Quantitative stability of fully random mixed-integer two-stage stochastic programs

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Abstract

Mixed-integer two-stage stochastic programs with fixed recourse matrix, random recourse costs, technology matrix, and right-hand sides are considered. Quantitative continuity properties of its optimal value and solution set are derived when the underlying probability distribution is perturbed with respect to an appropriate probability metric.

Keywords: Stochastic programming, two-stage, mixed-integer, stability, weak convergence, probability metric, discrepancy.
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1 Introduction

Mixed-integer two-stage stochastic programs model a variety of practical decision problems under stochastic uncertainty, e.g., in chemical engineering, power production, and trading planning [7, 12, 13]. The probability distribution of the stochastic programming model reflects the available knowledge on the randomness at the modeling stage. When solving such stochastic programming models, the probability distribution is approximately replaced in most cases by a discrete probability measure with finite support. Hence, perturbing or approximating the probability distribution of such models is an important issue for modeling, theory, and numerical methods in stochastic integer programming. While much is known on the structure and algorithms of mixed-integer two-stage stochastic programs (cf. the surveys [10, 11, 20, 21]), the available (quantitative) stability or statistical estimation results do not cover situations with stochastic costs (or prices) (cf. [6, 18] and the surveys [17, 20]).
Mixed-integer two-stage stochastic programs are of the form

$$\min \left\{ \int_{\Xi} f_0(x,\xi) dP(\xi) : x \in X \right\}, \tag{1.1}$$

where the (first-stage) feasible set $X \subseteq \mathbb{R}^m$ is closed, Ξ is a closed subset of $\mathbb{R}^s$, the function f_0 from $\mathbb{R}^m \times \Xi$ to the extended reals $\overline{\mathbb{R}}$ is a random lower semicontinuous

function, and P belongs to the set of all Borel probability measures $\mathcal{P}(\Xi)$ on Ξ. Recall that f_0 is a random lower semicontinuous function if its epigraphical mapping $\xi \mapsto \operatorname{epi} f_0(\cdot,\xi) := \{(x,r) \in \mathbb{R}^m \times \mathbb{R} : f_0(x,\xi) \le r\}$ is closed-valued and measurable. In (1.1), the function f_0 is of the form

$$f_0(x,\xi) = \langle c, x\rangle + \Phi(q(\xi), h(\xi) - T(\xi)x) \quad ((x,\xi) \in \mathbb{R}^m \times \Xi), \tag{1.2}$$

where $\Phi(u,t)$ denotes the optimal value of the (second-stage) mixed-integer program (with cost u and right-hand side t), and $q(\xi)$, $T(\xi)$ and $h(\xi)$ are the stochastic cost, technology matrix and right-hand side.

Throughout, let $v(P)$ and $S(P)$ denote the optimal value and solution set of (1.1), respectively. Our aim is to extend the quantitative continuity properties of $v(\cdot)$ and $S(\cdot)$ in [15, 17, 19] to cover situations with stochastic costs. To this end, we first recall properties of optimal value functions and solution sets of mixed-integer linear programs with parametric right-hand side. Then we derive novel continuity and growth results for optimal value functions of mixed-integer linear programs with parameters in right-hand sides and costs (Theorem 2.2). These results are used in Section 3 to obtain the desired quantitative stability result (Theorem 3.3) for fully random mixed-integer two-stage stochastic programs with fixed recourse. The relevant probability metric (3.4) on subsets of $\mathcal{P}(\Xi)$ and its relations to Fortet-Mourier metrics and polyhedral discrepancies are also discussed (Remark 3.5). The latter metrics may be used for designing moderately sized discrete approximations to P by optimal scenario reduction of discrete probability measures [8, 9].

2 Infima of mixed-integer linear programs

Consider the parametric mixed-integer linear program

$$\min\{\langle c_x, x\rangle + \langle c_y, y\rangle : A_x x + A_y y \le b, x \in \mathbb{Z}^n, y \in \mathbb{R}^{m-n}\} \tag{2.1}$$

with $c = (c_x, c_y) \in \mathbb{R}^m$ and $b \in \mathbb{R}^r$ playing the role of the parameters and $A = (A_x, A_y) \in \mathbb{Q}^{r\times m}$. Let $M(b)$, $\vartheta(b,c)$, and $\Sigma(b,c)$ denote the feasible set, optimal value, and solution set of (2.1), i.e.,

$$\begin{aligned} M(b) &:= \{(x,y) \in \mathbb{Z}^n \times \mathbb{R}^{m-n} : A(x,y) \le b\} \\ \vartheta(b,c) &:= \inf\{\langle c, (x,y)\rangle : (x,y) \in M(b)\} \\ \Sigma(b,c) &:= \{(x,y) \in M(b) : \langle c, (x,y)\rangle = \vartheta(b,c)\}. \end{aligned}$$

Let $\mathcal{K}$ denote the polyhedral cone $\{(x,y) \in \mathbb{R}^m : A_x x + A_y y \le 0\}$ and $\mathcal{K}^*$ its polar cone. Observe that $\vartheta(b,c)$ is finite for $b \in \mathcal{B} := \operatorname{dom} M$ and $c \in -\mathcal{K}^*$. Further, denote by $\operatorname{Pr}_x M(b)$ the projection of $M(b)$ onto the x-space, and, for $b^0 \in \mathcal{B}$, let

$$\mathcal{B}^*(b^0) := \{b \in \mathcal{B} : \operatorname{Pr}_x M(b) = \operatorname{Pr}_x M(b^0)\}$$

be the set of right-hand sides on which the projection of $M(b)$ onto the x-space is constant. It is well known (see [1, Chapter 5.6]) that the sets $\mathcal{B}^*(b^0)$ are continuity regions of the function $b \mapsto \vartheta(b,c)$. These regions are further characterized by the following result (Lemma 5.6.1 and 5.6.2 in [1]).

Lemma 2.1. *$\mathcal{B}$ is a connected set equal to the union of a countable family of convex polyhedral cones each of which is obtained by a translation of the r-dimensional cone $T := \{t \in \mathbb{R}^r : \exists y \in \mathbb{R}^{m-n}$ such that $t \geq A_y y\}$.*
For each $b^0 \in \mathcal{B}$, there exists $t^0 \in \mathcal{B}$ and a finite set $N \subseteq \mathbb{Z}^n \setminus \Pr_x M(b^0)$ such that

$$\mathcal{B}^*(b^0) = (t^0 + T) \setminus \bigcap_{z \in N} (A_x z + T).$$

If $\Pr_x M(b^0) = \mathbb{Z}^n$, then $N = \emptyset$ and $\mathcal{B}^*(b^0) = t^0 + T$ for some $t^0 \in \mathcal{B}$.

In the following we assume that the projection $\Sigma(b, c)$ of the solution set intersects with a ball of some radius K for all $(b, c) \in \mathcal{B} \times -\mathcal{K}^*$. This allows us to extend Lemma 2.3 in [19] and show local Lipschitz-continuity of the optimal value of (2.1) with respect to simultaneous perturbations of the right-hand side and the objective function coefficients where the right-hand side perturbation does not leave the continuity region $B^*(b)$. Otherwise, for arbitrary right-hand sides, a quasi-Lipschitz property of the value function $\vartheta(b, c)$ can be shown. For the proof of our next result we refer to the appendix.

Theorem 2.2. *1. Let $b \in \mathcal{B}$, $b' \in \mathcal{B}^*(b)$, and $c, c' \in -\mathcal{K}^*$. Assume that for some constant $k \geq 1$, $\Pr_x \Sigma(b', c) \cap \mathbb{B}(0, K) \neq \emptyset$ and $\Pr_x \Sigma(b', c') \cap \mathbb{B}(0, K) \neq \emptyset$. Then*

$$|\vartheta(b, c) - \vartheta(b', c')| \leq L_1 \max\{\|c\|, \|c'\|\}\|b - b'\| + L_2 \max\{\|b\|, \|b'\|, K\}\, \|c - c'\|,$$

where the constants L_1 and L_2 depend on A only.
2. Let $b, b' \in \mathcal{B}$ and $c, c' \in -\mathcal{K}^*$. Then

$$|\vartheta(b, c) - \vartheta(b', c')| \leq \max\{\|c\|, \|c'\|\}(\tilde{L}\|b - b'\| + 2\ell) + \tilde{L} \max\{\|b\|, \|b'\|\}\|c - c'\|,$$

where the constants $\tilde{L}$ and ℓ depend on A only.

Remark 2.3. Observe that for mixed-integer linear programs with bounded integer variables (e.g., mixed-binary programs), the integer part of the feasible points is uniformly bounded. This bound yields a suitable constant K.

The following Lemma is [4, Theorem 2.1] and can be found in similar form also in [2]. Together with Theorem 2.2 it is needed to prove Lemma 3.1.

Lemma 2.4. *Let $c \in -\mathcal{K}^*$. The mapping $b \mapsto \Sigma(b, c)$ is quasi-Lipschitz continuous on $\mathcal{B}$ with positive constants $\bar{L}_1$ and $\bar{L}_2$ not depending on b and c, i.e.,*

$$d_H(\Sigma(b, c), \Sigma(b', c)) \leq \bar{L}_1 \|b - b'\| + \bar{L}_2,$$

where d_H denotes the Hausdorff distance on subsets of $\mathbb{R}^m$.

3 Quantitative stability of mixed-integer two-stage stochastic programs

Let us consider the stochastic program

$$\min \left\{ \langle c, x \rangle + \int_{\Xi} \Phi(q(\xi), h(\xi) - T(\xi)x) P(d\xi) : x \in X \right\}, \tag{3.1}$$

where Φ is the infimum function of a mixed-integer linear program given by

$$\Phi(u,t) := \inf\left\{\langle u_1, y\rangle + \langle u_2, \bar{y}\rangle : Wy + \bar{W}\bar{y} \leq t, y \in \mathbb{Z}^{\hat{m}}, \bar{y} \in \mathbb{R}^{\bar{m}}\right\} \tag{3.2}$$

for all pairs $(u,t) \in \mathbb{R}^{\hat{m}+\bar{m}} \times \mathbb{R}^r$, and $c \in \mathbb{R}^m$, X is a closed subset of $\mathbb{R}^m$, Ξ a polyhedron in $\mathbb{R}^s$, W and $\bar{W}$ are $(r,\hat{m})$- and $(r,\bar{m})$-matrices, respectively, $q(\xi) \in \mathbb{R}^{\hat{m}+\bar{m}}$, $h(\xi) \in \mathbb{R}^r$, and the (r,m)-matrix $T(\xi)$ are affine functions of $\xi \in \mathbb{R}^s$, and $P \in \mathcal{P}(\Xi)$.
We need the following conditions to have the model (3.1) well-defined:
(B1) The matrices W and $\bar{W}$ have only rational elements.
(B2) For each pair $(x,\xi) \in X \times \Xi$ it holds that $h(\xi) - T(\xi)x \in \mathcal{T}$, where

$$\mathcal{T} := \left\{t \in \mathbb{R}^r : \exists (y,\bar{y}) \in \mathbb{Z}^{\hat{m}} \times \mathbb{R}^{\bar{m}} \text{ such that } Wy + \bar{W}\bar{y} \leq t\right\}.$$

(B3) For each $\xi \in \Xi$ the recourse cost $q(\xi)$ belongs to the dual feasible set

$$\mathcal{U} := \left\{u = (u_1,u_2) \in \mathbb{R}^{\hat{m}+\bar{m}} : \exists z \in \mathbb{R}^r_+ \text{ such that } W^\top z = u_1,\ \bar{W}^\top z = u_2\right\}.$$

(B4) The integer part of all second-stage feasible solutions is uniformly bounded for all $(u,t) \in \mathcal{U} \times \mathcal{T}$ (e.g., in case of mixed-binary programs).
(B5) $P \in \mathcal{P}_2(\Xi)$, i.e., $\int_\Xi \|\xi\|^2 P(d\xi) < +\infty$.
The conditions (B2) and (B3) mean *relatively complete recourse* and *dual feasibility*, respectively. We note that (B2) and (B3) imply $\Phi(u,t)$ to be finite for all $(u,t) \in \mathcal{U}\times\mathcal{T}$. The following additional properties of the value function Φ on $\mathcal{U} \times \mathcal{T}$ are important in the context of this paper.

Lemma 3.1. *Assume (B1)–(B4). Then there exists a countable partition of $\mathcal{T}$ into Borel subsets $\mathcal{B}_i$, i.e., $\mathcal{T} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i$ such that*

(i) each of the sets has a representation $\mathcal{B}_i = (b_i + T) \setminus \bigcup_{j=1}^{N_0}(b_{i,j} + T)$, where $b_i, b_{i,j} \in \mathbb{R}^r$ for $i \in \mathbb{N}$ and $j = 1,\ldots,N_0$, and $T := \{t \in \mathbb{R}^r : \exists y \geq 0$ such that $t \geq Wy\}$. Moreover there exists an $N_1 \in \mathbb{N}$ such that for any $t \in \mathcal{T}$ the ball $\mathbb{B}(t,1)$ in $\mathbb{R}^r$ is intersected by at most N_1 different subsets $\mathcal{B}_i$.

(ii) the restriction $\Phi|_{\mathcal{U}\times\mathcal{B}_i}$ of Φ to $\mathcal{U}\times\mathcal{B}_i$ has the property that there exists a constant $L > 0$ which does not depend on i, such that

$$|\Phi(u,t) - \Phi(\tilde{u},\tilde{t})| \leq L(\max\{1,\|t\|,\|\tilde{t}\|\}\|u-\tilde{u}\| + \max\{1,\|u\|,\|\tilde{u}\|\}\|t-\tilde{t}\|).$$

Furthermore, the function Φ is lower semicontinuous and piecewise polyhedral on $\mathcal{U}\times\mathcal{T}$ and there exist constants $D, d > 0$ such that it holds for all pairs $(u,t),(\tilde{u},\tilde{t}) \in \mathcal{U}\times\mathcal{T}$:

$$|\Phi(u,t) - \Phi(\tilde{u},\tilde{t})| \leq D(\max\{1,\|t\|,\|\tilde{t}\|\}(\|u-\tilde{u}\| + d) + \max\{1,\|u\|,\|\tilde{u}\|\}\|t-\tilde{t}\|).$$

The first part of Part 1 is Lemma 2.1. The second part is an extension of [19, Lemma 2.5] to the function $\Phi(u,t)$ since the relevant constants in its proof do not depend on the objective function as shown in Lemma 2.4. Part 2 and the quasi-Lipschitz property of Φ is Theorem 2.2.

The representation of Φ is given on countably many (possibly unbounded) Borel sets. This requires to incorporate the tail behavior of P and leads to the following representation of the function f_0.

Proposition 3.2. *Assume (B1)–(B4) and X be bounded. For each $R \geq 1$ and $x \in X$ there exist disjoint Borel subsets $\Xi_{j,x}^R$ of Ξ, $j = 1, \ldots, \nu$, whose closures are polyhedra with a uniformly bounded number of faces such that the function*

$$f_0(x,\xi) = \sum_{j=0}^{\nu} (\langle c, x\rangle + \Phi(q(\xi), h(\xi) - T(\xi)x)) \chi_{\Xi_{j,x}^R}(\xi) \quad ((x,\xi) \in X \times \Xi)$$

is Lipschitz continuous with respect to ξ on each $\Xi_{j,x}^R$, $j = 1, \ldots, \nu$, with some uniform Lipschitz constant. Here, $\Xi_{0,x}^R := \Xi \setminus \cup_{j=1}^{\nu} \Xi_{j,x}^R$ is contained in $\{\xi \in \mathbb{R}^s : \|\xi\| > R\}$ and ν is bounded by a multiple of R^r.

Proof: Since $q(\cdot)$, $h(\cdot)$ and $T(\cdot)$ are affine linear functions and X is bounded, there exists a constant $C > 0$ such that the estimate

$$\max\{\|q(\xi)\|_\infty, \|h(\xi) - T(\xi)x\|_\infty\} \leq C \max\{1, \|\xi\|_\infty\} \tag{3.3}$$

holds for each pair in $X \times \Xi$. Let $R \geq 1$ and $\mathcal{T}_R := \mathcal{T} \cap CR\mathbb{B}_\infty$, where $\mathbb{B}_\infty$ is the unit ball w.r.t. $\|\cdot\|_\infty$. As in [17, Proposition 34] there exist a number $\nu \in \mathbb{N}$ and disjoint Borel subsets $\{B_j\}_{j=1}^{\nu}$ of $CR\mathbb{B}_\infty$ such that their closures are polyhedra, their union contains $\mathcal{T}_R$, and ν is bounded above by κR^r, where the constant $\kappa > 0$ is independent of R. Now, let $x \in X$ and consider the following disjoint Borel subsets of Ξ:

$$\begin{aligned} \Xi_{j,x}^R &:= \{\xi \in \Xi : h(\xi) - T(\xi)x \in B_j,\ \|\xi\|_\infty \leq R\} \quad (j = 1, \ldots, \nu), \\ \Xi_{0,x}^R &:= \Xi \setminus \bigcup_{j=1}^{\nu} \Xi_{j,x}^R \subseteq \{\xi \in \Xi : \|\xi\|_\infty > R\}. \end{aligned}$$

Let $x \in X$ and $\xi, \xi' \in \Xi_{j,x}^R$ for some $j \in \{1, \ldots, \nu\}$. From Lemma 3.1 we obtain

$$\begin{aligned} |f_0(x,\xi) - f_0(x,\xi')| &= |\Phi(q(\xi), h(\xi) - T(\xi)x) - \Phi(q(\xi'), h(\xi') - T(\xi')x)| \\ &\leq L(\max\{1, \|q(\xi)\|_\infty, \|q(\xi')\|_\infty\}(\|h(\xi) - h(\xi')\|_\infty \\ &\quad + \|(T(\xi) - T(\xi'))x\|_\infty) + \max\{1, \|h(\xi) - T(\xi)x\|_\infty, \\ &\quad \|h(\xi') - T(\xi')x\|_\infty\}\|q(\xi) - q(\xi')\|_\infty) \\ &\leq LCR(\|h(\xi) - h(\xi')\|_\infty + \|(T(\xi) - T(\xi'))x\|_\infty \\ &\quad + \|q(\xi) - q(\xi')\|_\infty) \\ &\leq L_1 R\|\xi - \xi'\|_\infty, \end{aligned}$$

where we used (3.3) for $\xi, \xi' \in \Xi_{j,x}^R$, the affine linearity of $q(\cdot)$, $h(\cdot)$ and $T(\cdot)$, and the boundedness of X. We note that the constant L_1 does not depend on R. □

In order to state quantitative stability results for model (3.1), we consider the following probability metric on $\mathcal{P}_2(\Xi)$ for some $k \in \mathbb{N}$:

$$\zeta_{2,\mathrm{ph}_k}(P,Q) := \sup\left\{\left|\int_B f(\xi)(P-Q)(d\xi)\right| : f \in \mathcal{F}_2(\Xi), B \in \mathcal{B}_{\mathrm{ph}_k}(\Xi)\right\}. \tag{3.4}$$

Here, $\mathcal{B}_{\mathrm{ph}_k}(\Xi)$ denotes the set of all polyhedra being subsets of Ξ and having at most k faces. The set $\mathcal{F}_2(\Xi)$ contains all functions f from Ξ to $\mathbb{R}$ such that

$$|f(\xi) - f(\tilde{\xi})| \leq \max\{1, \|\xi\|, \|\tilde{\xi}\|\}\|\xi - \tilde{\xi}\|$$

holds for all $\xi, \tilde{\xi} \in \Xi$. Next we state the main result of our paper.

Theorem 3.3. *Let the conditions (B1)–(B5) be satisfied and X be compact. Then there exist constants $L > 0$ and $k \in \mathbb{N}$ such that*

$$\begin{aligned} |v(P) - v(Q)| &\leq L\phi_P(\zeta_{2,\mathrm{ph}_k}(P,Q)) \\ \emptyset \neq S(Q) &\subseteq S(P) + \Psi_P(L\phi_P(\zeta_{2,\mathrm{ph}_k}(P,Q)))\mathbb{B}, \end{aligned} \tag{3.5}$$

for each $Q \in \mathcal{P}_2(\Xi)$. The function ϕ_P on $\mathbb{R}_+$ is defined by $\phi_P(0) = 0$ and

$$\phi_P(t) := \inf_{R \geq 1} \left\{ R^{r+1}t + \int_{\{\xi \in \Xi : \|\xi\|_\infty > R\}} \|\xi\|_\infty^2 P(d\xi) \right\} \quad (t > 0), \tag{3.6}$$

and the conditioning function Ψ_P is given by

$$\Psi_P(\eta) := \eta + \psi_P^{-1}(2\eta) \quad (\eta \in \mathbb{R}_+), \tag{3.7}$$

where the growth function ψ_P on $\mathbb{R}_+$ is

$$\psi_P(\tau) := \min \left\{ \int_\Xi f_0(x,\xi)P(d\xi) - v(P) : d(x, S(P)) \geq \tau, x \in X \right\} \tag{3.8}$$

with inverse $\psi_P^{-1}(t) := \sup\{\tau \in \mathbb{R}_+ : \psi_P(\tau) \leq t\}$. If $P \in \mathcal{P}_p(\Xi)$ for $p > 2$, the estimate $\phi_P(t) \leq Ct^{\frac{p-2}{p+r-1}}$ holds for $t \geq 0$ and some constant $C > 0$.

Proof: Since the function Φ is lower semicontinuous on $\mathcal{U} \times \mathcal{T}$ (Lemma 3.1), f_0 is lower semicontinuous on $X \times \Xi$ and, hence, a random lower semicontinuous function [16, Example 14.31]. Using Lemma 3.1 we obtain the estimate

$$\begin{aligned} |f_0(x,\xi)| &\leq \|c\|\|x\| + D[\max\{1, \|h(\xi)\| + \|T(\xi)\|\|x\|\}(\|q(\xi)\| + d) \\ &\quad + \max\{1, \|q(\xi)\|\}(\|h(\xi)\| + \|T(\xi)\|\|x\|)] \\ &\leq C_1 \max\{1, \|\xi\|\}\|\xi\| \end{aligned}$$

for each pair $(x,\xi) \in X \times \Xi$ and some constant C_1. Hence, the objective function $\langle c, x\rangle + \int_\Xi \Phi(q(\xi), h(\xi) - T(\xi)x)Q(d\xi)$ is finite (if $Q \in \mathcal{P}_2(\Xi)$) and lower semicontinuous (due to Fatou's lemma). Since X is compact, the solution set $S(Q)$ is nonempty.
From Proposition 3.2 we know that, for each $R \geq 1$ and $x \in X$, there exist Borel subsets $\Xi_{j,x}^R$, $j = 1, \ldots, \nu$, of Ξ such that the function $f_{j,x}^R(\cdot) := f_0(x,\cdot)|_{\Xi_{j,x}^R}$ is Lipschitz continuous on $\Xi_{j,x}^R$ with constant $L_1 R$. We extend each function $f_{j,x}^R(\cdot)$ to the whole of Ξ by preserving the Lipschitz constant. Then we have $\frac{1}{L_1 R} f_{j,x}^R(\cdot) \in \mathcal{F}_2(\Xi)$. Furthermore, Proposition 3.2 implies that the closures of $\Xi_{j,x}^R$ are contained in $\mathcal{B}_{\mathrm{ph}_k}(\Xi)$ for some $k \in \mathbb{N}$, that the number ν is bounded above by κR^r, where the constant $\kappa > 0$ is

independent on R, and that $\Xi^R_{0,x} := \Xi \setminus \bigcup_{j=1}^{\nu} \Xi^R_{j,x}$ is a subset of $\{\xi \in \Xi : \|\xi\|_\infty > R\}$. For each $Q \in \mathcal{P}_2(\Xi)$ and $x \in X$ we obtain

$$\begin{aligned}\left|\int_\Xi f_0(x,\xi)(P-Q)(d\xi)\right| &= \left|\sum_{j=0}^{\nu}\int_{\Xi^R_{j,x}} f_0(x,\xi)(P-Q)(d\xi)\right| \\ &\le \sum_{j=1}^{\nu}\left|\int_{\Xi^R_{j,x}} f^R_{j,x}(\xi)(P-Q)(d\xi)\right| + I^R_x(P,Q) \\ &\le \nu L_1 R \sup_{\substack{f\in\mathcal{F}_2(\Xi)\\ j=1,\dots,\nu}}\left|\int_\Xi f(\xi)\chi_{\Xi^R_{j,x}}(\xi)(P-Q)(d\xi)\right| + I^R_x(P,Q),\end{aligned}$$

where the last summand on the right-hand side is given by

$$I^R_x(P,Q) := \left|\int_{\Xi^R_{0,x}} f_0(x,\xi)(P-Q)(d\xi)\right|.$$

Using $\nu \le \kappa R^r$ and arguing as in [17, Theorem 35] we continue

$$\left|\int_\Xi f_0(x,\xi)(P-Q)(d\xi)\right| \le \kappa L_1 R^{r+1}\zeta_{2,\mathrm{ph}_k}(P,Q) + I^R_x(P,Q).$$

For the term $I^R_x(P,Q)$ we use the estimate $|f_0(x,\xi)| \le C_1\|\xi\|^2$ for any pair $(x,\xi) \in X \times \{\xi \in \Xi : \|\xi\|_\infty > R\}$ and the norming constant C_2 such that $\|\xi\| \le C_2\|\xi\|_\infty$ holds for all $\xi \in \mathbb{R}^s$. We get

$$I^R_x(P,Q) \le C_1C_2^2\int_{\{\xi\in\Xi:\|\xi\|_\infty>R\}} \|\xi\|^2_\infty (P+Q)(d\xi).$$

Since the set $\{\xi \in \Xi : \|\xi\|_\infty > R\}$ can be covered by 2^s intersections of Ξ with open halfspaces (whose closures belong to $\mathcal{B}_{\mathrm{ph}_k}(\Xi)$), we have the estimate

$$\int_{\{\xi\in\Xi:\|\xi\|_\infty>R\}} \|\xi\|^2_\infty Q(d\xi) \le 2^s\zeta_{2,\mathrm{ph}_k}(P,Q) + \int_{\{\xi\in\Xi:\|\xi\|_\infty>R\}} \|\xi\|^2_\infty P(d\xi).$$

Hence, combining the last three estimates we get

$$\sup_{x\in X}\left|\int_\Xi f_0(x,\xi)(P-Q)(d\xi)\right| \le \kappa L_1 R^{r+1}\zeta_{2,\mathrm{ph}_k}(P,Q) + 2C_1C_2^2\int_{\{\xi\in\Xi:\|\xi\|>R\}} \|\xi\|^2_\infty P(d\xi)$$

for any $R \ge 1$. Taking the infimum with respect to $R \ge 1$ we obtain

$$\sup_{x\in X}\left|\int_\Xi f_0(x,\xi)(P-Q)(d\xi)\right| \le \hat{C}\phi_P(\zeta_{2,\mathrm{ph}_k}(P,Q))$$

with some constant $\hat{C} > 0$. Now, the result is a consequence of the Theorems 5 and 9 in [17]. If $\int_\Xi \|\xi\|^p dP(\xi) < \infty$ for some $p > 2$, it holds that $\int_{\{\xi\in\Xi:\|\xi\|_\infty>R\}} \|\xi\|^2_\infty dP(\xi) \le$

$R^{2-p}\int_\Xi \|\xi\|_\infty^p P(d\xi)$ by Markov's inequality. The desired estimate follows by inserting $R = t^{-\frac{1}{p+r-1}}$ for small $t > 0$ into the function whose infinum w.r.t. $R \geq 1$ is $\phi_P(t)$:

$$\phi_P(t) \leq t^{-\frac{r+1}{p+r-1}+1} + t^{\frac{p-2}{p+r-1}} \int_\Xi \|\xi\|_\infty^p P(d\xi) \leq C t^{\frac{p-2}{p+r-1}}. \quad \square$$

The boundedness condition on X may be relaxed if localized optimal values and solution sets are considered (see [17]). In case that the underlying distribution P and its perturbations Q have supports in some bounded subset Ξ of $\mathbb{R}^s$, the stability result improves slightly.

Corollary 3.4. *Let the conditions (B1)–(B4) be satisfied, $P \in \mathcal{P}(\Xi)$, X and Ξ be bounded. Then there exist constants $L > 0$ and $k \in \mathbb{N}$ such that*

$$\begin{aligned} |v(P) - v(Q)| &\leq L\zeta_{2,\mathrm{ph}_k}(P,Q) \\ \emptyset \neq S(Q) &\subseteq S(P) + \Psi_P(L\zeta_{2,\mathrm{ph}_k}(P,Q))\mathbb{B}, \end{aligned}$$

holds for each $Q \in \mathcal{P}(\Xi)$.

Proof: Since Ξ is bounded, we have $\mathcal{P}_2(\Xi) = \mathcal{P}(\Xi)$. Moreover, the function $\phi_P(t)$ (see (3.6)) can be estimated by $R^{r+1}t$ for some sufficiently large $R > 0$. Hence, Theorem 3.3 implies the assertion. $\square$

Remark 3.5. Since $\Xi \in \mathcal{B}_{\mathrm{ph}_k}(\Xi)$ for some $k \in \mathbb{N}$, we obtain from (3.4) by choosing $B := \Xi$ and $f \equiv 1$, respectively,

$$\max\{\zeta_2(P,Q), \alpha_{\mathrm{ph}_k}(P,Q)\} \leq \zeta_{2,\mathrm{ph}_k}(P,Q) \tag{3.9}$$

for all $P, Q \in \mathcal{P}_2(\Xi)$. Here, ζ_2 and α_{ph_k} denote the second order Fortet-Mourier metric [14, Section 5.1] and the polyhedral discrepancy

$$\begin{aligned} \zeta_2(P,Q) &:= \sup_{f \in \mathcal{F}_2(\Xi)} \left| \int_\Xi f(\xi)P(d\xi) - \int_\Xi f(\xi)Q(d\xi) \right| \\ \alpha_{\mathrm{ph}_k}(P,Q) &:= \sup_{B \in \mathcal{B}_{\mathrm{ph}_k}(\Xi)} |P(B) - Q(B)|, \end{aligned}$$

respectively. Hence, convergence with respect to ζ_{2,ph_k} implies weak convergence (see [3]), convergence of second order absolute moments, and convergence with respect to the polyhedral discrepancy α_{ph_k}. For bounded $\Xi \subset \mathbb{R}^s$ the estimate

$$\zeta_{2,\mathrm{ph}_k}(P,Q) \leq C_s \alpha_{\mathrm{ph}_k}(P,Q)^{\frac{1}{s+1}} \quad (P,Q \in \mathcal{P}(\Xi)) \tag{3.10}$$

(with some constant $C_s > 0$) can be derived by using the technique in the proof of [19, Proposition 3.1]. In view of (3.9), (3.10) the metric ζ_{2,ph_k} is stronger than α_{ph_k} in general, but in case of bounded Ξ both distances metrize the same topology on $\mathcal{P}(\Xi)$.

For more specific models (3.1), improvements of the above results may be obtained by exploiting specific recourse structures, i.e., by using additional information on the shape of the sets $\mathcal{B}_i$ in Lemma 3.1 and on the behavior of the (value) function Φ on these sets. This may lead to stability results with respect to probability metrics that are (much) weaker than ζ_{2,ph_k}. For example, if the recourse costs and the technology matrix are fixed and $h(\cdot)$ is of the form $h(\xi) := \xi$ (i.e., $r = s$), the relevant class $\mathcal{B}_{\mathrm{ph}_k}(\Xi)$ is completely characterized in [19, Section 3]. If, in addition, the model has pure integer recourse, the stability result is valid with respect to the Kolmogorov metric

$$d_{\mathrm{K}}(P,Q) := \sup_{z\in\mathbb{R}^s} |P((-\infty,z]) - Q((-\infty,z])|$$

on $\mathcal{P}(\Xi)$ if Ξ is bounded (see also [19, Proposition 3.4]).

A Proof of Theorem 2.2

Let $b \in \mathcal{B}$, $b' \in \mathcal{B}^*(b)$, and $c, c' \in -\mathcal{K}^*$ be given. To show local Lipschitz continuity of $\vartheta(b,c)$, we estimate

$$|\vartheta(b,c) - \vartheta(b',c')| \le |\vartheta(b,c) - \vartheta(b',c)| + |\vartheta(b',c) - \vartheta(b',c')|.$$

For the first difference we can proceed as for the proof of Lemma 2.3 in [19]. It is repeated here to keep the paper self-contained. We write (2.1) as

$$\min\{\langle c_x, x\rangle + \Psi(c_y, b - A_x x) : x \in \mathrm{Pr}_x M(b)\}$$

where $\Psi(c_y,\tilde{b}) := \min\{\langle c_y, y\rangle : A_y y \le \tilde{b}\}$. Since $\Psi(c_y,\tilde{b})$ is the optimal value function of a linear program and finite for $b \in \mathcal{B}$, $c' \in -\mathcal{K}^*$, there exist a finite number of matrices C_j, which depend on A_y only, such that $\Psi(c_y,\tilde{b}) = \max_j\langle \tilde{b}, C_j c_y\rangle$ (cf. [22]). Hence, for c_y fixed,

$$|\Psi(c_y,\tilde{b}) - \Psi(c_y,\tilde{b}')| \le \max_j \|C_j\|\|c_y\|\,\|\tilde{b} - \tilde{b}'\|.$$

Let now $(x,y) \in \Sigma(b,c)$ and $(x',y') \in \Sigma(b',c)$. Since $\mathrm{Pr}_x\, M(b) = \mathrm{Pr}_x\, M(b')$, we obtain

$$\vartheta(b,c) - \vartheta(b',c) \le \langle c_x, x'\rangle + \Psi(c_y, b - A_x x') - \langle c_x, x'\rangle - \Psi(c_y, b' - A_x x') \le L_1\|c\|\,\|b - b'\|,$$

where $L_1 := \max_j \|C_j\|$. Due to symmetry the same estimate holds for $\vartheta(b',c) - \vartheta(b,c)$.

To estimate $|\vartheta(b',c) - \vartheta(b',c')|$ we take $(x,y) \in \Sigma(b',c)$ and $(x',y') \in \Sigma(b',c')$ such that $x, x' \in \mathbb{B}(0,K)$, $y = C_j(b' - A_x x)$, and $y' = C_{j'}(b' - A_x x')$ for some indices j and j'. Since b' is fixed, it is $\vartheta(b',c) \le \langle c_x, x'\rangle + \langle c_y, y'\rangle$ and $\vartheta(b',c') \le \langle c'_x, x\rangle + \langle c'_y, y\rangle$. Hence,

$$|\vartheta(b',c) - \vartheta(b',c')| \le \max\{\|(x,y)\|, \|(x',y')\|\}\,\|c - c'\|.$$

Using $\|y\| \le L_1(\|b'\| + \|A_x\|\|x\|)$, $\|y'\| \le L_1(\|b'\| + \|A_x\|\|x'\|)$, and $\|x\|, \|x'\| \le K$, we obtain

$$|\vartheta(b',c) - \vartheta(b',c')| \le (K + L_1(\|b'\| + \|A_x\|K))\,\|c - c'\|$$

and let $L_2 := 1 + L_1\|A_x\| + L_1$.

To proof the second part of Theorem 2.2, we use the property that the distance of the optimal set of (2.1) to optimal points of its linear relaxation

$$\min\{\langle c_x, x\rangle + \langle c_y, y\rangle : A_x x + A_y y \leq b, (x,y) \in \mathbb{R}^m\} \tag{A.1}$$

can be bounded with a constant that is independent from the choice of b and c. This is stated in the following Lemma, which is Theorem 1.2 in [5].

Lemma A.1. *Let $b \in \mathcal{B}$, $c \in -\mathcal{K}^*$. Let $(\tilde{x}, \tilde{y})$ be an optimal point of* (A.1)*. Then there exists a solution point $(x,y) \in \Sigma(b,c)$, such that*

$$\|(x,y) - (\tilde{x}, \tilde{y})\| \leq \ell$$

for some constant ℓ depending on A only.

We are now ready to proof the quasi-Lipschitz continuity of the optimal value function $\vartheta(b,c)$. Since the optimal value function of a linear program is locally Lipschitz-continuous in its right-hand side and objective function coefficients, there exists a constant $\tilde{L}$ which depends on A only and optimal points $(\tilde{x}, \tilde{y})$ and $(\tilde{x}', \tilde{y}')$ of the linear program (A.1) with right-hand sides b and b' and objective function coefficients c and c', respectively, such that

$$|\langle c, (\tilde{x}, \tilde{y})\rangle - \langle c', (\tilde{x}', \tilde{y}')\rangle| \leq \tilde{L}\|c\|\|b - b'\| + \tilde{L}\|b'\|\|c - c'\|.$$

Next, by Lemma A.1, there exist points $(x,y) \in \Sigma(b,c)$ and $(x',y') \in \Sigma(b',c')$ with distance at most ℓ to $(\tilde{x}, \tilde{y})$ and $(\tilde{x}', \tilde{y}')$. Hence,

$$\begin{aligned} |\vartheta(b,c) - \vartheta(b',c')| &\leq |\langle c, (x,y)\rangle - \langle c, (\tilde{x}, \tilde{y})\rangle| + |\langle c, (\tilde{x}, \tilde{y})\rangle - \langle c', (\tilde{x}', \tilde{y}')\rangle| \\ &\quad + |\langle c', (x',y')\rangle - \langle c', (\tilde{x}', \tilde{y}')\rangle| \\ &\leq 2\ell \max\{\|c\|, \|c'\|\} + \tilde{L}\|c\|\|b - b'\| + \tilde{L}\|b'\|\|c - c'\|. \end{aligned}$$

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