := \rho(\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E}) = 0$ is an eigenvalue and the corresponding c -dominant eigenvalue of (E, A) is $\lambda = \mu - \alpha = -1$. This means that (E, A) is also c -stable.

LEMMA 4.6. Let (E, A) be a regular c -stable matrix pair. Then, for any $\alpha > 0$ we have that

$$B := -\alpha I + \hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E},$$

is a stable (regular) matrix. If, in addition, the matrix pair (E, A) is c -positive and the spectral projector $\hat{E}^D \hat{E}$ is non-negative, then there exists $\alpha > 0$ such that B is a $-M$ -matrix.

Proof. All finite eigenvalues of (E, A) are also eigenvalues of $\hat{E}^D \hat{A}$. The eigenvalue ∞ of (E, A) is mapped to the eigenvalue 0 of $\hat{E}^D \hat{A}$. For any finite eigenpair (λ, v) of (E, A) , we have

$$Bv = \hat{E}^D \hat{A}v = \lambda v.$$

Therefore, all finite, stable eigenvalues remain the same. For any eigenvector w corresponding to the eigenvalue ∞ of (E, A) , i.e., $Aw = 0$, we have by the definition of \hat{E} in Lemma 2.1 that

$$\hat{E}^D \hat{A}w = \hat{E}^D \hat{A} \hat{E}^D \hat{E}w = \hat{E}^D \hat{A} \hat{E}^D (\lambda E - A)^{-1} Aw = 0,$$

and hence,

$$Bw = -\alpha(I - \hat{E}^D \hat{E})w = -\alpha w.$$

Thus, w is now eigenvector corresponding to a negative eigenvalue $-\alpha$. Hence, all eigenvalues of B have negative real parts and therefore B is stable. If, in addition, the matrix pair (E, A) is c -positive and the spectral projector $\hat{E}^D \hat{E}$ is non-negative, then by Lemma 3.4 we have that there exists $\alpha > 0$ such that $T := \hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E} \geq 0$. By the generalised Perron-Frobenius Theorem 2.4 we have that $\rho(T)$ is an eigenvalue of T and $\rho(T) - \alpha$ is the finite eigenvalue of (E, A) with the largest real part and it is negative, since (E, A) is c -stable. Therefore, we have $\alpha > \rho(\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E})$ and B is a $-M$ -matrix. \square

In the following we generalise a Lyapunov-type stability condition from the standard case, see [20], to the descriptor case.

THEOREM 4.7. Let the matrix pair (E, A) be c -positive. If the spectral projector $P_r = \hat{E}^D \hat{E}$ is non-negative, then the pencil (E, A) is c -stable if and only if there exists a positive definite diagonal matrix Y such that $(\hat{E}^D \hat{A})^T Y + Y(\hat{E}^D \hat{A})$ is negative

definite on S_f^{def} .

For a d -positive matrix pair (E, A) , if the spectral projector $P_r = \hat{E}^D \hat{E}$ is non-negative, then (E, A) is d -stable if and only if there exists a positive definite diagonal matrix Y such that $(\hat{E}^D \hat{A})^T Y (\hat{E}^D \hat{A}) - Y$ is negative definite.

Proof. Continuous-time case:

“ \Rightarrow ” By Lemma 4.6, we have that the matrix

$$M := \alpha I - (\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E}),$$

is a regular M -matrix. Therefore, we have $M^{-1} \geq 0$. Let $\mathbf{1}$ denote the vector of all ones and set $x := M^{-T} \mathbf{1} > 0$ and $z := M^{-1} \mathbf{1} > 0$. Define the diagonal matrix Y by $Y := \text{diag}(y)$, where $y_i := x_i/z_i$. Then, since $\hat{E}^D \hat{E} v = v$, we have for all $v \in S_f^{def}$ that

$$\begin{aligned} v^T (\hat{E}^D \hat{A})^T Y v + v^T Y (\hat{E}^D \hat{A}) v &= \\ v^T (-\alpha I - (\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E}))^T Y v + v^T Y (-\alpha I - (\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E})) v &= \\ v^T (-M)^T Y v + v^T Y (-M) v. \end{aligned}$$

The matrix $-(M^T Y + Y M)$ is negative definite, which can be shown as in the standard case, see, e.g., [6], [20]. Hence, we have constructed a positive definite diagonal matrix Y such that $(\hat{E}^D \hat{A})^T Y + Y (\hat{E}^D \hat{A})$ is negative definite on S_f^{def} .

“ \Leftarrow ” By Theorem 4.4, we have that the c -dominant eigenvalue λ of (E, A) is real and unique. Hence, it suffices to show that λ is negative. Let v be an eigenvector corresponding to λ . Since by Theorem 2.4 the eigenpair (λ, v) is also an eigenpair of $\hat{E}^D \hat{A}$, we obtain

$$\begin{aligned} v^T (\hat{E}^D \hat{A})^T Y v + v^T Y (\hat{E}^D \hat{A}) v &= v^T \lambda Y v + v^T Y \lambda v = \\ 2\lambda v^T Y v &< 0, \end{aligned}$$

whereas $v^T Y v > 0$. Hence, $\lambda < 0$.

Discrete-time case:

“ \Rightarrow ” If $\hat{E}^D \hat{E} \geq 0$, for a positive system we also have $\hat{E}^D \hat{A} \geq 0$, see [9]. Since the matrix pair (E, A) is d -stable, we have $\rho_f(E, A) < 1$ and hence, the matrix

$$M := I - \hat{E}^D \hat{A},$$

is a regular M -matrix. Therefore, we have $M^{-1} > 0$. Set $x := M^{-T} \mathbf{1} > 0$ and define the diagonal matrix Y by $Y := \text{diag}(x)$. We show that $(\hat{E}^D \hat{A})^T Y (\hat{E}^D \hat{A}) - Y$ has negative row sums. Then, since the first product is non-negative and Y is diagonal, we have diagonal dominance and symmetry, and hence, negative definiteness. For the row sums we obtain

$$\begin{aligned} ((\hat{E}^D \hat{A})^T Y (\hat{E}^D \hat{A}) - Y) \mathbf{1} &= ((\hat{E}^D \hat{A})^T Y (\hat{E}^D \hat{A} - I + I) - Y) \mathbf{1} = \\ &= (\hat{E}^D \hat{A})^T (-Y M + Y) \mathbf{1} - x = \\ &= -(\hat{E}^D \hat{A})^T Y M \mathbf{1} + (\hat{E}^D \hat{A})^T x - x, \end{aligned}$$

where the first term is negative, since due to diagonal dominance and the fact that M is an M -matrix the matrix $Y M$ has positive row sums and $\hat{E}^D \hat{A} \geq 0$. Therefore, we obtain

$$((\hat{E}^D \hat{A})^T Y (\hat{E}^D \hat{A}) - Y) \mathbf{1} \leq -M x = -\mathbf{1} < 0.$$

Hence, we have shown that $(\hat{E}^D \hat{A})^T Y (\hat{E}^D \hat{A}) - Y$ has negative row sums, which completes the proof.

“ \Leftarrow ” By Theorem 4.4, we have that there exists a d -dominant eigenvalue λ of (E, A) that is non-negative and real. Hence, it suffices to show that λ is smaller than 1. Let v be an eigenvector corresponding to λ . Since by Theorem 2.4 (λ, v) is also an eigenpair of $\hat{E}^D \hat{A}$, we obtain

$$\begin{aligned} v^T (\hat{E}^D \hat{A})^T Y (\hat{E}^D \hat{A}) v - v^T Y v &= \lambda^2 v^T Y v - v^T Y v = \\ &= (\lambda^2 - 1) v^T Y v < 0, \end{aligned}$$

whereas $v^T Y v > 0$. Since λ is non-negative, we have $\lambda < 1$. \square

COROLLARY 4.8. *Let the matrix pair (E, A) be c -positive. If the spectral projector $P_r = \hat{E}^D \hat{E}$ is non-negative, then the matrix pair (E, A) is c -stable if and only if there exists a scalar $\alpha > 0$ such that $M := \alpha I - (\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E})$ and*

- all principal minors of M are positive;
- the coefficients of the characteristic polynomial of M are negative.

Let the matrix pair (E, A) be d -positive. If the spectral projector $P_r = \hat{E}^D \hat{E}$ is non-negative, then the matrix pair (E, A) is d -stable if and only if $\tilde{M} := I - \hat{E}^D \hat{A}$ and

- all principal minors of \tilde{M} are positive;
- the coefficients of the characteristic polynomial of \tilde{M} are negative.

Proof. By Lemma 4.6 there exists $\alpha > 0$ such that M is an M -matrix. By Theorem 4.7 also \tilde{M} is an M -matrix. Therefore, the assertions of the present Theorem follow directly from the M -matrix properties of M and \tilde{M} . \square

5. Non-negative solution of generalised Lyapunov equations. Consider the following generalised projected continuous-time Lyapunov equation [37]

$$E^T X A + A^T X E = -P_r^T G P_r, \quad (5.1)$$

where P_r is the spectral projection onto the right finite deflating subspace S_f^{def} of the pencil (E, A) .

THEOREM 5.1. *Let (E, A) be a regular matrix pair and let P_r be a spectral projector onto the finite deflating subspace with $P_r \geq 0$. If (E, A) is c -stable, then equation (5.1) has a unique solution for every matrix G . The solution is given by*

$$X = E_\nu^{-T} \left(\int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt \right) E_\nu^{-1}, \quad (5.2)$$

where E_ν is defined as in the matrix chain in (2.6). If G is symmetric positive (semi)definite, then X is symmetric positive semidefinite. If, in addition, the matrix pair (E, A) is c -positive and $G \geq 0$ and $P_r E_\nu^{-1} \geq 0$ then also $X \geq 0$.

Proof. We show that X as defined in (5.2) is solution of (5.1). Since (E, A) is c -stable, by Lemma 4.6 we have that for any $\alpha > 0$ the matrix

$$M := -\alpha I + \hat{E}^D \hat{A} + \hat{E}^D \hat{E}$$

is stable and $MP_r = P_rM = \hat{E}^D \hat{A}$. Therefore, we have that

$$\begin{aligned} E^T X E &= E^T E_\nu^{-T} \left(\int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt \right) E_\nu^{-1} E = \\ &= E^T E_\nu^{-T} \left(\int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt \right) (I - Q_0 - \dots - Q_{\nu-1}) = \\ &= \int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt = \\ &= \int_0^\infty e^{M^T t} P_r^T G P_r e^{M t} dt \end{aligned}$$

is a solution of the standard Lyapunov equation

$$(E^T X E)M + M^T (E^T X E) = -P_r^T G P_r.$$

On the other hand, we have

$$\begin{aligned} A^T X E &= A^T E_\nu^{-T} \left(\int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt \right) E_\nu^{-1} E \\ &= P_r^T A^T E_\nu^{-T} \left(\int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt \right) = \\ &= (\hat{E}^D \hat{A})^T \left(\int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt \right) = \\ &= M^T \left(\int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A}) t} dt \right) = \\ &= M^T (E^T X E), \end{aligned}$$

and analogously $E^T X A = (E^T X E)M$. Hence, if we plug X defined in (5.2) into equation (5.1), then we obtain

$$\begin{aligned} E^T X A + A^T X E &= (E^T X E)M + M^T (E^T X E) = \\ &= -P_r^T G P_r. \end{aligned}$$

If (E, A) is c-positive and $P_r \geq 0$, then $e^{(\hat{E}^D \hat{A})^T t} P_r \geq 0$. With $G \geq 0$ and $P_r E_\nu^{-1} \geq 0$ we obtain $X \geq 0$. \square

For the discrete-time case, consider the following generalised projected discrete-time Lyapunov equation [37]

$$A^T X A - E^T X E = -P_r^T G P_r, \quad (5.3)$$

where P_r is the spectral projection onto the right finite deflating subspace of the pencil (E, A) .

THEOREM 5.2. *Let (E, A) be a regular matrix pair and let P_r be a spectral projector onto the finite deflating subspace with $P_r \geq 0$. If (E, A) is d-stable, then equation (5.3) has a unique solution for every matrix G . The solution is given by*

$$X = E_\nu^{-T} \left(\sum_{t=0}^{\infty} ((\hat{E}^D \hat{A})^T)^t G (\hat{E}^D \hat{A})^t \right) E_\nu^{-1}, \quad (5.4)$$

where E_ν is defined as in the matrix chain in (2.6). If G is symmetric positive (semi)definite, than X is symmetric positive semidefinite. If, in addition, the matrix pair (E, A) is d -positive and $G \geq 0$ and $P_r E_\nu^{-1} \geq 0$ than also $X \geq 0$.

Proof. We show that X as defined in (5.4) is solution of (5.3). For X as defined in (5.4) we have that

$$\begin{aligned} E^T X E &= E^T E_\nu^{-T} \left(\sum_{t=0}^{\infty} ((\hat{E}^D \hat{A})^T)^t G (\hat{E}^D \hat{A})^t \right) E_\nu^{-1} E = \\ &= \sum_{t=0}^{\infty} ((\hat{E}^D \hat{A})^T)^t P_r^T G P_r (\hat{E}^D \hat{A})^t \end{aligned}$$

is a solution of the standard discrete-time Lyapunov equation

$$(\hat{E}^D \hat{A})^T (E^T X E) (\hat{E}^D \hat{A}) - (E^T X E) = -P_r^T G P_r.$$

On the other hand, we have

$$A^T X A = (\hat{E}^D \hat{A})^T \left(\sum_{t=0}^{\infty} ((\hat{E}^D \hat{A})^T)^t G (\hat{E}^D \hat{A})^t \right) (\hat{E}^D \hat{A}).$$

Hence, if we plug X into equation (5.3), then we obtain

$$\begin{aligned} A^T X A - E^T X E &= (\hat{E}^D \hat{A})^T (E^T X E) (\hat{E}^D \hat{A}) - (E^T X E) = \\ &= -P_r^T G P_r. \end{aligned}$$

If (E, A) is d -positive and $P_r \geq 0$, then we have that $\hat{E}^D \hat{A} \geq 0$, [9]. With $G \geq 0$ and $P_r E_\nu^{-1} \geq 0$ we obtain $X \geq 0$. \square

6. Stability of switched positive descriptor systems. The study of stability properties of switched systems is subject to ongoing research, see [25] and the references therein. Especially, in the case of standard positive systems, progress has been made on this subject due to the existence of a diagonal Lyapunov function, see, e.g., [30], [31] and the references therein. The existence of a common diagonal Lyapunov function of two positive systems, i.e. a diagonal positive definite matrix Y such that

$$\begin{aligned} A_1^T Y + Y A_1 &\text{ and} \\ A_2^T Y + Y A_2 &\end{aligned}$$

are negative definite, guarantees the stability of the switched system under arbitrary switching. In this section, we show how we can use the framework established throughout this paper in order to generalise these results to positive descriptor systems.

The following sufficient conditions for the existence of a common diagonal Lyapunov function in the standard case can be found, e.g., in [30], [31]. We give the proof for completeness.

THEOREM 6.1. *Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ be $-M$ -matrices, i.e., stable $-Z$ -matrices. Then, the following conditions are sufficient for the existence of a common diagonal Lyapunov function.*

1. $A_1 A_2^{-1}$ and $A_2^{-1} A_1$ are both M -matrices.

2. $A_1 A_2^{-1}$ and $A_2^{-1} A_1$ are both non-negative.

Proof. Define the diagonal matrix Y as in the proof of Theorem 4.7, i.e., $Y := \text{diag}(y)$ with $y_i := x_i/z_i$ and the vectors x and z are defined by $x := A_1^{-T} \mathbf{1}$ and $z := A_1^{-1} \mathbf{1}$, respectively. Then, Y is Lyapunov function for A_1 and we have to show that it is also Lyapunov function for A_2 . We have that

$$\begin{aligned} A_2^T Y + Y A_2 & \text{ is negative definite} \\ \Leftrightarrow Z(A_2^T A_1^{-T}) A_1^T X + Y(A_2 A_1^{-1}) A_1 Z & \text{ is negative definite,} \end{aligned}$$

where $X := \text{diag}(x)$ and $Z := \text{diag}(z)$. Furthermore, we have that the row sums of both summands are negative

$$\begin{aligned} Z(A_2^T A_1^{-T}) A_1^T X \mathbf{1} &= -Z(A_2^T A_1^{-T}) \mathbf{1} < 0, \\ X(A_2 A_1^{-1}) A_1 Z \mathbf{1} &= -X(A_2 A_1^{-1}) \mathbf{1} < 0, \end{aligned}$$

since $A_1 A_2^{-1}$ and $A_2^{-1} A_1$ are both M -matrices and therefore $(A_2^T A_1^{-T}) \geq 0$ and $(A_2 A_1^{-1}) \geq 0$. Due to the M -matrix properties we obtain diagonal dominance and hence negative definiteness. The second sufficient condition follows in an analogous manner. \square

The generalisation to positive descriptor systems uses Theorem 4.7 and is as follows.

THEOREM 6.2. *Let $(E_1, A_1), (E_2, A_2)$ be two regular c -stable matrix pairs and $\hat{E}_1^D \hat{E}_1 \geq 0$ and $\hat{E}_2^D \hat{E}_2 \geq 0$. Then there exist scalars $\alpha_1, \alpha_2 > 0$ such that*

$$\begin{aligned} M_1 &:= \alpha_1 I - \hat{E}_1^D \hat{A}_1 - \alpha \hat{E}_1^D \hat{E}_1, \quad \text{and} \\ M_2 &:= \alpha_2 I - \hat{E}_2^D \hat{A}_2 - \alpha \hat{E}_2^D \hat{E}_2 \end{aligned}$$

are M -matrices and each of the following conditions is sufficient for the existence of a common diagonal Lyapunov function.

1. $M_1 M_2^{-1}$ and $M_2^{-1} M_1$ are both M -matrices.
2. $M_1 M_2^{-1}$ and $M_2^{-1} M_1$ are both non-negative.

Proof. By Lemma 4.6, there exist scalars $\alpha_1, \alpha_2 > 0$ such that M_1, M_2 are M -matrices. The rest follows as in the proof of the standard case in Theorem 6.1. \square

7. Conclusions. In this paper, we have discussed positive descriptor systems in the continuous-time as well as in the discrete-time case. We have presented different characterisations of positivity and generalisations of stability criteria for the case of positive descriptor systems. We have shown that if the spectral projector onto the right finite deflating subspace of the matrix pair (E, A) is non-negative, then all stability criteria for standard positive systems take a comparably simple form in the positive descriptor case. Furthermore, we have provided sufficient conditions that guarantee entry-wise non-negativity of solutions of generalised projected Lyapunov equations. As an application of the framework established throughout this paper, we have shown how stability criteria of switched standard positive systems can be extended to the descriptor case.

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