Perturbation analysis of Lagrangian invariant subspaces of symplectic matrices

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Abstract

Lagrangian invariant subspaces for symplectic matrices play an important role in the numerical solution of discrete time, robust and optimal control problems. The sensitivity (perturbation) analysis of these subspaces, however, is a difficult problem, in particular, when the eigenvalues are on or close to some critical regions in the complex plane, such as the unit circle.

We present a detailed perturbation analysis for several different cases of real and complex symplectic matrices. We analyze stability and conditional stability as well as the index of stability for these subspaces.

Key Words: symplectic matrix, Hamiltonian matrix, Lagrangian invariant subspace, stability, conditional stability, perturbation analysis.

Mathematics Subject Classification: 15A63, 15A21, 93B35, 93C73.

1 Introduction

Let $\mathbb{F}$ denote either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$ and let I_n denote the $n \times n$ identity matrix. Setting

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{1.1}$$

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a matrix $S \in \mathbb{F}^{2n\times 2n}$ is called *symplectic* if $S^T JS = J$, where the superscript ^{T} denotes the transposed matrix or vector. In the case $\mathbb{F} = \mathbb{C}$, in many sources also matrices $S \in \mathbb{C}^{2n\times 2n}$ satisfying $S^*JS = J$ are called symplectic, where the superscript * denotes the conjugate transposed matrix or vector.

In this paper, we consider more general matrices. If $J \in \mathbb{F}^{2n\times 2n}$ is a skew-symmetric invertible matrix, then a matrix $S \in \mathbb{F}^{2n\times 2n}$ is called *J-symplectic* if $S^T JS = J$. In the case $\mathbb{F} = \mathbb{C}$, it makes no difference whether we consider skew-Hermitian or Hermitian invertible matrices $J \in \mathbb{C}^{2n\times 2n}$, since the equality $S^*JS = J$ remains true if we replace J by iJ. When referring to invertible Hermitian matrices $J \in \mathbb{C}^{m\times m}$, where m is allowed to be odd, matrices $S \in \mathbb{C}^{m\times m}$ are called *J-unitary* if $S^*JS = J$. We will adapt this terminology in this paper, but, having in mind applications where J is as in (1.1), we will always assume that $m = 2n$ is even.

We make use of the standard bilinear and (in the case $\mathbb{F} = \mathbb{C}$) sesquilinear forms on $\mathbb{F}^{2n}$:

$$\langle x, y\rangle = \sum_{j=1}^{2n} x_j y_j, \quad x = [x_1, \ldots, x_{2n}]^T, \quad y = [y_1, \ldots, y_{2n}]^T \in \mathbb{F}^{2n},$$

$$\langle x, y\rangle_* = \sum_{j=1}^{2n} x_j \overline{y_j}, \quad x = [x_1, \ldots, x_{2n}]^T, \quad y = [y_1, \ldots, y_{2n}]^T \in \mathbb{C}^{2n}.$$

Definition 1.1 *Let $J \in \mathbb{F}^{2n\times 2n}$ be either skew-symmetric and invertible (or in the complex case only, Hermitian and invertible, respectively).*

1. *A subspace $\mathcal{M} \subseteq \mathbb{F}^{2n}$ is called* J-Lagrangian *if* $\dim \mathcal{M} = n$ *and*

   $$\langle Jx, y\rangle = 0 \quad \text{for all } x, y \in \mathcal{M}$$

   or in the case of Hermitian J if $\langle Jx, y\rangle_* = 0 \quad \text{for all } x, y \in \mathcal{M}$.

2. *For a J-symplectic (or J-unitary, in the case of Hermitian J) matrix $S \in \mathbb{F}^{2n\times 2n}$, we denote by $\mathcal{IL}(S, J)$ the set of all S-invariant J-Lagrangian subspaces.*

We will always assume in this paper that if J is Hermitian and invertible (complex case), then J has n positive and n negative eigenvalues (counted with multiplicities). This assumption guarantees that the set of J-Lagrangian subspaces is not empty.

It is well known that J-symplectic matrices, and in particular their invariant Lagrangian subspaces, play a key role in many applied problems. In the next subsections we present several such problems.

1.1 Discrete time optimal control

The classical application, where symplectic matrices and invariant Lagrangian subspaces arise, is the discrete time optimal control problem to minimize

$$\sum_{j=0}^{\infty} (x_j^* Q x_j + u_j^* R u_j) \tag{1.2}$$

subject to the discrete time control problem

$$x_{k+1} = Ax_k + Bu_k,$$

with x_0 given. Here the coefficients are assumed to satisfy $Q = Q^* \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$, and $R = R^* \in \mathbb{F}^{m\times m}$. In classical linear quadratic optimal control, the matrix Q in (1.2) is real symmetric (or Hermitian) positive semidefinite, with R being positive definite [33, 24].

Applying discrete time variational calculus (the discrete time version of the Pontryagin maximum principle) [33], this problem yields as a necessary condition the discrete boundary value problem

$$\begin{bmatrix} I & 0 & 0 \\ 0 & -A^* & 0 \\ 0 & -B^* & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \mu_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ Q & -I & S \\ S^* & 0 & R \end{bmatrix} \begin{bmatrix} x_k \\ \mu_k \\ u_k \end{bmatrix}, \tag{1.3}$$

with boundary values

$$x_0 = x^0, \quad \lim_{k\to\infty} \mu_k = 0.$$

Here μ_k represents a sequence of Lagrange multipliers. Inserting the equation for u_k leads to the reduced boundary value problem

$$\begin{bmatrix} I & BR^{-1}B^* \\ 0 & -(A - BR^{-1}S^*)^* \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \mu_{k+1} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}S^* & 0 \\ Q - SR^{-1}S^* & -I \end{bmatrix} \begin{bmatrix} x_k \\ \mu_k \end{bmatrix},$$

and if, furthermore, $\widetilde{A} := A - BR^{-1}S^*$ is invertible, then the problem can be written as

$$\begin{bmatrix} x_{k+1} \\ \mu_{k+1} \end{bmatrix} = S \begin{bmatrix} x_k \\ \mu_k \end{bmatrix},$$

where the matrix

$$S = \begin{bmatrix} \widetilde{A} + BR^{-1}B^*(\widetilde{A}^{-1})^*(Q - SR^{-1}S^*) & -BR^{-1}B^*(\widetilde{A}^{-1})^* \\ -(\widetilde{A}^{-1})^*(Q - SR^{-1}S^*) & +(\widetilde{A}^{-1})^* \end{bmatrix}$$

satisfies $S^TJS = J$ in the real case and $S^*JS = J$ in the complex case and J is as in (1.1).

The solution of the boundary value problem can be achieved by computing a Lagrangian invariant subspace of S that allows to decouple the state sequence $\{x_k\}$ from the Lagrange multiplier sequence $\{\mu_k\}$. In general, from the numerical point of view it is better to solve the generalized eigenvalue problem for the matrix pencil associated with the original boundary value problem, but in many applications the problem is solved by computing an appropriate J-Lagrangian subspace of S, see e.g. [24, 33].

1.2 High speed trains

A project of the company SFE GmbH in Berlin investigates rail traffic noise caused by high speed trains [18, 19]. The vibration of an infinite rail track is simulated and analyzed to obtain information on the development of noise between wheel and rail. Discretizing the problem using classical finite element techniques and using the periodicity of the rail leads to the complex eigenvalue problem

$$P(\kappa)y = \frac{1}{\kappa}(A_1^T + \kappa A_0 + \kappa^2 A_1)y = 0,$$

where A_0 is complex symmetric. Such eigenvalue problems are called *palindromic* in [30], since (except for transposition) the coefficients are the same if the order is reversed. It should be noted that A_1 is highly rank deficient and due to the underlying physical properties this problem has no eigenvalues $1, -1$. However, eigenvalues close to 1 and -1 occur in practice, see [19] for more details. With these properties, structure preserving linearization as introduced in [30] leads to a generalized palindromic eigenvalue problem $(\lambda B^T + B)z = 0$ or equivalently $Tz = \lambda z$ with $T = (B^{-1})^T B$, which according to [20, 51] is similar to a symplectic matrix. A similar palindromic eigenvalue problem is derived in the context of discrete time optimal control, and is studied in [49].

1.3 Discrete time H_∞-control

Another important application of symplectic matrices is the design of suboptimal H_∞-controllers, based on the solution of discrete-time algebraic Riccati equations (see [9, 17] for the continuous time case). The discrete H_∞-control problem was first considered in [53].

Consider a linear discrete-time system, described by the equations

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k, \\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k, \\ y_k &= C_2 x_k + D_{21} w_k + D_{22} u_k, \end{aligned} \tag{1.4}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^{m_1}$ is an exogenous input vector (the disturbance), $u_k \in \mathbb{R}^{m_2}$ is the control input vector, $z_k \in \mathbb{R}^{p_1}$ is a controlled vector, and $y_k \in \mathbb{R}^{p_2}$ is a measurement vector. The transfer function matrix of the system is denoted by

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} \hat{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

The *H_∞-suboptimal discrete-time control problem* is to find an internally stabilizing controller $K(z)$ such that, for a pre-specified positive value of γ, the inequality

$$\|T_{zw}(z)\|_\infty < \gamma$$

is satisfied, where $T_{zw}(z)$ is the transfer function from w to z, given by

$$T_{zw}(z) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

In the H_∞*-optimization control problem* one tries to find the infimum for γ. This infimum is usually computed numerically by a root-finding procedure involving the computation of the solution of two discrete-time Riccati equations. Since often the norm of these Riccati solutions approaches ∞ when γ approaches the optimal value, it is preferable to compute Lagrangian J-invariant subspaces of the associated symplectic matrices, see [35]. It is important to note that close to the optimal value of γ typically eigenvalues of the symplectic matrices approach the unit circle and hence a detailed perturbation theory is necessary to determine whether the Lagrangian invariant subspaces have been computed in a sufficiently accurate manner. A similar approach is carried out in the continuous time case where the computation of the optimal γ is improved by computing J-Lagrangian invariant subspaces of Hamiltonian matrices [55] or, what is even better, structured matrix pencils, see [3, 4].

1.4 Notation

In the following we denote by $\mathcal{J}_m(\lambda)$ an upper triangular $m \times m$ Jordan block with eigenvalue λ. By $\mathcal{J}_m(a \pm ib)$ we denote a quasi-upper triangular $m \times m$ real Jordan block with nonreal complex conjugate eigenvalues $\lambda, \overline{\lambda}$, i.e. m is even and the 2×2 blocks on the main block diagonal of $\mathcal{J}_m(a \pm ib)$ have the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

where $\lambda = a+ib$, $a, b \in \mathbb{R}$, $b \neq 0$. The transpose and the conjugate transpose of a matrix A will be denoted A^T and A^*, respectively, and we use the shorthand $A^{-T} := (A^{-1})^T$, $A^{-*} := (A^{-1})^*$.

The spectrum of a matrix $A \in \mathbb{F}^{2n \times 2n}$, i.e., the set of eigenvalues including possibly nonreal eigenvalues of real matrices, is denoted by $\sigma(A)$. We use a fixed matrix norm $\|\cdot\|$ throughout, namely, the spectral norm $\|\cdot\|_2$, i.e., the largest singular value.

We let

$$\mathcal{R}(A; \lambda) := \mathrm{Ker}\,(A - \lambda I)^{2n} \subseteq \mathbb{F}^{2n}$$

stand for the *root subspace* of a real matrix $A \in \mathbb{F}^{2n \times 2n}$ corresponding to an eigenvalue $\lambda \in \mathbb{F}$. If A is a real matrix, we also let

$$\mathcal{R}(A; a \pm ib) := \mathrm{Ker}\,\left(A^2 - 2aA + (a^2 + b^2)I\right)^{2n} \subseteq \mathbb{R}^{2n}$$

stand for the *root subspace* of A corresponding to a pair of nonreal complex conjugate eigenvalues $a \pm ib$ of A (here $a, b \in \mathbb{R}$, $b \neq 0$).

A block diagonal matrix with diagonal blocks $X_1, \ldots, X_q$ (in that order) is denoted by $X_1 \oplus X_2 \oplus \cdots \oplus X_q$.

If $\mathcal{M} \subseteq \mathbb{F}^m$ is a subspace, we denote by $\mathcal{M}^\perp$ the orthogonal complement of $\mathcal{M}$ with respect to the standard Euclidean metric in $\mathbb{F}^m$. $\mathbb{N}$ stands for the set of positive integers.

For most of the paper, we consider the following three cases separately.

$$\begin{aligned} \mathbb{F} &= \mathbb{R}, \text{ and } J \in \mathbb{R}^{2n\times 2n} \text{ is skew-symmetric and invertible;} && \text{(I)} \\ \mathbb{F} &= \mathbb{C}, \text{ and } J \in \mathbb{C}^{2n\times 2n} \text{ is skew-symmetric and invertible;} && \text{(II)} \\ \mathbb{F} &= \mathbb{C}, \text{ and } J \in \mathbb{C}^{2n\times 2n} \text{ is Hermitian with } n \text{ positive} && \text{(III)} \\ & \quad \text{and } n \text{ negative eigenvalues (counted with multiplicities).} \end{aligned}$$

1.5 Overview of the paper

We study various stability properties of S-invariant, J-Lagrangian subspaces under small perturbations of S and of both S and J, and some applications of these stability properties.

We start in Section 2 with a review of various stability concepts of invariant subspaces of matrices without symmetries. In Section 3 we begin our analysis of Lagrangian invariant subspaces by adapting the general concepts of stable invariant subspaces to the case of structured perturbations relevant to symplectic matrices and Lagrangian subspaces. The subsequent sections 4, 5, and 6 represent the core of the paper. Here, we provide detailed analysis, and in many cases also characterizations, of invariant Lagrangian subspaces with various stability properties. In Section 4 we study real J-symplectic matrices (case (I)), in Section 5 we study the case (II) of complex J-symplectic matrices, whereas Section 6 is devoted to complex J-unitary matrices (case (III)). We bring out much of the necessary background, including canonical forms, to make the presentation reasonably self-contained. Each of the sections 4, 5, and 6 is concluded with examples that illustrate the results of the respective section, as well as similarities and differences between the three situations. Particular cases of stable invariant Lagrangian subspaces with certain spectral properties, namely, such that the spectrum of the restriction to the subspace lies entirely inside the closed unit disk or entirely outside the open unit disk, are highlighted in Section 7. Such invariant Lagrangian subspaces play a key role in many applications.

For the readers' convenience and ease of reference, many sections are divided into subsections.

2 Stability of invariant subspaces

In this section we recall some general concepts of stability (introduced and first studied in [1, 8]) and α-stability (introduced and studied in [44, 46]), for invariant subspaces of general matrices. Many other notions of stability of invariant subspaces of matrices, as well as of linear bounded operators in infinite dimensional Banach spaces have been studied since; we mention here only a representative sample of books where this material is treated in some way and where further references can be found, [1, 15, 13, 22, 48], and papers which are most relevant to the present paper [38, 40, 42, 44, 46, 47], see also [23] for the related analysis of the Hamiltonian Schur form.

To study the stability properties of invariant subspaces, we make use of the *gap* between two subspaces $\mathcal{M} \subseteq \mathbb{F}^m$ and $\mathcal{N} \subseteq \mathbb{F}^m$, which is defined by

$$\text{gap}\,(\mathcal{M}, \mathcal{N}) := \|P_{\mathcal{M}} - P_{\mathcal{N}}\|,$$

where $P_{\mathcal{M}}$ is the orthogonal (with respect to the standard Euclidean inner product in $\mathbb{F}^m$) projection onto $\mathcal{M}$, and where $\|\cdot\|$ stands for the operator norm (the largest singular value). This notion is well-known in the literature, see, for example, [15, 52] for basic properties of the gap in the context of finite dimensional vector spaces. The following fact will be especially useful.

Lemma 2.1 *If $Q \in \mathbb{F}^{m\times m}$ is invertible and $\mathcal{M}, \mathcal{N} \subseteq \mathbb{F}^m$ are two subspaces, then*

$$\frac{1}{\kappa} \cdot \text{gap}\,(Q\mathcal{M}, Q\mathcal{N}) \leq \text{gap}\,(\mathcal{M}, \mathcal{N}) \leq \kappa \cdot \text{gap}\,(Q\mathcal{M}, Q\mathcal{N}),$$

where $\kappa := \|Q\| \cdot \|Q^{-1}\|$ is the condition number of Q.

For the proof, observe that $QP_{\mathcal{M}}Q^{-1}$, resp. $QP_{\mathcal{N}}Q^{-1}$, is a (not necessarily orthogonal) projection onto $Q\mathcal{M}$, resp. $Q\mathcal{N}$, and therefore (see for example [15, Theorem 13.1.1])

$$\text{gap}\,(Q\mathcal{M}, Q\mathcal{N}) \leq \|QP_{\mathcal{M}}Q^{-1} - QP_{\mathcal{N}}Q^{-1}\|,$$

which leads to

$$\text{gap}\,(Q\mathcal{M}, Q\mathcal{N}) \leq \|Q\| \cdot \|P_{\mathcal{M}} - P_{\mathcal{N}}\| \cdot \|Q^{-1}\| = \kappa \cdot \text{gap}\,(\mathcal{M}, \mathcal{N}),$$

as required.

Lemma 2.1 will allow us in many cases to use various canonical forms for J-symplectic matrices in proofs of our main results (Sections 4, 5, 6).

Definition 2.2 *Let $X \in \mathbb{F}^{m\times m}$ and let $\mathcal{M} \subseteq \mathbb{F}^m$ be an X-invariant subspace.*

1) $\mathcal{M}$ is said to be stable *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$X' \in \mathbb{F}^{m\times m} \quad \textit{and} \quad \|X - X'\| < \delta, \tag{2.1}$$

then there exists an X'-invariant subspace $\mathcal{M}' \in \mathbb{F}^m$ such that $\text{gap}\,(\mathcal{M}, \mathcal{M}') < \varepsilon$.

*2) $\mathcal{M}$ is said to be α-*stable *if there exist constants $\delta, K > 0$ (depending on X only) such that if (2.1) holds, then there exists an X'-invariant subspace $\mathcal{M}' \in \mathbb{F}^m$ such that*

$$\text{gap}\,(\mathcal{M}, \mathcal{M}') \leq K\big(\|X - X'\|\big)^{\frac{1}{\alpha}}.$$

3) We say that $\alpha \geq 1$ is the index of stability *of $\mathcal{M}$ if $\mathcal{M}$ is α-stable and is not β-stable for any β such that $1 \leq \beta < \alpha$.*

The following theorem is basic in the study of stable invariant subspaces. It was proved in [1, 8] in the complex case, and in [2] in the real case.

Theorem 2.3 *Let $X \in \mathbb{F}^{m\times m}$ and let $\mathcal{M} \subseteq \mathbb{F}^m$ be an X-invariant subspace.*

(a) *If $\mathbb{F} = \mathbb{C}$, then $\mathcal{M}$ is stable if and only if for every eigenvalue λ of X such that* $\dim \operatorname{Ker}(X - \lambda I) > 1$*, we have either*

$$\mathcal{M} \cap \mathcal{R}(X;\lambda) = \{0\} \quad \text{or} \quad \mathcal{M} \supseteq \mathcal{R}(X;\lambda);$$

(b) *If $\mathbb{F} = \mathbb{R}$, then $\mathcal{M}$ is stable if and only if*

(1) *for every pair of nonreal complex conjugate eigenvalues $\lambda \pm i\mu$ of X such that*

$$\dim \operatorname{Ker}(X^2 - 2\lambda X + (\lambda^2 + \mu^2)I) > 2$$

we have either

$$\mathcal{M} \cap \mathcal{R}(X;\lambda \pm i\mu) = \{0\} \quad \text{or} \quad \mathcal{M} \supseteq \mathcal{R}(X;\lambda \pm i\mu);$$

(2) *for every real eigenvalue λ of X such that* $\dim \operatorname{Ker}(X - \lambda I) > 1$*, we have either*

$$\mathcal{M} \cap \mathcal{R}(X;\lambda) = \{0\} \quad \text{or} \quad \mathcal{M} \supseteq \mathcal{R}(X;\lambda);$$

(3) *for every real eigenvalue λ of X such that* $\dim \operatorname{Ker}(X - \lambda I) = 1$ *and* $\dim \mathcal{R}(X;\lambda)$ *is even, the dimension of $\mathcal{M} \cap \mathcal{R}(X;\lambda)$ is also even.*

Given integers k, m, where $0 \leq k \leq m$, $m > 0$, we define

$$\alpha_{\mathbb{C}}(m,k) := \begin{cases} 1 & \text{if } k = 0 \text{ or } k = m; \\ m-1 & \text{if } 1 \leq k \leq m-1 \text{ and there exist } k \text{ distinct } m\text{-th} \\ & \text{roots of unity that sum up to } 0; \\ m & \text{in all other cases.} \end{cases} \tag{2.2}$$

Note that $\alpha_{\mathbb{C}}(m,k) = \alpha_{\mathbb{C}}(m,m-k)$. The following results on α-stability of invariant subspaces for complex matrices were proved in [46].

Theorem 2.4 *Let $\mathcal{J}_m(\lambda)$ be an $m \times m$ Jordan block with an eigenvalue $\lambda \in \mathbb{C}$. Then the (unique) k-dimensional $\mathcal{J}_m(\lambda)$-invariant subspace is stable and its index of stability is $\alpha_{\mathbb{C}}(m,k)$.*

Theorem 2.5 *Let $X \in \mathbb{C}^{m\times m}$, let $\lambda_1, \dots, \lambda_p$ be the pairwise distinct eigenvalues of X, and let $\mathcal{M} \subseteq \mathbb{C}^m$ be a stable X-invariant subspace. Then the index of stability of $\mathcal{M}$ is equal to*

$$\max_{j=1,2,\dots,p} \left\{ \alpha_{\mathbb{C}} \Big(\dim \mathcal{R}(X;\lambda_j), \dim \big(\mathcal{R}(X;\lambda_j) \cap \mathcal{M} \big) \Big) \right\}.$$

We need some additional notation to state the analogue of Theorem 2.5 for the real case. A finite set of complex numbers $\{\zeta_1, \cdots, \zeta_m\}$ will be called *zero sum self conjugate* if $\zeta_1 + \cdots + \zeta_m = 0$, and if the non-real elements of the set can be arranged in pairs of complex conjugate numbers. For two integers k and m, with $0 \le k \le m$, $m > 0$, we define

$$\alpha_{\mathbb{R}}(m,\ k) := \begin{cases} 1 & \text{if } k = 0 \text{ or } k = m; \\[2ex] m & \begin{array}{ll} \multicolumn{2}{l}{\text{if one of the following three cases holds:}} \\ \text{(i)} & 0 < k < m,\ m \text{ is odd and there is no zero sum} \\ & \text{self conjugate set of } k \text{ distinct } m\text{-th roots of } 1; \\ \text{(ii)} & m \text{ is even and } k \text{ is odd;} \\ \text{(iii)} & m \text{ is even and divisible by 4, } k \text{ is also even but not} \\ & \text{divisible by 4, and there is no zero sum self} \\ & \text{conjugate set of } k \text{ distinct } m\text{-th roots of } -1; \end{array} \\[2ex] m-1 & \text{in all other cases.} \end{cases} \tag{2.3}$$

For the real case, then in [43] the following result was proved.

Theorem 2.6 *Let $X \in \mathbb{R}^{m\times m}$ and let $\mathcal{M} \subseteq \mathbb{R}^m$ be a stable X-invariant subspace. Let $\lambda_1, \dots, \lambda_p$ denote the pairwise distinct real eigenvalues of X, and $\mu_1 \pm i\nu_1, \dots, \mu_q \pm i\nu_q$, $\mu_j, \nu_j \in \mathbb{R}$, $\nu_j \neq 0$, the pairwise distinct pairs of nonreal complex conjugate eigenvalues of X. Then the index of stability of $\mathcal{M}$ is equal to the largest of the two numbers*

$$\max_{j=1,2,\dots,p} \left\{ \alpha_{\mathbb{R}} \Big(\dim \mathcal{R}(X;\lambda_j), \dim \big(\mathcal{R}(X;\lambda_j) \cap \mathcal{M} \big) \Big) \right\}$$

and

$$\max_{j=1,2,\dots,q} \left\{ \alpha_{\mathbb{C}} \left(\frac{\dim \mathcal{R}(X;\mu_j \pm i\nu_j)}{2},\ \frac{\dim \left(\mathcal{R}(X;\mu_j \pm i\nu_j) \cap \mathcal{M} \right)}{2} \right) \right\}.$$

(The maximum of an empty set of numbers is here taken to be equal to 1*.)*

After recalling the stability results for general matrices, in the next section we present some basic results on the stability of J-Lagrangian subspaces.

3 Stability of J-Lagrangian invariant subspaces

We begin our analysis of Lagrangian invariant subspaces by adapting Definition 2.2 to the case of structured perturbations relevant to symplectic matrices and Lagrangian subspaces.

Definition 3.1 *Consider the cases* (I) *or* (II) *with* $S \in \mathbb{F}^{2n\times 2n}$ *being* J*-symplectic (resp., the case* (III) *with* $S \in \mathbb{C}^{2n\times 2n}$ *being* J*-unitary). Moreover, let* $\mathcal{M} \in \mathcal{IL}(S, J)$*.*

1. *The subspace* $\mathcal{M}$ *is called* stable, *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that if* J' *is skew-symmetric (resp., Hermitian with* n *positive and* n *negative eigenvalues) and if* S' *is* J'*-symplectic (resp.,* J'*-unitary) satisfying*
$$\|S - S'\| + \|J - J'\| < \delta,$$
then there exists $\mathcal{M}' \in \mathcal{IL}(S', J')$ *with the property that* $\operatorname{gap}(\mathcal{M}, \mathcal{M}') < \varepsilon$*.*

2. *The subspace* $\mathcal{M}$ *is called* J-stable, *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that if* S' *is* J*-symplectic (resp.,* J*-unitary) satisfying*
$$\|S - S'\| < \delta,$$
then there exists $\mathcal{M}' \in \mathcal{IL}(S', J)$ *with the property that* $\operatorname{gap}(\mathcal{M}, \mathcal{M}') < \varepsilon$*.*

3. $\mathcal{M}$ *is called* conditionally stable, *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that if* J' *is skew-symmetric (resp., Hermitian with* n *positive and* n *negative eigenvalues) and* S' *is* J'*-symplectic (resp.,* J'*-unitary) satisfying*
$$\|S - S'\| + \|J - J'\| < \delta,$$
then either $\mathcal{IL}(S', J') = \emptyset$ *or there exists* $\mathcal{M}' \in \mathcal{IL}(S', J')$ *with the property that* $\operatorname{gap}(\mathcal{M}, \mathcal{M}') < \varepsilon$*.*

4. $\mathcal{M}$ *is called* conditionally J-stable, *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that if* S' *is* J*-symplectic (resp.,* J*-unitary) satisfying*
$$\|S - S'\| < \delta,$$
then either $\mathcal{IL}(S', J) = \emptyset$ *or there exists* $\mathcal{M}' \in \mathcal{IL}(S', J)$ *with the property that* $\operatorname{gap}(\mathcal{M}, \mathcal{M}') < \varepsilon$*.*

5. *For a fixed* $\alpha \geq 1$*, the subspace* $\mathcal{M}$ *is called* α-stable, *if there exists* $\delta, K > 0$ *such that if* J' *is skew-symmetric (resp., Hermitian with* n *positive and* n *negative eigenvalues) and* S' *is* J'*-symplectic (resp.,* J'*-unitary) satisfying*
$$\|S - S'\| + \|J - J'\| < \delta,$$
then there exists $\mathcal{M}' \in \mathcal{IL}(S', J')$ *such that*
$$\operatorname{gap}(\mathcal{M}, \mathcal{M}') \leq K\big(\|S - S'\| + \|J - J'\|\big)^{\frac{1}{\alpha}}.$$

The concepts of J-α-stability, conditional α-stability, *and* conditional J-α-stability *are defined in the obvious way.*

Remark 3.2 The following obvious implications hold for subspaces $\mathcal{M} \in \mathcal{IL}(S, J)$:
$$\alpha\text{−stability} \overset{\alpha\leq\beta}{\Rightarrow} \beta\text{−stability} \ \Rightarrow\ \text{stability} \ \Rightarrow\ \text{conditional stability}.$$

3.1 Stability and J-stability

In this subsection, we will relate stability (in any sense) with J-stability (in the same sense). We start with the following lemma.

Lemma 3.3 (a) *Let $J \in \mathbb{F}^{2n\times 2n}$ be an invertible skew-symmetric matrix. Then there exist positive constants δ and K such that the implication*

$$J' \in \mathbb{F}^{2n\times 2n} \ \text{ skew-symmetric}, \quad \|J' - J\| < \delta \quad \Longrightarrow \quad J' = X^T J X$$

holds for some invertible $X \in \mathbb{F}^{2n\times 2n}$ satisfying the inequality $\|X - I\| \le K\|J' - J\|$.

(b) *Let $\mathbb{F} = \mathbb{C}$, and let $J \in \mathbb{C}^{2n\times 2n}$ be an invertible Hermitian matrix. Then there exist positive constants δ and K such that the implication*

$$J' \in \mathbb{C}^{2n\times 2n} \ \text{ Hermitian}, \quad \|J' - J\| < \delta \quad \Longrightarrow \quad J' = X^T J X$$

holds for some invertible $X \in \mathbb{C}^{2n\times 2n}$ satisfying the inequality $\|X - I\| \le K\|J' - J\|$.

Proof. The proof of part (b) follows easily by using the Lagrange algorithm for reduction of a sesquilinear form to a sum of squares (see [27], for example), or use a more direct approach of [36]. For part (a) use a standard reduction of a skew-symmetric form to a canonical structure (see, for example, [21, Section V.10]). $\square$

It should be noted that we do not need Lemma 3.3 (b) in its full generality but we will apply it only to the case that J has an equal number of negative and positive eigenvalues.

Theorem 3.4 *Consider cases* (I) *or* (II) *and let $S \in \mathbb{F}^{2n\times 2n}$ be J-symplectic (or consider case* (III) *and let $S \in \mathbb{C}^{2n\times 2n}$ be J-unitary, respectively). Then the subspace $\mathcal{M} \in \mathcal{IL}(S, J)$ is J-stable in any sense if and only if $\mathcal{M}$ is stable in the same sense.*

Proof. We give the proof for α-stability and for J-symplectic matrices, i.e., for the case that (I) or (II) holds. In all other cases the proof is similar. The 'if'-part is obvious. So let $\mathcal{M} \in \mathcal{IL}(S, J)$ be J-α-stable, where $J \in \mathbb{F}^{2n\times 2n}$ is an invertible skew-symmetric matrix. By assumption, there exist constants $\delta', K' > 0$ such that if S_0 is J-symplectic and

$$\|S - S_0\| < \delta',$$

then there exists $\mathcal{M}_0 \in \mathcal{IL}(S_0, J)$ such that

$$\text{gap}\,(\mathcal{M}, \mathcal{M}_0) \le K'\big(\|S - S_0\|\big)^{\frac{1}{\alpha}}. \tag{3.1}$$

Now assume that the matrix $S' \in \mathbb{F}^{2n\times 2n}$ is J'-symplectic, where $J' \in \mathbb{F}^{2n\times 2n}$ is skew-symmetric and invertible, and where

$$\delta_0 := \|J' - J\| + \|S' - S\|.$$

Here, δ_0 has to satisfy several restrictions in terms of inequalities. The first restriction is that $\delta_0 \leq \delta$, where δ is taken from Lemma 3.3. Then, by Lemma 3.3, there exists an invertible matrix X such that $J' = X^T J X$ and the inequality $\|X - I\| \leq K\|J' - J\|$ holds. Moreover, the matrix $S_0 := XS'X^{-1}$ is J-symplectic and satisfies

$$\|S_0 - S\| = \|XS'X^{-1} - S\| \leq \|X\| \cdot \|S'\| \cdot \|X^{-1} - I\| \ + \ \|X - I\| \cdot \|S'\| \ + \ \|S' - S\|.$$

Using the inequalities

$$\|X - I\| \leq K\delta_0, \quad \|X\| \leq 1 + K\delta_0, \quad \|I - X^{-1}\| \leq \frac{K\delta_0}{(1 - K\delta_0)},$$

where K is taken from Lemma 3.3, we obtain that

$$\|S_0 - S\| < (1 + K\delta_0)\big(\|S\| + \delta_0\big)\frac{K\delta_0}{(1 - K\delta_0)} \ + \ K\delta_0\big(\|S\| + \delta_0\big) \ + \ \delta_0 \ \leq \ K_1\delta_0, \quad (3.2)$$

where the positive constant K_1 depends only on $\|S\|$ and K.

Suppose now that δ_0 has been chosen so that the right hand side of (3.2) is less than δ'. Then (3.1) holds for some $\mathcal{M}_0 \in \mathcal{IL}(S_0, J)$. So we obtain

$$\operatorname{gap}(\mathcal{M}, \mathcal{M}_0) \ \leq \ K' K_1^{\frac{1}{\alpha}} \delta_0^{\frac{1}{\alpha}} \ = \ K' K_1^{\frac{1}{\alpha}} \big(\|J' - J\| + \|S' - S\|\big)^{\frac{1}{\alpha}}. \qquad (3.3)$$

Let $\mathcal{M}' := X^{-1}(\mathcal{M}_0)$. Then $\mathcal{M}' \in \mathcal{IL}(S', J')$ and

$$\operatorname{gap}(\mathcal{M}', \mathcal{M}_0) \leq \|X^{-1}PX - P\|,$$

where P is the orthogonal projection on $\mathcal{M}_0$ (this is a standard property of the gap metric, see [15, Theorem 13.1]). Arguing as in the proof of inequality (3.2), we see that

$$\operatorname{gap}(\mathcal{M}', \mathcal{M}_0) \ < \ K_2\delta_0 \ = \ K_2\left(\|J' - J\| + \|S' - S\|\right), \qquad (3.4)$$

for some constant K_2 which depends on K only. Combining (3.3) and (3.4), we obtain

$$\operatorname{gap}(\mathcal{M}', \mathcal{M}) \ \leq \ K_3\big(\|J' - J\| + \|S' - S\|\big)^{\frac{1}{\alpha}},$$

as required. $\square$

With Theorem 3.4 in hands, there is no need to distinguish between stability and J-stability. In the following, we will only mention stability results, since it is clear that corresponding results for J-stability can always be stated. On the other hand, we will often use the equivalence of stability and J-stability in proofs without further notice.

3.2 Localization principle and index of stability

An important tool in the stability analysis of invariant subspaces is the so-called *localization principle*. It will often allow us to reduce the proofs to the case when the matrices S have minimal (by inclusion) spectra.

Before stating the theorem, we make a useful remark concerning J-orthogonality of root subspaces. Two subspaces $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathbb{F}^{2n}$ are said to be *J-orthogonal* if $\langle Jx, y\rangle = 0$ (in cases (I) and (II)), or $\langle Jx, y\rangle_* = 0$ (in case (III)), for all $x \in \mathcal{M}_1$ and all $y \in \mathcal{M}_2$.

Remark 3.5 For J-symplectic matrices (in cases (I) and (II)) or J-unitary matrices (in case (III)), there are certain J-orthogonality relations between root subspaces of such matrices. We have in mind the following statement, for instance:

If $S \in \mathbb{C}^{2n\times 2n}$ is J-unitary, and if σ_1, σ_2 are nonempty parts of the spectrum of S such that $\sigma_1 \cap \sigma_2 = \emptyset$ and the sum, denoted $\mathcal{N}_j$, of the root subspaces of S corresponding to σ_j is J-nondegenerate in the sense that zero is the only vector in cN_j which is J-orthogonal to every vector in $\mathcal{N}_j$, for $j = 1, 2$, then the subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ are J-orthogonal.

This statement can be found in, e.g., [5, 16] and follows, as well as analogous statements for the cases (I) and (II), from the respective canonical forms, to be presented later in the paper. These statements could also be proved independently, but we refrain from doing so.

Theorem 3.6 (Localization principle) *Consider one of the cases* (I)–(III) *and let $S \in \mathbb{F}^{2n\times 2n}$. Suppose that*

$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_k \quad \text{and} \quad S = S_1 \oplus S_2 \oplus \cdots \oplus S_k, \tag{3.5}$$

where $J_i, S_i \in \mathbb{F}^{n_i \times n_i}$, $\sum_{i=1}^k n_i = 2n$, and where

$$\sigma(S_i) \cap \sigma(S_j) = \emptyset \quad \text{for} \quad i \neq j, \quad i, j = 1 \ldots, k. \tag{3.6}$$

Assume that a given subspace $\mathcal{M}$ is S-invariant and J-Lagrangian. Then n_i is even for $i = 1, \ldots, k$ and

$$\mathcal{M} = \widetilde{\mathcal{M}}_1 \dot{+} \widetilde{\mathcal{M}}_2 \dot{+} \cdots \dot{+} \widetilde{\mathcal{M}}_k, \tag{3.7}$$

where $\widetilde{\mathcal{M}}_i = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} : x_j \in \mathbb{F}^{n_j},\ j = 1, \ldots, k;\ x_j = 0 \text{ if } j \neq i;\ x_i \in \mathcal{M}_i \right\}$,

and where the subspace $\mathcal{M}_i$ is S_i-invariant and J_i-Lagrangian, for $i = 1, 2, \ldots, k$.

Moreover, in case S is J-symplectic (if (I)*) or* (II) *holds), or in case S is J-unitary (if* (III) *holds), then $\mathcal{M}$ is stable in any of the senses defined in Definition* 3.1 *if and only if $\mathcal{M}_i$ is stable in the same sense for $i = 1, 2, \ldots, k$.*

Proof. For notational simplicity, we assume $k = 2$ in the proof. Clearly, since S_1 and S_2 have no common eigenvalues, $\mathcal{M}$ is of the form (3.7) as in the assertion, where $\mathcal{M}_i$ is S_i-invariant. Evidently, the subspaces $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are J-orthogonal. From the fact that $\mathcal{M}$ is J-Lagrangian it follows that $\widetilde{\mathcal{M}}_i$ is J_i-Lagrangian for both $i = 1, 2$. In particular, this implies that n_i is even for $i = 1, 2$. (Notice that the decomposition (3.5) of J and the standing hypothesis that J is invertible already guarantees that the size of each J_i is even in cases (I) and (II).)

For the remainder of the proof, we only consider the symplectic case, i.e., the case that (I) or (II) holds. (In the case that (III) holds, the proof is virtually the same).

The "only if" part. Let S be J-symplectic, and assume that $\mathcal{M}$ is stable (in one of the senses given in Definition 3.1). Let S_i' be J_i'-symplectic and suppose these matrices are small perturbations of S_i and J_i (with the additional condition that there exists an S_i'-invariant J_i'-Lagrangian subspace if we are in the conditional case, or with the extra assumption that $J_i = J_i'$ if needs be). Form $S' = S_1' \oplus S_2'$ and $J' = J_1' \oplus J_2'$. Then these matrices are small perturbations of S and J, respectively. Since $\mathcal{M}$ is stable, there is an S'-invariant J-Lagrangian subspace $\mathcal{M}'$ close to $\mathcal{M}$ (in the sense indicated). Now S_1 and S_2 have no common eigenvalues, and hence, provided the perturbation is small enough, neither have S_1' and S_2'. Then $\mathcal{M}'$ decomposes as the direct sum of subspaces in the same way as $\mathcal{M}$ does. Moreover, it is easy to see that the inequality

$$\sup_{i=1,2} \operatorname{gap}(\mathcal{M}_i, \mathcal{M}_i') \leq \operatorname{gap}(\mathcal{M}, \mathcal{M}')$$

holds. From this inequality, we obtain that each $\mathcal{M}_i$ is stable in the same sense as $\mathcal{M}$ is.

The "if" part. We will give the proof for the case of stability, the other cases may be done similarly (for α-stability and its variations, one uses the perturbation theory for invariant subspaces, see for example [52, Theorem V.2.7]). Thus, let each of the $\mathcal{M}_i$'s be stable, and we have to prove that $\mathcal{M}$ is stable. By Theorem 3.4 it suffices to prove that $\mathcal{M}$ is J-stable. We consider a J-symplectic (or J-unitary as the case may be) matrix S' such that $\|S' - S\| < \delta$, where $\delta > 0$ is chosen sufficiently small.

Now let γ be a simple closed rectifiable contour such that $\sigma(S_1)$ is inside γ and $\sigma(S_2)$ is outside γ. For δ sufficiently small, γ will split the spectrum of S' as well. Let P' be the projection onto the sum of root subspaces of S' inside γ (the range of P') along the sum of root subspaces of S outside γ (the kernel of P'). Because of Remark 3.5, $\operatorname{Range} P'$ and $\operatorname{Ker} P'$ are J-orthogonal to each other. From now on argue as in the proof of [37, Theorem 3.1]. $\square$

To obtain more precise statements concerning α-stability, we recall the concept of *index of stability*, introduced and first studied in [41, 46].

Definition 3.7 *Consider the cases* (I) *or* (II) *and let $S \in \mathbb{F}^{2n\times 2n}$ be J-symplectic (resp., consider the case* (III) *and let $S \in \mathbb{C}^{2n\times 2n}$ be J-unitary). Suppose that the subspace $\mathcal{M} \in \mathcal{IL}(S, J)$ is α-stable for some $\alpha \geq 1$. We say that $\alpha_0 \geq 1$ is the* index of stability *of $\mathcal{M}$ if $\mathcal{M}$ is α_0-stable but is not α-stable for any α with $1 \leq \alpha < \alpha_0$.*

Analogously, the *index of conditional stability*, *the index of J-stability*, and *the index of conditional J-stability* are defined. By Theorem 3.4 the indices of stability and of J-stability coincide, and the indices of conditional stability and of conditional J-stability coincide as well.

We do not know whether (or not) the index of stability, or the index of conditional stability, always exists (provided the S-invariant J-Lagrangian subspace is α-stable, or conditionally α-stable, to start with). In all our statements, when a value of an index is given, it will be understood that the existence of the index is implied by the hypotheses of a statement. Note that α-stability with $\alpha = 1$ is often termed Lipschitz stability in the literature. We relate now the index of stability to the localization principle.

Theorem 3.8 *Under the hypotheses and notation of Theorem* 3.6*, suppose the subspace $\mathcal{M}_i$ is α-stable, resp., conditionally α-stable, with the index of stability, resp., of conditional stability, equal to κ_i, for $i = 1, 2, \ldots, k$. Then the index of stability of $\mathcal{M}$ is equal to* $\max\{\kappa_1, \ldots, \kappa_k\}$.

Proof. Theorem 3.8 follows from the proof of Theorem 3.6. □

In this section we have studied some general concepts for stability of J-Lagrangian invariant subspaces. We will now study the three cases (I)–(III) separately.

4 Case (I): real J-symplectic matrices

In this section we study the case (I), i.e., $J \in \mathbb{R}^{2n\times 2n}$ is a fixed invertible and skew-symmetric matrix, and $S \in \mathbb{R}^{2n\times 2n}$ is J-symplectic. All matrices in this section are assumed to be real. For the proofs of the main results of this section, it will be convenient first to formulate and prove corresponding results for J-Hamiltonian matrices that will be introduced in Subsection 4.1, where we also relate J-Hamiltonian and J-symplectic matrices via the Cayley transform and present canonical forms. Then Subsection 4.2 is dedicated to the analysis of Lagrangian subspaces for J-Hamiltonian matrices and Subsection 4.3 to the corresponding analysis in the J-symplectic case. Finally, we illustrate the main results with the help of a few examples in Subsection 4.4.

4.1 Canonical forms

A matrix $A \in \mathbb{R}^{2n\times 2n}$ is called *J-Hamiltonian* if $A^T J + JA = 0$. We first present the canonical form of real J-Hamiltonian matrices which is available in many sources, see, e.g., [24, 29], or [26, 54] in the framework of pairs of symmetric and skew-symmetric matrices. The following special notation is used:

$$\Sigma_k = \begin{bmatrix} 0 & \cdots & 0 & (-1)^0 \\ \vdots & \iddots & (-1)^1 & 0 \\ 0 & \iddots & & \vdots \\ (-1)^{k-1} & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & 0 \end{bmatrix} = (-1)^{k-1}\Sigma_k^T, \quad (4.1)$$

Thus, Σ_k is symmetric if k is odd, and skew-symmetric if k is even. Moreover, we use the skew-symmetric matrices $\Sigma_k \otimes \Sigma_2^k$. Here, $\otimes$ denotes the Kronecker (tensor) product $[a_{ij}] \otimes B = [a_{ij}B]$. For example, for $k = 1, 2, 3$, we have

$$\Sigma_1 \otimes \Sigma_2^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Sigma_2 \otimes \Sigma_2^2 = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}, \quad \Sigma_3 \otimes \Sigma_2^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 4.1 *Let A be real J-Hamiltonian. Then there exists a real, invertible matrix P such that $P^{-1}AP$ and P^TJP are block diagonal matrices*

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_s, \quad P^TJP = J_1 \oplus \cdots \oplus J_s, \quad (4.2)$$

where each diagonal block (A_j, J_j) is of one of the following five types:

(i) $$A_j = \mathcal{J}_{2n_1}(0) \oplus \cdots \oplus \mathcal{J}_{2n_p}(0), \quad J_j = \kappa_1\Sigma_{2n_1} \oplus \cdots \oplus \kappa_p\Sigma_{2n_p},$$

where $\kappa_1, \ldots, \kappa_p \in \{+1, -1\}$;

(ii) $$A_j = \begin{bmatrix} \mathcal{J}_{2m_1+1}(0) & 0 \\ 0 & -\mathcal{J}_{2m_1+1}(0)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{2m_q+1}(0) & 0 \\ 0 & -\mathcal{J}_{2m_q+1}(0)^T \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{2m_1+1} \\ -I_{2m_1+1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2m_q+1} \\ -I_{2m_q+1} & 0 \end{bmatrix};$$

(iii) $$A_j = \begin{bmatrix} \mathcal{J}_{\ell_1}(a) & 0 \\ 0 & -\mathcal{J}_{\ell_1}(a)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\ell_r}(a) & 0 \\ 0 & -\mathcal{J}_{\ell_r}(a)^T \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{\ell_1} \\ -I_{\ell_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_r} \\ -I_{\ell_r} & 0 \end{bmatrix},$$

where $a > 0$, and the number a, the total number $2r$ of Jordan blocks, and the sizes $\ell_1, \ldots, \ell_r$ depend on the particular diagonal block (A_j, J_j);

(iv) $A_j = \begin{bmatrix} \mathcal{J}_{2k_1}(a \pm ib) & 0 \\ 0 & -\mathcal{J}_{2k_1}(a \pm ib)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{2k_s}(a \pm ib) & 0 \\ 0 & -\mathcal{J}_{2k_s}(a \pm ib)^T \end{bmatrix}$,

$$J_j = \begin{bmatrix} 0 & I_{2k_1} \\ -I_{2k_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2k_s} \\ -I_{2k_s} & 0 \end{bmatrix},$$

where $a, b > 0$, and again the numbers a and b, the total number $2s$ of Jordan blocks, and the sizes $2k_1, \ldots, 2k_s$ depend on (A_j, J_j);

(v) $A_j = \mathcal{J}_{2h_1}(\pm ib) \oplus \cdots \oplus \mathcal{J}_{2h_t}(\pm ib)$, $\quad J_j = \eta_1(\Sigma_{h_1} \otimes \Sigma_2^{h_1}) \oplus \cdots \oplus \eta_t(\Sigma_{h_t} \otimes \Sigma_2^{h_t})$,

where $b > 0$ and $\eta_1, \ldots, \eta_t$ are signs ± 1. Again, the parameters b, t, $h_1, \ldots, h_t$, and $\eta_1, \ldots, \eta_t$ depend on the particular diagonal block (A_j, J_j).

There is at most one block each of type (i) *and* (ii)*. Furthermore, two blocks A_i and A_j of one of the types* (iii)–(v) *have nonintersecting spectra if $i \neq j$. Moreover, the form* (4.2) *is uniquely determined by the pair (A, J), up to a simultaneous permutation of diagonal blocks in the right hand sides of* (4.2)*.*

We see from Theorem 4.1 that there are signs $\kappa_i, \eta_j \in \{+1, -1\}$ associated with each even partial multiplicity of the zero eigenvalue and with each partial multiplicity corresponding to purely imaginary eigenvalues ib of A with $b > 0$. These signs are said to form the *sign characteristic* of the pair (A, J). This terminology was introduced in [12] in the context of selfadjoint matrices with respect to complex sesquilinear indefinite inner products, see also [11, 29].

It is well-known that J-Hamiltonian and J-symplectic matrices as well as their sign characteristics are closely related via the *Cayley transform.* If $M, N \in \mathbb{R}^{m \times m}$, if 1 is not an eigenvalue of M, and -1 is not an eigenvalue of N, then we set

$$\mathcal{C}_1(M) := (I - M)^{-1}(I + M), \quad \mathcal{C}_{-1}(N) := (I + N)^{-1}(I - N).$$

The following lemma is well known. Its proof can be found in many sources, see, e.g., [16, 24, 34].

Lemma 4.2 (Cayley transform) *Let $S \in \mathbb{R}^{2n \times 2n}$ be J-symplectic.*

(a) *If 1 is not an eigenvalue of S, then the matrix $A_1 := \mathcal{C}_1(S) = (I - S)^{-1}(I + S)$ is J-Hamiltonian and $+1, -1$ are not eigenvalues of A_1. Moreover, we have*

$$S = \mathcal{C}_1^{-1}(A_1) = (A_1 - I)(A_1 + I)^{-1}.$$

(b) *If -1 is not an eigenvalue of S, then the matrix $A_2 := \mathcal{C}_{-1}(S) = (I + S)^{-1}(I - S)$ is J-Hamiltonian and $+1, -1$ are not eigenvalues of A_2. Moreover, we have*

$$S = \mathcal{C}_{-1}^{-1}(A_2) = (I - A_2)(A_2 + I)^{-1}.$$

Using Theorem 4.1 and the Cayley transform, we obtain a canonical form for real J-symplectic matrices. Again, such canonical forms (in various presentations) are known, see [10, 28, 29, 31, 50]. Nevertheless, we provide an independent presentation, because we need a special version that allows us to easily relate the sign characteristics of J-Hamiltonian and J-symplectic matrices via the Cayley transform.

We need additional notation that we adopt from [50]. For $\varepsilon \in \{+1, -1\}$ let

$$\mathcal{T}_k(\varepsilon) := \mathcal{C}_{-\varepsilon}^{-1}\big(\mathcal{J}_k(0)\big) := \varepsilon \begin{bmatrix} 1 & (-1)2 & (-1)^2 2 & \dots & (-1)^{k-1}2 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & (-1)^2 2 \\ \vdots & & \ddots & \ddots & (-1)2 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{4.3}$$

and for $\omega \in \mathbb{C}$, $|\omega| = 1$, $\mathrm{Im}(\omega) > 0$, let $b = i(\omega+1)/(\omega-1)$ and

$$Q_1 = \frac{1}{b^2+1}\begin{bmatrix} b^2-1 & 2b \\ -2b & b^2-1 \end{bmatrix}, \quad Q_k = 2\left(\frac{-1}{b^2+1}\begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix}\right)^k, \quad k \geq 2,$$

$$\text{and} \quad \mathcal{T}_{2k}(\omega, \overline{\omega}) = \begin{bmatrix} Q_1 & Q_2 & \dots & Q_{k-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Q_2 \\ 0 & \dots & 0 & Q_1 \end{bmatrix}, \quad k \geq 1. \tag{4.4}$$

Clearly, $\mathcal{T}_k(\varepsilon)$ is similar to one Jordan block of size k associated with the eigenvalue ε and the real matrix $\mathcal{T}_{2k}(\omega, \overline{\omega})$ is similar (over the complex field) to a matrix with two Jordan blocks of size k associated with the unimodular eigenvalues ω and $\overline{\omega}$.

Theorem 4.3 *Let $S \in \mathbb{R}^{2n\times 2n}$ be J-symplectic. Then there exists a nonsingular matrix P such that $P^{-1}SP$ and P^TJP are block diagonal matrices*

$$P^{-1}SP = S_1 \oplus \cdots \oplus S_s, \quad P^TJP = J_1 \oplus \cdots \oplus J_s, \tag{4.5}$$

where each diagonal block (S_j, J_j) is of one of the following five types:

(i) $$S_j = \mathcal{T}_{2n_1}(\varepsilon) \oplus \cdots \oplus \mathcal{T}_{2n_p}(\varepsilon), \quad J_j = \kappa_1\Sigma_{2n_1} \oplus \cdots \oplus \kappa_p\Sigma_{2n_p},$$

where $\varepsilon, \kappa_1, \dots, \kappa_p \in \{+1, -1\}$, and the number ε and the parameters $2n_1, \dots, 2n_p$, and $\kappa_1, \dots, \kappa_p$ depend on the particular block (S_j, J_j);

(ii) $$S_j = \begin{bmatrix} \mathcal{T}_{2m_1+1}(\varepsilon) & 0 \\ 0 & \big(\mathcal{T}_{2m_1+1}(\varepsilon)\big)^{-T} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{T}_{2m_q+1}(\varepsilon) & 0 \\ 0 & \big(\mathcal{T}_{2m_q+1}(\varepsilon)\big)^{-T} \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{2m_1+1} \\ -I_{2m_1+1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2m_q+1} \\ -I_{2m_q+1} & 0 \end{bmatrix},$$

where $\varepsilon \in \{+1, -1\}$ and, again, the number ε and the sizes $2m_1+1, \dots, 2m_q+1$ depend on the particular diagonal block (S_j, J_j);

(iii) $$S_j = \begin{bmatrix} \mathcal{J}_{\ell_1}(\lambda) & 0 \\ 0 & \left(\mathcal{J}_{\ell_1}(\lambda)\right)^{-T} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\ell_r}(\lambda) & 0 \\ 0 & \left(\mathcal{J}_{\ell_r}(\lambda)\right)^{-T} \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{\ell_1} \\ -I_{\ell_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_r} \\ -I_{\ell_r} & 0 \end{bmatrix},$$

where $|\lambda| < 1$, *and the number* λ, *the total number* $2r$ *of Jordan blocks, and the sizes* $\ell_1, \dots, \ell_r$ *depend on the particular diagonal block* (S_j, J_j)*;*

(iv) $$S_j = \begin{bmatrix} \mathcal{J}_{2k_1}(\lambda \pm i\mu) & 0 \\ 0 & \left(\mathcal{J}_{2k_1}(\lambda \pm i\mu)\right)^{-T} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{2k_s}(\lambda \pm i\mu) & 0 \\ 0 & \left(\mathcal{J}_{2k_s}(\lambda \pm i\mu)\right)^{-T} \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{2k_1} \\ -I_{2k_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2k_s} \\ -I_{2k_s} & 0 \end{bmatrix},$$

where $|\lambda| < 1$, $\mu > 0$, *and again, the numbers* λ *and* μ, *the total number* $2s$ *of Jordan blocks, and the sizes* $2k_1, \dots, 2k_s$ *depend on* (S_j, J_j)*;*

(v) $S_j = \mathcal{T}_{2h_1}(\omega, \overline{\omega}) \oplus \cdots \oplus \mathcal{T}_{2h_t}(\omega, \overline{\omega})$, $\quad J_j = \eta_1(\Sigma_{h_1} \otimes \Sigma_2^{h_1}) \oplus \cdots \oplus \eta_t(\Sigma_{h_t} \otimes \Sigma_2^{h_t})$,

where $\omega \in \mathbb{C}$, $|\omega| = 1$, $\mathrm{Im}(\omega) > 0$, *and* $\eta_1, \dots, \eta_t \in \{+1, -1\}$. *Again, the numbers* ω, t, $h_1, \dots, h_t$, *and* $\eta_1, \dots, \eta_t$ *depend on the particular diagonal block* (S_j, J_j).

There is at most one block of type (i) *(and at most one block of type* (ii)*, respectively) associated with the same eigenvalue. Furthermore, two blocks* S_i *and* S_j *of one of the types* (iii)–(v) *have nonintersecting spectra if* $i \neq j$. *Moreover, the form* (4.5) *is uniquely determined by the pair* (S, J)*, up to a simultaneous permutation of diagonal blocks in the right hand sides of* (4.5)*.*

Proof. If 1 is an eigenvalue of S, then without loss of generality we may assume that $S = S_{11} \oplus S_{22}$, where $S_{11} \in \mathbb{R}^{m \times m}$, $\sigma(S_{11}) = \{1\}$, and $\sigma(S_{11}) \cap \sigma(S_{22}) = \emptyset$. (Otherwise, apply a transformation $(S, J) \mapsto (P^{-1}SP, P^TJP)$.) Partition J conformably, i.e.,

$$J = \begin{bmatrix} J_{11} & J_{12} \\ -J_{12}^T & J_{22} \end{bmatrix}, \quad J_{11} \in \mathbb{R}^{m \times m}.$$

Then the equality $S^TJS = J$ implies, in particular, that $S_{11}^T J_{12} S_{22} = J_{12}$, or, equivalently, $J_{12}S_{22} = S_{11}^{-T}J_{12}$. This is a Sylvester equation that only has the trivial solution $J_{12} = 0$, because the spectra of S_{11} and S_{22} do not intersect, a well-known fact, see, e.g., [27]. Then, since J is invertible and skew-symmetric, so must be J_{11} and J_{22}, i.e., m is necessarily even. For the remainder of the proof, we may consider the blocks (S_{ii}, J_{ii}), $i = 1, 2$ separately, i.e., we may assume that either $\sigma(S) = \{1\}$ or that S does not have the eigenvalue 1.

Case (1): Suppose that $1 \notin \sigma(S)$. Then $A = \mathcal{C}_1(S) = (I - S)^{-1}(I + S)$ is J-Hamiltonian by Lemma 4.2. Let P be invertible such that $P^{-1}AP$ is in the canonical form of Theorem 4.1. Since with $P^{-1}AP$ also $P^{-1}SP = \mathcal{C}_1^{-1}(P^{-1}AP)$ is block diagonal, it is sufficient to consider the blocks of different types in the canonical form of Theorem 4.1 separately. Thus, without loss of generality assume that $A = A_i$, where A_i is of one of the types (i)–(v) of Theorem 4.1.

(i) If A is a block of type (i), then a straightforward computation reveals that S is a block of type (i), where $\varepsilon = -1$ and where the parameters $2n_1, \ldots, 2n_p$ and $\kappa_1, \ldots, \kappa_p$ coincide with those in Theorem 4.1.

(ii) If A is a block of type (ii), then a straightforward computation shows that

$$S = \begin{bmatrix} \mathcal{T}_{2m_1+1}(-1) & 0 \\ 0 & \big(\mathcal{T}_{2m_1+1}(-1)\big)^{-T} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{T}_{2m_q+1}(-1) & 0 \\ 0 & \big(\mathcal{T}_{2m_q+1}(-1)\big)^{-T} \end{bmatrix}.$$

Let P_{2m_i+1} be invertible such that $P_{2m_i+1}^{-1}\mathcal{T}_{2m_i+1}(-1)P_{2m_i+1} = \mathcal{J}_{2m_i+1}(-1)$ and set

$$P = \begin{bmatrix} P_{2m_1+1} & 0 \\ 0 & P_{2m_1+1}^{-T} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} P_{2m_q+1} & 0 \\ 0 & P_{2m_q+1}^{-T} \end{bmatrix}.$$

Then $P^TJP = J$ and we obtain that $(P^{-1}SP, P^TJP)$ is a block of type (ii) with $\varepsilon = -1$.

(iii) If A is the block of type (iii), then S consists of diagonal blocks of the form

$$\mathcal{C}_1\big(\mathcal{J}_{\ell_i}(a)\big) \oplus \mathcal{C}_1\big(-\mathcal{J}_{\ell_i}(a)^T\big), \quad a > 0.$$

It is straightforward to verify that $\mathcal{C}_1(\mathcal{J}_{\ell_i}(a))$ is similar to a Jordan block $\mathcal{J}_{\ell_i}(\lambda)$, where $\lambda = (1-a)/(1+a) < 1$. Thus, as in the previous case, we find a transformation matrix P such that $(P^{-1}SP, P^TJP)$ is a block of type (iii), where $\lambda = (1-a)/(1+a) < 1$ and the other parameters coincide with those in Theorem 4.1.

(iv) If A is the block of type (iv), then analogous to the previous case, we find that there is a transformation matrix P such that $(P^{-1}SP, P^TJP)$ is a block of type (iv) associated with the eigenvalues $\lambda \pm i\mu$, where

$$\lambda = \mathrm{Re}\left(\frac{a+ib-1}{a+ib+1}\right) = \frac{a^2+b^2-1}{a^2+b^2+2a+1}, \quad \mu = \mathrm{Im}\left(\frac{a+ib-1}{a+ib+1}\right) = \frac{2b}{a^2+b^2+2a+1},$$

and the other parameters coincide with those in Theorem 4.1.

(v) If A is the block of type (v), then a straightforward but tedious computation (see [50] for complete details) reveals that (S, J) is a block of type (v) associated with the eigenvalues $\omega, \overline{\omega}$, where

$$\omega = \frac{b^2 - 1 + 2bi}{b^2 + 1}$$

and the other parameters coincide with those in Theorem 4.1.

Case (2): $\sigma(S) = \{1\}$. Then, $A = (I+S)^{-1}(I-S)$ is J-Hamiltonian by Lemma 4.2, and analogous to Case (1), we may assume that A is either a block of type (i) or of type (ii). Then $S = \mathcal{C}_{-1}(A)$ and we obtain that S is either a block of type (i) or (after an appropriate transformation $(S, J) \mapsto (P^{-1}SP, P^T JP)$) a block of type (ii) with $\varepsilon = 1$ and where the other parameters coincide with those in Theorem 4.1. □

We see from Theorem 4.3 that there are signs ± 1 associated with each even partial multiplicity corresponding to the eigenvalues $+1$ and -1 of S in the real Jordan form of S, as well as to each partial multiplicity corresponding to the real Jordan of complex conjugate pairs of nonreal unimodular eigenvalues of S. As for J-Hamiltonian matrices, these signs are said to form the *sign characteristic* of the pair (S, J). As a by-product from the proof of Theorem 4.1, we obtain that the sign characteristics of a symplectic matrix S and its Cayley transform are related in a simple way.

Lemma 4.4 *If* 1 *(resp.* -1*) is not an eigenvalue of* S*, then the sign characteristics of* S *as a* J*-symplectic matrix and of* $A = \mathcal{C}_1(S)$ *(resp.* $A = \mathcal{C}_{-1}(S)$*) as a* J*-Hamiltonian matrix are related as follows:*

(a) *If* $2n_1, \ldots, 2n_p$ *are the even partial multiplicities of the eigenvalue* -1 *(resp.* 1*) of* S *with corresponding signs* $\kappa_1, \ldots, \kappa_p$ *and if* $2\tilde{n}_1, \ldots, 2\tilde{n}_{\tilde{p}}$ *are the even partial multiplicities of the eigenvalue* 0 *of* A *with corresponding signs* $\tilde{\kappa}_1, \ldots, \tilde{\kappa}_{\tilde{p}}$*, then* $p = \tilde{p}$ *and there exists a permutation* π *on* $\{1, 2, \ldots, p\}$ *such that*

$$2n_i = 2\tilde{n}_{\pi(i)} \quad \text{and} \quad \kappa_i = \tilde{\kappa}_{\pi(i)}, \quad i = 1, \ldots, p.$$

(b) *If* $h_1, \ldots, h_t$ *are the partial multiplicities of the unimodular eigenvalue* ω *of* S*, where* $\mathrm{Im}(\omega) > 0$*, with corresponding signs* $\eta_1, \ldots, \eta_t$ *and if* $\tilde{h}_1, \ldots, \tilde{h}_{\tilde{t}}$ *are the partial multiplicities of the purely imaginary eigenvalue* $(1+\omega)/(1-\omega)$ *of* A *with corresponding signs* $\tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{t}}$ *then* $t = \tilde{t}$ *and there exists a permutation* π *such that*

$$h_i = \tilde{h}_{\pi(i)} \quad \text{and} \quad \eta_i = \tilde{\eta}_{\pi(i)}, \quad i = 1, \ldots, t.$$

4.2 J-Hamiltonian matrices: Stability of Lagrangian subspaces

In this subsection, we present results on stability of invariant Lagrangian subspaces of Hamiltonian matrices. The main part of these results was proved already in [40]. We start with a criterion for existence of invariant Lagrangian subspaces (Theorem 3.1 in [40]).

Theorem 4.5 *Let* A *be a real* J*-Hamiltonian matrix. Then there exists a (real)* A*-invariant* J*-Lagrangian subspace if and only if for every nonzero purely imaginary eigenvalue* ib*,* $b > 0$*, of* A*, the number of odd partial multiplicities corresponding to* ib *is even, and the signs in the sign characteristic of* A *that correspond to these odd partial multiplicities sum up to zero.*

We also recall a result on the uniqueness of invariant J-Lagrangian subspaces (Theorem 3.2 in [40]).

Theorem 4.6 *Let A be a real J-Hamiltonian matrix. Then there exists a unique (real) A-invariant J-Lagrangian subspace $\mathcal{M}$ with $\sigma(S|_{\mathcal{M}})$ contained in the closed left half plane if and only if the following conditions are satisfied:*

(a) *the eigenvalue zero of A only has even partial multiplicities, say $2n_1, \ldots, 2n_p$, and if $\kappa_1, \ldots, \kappa_p$ are the corresponding signs, then*

$$(-1)^{n_1}\kappa_1 = (-1)^{n_2}\kappa_2 = \cdots = (-1)^{n_p}\kappa_p.$$

(b) *for every eigenvalue ib, $b > 0$, of A, all partial multiplicities are even and the signs in the sign characteristic of A corresponding to ib are equal (however, for $b_1 \neq b_2$, $b_1, b_2 > 0$, the signs corresponding to ib_1 and ib_2 need not be the same);*

In this case, the subspace $\mathcal{M}$ is conditionally stable.

The definitions of various types of stability of invariant Lagrangian subspaces of real J-Hamiltonian matrices are completely analogous to the definitions given in Section 3 for invariant Lagrangian subspaces for real J-symplectic matrices (one just substitutes "Hamiltonian" for "symplectic").

Let A be a real J-Hamiltonian matrix, and let $\lambda_1, \ldots, \lambda_p$ be are all pairwise distinct positive eigenvalues of A, and let $\mu_1 \pm i\nu_1, \ldots, \mu_q \pm i\nu_q$ ($\mu_j, \nu_j > 0$) be are all distinct pairs of nonreal complex conjugate eigenvalues of A with positive real parts.

If $\mathcal{M} \subseteq \mathbb{R}^{2n}$ is a subspace then we introduce

$$\Gamma_{\mathcal{M}}(A) = \max\left\{ \max_{j=1,2,\ldots,p} \left\{\alpha_{\mathbb{R}}\big(\dim \mathcal{R}(A;\lambda_j), \dim\left(\mathcal{R}(A;\lambda_j) \cap \mathcal{M}\right)\big)\right\}, \right.$$
$$\left. \max_{j=1,2,\ldots,q} \left\{\alpha_{\mathbb{C}}\left(\frac{\dim \mathcal{R}(A;\mu_j \pm i\nu_j)}{2}, \frac{\dim\left(\mathcal{R}(A;\mu_j \pm i\nu_j) \cap \mathcal{M}\right)}{2}\right)\right\} \right\} \tag{4.6}$$

with the understanding that the maximum of the empty set is taken to be equal 1.

Theorem 4.7 *Let A be a real J-Hamiltonian matrix.*

(i) *There exists a conditionally stable A-invariant J-Lagrangian subspace if and only if the statements* (a) *and* (b) *of Theorem* 4.6 *are satisfied.*

(ii) *There exists a stable A-invariant J-Lagrangian subspace if and only if A has no purely imaginary or zero eigenvalues.*

(iii) *If the conditions in* (i)*, resp. in* (ii)*, are satisfied, then an A-invariant J-Lagrangian subspace $\mathcal{M}$ is conditionally stable, resp. stable, if and only if the following properties are satisfied:*

(c) *For every nonzero real eigenvalue λ of A with geometric multiplicity greater than one, either $\mathcal{M} \cap \mathcal{R}(A;\lambda) = \{0\}$ or $\mathcal{M} \supseteq \mathcal{R}(A;\lambda)$ holds. In this case*

$$\mathcal{M} \cap \mathcal{R}(A;\lambda) = \{0\} \quad \textit{if and only if} \quad \mathcal{M} \supseteq \mathcal{R}(A;-\lambda).$$

(d) *For every nonzero real eigenvalue λ of A with geometric multiplicity one and even algebraic multiplicity, the subspace $\mathcal{M}\cap\mathcal{R}(A;\lambda)$ is even dimensional. In this case,*

$$\mathcal{M} \cap \mathcal{R}(A;-\lambda) = \Big(J\big(\mathcal{M} \cap \mathcal{R}(A;\lambda)\big)\Big)^{\perp} \cap \mathcal{R}(A;-\lambda).$$

(e) *For every pair of nonreal complex conjugate eigenvalues $a \pm ib$ of A with nonzero real part a such that the geometric multiplicity of $a+ib$ is greater than one, either $\mathcal{M} \cap \mathcal{R}(A;a\pm ib) = \{0\}$ or $\mathcal{M} \supseteq \mathcal{R}(A;a\pm ib)$ holds. In this case*

$$\mathcal{M} \cap \mathcal{R}(A;a\pm ib) = \{0\} \quad \textit{if and only if} \quad \mathcal{M} \supseteq \mathcal{R}(A;-a\pm ib).$$

(iv) *Suppose that A has no purely imaginary or zero eigenvalues and let $\mathcal{M}$ be a stable A-invariant J-Lagrangian subspace. Then the index of stability of $\mathcal{M}$ is equal to $\Gamma_{\mathcal{M}}(A)$, where $\Gamma_{\mathcal{M}}(A)$ is given in* (4.6).

Proof. Parts (i) - (iii) follow from [40, Theorem 3.4] combined with [40, Proposition 3.3]. For the proof of (iv), by Theorem 3.6 (applied to a J-Hamiltonian rather than J-symplectic matrices), we may assume that either

(1) $\sigma(A) = \{\lambda, -\lambda\}$, where $\lambda \in \mathbb{R} \setminus \{0\}$, or

(2) $\sigma(A) = \{a \pm ib, -a \pm ib\}$, where $a, b > 0$.

In either of these two cases, the canonical form of Theorem 4.1 shows that without loss of generality we may further assume that

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & -A_0^T \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix},$$

where $\sigma(A_0) = \{\lambda\}$ or $\sigma(A_0) = \{a \pm ib\}$, as the case may be. Clearly, any A-invariant J-Lagrangian subspace $\mathcal{M}$ has the form

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_0 \\ (\mathcal{M}_0)^{\perp} \end{bmatrix},$$

where $\mathcal{M}_0$ is an arbitrary A_0-invariant subspace. It is easy to see that the index of stability of the A-invariant J-Lagrangian subspace $\mathcal{M}$ coincides with the index of stability of $\mathcal{M}_0$ as an A_0-invariant subspace. It remains to apply Theorem 2.6 to finish the proof. $\square$

Determination of the index of conditional stability of a conditionally stable A-invariant J-Lagrangian subspace is a challenging problem, and we do not have a complete solution. We present one result in this direction in a particular (but generic) case when the geometric multiplicity of every purely imaginary or zero eigenvalue of A is equal to one, see also [6, 7]. For the proof we need the following lemma.

Lemma 4.8 *Let $A \in \mathbb{R}^{2n\times 2n}$ be a J-Hamiltonian matrix such that $\sigma(A) \subseteq i\mathbb{R}$ and the geometric multiplicity of every eigenvalue of A is equal to* 1*. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every J-Hamiltonian matrix B that satisfies $\|B - A\| < \delta$ and has an invariant J-Lagrangian subspace, there is also a B-invariant J-Lagrangian subspace $\mathcal{N}$ with the property that* $\operatorname{gap}(\mathcal{N}, \mathcal{M}) < \varepsilon$.

Proof. To give a proof by contradiction, assume that there exists $\varepsilon_0 > 0$ such that for some sequence of J-Hamiltonian matrices $B_m \to A$, as $m \to \infty$, we have that the set of B_m-invariant J-Lagrangian subspaces is nonempty, and the inequality

$$\operatorname{gap}(\mathcal{N}_m, \mathcal{M}) \geq \varepsilon_0 \tag{4.7}$$

holds for every B_m-invariant J-Lagrangian subspace $\mathcal{N}_m$. Selecting one such $\mathcal{N}_m$ for every m, and passing to a subsequence if necessary, we may assume that

$$\lim_{m\to\infty} \operatorname{gap}(\mathcal{N}_m, \mathcal{N}) = 0.$$

(Here the compactness of the set of subspaces of a finite dimensional real vector space in the gap metric was used.) The subspace $\mathcal{N}$ is easily seen to be J-Lagrangian and (in view of $B_m \to A$) A-invariant; however, in view of (4.7), we have also $\mathcal{N} \neq \mathcal{M}$. But given the assumption $\sigma(A) \subseteq i\mathbb{R}$ on the spectrum of A, it follows from Theorem 4.6 that $\mathcal{M}$ is unique as an A-invariant J-Lagrangian subspace, a contradiction. □

Using this lemma we can prove the following result.

Theorem 4.9 *Let $A \in \mathbb{R}^{2n\times 2n}$ be a J-Hamiltonian matrix. Assume that the geometric multiplicity of every purely imaginary or zero eigenvalue of A is equal to one. Denote by κ the largest partial multiplicity of any purely imaginary or zero eigenvalue of A, and let $\kappa = 1$ if $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $\mathcal{M}$ be a conditionally stable A-invariant J-Lagrangian subspace (in particular, the partial multiplicity corresponding to every purely imaginary or zero eigenvalue is even, by Theorem* 4.7*). Then:*

(i) *The index of conditional stability of $\mathcal{M}$ does not exceed* $\max\{\Gamma_{\mathcal{M}}(A), \kappa\}$.

(ii) *If $\kappa = 2$, then the index of conditional stability of $\mathcal{M}$ is equal to* $\max\{\Gamma_{\mathcal{M}}(A), 2\}$.

Proof. We first consider part (i). By Theorem 3.6, applied to J-Hamiltonian matrices, we may assume without loss of generality that one of the following cases occurs:

(1) $\sigma(A) \subseteq i\mathbb{R}$;

(2) $\sigma(A) = \{\lambda, -\lambda\}$, where $\lambda \in \mathbb{R} \setminus \{0\}$;

(3) $\sigma(A) = \{a \pm ib, -a \pm ib\}$, where $a, b > 0$.

In the cases (2) and (3), by Theorem 4.7 conditional stability of $\mathcal{M}$ coincides with (unconditional) stability, and we are done by Theorem 4.7 (iv).

In case (1), observe that by Theorem 4.6 the A-invariant J-Lagrangian subspace is unique, and therefore must coincide with $\mathcal{M}$.

On the other hand, it has been shown in [44, Theorem 2.4] that the subspace $\mathcal{M}$ is *strongly* κ-stable as a real invariant subspace of the real matrix A (see [39, 44, 47] for the concept and properties of strong stability and strong κ-stability). In view of Lemma 4.8, this means that for every J-Hamiltonian B which is sufficiently close to A and has a B-invariant J-Lagrangian subspace, there also exists a B-invariant J-Lagrangian subspace $\mathcal{N}$ with the property that

$$\operatorname{gap}(\mathcal{M}, \mathcal{N}) \leq K \|B - A\|^{\frac{1}{\kappa}},$$

where K is a constant. This proves that the index of conditional stability of $\mathcal{M}$ does not exceed κ.

For the proof of part (ii), we may assume that either

(a) A is in real Jordan form with $\sigma(A) = \{\pm ib\}$, where $b > 0$, or

(b) A is a nilpotent Jordan block.

In case (b) we may assume that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{M} = \operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $\varepsilon > 0$ consider

$$A(\varepsilon) = \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix}, \quad \mathcal{M}_{\pm}(\varepsilon) = \operatorname{span} \begin{bmatrix} 1 \\ \pm\varepsilon \end{bmatrix}$$

Then there are two $A(\varepsilon)$-invariant J-Lagrangian subspaces, they are both real and coincide with $\mathcal{M}_{\pm}(\varepsilon)$. We see that $\operatorname{gap}\big(\mathcal{M}, \mathcal{M}_{\pm}(\varepsilon)\big) \sim \sqrt{\varepsilon} = \|A - A(\varepsilon)\|^{\frac{1}{2}}$ and thus, the index of conditional stability is larger than or equal to 2 and hence it is actually equal to 2.

For the case (a) we may assume that

$$A = \begin{bmatrix} 0 & b & 1 & 0 \\ -b & 0 & 0 & 1 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{M} = \operatorname{span}\{e_1, e_2\}.$$

Consider the perturbation $A(\varepsilon)$ of A obtained by putting $\varepsilon > 0$ in the entries (3,1) and (4,2). Then $A(\varepsilon)$ is J-Hamiltonian, and the eigenvalues of $A(\varepsilon)$ are $\pm\sqrt{\varepsilon}+bi, \pm\sqrt{\varepsilon}-bi$, with eigenvectors of the form $\begin{bmatrix} 1 & i & \pm\sqrt{\varepsilon} & \pm\sqrt{\varepsilon}i \end{bmatrix}^T$, and $\begin{bmatrix} 1 & -i & \pm\sqrt{\varepsilon} & \mp\sqrt{\varepsilon}i \end{bmatrix}^T$. There are just two real $A(\varepsilon)$-invariant subspaces of dimension two, and they happen to be Lagrangian as well, namely

$$\mathcal{M}_+(\varepsilon) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \sqrt{\varepsilon} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \sqrt{\varepsilon} \end{bmatrix} \right\}, \quad \mathcal{M}_-(\varepsilon) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -\sqrt{\varepsilon} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{\varepsilon} \end{bmatrix} \right\}.$$

Again, we observe that $\operatorname{gap}(\mathcal{M}, \mathcal{M}_\pm(\varepsilon)) \sim \sqrt{\varepsilon} = \|A - A(\varepsilon)\|^{\frac{1}{2}}$. So the index of conditional stability is again larger than or equal to two. As it was already shown to be less than or equal to two, it must be equal to two. $\Box$

Having presented stability results for J-Lagrangian subspaces of J-Hamiltonian matrices, in the next subsection we easily obtain corresponding results for J-symplectic matrices.

4.3 Stability of Lagrangian subspaces of J-symplectic matrices

In this subsection, we present conditions for existence of invariant Lagrangian subspaces of a symplectic matrix $S \in \mathbb{R}^{2n \times 2n}$. The results are direct consequences of the corresponding results in Section 4.2. Indeed, applying Theorem 3.6, we may assume that either $-1 \notin \sigma(S)$ or $\sigma(S) = \{-1\}$. Then applying the Cayley transform from Lemma 4.2 to S and keeping in mind the relations for the sign characteristics of S and of $\mathcal{C}_{\pm 1}(S)$ given in Lemma 4.4, we obtain the following results as immediate consequences of Theorems 4.5, 4.6, 4.7, and 4.9.

Theorem 4.10 *Let S be a real J-symplectic matrix. Then there exists a (real) S-invariant J-Lagrangian subspace if and only if for every eigenvalue $\omega \in \mathbb{C} \setminus \mathbb{R}$, $|\omega| = 1$, $\operatorname{Im}(\omega) > 0$ of S, the number of odd partial multiplicities corresponding to ω is even, and the signs in the sign characteristic of S that correspond to these odd partial multiplicities sum up to zero.*

Theorem 4.11 *Let S be a real J-symplectic matrix. Then there exists a unique (real) S-invariant J-Lagrangian subspace $\mathcal{M}$ with $\sigma(S|_{\mathcal{M}})$ contained in the closed unit disc if and only if the following conditions are satisfied:*

(a) *the eigenvalue 1 of S only has even partial multiplicities, say $2n_1, \ldots, 2n_p$, and if $\kappa_1, \ldots, \kappa_p$ are the corresponding signs, then*

$$(-1)^{n_1}\kappa_1 = (-1)^{n_2}\kappa_2 = \cdots = (-1)^{n_p}\kappa_p.$$

(b) *the eigenvalue -1 of S only has even partial multiplicities, say $2n'_1, \ldots, 2n'_{p'}$, and if $\kappa'_1, \ldots, \kappa'_{p'}$ are the corresponding signs, then*

$$(-1)^{n'_1}\kappa'_1 = (-1)^{n'_2}\kappa'_2 = \cdots = (-1)^{n'_{p'}}\kappa'_{p'}.$$

(c) *for every eigenvalue* $\omega \in \mathbb{C} \setminus \mathbb{R}$, $|\omega| = 1$, $\mathrm{Im}(\omega) > 0$ *of* S, *all partial multiplicities are even and the signs in the sign characteristic of* S *corresponding to* ω *are equal (however, for* $\omega_1 \neq \omega_2$ *with* $|\omega_1| = |\omega_2| = 1$, $\mathrm{Im}(\omega_1) > 0$, $\mathrm{Im}(\omega_2) > 0$, *the signs corresponding to* ω_1 *and* ω_2 *need not be the same);*

In this case, the subspace $\mathcal{M}$ *is conditionally stable.*

We need additional notation for the next result that provides complete description of stable and conditionally stable S-invariant J-Lagrangian matrices, for real J-symplectic matrices S. In analogy to (4.6), if $\lambda_1, \ldots, \lambda_p$ are all distinct real eigenvalues of S with modulus greater than 1, and if $\mu_1 \pm i\nu_1, \ldots, \mu_q \pm i\nu_q$ ($\mu_j, \nu_j > 0$) are all distinct pairs of nonreal complex conjugate eigenvalues of S with modulus greater than 1, then we introduce

$$\Gamma_{\mathcal{M}}(S) = \max \left\{ \max_{j=1,2,\ldots,p} \left\{ \alpha_{\mathbb{R}} \big(\dim \mathcal{R}(S;\lambda_j), \dim \left(\mathcal{R}(S;\lambda_j) \cap \mathcal{M} \right) \big) \right\}, \right. \tag{4.8}$$
$$\left. \max_{j=1,2,\ldots,q} \left\{ \alpha_{\mathbb{C}} \left(\frac{\dim \mathcal{R}(S;\mu_j \pm i\nu_j)}{2}, \frac{\dim \left(\mathcal{R}(S;\mu_j \pm i\nu_j) \cap \mathcal{M} \right)}{2} \right) \right\} \right\}$$

with the understanding that the maximum of the empty set is taken to be equal 1.

Theorem 4.12 *Let* S *be a real* J*-symplectic matrix.*

(i) *There exists a conditionally stable* S*-invariant* J*-Lagrangian subspace if and only if the conditions* (a), (b), *and* (c) *of Theorem* 4.11 *are satisfied.*

(ii) *There exists a stable* S*-invariant* J*-Lagrangian subspace if and only if* S *has no unimodular eigenvalues.*

(iii) *If the conditions of* (i), *resp. of* (ii), *are satisfied, then an* S*-invariant* J*-Lagrangian subspace* $\mathcal{M}$ *is conditionally stable, resp. stable, if and only if the following properties are satisfied:*

(d) *For every real eigenvalue* $\lambda \neq \pm 1$ *of* S *with geometric multiplicity greater than one, either* $\mathcal{M} \cap \mathcal{R}(S;\lambda) = \{0\}$ *or* $\mathcal{M} \supseteq \mathcal{R}(S;\lambda)$ *holds; in this case*

$$\mathcal{M} \cap \mathcal{R}(S;\lambda) = \{0\} \quad \textit{if and only if} \quad \mathcal{M} \supseteq \mathcal{R}(S;-\lambda).$$

(e) *For every real eigenvalue* $\lambda \neq \pm 1$ *of* S *with geometric multiplicity one and even algebraic multiplicity, the subspace* $\mathcal{M} \cap \mathcal{R}(S;\lambda)$ *is even dimensional. In this case,*

$$\mathcal{M} \cap \mathcal{R}(S;-\lambda) = \Big(J\big(\mathcal{M} \cap \mathcal{R}(S;\lambda)\big)\Big)^{\perp} \cap \mathcal{R}(S;-\lambda).$$

(f) *For every pair of complex conjugate eigenvalues* $a \pm ib$ *of* A *with* $|a+ib| \neq 1$, $b \neq 0$ *such that the geometric multiplicity of* $a+ib$ *is greater than one, either* $\mathcal{M} \cap \mathcal{R}(S; a \pm ib) = \{0\}$ *or* $\mathcal{M} \supseteq \mathcal{R}(S; a \pm ib)$ *holds. In this case*

$$\mathcal{M} \cap \mathcal{R}(S; a \pm ib) = \{0\} \quad \textit{if and only if} \quad \mathcal{M} \supseteq \mathcal{R}(S; -a \pm ib).$$

(iv) *Assume that* S *has no unimodular eigenvalues. Let* $\mathcal{M}$ *be a stable* S*-invariant* J*-Lagrangian subspace. Then the index of stability of* $\mathcal{M}$ *is equal to* $\Gamma_{\mathcal{M}}(S)$*, where* $\Gamma_{\mathcal{M}}(S)$ *is given in* (4.8).

Finally, we present a result on the index of conditional stability.

Theorem 4.13 *Let* $S \in \mathbb{R}^{2n \times 2n}$ *be a* J*-symplectic matrix. Assume that the geometric multiplicity of every unimodular eigenvalue of* S *is equal to one. Denote by* κ *the largest partial multiplicity of any unimodular eigenvalue of* S*, and let* $\kappa = 1$ *if* S *has no unimodular eigenvalues. Let* $\mathcal{M}$ *be a conditionally stable* S*-invariant* J*-Lagrangian subspace (in particular, the partial multiplicity corresponding to every unimodular eigenvalue is even, by Theorem* 4.12*). Then:*

(i) *The index of conditional stability of* $\mathcal{M}$ *does not exceed* $\max\{\Gamma_{\mathcal{M}}(S), \kappa\}$.

(ii) *If* $\kappa = 2$*, then the index of conditional stability of* $\mathcal{M}$ *is equal to* $\max\{\Gamma_{\mathcal{M}}(S), 2\}$.

4.4 Examples

We illustrate the results of this section with some examples.

Example 4.14 Consider the real matrices

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{S} = \begin{bmatrix} c & b \\ a & d \end{bmatrix}, \quad S_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}, \quad S_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix},$$

where c, d are close to one, $a, b, \delta, \varepsilon$ are close to zero, and $dc - ab = 1$. Every real 2×2 matrix is J-symplectic if and only if its determinant equals one, so $S, \widetilde{S}, S_\varepsilon$, and S_δ are all J-symplectic. Moreover, every one-dimensional subspace of $\mathbb{R}^2$ is S-invariant J-Lagrangian. For $c + d < 2$ the matrix $\widetilde{S}$ has a pair of complex conjugate nonreal eigenvalues and therefore, there are no real $\widetilde{S}$-invariant J-Lagrangian subspaces. Consequently, there are no (unconditionally) stable S-invariant J-Lagrangian subspaces. On the other hand, there exists a unique S_ε-invariant J-Lagrangian subspace $\mathcal{M}_\varepsilon$ and a unique S_δ-invariant J-Lagrangian subspace $\mathcal{M}_\delta$, where

$$\mathcal{M}_\varepsilon = \operatorname{Span} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathcal{M}_\delta = \operatorname{Span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It follows that S does not have any conditionally stable invariant J-Lagrangian subspaces, as it also follows from Theorem 4.12.

Example 4.15 Consider the real matrices

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \widetilde{S} = \begin{bmatrix} -c & b \\ a & -d \end{bmatrix},$$

where a, c, d are close to one, b is close to zero, and $dc - ab = 1$. Then S and $\widetilde{S}$ are J-symplectic. The only S-invariant J-Lagrangian subspace is

$$\mathcal{M} = \operatorname{Span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For $c + d > 2$ the matrix $\tilde{S}$ has a pair of complex conjugate nonreal eigenvalues and therefore there is no real invariant Lagrangian subspace. If $c + d < 2$, then $\tilde{S}$ has two distinct real eigenvalues. Here, $\mathcal{M}$ is 2-conditionally stable by Theorem 4.13.

Theorem 4.12 shows that the signs corresponding to the partial multiplicities of the eigenvalues 1 and -1 of S have an important effect on the stability of invariant J-Lagrangian subspaces. We illustrate this with the following example.

Example 4.16 Consider the real matrices

$$S_\varepsilon = \begin{bmatrix} -1 & 0 & \varepsilon & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Here, S_ε has two Jordan blocks of size two with opposite signs in the sign characteristic if $\varepsilon = 1$, and with equal signs if $\varepsilon = -1$. Two corresponding Jordan chains are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\varepsilon \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

We consider two types of perturbations that will both depend on parameters. Set

$$\widehat{S}_\varepsilon(\mu) := \begin{bmatrix} -1 & 0 & \varepsilon & 0 \\ 0 & -(1-\mu)^{-1/2} & 0 & -(1-\mu)^{-1/2} \\ 0 & 0 & -1 & 0 \\ 0 & -\mu(1+\mu)^{-1/2} & 0 & -(1-\mu)^{-1/2} \end{bmatrix}, \quad \text{where } 0 < \mu < 1. \tag{4.9}$$

The eigenvalues of $\widehat{S}_\varepsilon(\mu)$ are -1 and $-\sqrt{\frac{1}{1+\mu}} \pm \sqrt{\frac{\mu}{1+\mu}}$. The corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \pm\sqrt{\mu} \end{bmatrix}.$$

Corresponding to the eigenvalue -1, there is a Jordan block of order two. This means that there are two invariant J-Lagrangian subspaces for $\widehat{S}_\varepsilon(\mu)$, namely

$$\mathcal{M}_\pm = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \pm\sqrt{\mu} \end{bmatrix} \right\}.$$

Let $\mu \to 0$ to see that the only candidate for a conditionally stable invariant J-Lagrangian subspace for S_ε is the space

$$\mathcal{M}_0 := \operatorname{Range} \begin{bmatrix} I_2 \\ 0 \end{bmatrix}.$$

Let $2 > \beta > 0$ and set

$$S_\varepsilon(\beta) = \begin{bmatrix} -1 & 0 & \varepsilon & 0 \\ 0 & -1 & 0 & -1 \\ \beta & -\beta & -1-\varepsilon\beta & -\beta \\ -\beta & \beta & \varepsilon\beta & -1+\beta \end{bmatrix}.$$

Then $S_\varepsilon(\beta)$ is symplectic. If $\varepsilon = 1$ then the Jordan canonical form consists of only one Jordan block of size four with eigenvalue -1 (if $\beta \neq 0$), and the only two-dimensional invariant subspace of $S_\varepsilon(\beta)$ is given by

$$\mathcal{M}_1 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and this subspace is J-Lagrangian. Observe that $\mathcal{M}_1$ is also invariant for $S_\varepsilon = S_\varepsilon(0)$. Letting $\beta \to 0$, we see that the only possible candidate for a stable S_ε-invariant J-Lagrangian subspace would be this subspace. However, $\mathcal{M}_1$ is not equal to $\mathcal{M}_0$ and hence we conclude that there is no stable S_ε-invariant J-Lagrangian subspace in this case.

On the other hand, if $\varepsilon = -1$, then $S_\varepsilon(\beta)$ has two simple unimodular eigenvalues $\beta - 1 \pm i\sqrt{2\beta - \beta^2}$ and the eigenvalue -1 with algebraic multiplicity two, i.e., there does not exist an $S_\varepsilon(\beta)$-invariant J-Lagrangian subspace. So $\mathcal{M}_0$ is still a candidate for a conditionally stable invariant J-Lagrangian subspace, and Theorem 4.12 confirms that, indeed, this is the case.

In this section we have presented stability results for real J-Hamiltonian and J-symplectic matrices. In the next section we discuss complex J-symplectic matrices.

5 Case (II): Complex J-symplectic matrices

In this section we assume $\mathbb{F} = \mathbb{C}$, and J will stand for an invertible complex skew-symmetric $2n \times 2n$ matrix. For the proof of the main results on stability of invariant Lagrangian subspaces, we need a canonical form for J-symplectic matrices as presented in [32]. We need additional notation again, so for $\varepsilon \in \{+1, -1\}$ let

$$\widetilde{\mathcal{T}}_k(\varepsilon) = \begin{bmatrix} \varepsilon & 1 & r_2 & \dots & r_{k-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & r_2 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \varepsilon \end{bmatrix}, \tag{5.1}$$

where $r_k = 0$ for odd k and the parameters r_k for even k are real and uniquely determined by the recursive formula

$$r_2 = \frac{1}{2}\varepsilon, \quad r_k = -\frac{1}{2}\varepsilon \left(\sum_{\nu=1}^{\frac{k}{2}-1} r_{2\cdot\nu} r_{2\cdot(\frac{k}{2}-\nu)} \right), \quad 4 \le k \le n_j; \tag{5.2}$$

Also recall that Σ_k denotes the $k \times k$ matrix with alternating signs on the anti-diagonal as in (4.1).

Theorem 5.1 *Let $J \in \mathbb{C}^{2n\times 2n}$ be invertible and skew-symmetric and let $S \in \mathbb{C}^{2n\times 2n}$ be J-symplectic. Then there exists a nonsingular matrix Q such that $Q^{-1}SQ$ and Q^TJQ are block diagonal matrices*

$$Q^{-1}SQ = S_1 \oplus \cdots \oplus S_p, \quad Q^TJQ = J_1 \oplus \cdots \oplus J_p, \tag{5.3}$$

where each diagonal block (S_j, J_j) is of one of the following three types:

(i) $$S_j = \widetilde{\mathcal{T}}_{2n_1}(\varepsilon) \oplus \cdots \oplus \widetilde{\mathcal{T}}_{2n_p}(\varepsilon), \quad J_j = \Sigma_{2n_1} \oplus \cdots \oplus \Sigma_{2n_p},$$

where the number $\varepsilon \in \{-1, +1\}$ and the parameters $2n_1, \dots, 2n_p$ depend on the particular block (S_j, J_j);

(ii) $$S_j = \begin{bmatrix} \mathcal{J}_{2m_1+1}(\varepsilon) & 0 \\ 0 & \left(\mathcal{J}_{2m_1+1}(\varepsilon)\right)^{-T} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{2m_q+1}(\varepsilon) & 0 \\ 0 & \left(\mathcal{J}_{2m_q+1}(\varepsilon)\right)^{-T} \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{2m_1+1} \\ -I_{2m_1+1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2m_q+1} \\ -I_{2m_q+1} & 0 \end{bmatrix},$$

where $\varepsilon = \pm 1$, and the number ε, the total number $2q$ of Jordan blocks, and the sizes $2m_1 + 1, \dots, 2m_r + 1$ depend on the particular diagonal block (S_j, J_j);

(iii)
$$S_j = \begin{bmatrix} \mathcal{J}_{\ell_1}(\lambda) & 0 \\ 0 & \left(\mathcal{J}_{\ell_1}(\lambda)\right)^{-T} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\ell_r}(\lambda) & 0 \\ 0 & \left(\mathcal{J}_{\ell_r}(\lambda)\right)^{-T} \end{bmatrix},$$
$$J_j = \begin{bmatrix} 0 & I_{\ell_1} \\ -I_{\ell_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_r} \\ -I_{\ell_r} & 0 \end{bmatrix},$$

where λ satisfies the conditions $|\lambda| \leq 1$ and the imaginary part of λ is positive if $|\lambda| = 1$, and the number λ, the total number $2r$ of Jordan blocks, and the sizes $\ell_1, \ldots, \ell_r$ depend on the particular diagonal block (S_j, J_j);

There is at most one block of type (i) *(and at most one block of type* (ii)*, respectively) associated with the same eigenvalue. Furthermore, two blocks S_i and S_j of type* (iii) *have nonintersecting spectra if $i \neq j$. Moreover, the form* (5.3) *is uniquely determined by the pair (S, J), up to a simultaneous permutation of diagonal blocks in the right hand sides of* (5.3)*.*

5.1 Existence and stability of invariant Lagrangian subspaces

Theorem 5.1 has an immediate consequence concerning the existence of invariant Lagrangian subspaces which is very different from the real J-symplectic case.

Corollary 5.2 *Let $S \in \mathbb{C}^{2n\times 2n}$ be J-symplectic. Then there exists an S-invariant J-Lagrangian subspace with $\sigma(S|_{\mathcal{M}})$ contained in the closed unit disc. In particular, conditional stability of S-invariant J-Lagrangian subspaces coincides with their (unconditional) stability.*

We then obtain the following result on the stability of J-invariant subspaces.

Theorem 5.3 *Let $S \in \mathbb{C}^{2n\times 2n}$ be a J-symplectic matrix. Then the following assertions are equivalent:*

(1) *there exist a stable S-invariant J-Lagrangian subspace;*

(2) *there exists a conditionally stable S-invariant J-Lagrangian subspace;*

(3) $\dim \operatorname{Ker}\,(S - I) \leq 1$ *and* $\dim \operatorname{Ker}\,(S + I) \leq 1$.

In case one (or every) of the assertions (1)–(3) *holds, the following statements are equivalent for an S-invariant J-Lagrangian subspace $\mathcal{M}$:*

(4) *$\mathcal{M}$ is stable;*

(5) *$\mathcal{M}$ is conditionally stable;*

(6) $\mathcal{M} \cap \mathcal{R}(S; \lambda_j) = \mathcal{R}(S; \lambda_j)$ or $\mathcal{M} \cap \mathcal{R}(S; \lambda_j) = \{0\}$, *whenever* $\dim \operatorname{Ker}\,(S - \lambda_j I) > 1$.

Proof. The equivalence of (4), (5), and (6) (under the hypothesis that (1)–(3) hold), as well as the equivalence of (1), (2), and (3), follows from Corollary 5.2.

We show next, that each of (1), (2) implies (3). If neither 1 nor -1 is an eigenvalue of S, then there is nothing to show. Otherwise, by Theorems 3.6 and 5.1, we may assume without loss of generality that $\sigma(S) = \{1\}$ or $\sigma(S) = \{-1\}$. We will only consider the case $\sigma(S) = \{1\}$, the other case follows analogously or by passing to $-S$. Now assume that $\dim \mathrm{Ker}\,(S - I) > 1$ and let $\mathcal{M}$ be a stable S-invariant J-Lagrangian subspace. Then $\mathcal{M}$ is, in particular, stable as an invariant subspace of S and, by Theorem 2.3, we either have $\mathcal{M} = \{0\}$ or $\mathcal{M} = \mathbb{C}^{2n}$, because the fact that 1 is the only eigenvalue of S implies that $\mathcal{R}(S;1) = \mathbb{C}^{2n}$. In both cases, $\mathcal{M}$ is not J-Lagrangian which contradicts the assumption. Thus, we must have $\dim \mathrm{Ker}\,(S - I) \leq 1$.

Conversely, assume that (3) holds. Divide the set of eigenvalues of S different from ± 1 into two disjoint sets $\sigma_1(S)$ and $\sigma_2(S)$ so that

$$\lambda \in \sigma_1(S) \quad \text{if and only if} \quad \lambda^{-1} \in \sigma_2(S).$$

The canonical form of Theorem 5.1 shows that this is possible. Let $\mathcal{M}_1$ be the unique S-invariant subspace such that $\sigma(S|_{\mathcal{M}_1}) \subseteq \{1\}$ and

$$\dim \mathcal{M}_1 = \frac{1}{2} \cdot \dim \mathcal{R}(S;1),$$

and let $\mathcal{M}_{-1}$ be the unique S-invariant subspace such that $\sigma(S|_{\mathcal{M}_{-1}}) \subseteq \{-1\}$ and

$$\dim \mathcal{M}_{-1} = \frac{1}{2} \cdot \dim \mathcal{R}(S;-1).$$

Then the canonical form of Theorem 5.1 shows that the subspace

$$\mathcal{M} := \mathcal{M}_1 \dot{+} \mathcal{M}_{-1} \dot{+} \sum_{\lambda \in \sigma_1(S)} \mathcal{R}(S;\lambda)$$

is an S-invariant J-Lagrangian subspace. Moreover, by Theorem 2.3, $\mathcal{M}$ is stable as an S-invariant subspace, and a fortiori also stable as an S-invariant J-Lagrangian subspace. (Indeed, the canonical form (4.5) together with the spectral properties of S imply that if $\widetilde{S}$ is a J-symplectic matrix sufficiently close to S and $\widetilde{\mathcal{M}}$ is the perturbed $\widetilde{S}$-invariant subspace corresponding to $\mathcal{M}$, then $\widetilde{\mathcal{M}}$ is J-Lagrangian.) Thus, (1) and consequently also (2) holds.

A similar argument shows that if (3) holds, then (6) implies each of (4) and (5), and if one of (4), (5) holds then by Theorem 2.3 part (a) we obtain that also (6) is satisfied. □

An interesting consequence of Theorem 5.3 is that existence of stable invariant Lagrangian subspaces persists under small perturbations: If $S \in \mathbb{C}^{2n \times 2n}$ is a J-symplectic matrix with a stable S-invariant J-Lagrangian subspace, then there exists $\delta > 0$ with the property that there exists a stable S'-invariant J'-Lagrangian subspace for every J'-symplectic matrix $S' \in \mathbb{C}^{2n\times 2n}$ and every complex skew-symmetric matrix J' satisfying the inequalities

$$\|S' - S\| + \|J' - J\| < \delta.$$

5.2 α-stability

Next, we study α-stability of S-invariant J-Lagrangian subspaces $\mathcal{M}$. Once again, we need additional notation. If $\lambda_1, \lambda_2, \ldots, \lambda_t$ denote all pairwise distinct eigenvalues of S different from ± 1, then we introduce

$$\Gamma^{\mathbb{C}}_{\mathcal{M}}(S) := \max_{j=1,\ldots,t} \alpha_{\mathbb{C}}\Big(\dim \mathcal{R}(S;\lambda_j), \dim\left(\mathcal{R}(S;\lambda_j)\cap\mathcal{M}\right)\Big), \tag{5.4}$$

where $\alpha_{\mathbb{C}}(\cdot,\cdot)$ is as in (2.2).

Theorem 5.4 *Let $S \in \mathbb{C}^{2n\times 2n}$ be J-symplectic, let n_+ and n_- be the algebraic multiplicities of the eigenvalues 1 and -1 of S, respectively (where we allow $n_+ = 0$ and/or $n_- = 0$), and let $\lambda_1, \lambda_2, \ldots, \lambda_t$ denote all pairwise distinct eigenvalues of S different from ± 1. Moreover, let $\mathcal{M}$ be a stable S-invariant J-Lagrangian subspace.*

(i) *If $1, -1 \notin \sigma(S)$, then the index of stability of $\mathcal{M}$ is $\Gamma^{\mathbb{C}}_{\mathcal{M}}(S)$.*

(ii) *If 1 and/or -1 are eigenvalues of S and*

$$\alpha_- := \max\left\{2,\ n_+ - 1,\ n_- - 1,\ \Gamma^{\mathbb{C}}_{\mathcal{M}}(S)\right\}, \tag{5.5}$$
$$\alpha_+ := \max\left\{n_+,\ n_-,\ \Gamma^{\mathbb{C}}_{\mathcal{M}}(S)\right\}, \tag{5.6}$$

then $\mathcal{M}$ is not β-stable for any $\beta < \alpha_-$, and $\mathcal{M}$ is β-stable for any $\beta \geq \alpha_+$.

Proof. By Theorems 5.3 and 5.1, the geometric multiplicity of eigenvalues ± 1 (if indeed they are eigenvalues) of S is equal to one, and $n_\pm$ are even. From the canonical form of Theorem 5.1 it is easy to see that

$$\dim\left(\mathcal{M}\cap\mathcal{R}(S;\pm 1)\right) = \frac{1}{2}n_\pm.$$

By the main result of [47], $\mathcal{M}$ is strongly α_+-stable as an S-invariant subspace (where α_+ is defined as in (5.6) even if ± 1 are not eigenvalues of S); see [39, 44, 47]. Namely, every matrix S' (irrespective of its symplectic properties) and every of its invariant subspaces $\mathcal{N}$ satisfy the inequality

$$\operatorname{gap}(\mathcal{M},\mathcal{N}) \leq K\|S' - S\|^{\frac{1}{\alpha_+}},$$

provided S' is sufficiently close to S and the obvious necessary condition is fulfilled concerning dimensions of intersection of $\mathcal{N}$ with spectral invariant subspaces of S', where the positive constant K is independent on S' and $\mathcal{N}$. Hence $\mathcal{M}$ is a fortiori α_+-stable as an S-invariant J-Lagrangian subspace.

It remains to show that $\mathcal{M}$ is not β-stable as an S-invariant J-Lagrangian subspace for any $1 \le \beta < \Gamma^{\mathbb{C}}_{\mathcal{M}}(S)$ if the assumption of (i) holds, or for any $1 \le \beta < \alpha_-$ if the assumption of (ii) holds. First, we consider the case (i), that is, $\pm 1 \notin S$, and denote for brevity $\gamma := \Gamma^{\mathbb{C}}_{\mathcal{M}}(S)$. If $\gamma = 1$ then there is nothing to show, so assume $\gamma \ge 2$ and let λ be an eigenvalue of S such that

$$\gamma = \alpha_{\mathbb{C}}\Big(\dim \mathcal{R}(S;\lambda), \dim\big(\mathcal{R}(S;\lambda) \cap \mathcal{M}\big)\Big).$$

Using the canonical form of Theorem 5.1, we may assume without loss of generality that

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_1^{-T} \end{bmatrix} \oplus S_2, \quad J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \oplus J_2, \tag{5.7}$$

where $\sigma(S_1) = \{\lambda\}$ and $\lambda, \lambda^{-1} \notin \sigma(S_2)$. This is achieved by a change of basis via a nonsingular matrix Q, which will be fixed throughout this part of the proof. The subspace $\mathcal{R}(S;\lambda) \cap \mathcal{M}$ is nontrivial, because otherwise, we would have $\gamma = 1$ which contradicts $\gamma \ge 2$. Thus, in view of the form (5.7), we have

$$\mathcal{R}(S;\lambda) \cap \mathcal{M} = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathcal{M}_0 \right\}$$

for some S_1-invariant subspace $\mathcal{M}_0$. With this notation we have

$$\mathcal{M} = \mathcal{M}_0 \oplus \big(\mathcal{M}_0^{\perp} \cap \mathcal{R}(S;\lambda^{-1})\big) \oplus \mathcal{M}'$$

for some fixed S_2-invariant J_2-Lagrangian subspace $\mathcal{M}'$. Moreover, $\mathcal{R}(S;\lambda) \cap \mathcal{M}$ is not β-stable by Theorem 2.5. Thus, there exists a sequence $\{T_p\}_{p=1}^{\infty}$, $T_p \in \mathbb{C}^{m\times m}$, such that

$$\lim_{p\to\infty} \|T_p - S_1\| = 0 \quad \text{and} \quad \lim_{p\to\infty} \frac{\min\{\operatorname{gap}(\mathcal{M}_0, \mathcal{N}_p)\}}{\|T_p - S_1\|^{1/\beta}} = \infty, \tag{5.8}$$

where the minimum is taken over all T_p-invariant subspaces $\mathcal{N}_p$. Let

$$S^{(p)} = \begin{bmatrix} T_p & 0 \\ 0 & (T_p)^{-T} \end{bmatrix} \oplus S_2, \quad p = 1, 2, \dots.$$

Clearly, $S^{(p)}$ is J-symplectic. Now we consider the invariant Lagrangian subspaces of $S^{(p)}$. Since $\lambda \ne \pm 1$, we may assume that T_p and $(T_p)^{-T}$ do not have a common eigenvalue. Then all $S^{(p)}$-invariant J-Lagrangian subspaces are of the form

$$\mathcal{N}^{(p)} = \mathcal{N}_p \oplus \big(\mathcal{N}_p^{\perp} \cap \mathcal{R}(S;\lambda^{-1})\big) \oplus \mathcal{N}',$$

where $\mathcal{N}_p$ is T_p invariant and $\mathcal{N}'$ is S_2-invariant and J_2-Lagrangian. Because of the fact that all decompositions are with respect to the same basis (given by the columns of the nonsingular matrix Q) in view of Lemma 2.1 we have an estimate

$$\operatorname{gap}(\mathcal{M}, \mathcal{N}^{(p)}) \ge \kappa \cdot \operatorname{gap}(\mathcal{M}_0, \mathcal{N}_p),$$

where $\kappa > 0$ is a positive constant independent of p. From this and (5.8) one sees that

$$\lim_{p\to\infty} \frac{\min\{\operatorname{gap}(\mathcal{M}, \mathcal{N}^{(p)})\}}{\|S^{(p)} - S\|^{1/\beta}} = \infty,$$

where the minimum is taken over all T_p-invariant subspaces $\mathcal{N}_p$, and all S_2-invariant J_2-Lagrangian subspaces $\mathcal{N}'$. Thus, $\mathcal{M}$ is not β-stable as an S-invariant J-Lagrangian subspace.

Next, we assume that we are in case (ii), i.e., either 1 or -1 or both are eigenvalues of S. If $\alpha_- = \Gamma^{\mathbb{C}}_{\mathcal{M}}(S)$, then we argue as in case (i). Thus, assume $\alpha_- = \max\{2, n_+\}$.(The argument in the case $\alpha_- = \max\{2, n_-\}$ is completely analogous). Then, in particular, we have $n_+ \geq 2$. Using the canonical form of Theorem 5.1 and that the geometric multiplicity of the eigenvalue 1 is necessarily equal to one by Theorem 5.3, we may assume that

$$S = \widetilde{\mathcal{T}}_{n_+} \oplus S_1, \quad J = \Sigma_{n_+} \oplus J_1,$$

where $1 \notin \sigma(S_1)$ and $\widetilde{\mathcal{T}}_{n_+}$ is as in (5.1).

For notational simplicity, we assume further that $S = \widetilde{\mathcal{T}}_{n_+}$, $J = \Sigma_{n_+}$ (if this is not the case, argue as in the preceding paragraph). Observe that all parameters in S, J are real and that $\mathcal{M}$ is spanned by $e_1, \ldots, e_{n_+/2}$. Thus, for the moment, let us consider (S, J) as a real pair, where $\mathcal{M}_{\mathbb{R}} := \mathcal{M} \cap \mathbb{R}^{2n}$ is a real S-invariant J-Lagrangian subspace. Clearly, we have $2, n_+ - 1 \leq \alpha_- \leq \Sigma_{\mathcal{M}_{\mathbb{R}}}$, where $\Sigma_{\mathcal{M}_{\mathbb{R}}}$ is defined as in (4.8). Hence, Theorem 4.12 (iv) implies that $\mathcal{M}_{\mathbb{R}}$ is not β-stable for any $\beta < \alpha_-$ as a real S-invariant J-Lagrangian subspace. But then, $\mathcal{M}$ is a fortiori not β-stable as a complex S-invariant J-Lagrangian subspace. □

Obviously, if 1 or -1 or both are eigenvalues of S, then Theorem 5.4 does not determine the index of stability of $\mathcal{M}$ in all cases. Thus, the following open problem is natural:

Problem 5.5 For every stable S-invariant J-Lagrangian subspace $\mathcal{M}$, where S is a complex J-symplectic matrix such that 1 or -1 or both are eigenvalues of S, determine the index of stability of $\mathcal{M}$.

In the following particular case, the result of Theorem 5.4 does give an exact description of the index of stability:

Corollary 5.6 *Let $S \in \mathbb{C}^{2n\times 2n}$ be J-symplectic such that* 1 *or* -1 *or both are eigenvalues of S. Furthermore, assume that if* 1 *or* -1 *is an eigenvalue of S then its geometric multiplicity is* 1 *and its algebraic multiplicity is* 2, *and let $\mathcal{M}$ be a stable S-invariant J-Lagrangian subspace. Then the index of stability of $\mathcal{M}$ is equal to* $\max\{2, \Gamma^{\mathbb{C}}_{\mathcal{M}}(S)\}$.

5.3 Examples

We present some examples here to illustrate the main results of this section and to highlight the differences is the results compared to the real case.

Example 5.7 Consider the matrices

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} c & b \\ a & d \end{bmatrix},$$

where a, b, c, d are complex numbers. The matrix S is J-symplectic if and only if $\det S = 1$. Unless $S = \pm I$, there exist stable S-invariant J-Lagrangian subspaces, and moreover every eigenvector x of S spans a one-dimensional stable S-invariant J-Lagrangian subspace. Furthermore, $\operatorname{Span}\{x\}$ is 1-stable, i.e., Lipschitz stable, if x corresponds to the eigenvalue of S different from ± 1, and the index of stability of $\operatorname{Span}\{x\}$ is equal 2 if x corresponds to the eigenvalue ± 1. In the exceptional case $S = \pm I$ there do not exist stable S-invariant J-Lagrangian subspaces.

Example 5.8 Consider

$$S_\varepsilon = \begin{bmatrix} -1 & 0 & \varepsilon & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

where $\varepsilon = \pm 1$. As in Example 4.16, one verifies that for $\varepsilon = 1$, there are no stable S-invariant J-Lagrangian subspaces. The case $\varepsilon = -1$ is easily reduced over the complex field to the case $\varepsilon = 1$. Indeed, letting $Q = \operatorname{diag}(i, 1, -i, 1)$, the identities

$$Q^T J Q = J, \quad Q^{-1} S_{-1} Q = S_1$$

are satisfied. Thus, we conclude that there are no (conditionally) stable S-invariant J-Lagrangian subspaces also in the case $\varepsilon = -1$, as predicted by Theorem 5.3.

In this section we have studied the case of complex J-symplectic matrices. In the next section we finally study the complex J-unitary case.

6 Case (III): Complex J-unitary matrices

In this section we study the case (III), i.e., $J \in \mathbb{C}^{2n\times 2n}$ is a fixed invertible and Hermitian matrix with exactly n positive and n negative eigenvalues, and $S \in \mathbb{C}^{2n\times 2n}$ is J-unitary. Most of the results in this section are direct consequences of the corresponding results for J-selfadjoint matrices. Recall that a matrix $A \in \mathbb{C}^{2n\times 2n}$ is called J-selfadjoint if $A^* J = JA$.

6.1 Canonical forms

We start with a canonical form for J-selfadjoint matrices that can be found in many sources, see, e.g., [16, 24, 25, 29]. Once more, we need additional notation for this form and the following canonical form for J-unitary matrices. So, we set

$$R_k = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}, \quad T_k^{\mathbb{C}}(\mu) = \mu \begin{bmatrix} 1 & 2i & 2i^2 & \cdots & 2i^{k-1} \\ 0 & 1 & 2i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2i^2 \\ \vdots & & \ddots & \ddots & 2i \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}, \quad \mu \in \mathbb{C}.$$

Theorem 6.1 *Let A be J-selfadjoint. Then there exists an invertible matrix P such that $P^{-1}AP$ and P^*JP are block diagonal matrices*

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_s, \quad P^*JP = J_1 \oplus \cdots \oplus J_s, \tag{6.1}$$

where each diagonal block (A_j, J_j) is of one of the following two types:

(i) $$A_j = \mathcal{J}_{n_1}(a) \oplus \cdots \oplus \mathcal{J}_{n_p}(a), \quad J_j = \kappa_1 R_{n_1} \oplus \cdots \oplus \kappa_p R_{n_p},$$

where $a \in \mathbb{R}$ and where $\kappa_1, \dots, \kappa_p \in \{1, -1\}$, and the number a, the total number p of Jordan blocks, the sizes $n_1, \dots, n_p$, and the signs $\kappa_1, \dots, \kappa_p$ depend on the particular diagonal block (A_j, J_j);

(ii) $$A_j = \begin{bmatrix} \mathcal{J}_{\ell_1}(a+ib) & 0 \\ 0 & \mathcal{J}_{\ell_1}(a-ib)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\ell_r}(a+ib) & 0 \\ 0 & \mathcal{J}_{\ell_r}(a-ib)^T \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{\ell_1} \\ I_{\ell_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_r} \\ I_{\ell_r} & 0 \end{bmatrix},$$

where $a \in \mathbb{R}, b > 0$, and again the numbers a and b, the total number $2r$ of Jordan blocks, and the sizes $\ell_1, \dots, \ell_r$ depend on (A_j, J_j).

Two blocks A_i and A_j of one of the types (i)–(ii) *have nonintersecting spectra if $i \neq j$. Moreover, the form* (6.1) *is uniquely determined by the pair (A, J), up to a simultaneous permutation of diagonal blocks in the right hand sides of* (6.1).

We see from Theorem 4.1 that there are signs $\kappa_i, \eta_j \in \{+1, -1\}$ associated with each partial multiplicity corresponding to real eigenvalues a of A. Once again, these signs are said to form the *sign characteristic* of the pair (A, J). A number of versions of a canonical form for J-unitary matrices are available in the literature; we present here the form developed in [14].

Theorem 6.2 *Let $S \in \mathbb{C}^{2n \times 2n}$ be J-unitary. Then there exists a nonsingular matrix P such that $P^{-1}SP$ and P^*JP are block diagonal matrices*

$$P^{-1}SP = S_1 \oplus \cdots \oplus S_s, \quad P^*JP = J_1 \oplus \cdots \oplus J_s, \tag{6.2}$$

where each diagonal block (S_j, J_j) is of one of the following two types:

(i) $$S_j = \mathcal{T}^{\mathbb{C}}_{n_1}(\mu) \oplus \cdots \oplus \mathcal{T}^{\mathbb{C}}_{n_p}(\mu), \quad J_j = \kappa_1 R_{n_1} \oplus \cdots \oplus \kappa_p R_{n_p},$$

where $|\mu| = 1$ and where $\kappa_1, \dots, \kappa_p$ are signs ± 1, and the number μ and the parameters $n_1, \dots, n_p$, and $\kappa_1, \dots, \kappa_p$ depend on the particular block (S_j, J_j);

(ii) $$S_j = \begin{bmatrix} \mathcal{J}_{\ell_1}(\lambda) & 0 \\ 0 & \big(\mathcal{J}_{\ell_1}(\lambda)\big)^{-*} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\ell_r}(\lambda) & 0 \\ 0 & \big(\mathcal{J}_{\ell_r}(\lambda)\big)^{-*} \end{bmatrix},$$

$$J_j = \begin{bmatrix} 0 & I_{\ell_1} \\ I_{\ell_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_r} \\ I_{\ell_r} & 0 \end{bmatrix},$$

where $|\lambda| < 1$, and the number λ, the total number $2r$ of Jordan blocks, and the sizes $\ell_1, \dots, \ell_r$ depend on the particular diagonal block (S_j, J_j).

Two blocks S_i and S_j of one of the types (i)–(ii) *have nonintersecting spectra if $i \neq j$. Moreover, the form* (6.2) *is uniquely determined by the pair (S, J), up to a simultaneous permutation of diagonal blocks in the right hand sides of* (6.2).

Once again, the signs $\kappa_1, \dots, \kappa_p$ are said to form the *sign characteristic* of the J-unitary matrix S. The proof of Theorem 6.2 is based on use of the Cayley transform (or, more precisely, Möbius transform)

$$\mathcal{C}_{\eta,w}(z) = \frac{\eta(z - \overline{w})}{z - w}, \quad |\eta| = 1, \quad w \notin \mathbb{R}, \quad \text{with} \quad \mathcal{C}^{-1}_{\eta,w}(z) = \frac{wz - \overline{w}\eta}{z - \eta}, \tag{6.3}$$

see [16] for its basic properties. The proof of Theorem 6.2 then follows the same lines as the proof of Theorem 4.3 and will therefore not be reproduced here. The first part of the proof can actually be found in slight variation in [16]. The Cayley transform (6.3) not only relates J-selfadjoint and J-unitary matrices, but also their sign characteristics.

Lemma 6.3 *Let $S \in \mathbb{C}^{2n \times 2n}$ be J-unitary, let $\eta \in \mathbb{C}$, $|\eta| = 1$ be a unimodular number which is not an eigenvalue of S and let $w \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$A = \mathcal{C}^{-1}_{\eta,w}(S) = (wS - \overline{w}\eta I)(S - \eta I)^{-1}$$

is J-selfadjoint and w is not an eigenvalue of A. Moreover,

$$S = \mathcal{C}_{\eta,w}(A) = \eta(A - \overline{w}I)(A - wI)^{-1}$$

and the sign characteristics of S as a J-unitary matrix and of A as a J-selfadjoint matrix are related as follows:

If $n_1, \dots, n_p$ are the partial multiplicities of the unimodular eigenvalue μ of S with corresponding signs $\kappa_1, \dots, \kappa_p$ and if $\tilde{n}_1, \dots, \tilde{n}_{\tilde{p}}$ are the partial multiplicities of the real eigenvalue $\mathcal{C}^{-1}_{\eta,w}(\mu) = (w\mu - \overline{w}\eta)/(\mu - \eta)$ of A with corresponding signs $\tilde{\kappa}_1, \dots, \tilde{\kappa}_{\tilde{p}}$, then $p = \tilde{p}$ and there exists a permutation π of $\{1, 2, \dots, p\}$ such that

$$n_i = \tilde{n}_{\pi(i)} \quad \text{and} \quad \kappa_i = \tilde{\kappa}_{\pi(i)}, \quad i = 1, \dots, p.$$

For the proof of Lemma 6.3 see the proof of [16, Theorem 5.15.5].

6.2 Existence and stability of invariant Lagrangian subspaces

We now turn to the question of existence of invariant J-Lagrangian subspaces. The answer is given in [38, Theorem 5.1] in the context of J-selfadjoint matrices. Applying Lemma 6.3, we immediately obtain the following corresponding result for J-unitary matrices.

Theorem 6.4 *Let S be a J-unitary matrix. There exists an S-invariant J-Lagrangian subspace if and only if for every unimodular eigenvalue $\omega \in \mathbb{C}$, $|\omega| = 1$, the number of odd partial multiplicities corresponding to ω is even, and the signs in the sign characteristic of S that correspond to these odd partial multiplicities sum up to zero.*

Next, we consider stability results of invariant J-Lagrangian subspaces. Once again, we need special notation before we are able to state results concerning α-stability of S-invariant J-Lagrangian subspaces $\mathcal{M}$. So, if $\lambda_1, \ldots, \lambda_r$ are all distinct nonunimodular eigenvalues of S, then we denote

$$\Theta_{\mathcal{M}}(S) := \max_{j=1,\ldots,r} \alpha_{\mathbb{C}}\Big(\dim \mathcal{R}(S;\lambda_j), \dim\left(\mathcal{R}(S;\lambda_j) \cap \mathcal{M}\right)\Big), \tag{6.4}$$

where $\alpha_{\mathbb{C}}(\cdot,\cdot)$ is as in (2.2) with the understanding that the maximum of the empty set is taken to be equal to one.

Theorem 6.5 *Let $S \in \mathbb{C}^{2n\times 2n}$ be J-unitary. Then there exists a stable S-invariant J-Lagrangian subspace if and only if S has no unimodular eigenvalues.*

In this case, an S-invariant J-Lagrangian subspace $\mathcal{M} \subseteq \mathbb{C}^{2n}$ is stable if and only if

$$\mathcal{M} \cap \mathcal{R}(S;\lambda_0) = \{0\} \quad \text{or} \quad \mathcal{R}(S;\lambda_0) \subseteq \mathcal{M}$$

for every eigenvalue λ_0 of S having geometric multiplicity greater than 1. *Moreover, every stable S-invariant J-Lagrangian subspace $\mathcal{M}$ has the index of stability which is equal to $\Theta_{\mathcal{M}}(S)$ as in* (6.4).

Proof. Assume that λ_0 is a unimodular eigenvalue of S. A perturbation theory of unimodular eigenvalues of J-unitary shows that there is a sequence of J-unitary matrices S_m, $m = 1, 2, \ldots$, such that $\lim_{m\to\infty} S_m = S$ and every S_m has at least one simple (i.e., of algebraic multiplicity one) unimodular eigenvalue in a vicinity of λ_0. Let us verify this statement in the context of J-selfadjoint matrices $X \in \mathbb{C}^{2n\times 2n}$ and their real eigenvalues μ_0. (The statement follows from more general results of [37]; we provide here a simpler independent proof.) In view of the canonical form (6.1), we need to consider only the case when $\mu_0 = 0$ and

$$X = \mathcal{J}_{n_1}(0) \oplus \cdots \oplus \mathcal{J}_{n_p}(0), \quad J = \kappa_1 R_{n_1} \oplus \cdots \oplus \kappa_p R_{n_p}.$$

If $Q = e_1^T e_{n_1}$ is the $2n \times 2n$ matrix having 1 in the $(n_1, 1)$-position and zeros in all other positions, then $X + (1/m)Q$, for almost all $m = 1, 2, \ldots$, is J-selfadjoint and has either one (if p is odd) or two (if p is even) simple real eigenvalues in a vicinity of zero. Applying the Cayley transform from Lemma 6.3, we obtain the corresponding statement for J-unitary matrices.

It follows from Theorem 6.4 that S_m has no invariant J-Lagrangian subspaces for almost all $m = 1, 2, \ldots$. Therefore, S has no stable invariant Lagrangian subspaces.

Suppose now that S has no unimodular eigenvalues. Denote by $\mathcal{M}_+$ the sum of root subspaces of S that correspond to eigenvalues outside of the unit circle. It is easy to see from the canonical form (6.2) that for every S-invariant subspace $\mathcal{M}' \subseteq \mathcal{M}_+$ there is a unique S-invariant J-Lagrangian subspace $\mathcal{M}$ such that $\mathcal{M} \cap \mathcal{M}_+ = \mathcal{M}'$. Indeed, applying a transformation $(S, J) \mapsto (T^{-1}ST, T^*JT)$ for some invertible $T \in \mathbb{C}^{2n\times 2n}$, we may assume that

$$S = \begin{bmatrix} S_+ & 0 \\ 0 & S_+^{-*} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix},$$

where the spectrum of S_+ is outside of the unit circle. Then

$$\mathcal{M}_+ = \mathbb{C}^n \oplus \{0\} \quad \text{and} \quad \mathcal{M} = \mathcal{M}' \oplus (\mathcal{M}')^\perp.$$

It follows that stability of $\mathcal{M}$ as an S-invariant J-Lagrangian subspace coincides with stability of $\mathcal{M}'$ and $(\mathcal{M}')^\perp$ as S-invariant subspaces. Now the result follows easily from Theorems 2.3 and 2.5. □

The problem of conditionally stable S-invariant J-Lagrangian subspaces is more subtle as the following results shows. It is obtained directly from [38, Theorem 0.2] using Lemma 6.3.

Theorem 6.6 *Let $S \in \mathbb{C}^{2n\times 2n}$ be J-unitary. There exists a conditionally stable S-invariant J-Lagrangian subspace if and only if every unimodular eigenvalue ω of S has only even partial multiplicities, and all the signs in the sign characteristic of S corresponding to ω are equal (however, for $\omega_1 \neq \omega_2$ with $|\omega_1| = |\omega_2| = 1$, the signs corresponding to ω_1 and ω_2 need not be the same).*

In this case, an S-invariant J-Lagrangian subspace $\mathcal{M}$ is conditionally stable if and only if for every nonunimodular eigenvalue μ of S with $\dim\big(\mathrm{Ker}\,(S - \mu I)\big) > 1$, either $\mathcal{M} \supseteq \mathcal{R}(S; \mu)$ or $\mathcal{M} \cap \mathcal{R}(S; \mu) = \{0\}$ holds.

Our next result considers conditional α-stability.

Theorem 6.7 *Let $S \in \mathbb{C}^{2n\times 2n}$ be J-unitary, and assume that there exists a conditionally stable S-invariant J-Lagrangian subspace $\mathcal{M}$. Assume furthermore that the geometric multiplicity of every unimodular eigenvalue of S (if any) is equal to one.*

(i) *If S has no unimodular eigenvalues, then the index of conditional stability of $\mathcal{M}$ is $\Theta_{\mathcal{M}}(S)$.*

(ii) *If S has unimodular eigenvalues, and if*

$$\alpha_- := \max\{2, n_1 - 1, \dots, n_r - 1,\ \Theta_{\mathcal{M}}(S)\}, \tag{6.5}$$
$$\alpha_+ := \max\{n_1, \dots, n_r, \Theta_{\mathcal{M}}(S)\}, \tag{6.6}$$

where $n_1, \dots, n_r$ are the algebraic multiplicities of all the unimodular eigenvalues, then $\mathcal{M}$ is not β-stable for any $\beta < \alpha_-$, and $\mathcal{M}$ is β-stable for every $\beta \geq \alpha_+$.

Theorem 6.7 follows from a result in [45] in the context of J-selfadjoint matrices. The passage to J-unitaries as needed for Theorem 6.7 is easily done using Lemma 6.3.

6.3 Examples

As in the previous two sections, we conclude with illustrative examples.

Example 6.8 Consider the matrices

$$J = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, d are complex numbers such that $a\overline{c}, b\overline{d} \in \mathbb{R}$ and $\overline{a}d - b\overline{c} = 1$. Then S is J-unitary. Unless S has one (and then, counting multiplicities, necessarily two) eigenvalue on the unit circle, there exists stable J-Lagrangian subspaces. Moreover, every eigenvector spans a one-dimensional 1-stable, i.e., Lipschitz stable, J-Lagrangian subspace.

Example 6.9 Consider the matrices

$$S_\varepsilon = \begin{bmatrix} -1 & 0 & \varepsilon & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \varepsilon = \pm 1$$

from Examples 4.16 and 5.8. As in the cases (I) and (II), there are no stable S-invariant J-Lagrangian subspaces for the case $\varepsilon = 1$. However, Theorem 6.6 predicts that there exists a conditionally stable S-invariant J-Lagrangian subspace for the case $\varepsilon = -1$. Indeed, the argument used in Example 5.8 does not work in this case, because S_1 and S_{-1} have different sign characteristics, and consequently, there does not exist an invertible matrix Q such that $Q^*JQ = J$ and $Q^{-1}S_{-1}Q = S_1$. Considering the perturbation as in (4.9), we conclude as in Example 4.16 that the only candidate for a conditionally stable invariant J-Lagrangian subspace for S_{-1} is the space

$$\mathcal{M}_0 := \text{Range}\begin{bmatrix} I_2 \\ 0 \end{bmatrix}.$$

Now let us return to the J-symplectic matrix which was used to show that this space was not conditionally stable in the real case, i.e., consider

$$S(\beta) = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ \beta & -\beta & -1+\beta & -\beta \\ -\beta & \beta & -\beta & -1+\beta \end{bmatrix}, \quad 0 < \beta < 2,$$

with simple eigenvalues $\beta-1 \pm i\sqrt{2\beta-\beta^2}$ and eigenvalue -1 (with algebraic multiplicity 2). In contrast to the real case of Example 4.16, there exist exactly two (complex) $S(\beta)$-invariant J-Lagrangian subspaces, namely,

$$\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -\beta \mp i\sqrt{2\beta-\beta^2} \\ \beta \pm i\sqrt{2\beta-\beta^2} \end{bmatrix} \right\}.$$

Letting $\beta \to 0$, we see that $\mathcal{M}_0$ is still a candidate for a conditionally stable subspace and Theorem 6.6 states that this candidate wins the election.

7 Invariant Lagrangian subspaces with spectrum location

In this section we consider stability of S-invariant J-Lagrangian subspaces $\mathcal{M}$ with the special property that the spectrum of the restriction $S|_{\mathcal{M}}$ lies either entirely inside or entirely outside the unit circle. These cases are important in applications, see [22]; for instance, $\sigma(S|_{\mathcal{M}})$ being inside the unit circle is an indication that the underlying discrete system is stable.

It will be convenient to introduce the following terminology: An S-invariant J-Lagrangian subspace $\mathcal{M}$ is said to be (S,J)-*inner*, resp., (S,J)-*outer*, if $|\lambda| \le 1$, resp., $|\lambda| \ge 1$, for every $\lambda \in \sigma(S|_{\mathcal{M}})$. As it follows from the well known description of the gap between subspaces in terms of limits of vectors (see [15, Theorem 13.4.2]), the sets of (S,J)-inner and of (S,J)-outer subspaces (for fixed J and S) are closed in the gap metric. Parts of the following result follow from the corresponding result for real Hamiltonian matrices in [23].

Theorem 7.1 *Consider the cases* (I) *or* (II) *and let* $S \in \mathbb{F}^{2n\times 2n}$ *be* J*-symplectic (or consider the case* (III) *and let* $S \in \mathbb{C}^{2n\times 2n}$ *be* J*-unitary, respectively).*

(a) *If* $\mathcal{IL}(S,J)$ *is not empty, then there exist an* (S,J)*-outer subspace* $\mathcal{M}_+$ *and an* (S,J)*-inner subspace* $\mathcal{M}_-$.

(b) *The* (S,J)*-outer subspace* $\mathcal{M}_+$ *is unique if and only if the* (S,J)*-inner subspace* $\mathcal{M}_-$ *is unique.*

(c) *If the (S, J)-outer subspace $\mathcal{M}_+$, or, equivalently, the (S, J)-inner subspace $\mathcal{M}_-$ is unique, then $\mathcal{M}_+$ and $\mathcal{M}_-$ are conditionally stable S-invariant J-Lagrangian subspaces.*

Proof. Parts (a) and (b) follow from the canonical forms (see Theorem 4.3 for the real symplectic case, Theorem 5.1 for the complex symplectic case, and Theorem 6.2 for the complex unitary case).

For the part (c) we argue by contradiction. Assume $\mathcal{M}_+$ is not conditionally stable. Then there exists $\varepsilon > 0$, a sequence S_m of J-symplectic matrices, and a sequence $\mathcal{M}_m$ of (S_m, J)-outer subspaces such that $S_m \to S$, but

$$\text{gap}\,(\mathcal{M}_m, \mathcal{M}_+) > \varepsilon \quad \text{for } m = 1, 2, \ldots. \tag{7.1}$$

By the compactness property of the set of subspaces in a finite dimensional real or complex vector space (see, e.g., [15, Chapter 13]) the sequence $\mathcal{M}_m$ must have a convergent subsequence, say, with limit $\mathcal{M}'$. Then $\mathcal{M}'$ is an (S, J)-outer subspace. By the uniqueness of $\mathcal{M}_+$ we must have $\mathcal{M}' = \mathcal{M}_+$ which contradicts (7.1). $\square$

Specializing the results of Section 5 yields the following corollary.

Corollary 7.2 *Let $S \in \mathbb{C}^{2n\times 2n}$ be J-symplectic.*

(a) *The following statements are equivalent:*

  (1) *There exists a conditionally stable (S, J)-inner subspace.*

  (2) *There exists a conditionally stable (S, J)-outer subspace.*

  (3) *There exists a stable (S, J)-inner subspace.*

  (4) *There exists a stable (S, J)-outer subspace.*

  (5) $\dim \text{Ker}\,(S - I) \le 1$ *and* $\dim \text{Ker}\,(S + I) \le 1$.

(b) *Assume that the conditions (1)–(5) hold, or equivalently, any one of these conditions holds. Define $\alpha_+ = 1$ if $S \pm I$ are invertible, and $\alpha_+ = \max\{n_+, n_-\}$ if at least one of the two matrices $S \pm I$ is not invertible, where n_+, resp. n_-, stand for the algebraic multiplicity of the eigenvalue 1, resp. -1, of S. Then there exist α_+-stable (S, J)-inner subspaces and α_+-stable (S, J)-outer subspaces. In fact, if an (S, J)-inner or (S, J)-inner subspace $\mathcal{M}$ is such that either $\mathcal{M} \cap \mathcal{R}(S, \lambda) = \{0\}$ or $\mathcal{M} \supseteq \mathcal{R}(S, \lambda)$ holds for every unimodular eigenvalue λ of S different from ± 1, then $\mathcal{M}$ is α_+-stable.*

(c) *Assume that the conditions (1)–(5) hold. Define $\alpha_- = 1$ if $S \pm I$ are invertible and $\alpha_- = \max\{2, n_+ - 1, n_- - 1\}$, where $n_\pm$ are defined as in part (b). Then no (S, J)-inner subspace nor (S, J)-outer subspace is stable for any $\beta < \alpha_-$.*

Note that by Theorem 5.1 the integers n_+ and n_- in Corollary 7.2 are even, and possibly zero.

We consider now the case (I), i.e, $\mathbb{F} = \mathbb{R}$ and real J-symplectic matrices, where $J \in \mathbb{R}^{2n\times 2n}$ is skew-symmetric and invertible. Then the results of Subsection 4.3 lead to the following corollary regarding outer and inner subspaces.

Corollary 7.3 *Let S be a real J-symplectic matrix.*

(i) *There exists a conditionally stable (S, J)-inner subspace, or, equivalently, there exists a conditionally stable (S, J)-outer subspace, if and only if the conditions* (a), (b), *and* (c) *of Theorem* 4.11 *are satisfied.*

(ii) *There exists a stable (S, J)-inner subspace, or, equivalently, there exists a stable (S, J)-outer subspace, if and only if S has no unimodular eigenvalues.*

(iii) *If the conditions of* (i)*, resp. of* (ii)*, are satisfied, then there is a unique (S, J)-inner subspace, there is a unique (S, J)-outer subspace, and both subspaces are conditionally stable, resp. Lipschitz stable.*

(iv) *Assume that the geometric multiplicity of every unimodular eigenvalue of S is equal to one. Denote by κ the largest partial multiplicity of any unimodular eigenvalue of S, and let $\kappa = 1$ if S has no unimodular eigenvalues. Then the indices of conditional stability of the (S, J)-inner and of the (S, J)-outer subspace do not exceed κ. If, in addition, $\kappa = 2$, then these indices are equal to* 2.

Finally, we consider complex J-unitary matrices. The results of Section 6 when specialized to the case of inner and outer subspaces yield:

Corollary 7.4 *Let $S \in \mathbb{C}^{2n\times 2n}$ be a J-unitary matrix. Then:*

(a) *There exists an (S, J)-inner subspace, equivalently an (S, J)-outer subspace, if and only if for every unimodular eigenvalue $\omega \in \mathbb{C}$, $|\omega| = 1$, the number of odd partial multiplicities corresponding to ω is even, and their signs in the sign characteristic of S sum up to zero.*

(b) *An (S, J)-inner subspace, or an (S, J)-outer subspace, is stable if and only if S has no unimodular eigenvalues, and in this case the subspace is Lipschitz stable.*

(c) *An (S, J)-inner subspace, or an (S, J)-outer subspace is conditionally stable if and only if every unimodular eigenvalue ω of S has only even partial multiplicities, and all the signs in the sign characteristic of S corresponding to ω are equal.*

(d) *Assume in addition that S has unimodular eigenvalues and that the geometric multiplicity of every unimodular eigenvalue of S is equal to* 1*, whereas the algebraic multiplicities $n_1, \ldots, n_r$ of unimodular eigenvalues of S are all even. Then the (S, J)-inner subspace, or the (S, J)-outer subspace, is β-stable for every $\beta \geq \max\{n_1, \ldots, n_r\}$, and is not β-stable for any $\beta < \max\{2, n_1 - 1, \ldots, n_r - 1\}$.*

8 Conclusion

We have studied the perturbation analysis for J-Lagrangian invariant subspaces of symplectic matrices. We have analyzed different stability concepts in the real and complex case. The results have been illustrated with several examples.

Determination of the index of (conditional) stability in all cases remains a challenging open problem.

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