Passivation of LTI systems

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Abstract

In this paper we consider a passivation procedure for linear time-invariant systems. This procedure is based on the spectral properties of related Hamiltonian matrices. We also present a structure-preserving algorithm for computing the imaginary eigenvalues and the corresponding eigenvectors of Hamiltonian matrices. Numerical examples are given.

1 Introduction

Consider a linear time-invariant (LTI) system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t) + Du(t), \end{array} \tag{1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is an input and $y(t) \in \mathbb{R}^p$ is an output. We will assume that system (1) is *asymptotically stable*, i.e., all eigenvalues of A lie in the open left half-plane. A *transfer function* of (1) is given by

$$H(s) = C(sI - A)^{-1}B + D.$$

It describes the input-output relation of system (1) in the frequency domain. For simplicity, we will denote system (1) also by $H = [\,A,\, B,\, C,\, D\,]$.

Passivity is a crucial concept in system design, because the interconnection of passive multiport systems is guaranteed to be stable [18]. The asymptotically stable system (1) is *passive* if its transfer function $H(s)$ is *positive real*, i.e., $H(j\omega)^* + H(j\omega)$ is positive semidefinite for all $\omega \in \mathbb{R}$. In the case of the scattering representation, e.g., [16], system (1) is *passive* if its transfer function $H(s)$ is *bounded real*, i.e., $I - H(j\omega)^*H(j\omega)$ is positive definite for all $w \in \mathbb{R}$. In the following we will consider only this type of passivity that is also known as *contractivity* [1]. Note that the bounded realness can also be written as

$$\sigma_{\max}(H(j\omega)) \le 1 \quad \text{for all } \omega \in \mathbb{R}.$$

More general, an asymptotically stable system (1) is said to be γ_*-passive, if all singular values of $H(j\omega)$ are not greater than γ_*. For $\gamma_* = 1$, this is the original definition.

In this paper, we discuss the following two problems:

- Check whether system (1) is γ_*-passive. In particular, determine the frequencies, where the passivity condition $\sigma_{\max}(H(j\omega)) \le \gamma_*$ is violated.
- For a non-passive system, find another one that is 'close by' and passive.

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In Section 2 we introduce some notation and used results. Section 3 deals with the passivity check, while in Section 4 the passivation scheme is discussed. The described algorithms have been implemented. Numerical results are presented in Section 5. We want to note that, apart from Section 3.1, most of the theory is along the lines of [10, 11].

Notations. We will denote by $\mathbb{R}^{n,m}$ and $\mathbb{C}^{n,m}$ the space of $n \times m$ real and complex matrices, respectively. The real and imaginary parts of $s \in \mathbb{C}$ are denoted by $\Re e(s)$ and $\Im m(s)$, respectively, and $j = \sqrt{-1}$. The matrices A^T and $\bar{A}$ denote, respectively, the transpose and the complex conjugate of A, $A^* = \bar{A}^T$ and $A^{-T} = (A^{-1})^T$. An identity matrix of order n is denoted by I_n or simply by I, and e_i denotes the i-th column of I. A positive scalar σ is said to be a singular value of a matrix $A \in \mathbb{C}^{n,m}$ if σ^2 is an eigenvalue of A^*A. The largest singular value of A is denoted by $\sigma_{\max}(A)$. We use $\|\cdot\|_2$ to denote the Euclidean vector norm and the spectral matrix norm.

2 Preliminaries

For matrices $A \in \mathbb{C}^{n,m}$ and $B \in \mathbb{C}^{k,l}$, the Kronecker product $A \otimes B \in \mathbb{C}^{nk,ml}$ is defined via

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}.$$

Using the Kronecker product, the matrix relation $Y = BXA^T$ can be rewritten in an equivalent form $\mathrm{vec}(Y) = (A \otimes B)\mathrm{vec}(X)$, where $\mathrm{vec}(\cdot)$ denotes the vectorization operator, i.e., stacking the columns of a matrix upon each other to get a single vector [14]. The Frobenius norm of a matrix is defined as $\|A\|_F = \|\mathrm{vec}(A)\|_2$.

We will need the following spectral perturbation result, see [19]. Let λ be a simple eigenvalue of a matrix $A \in \mathbb{R}^{n,n}$ and let $A_\epsilon = A + \epsilon A'$ be a perturbed matrix with $A' \in \mathbb{R}^{n,n}$ and $\epsilon \in \mathbb{R}$. Then for sufficiently small ϵ, an eigenvalue λ_ϵ of A_ϵ can be written as

$$\lambda_\epsilon = \lambda + \epsilon\,\kappa + \mathcal{O}(\epsilon^2), \tag{2}$$

where the coefficient κ is given by

$$\kappa = \left.\frac{d\lambda_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \frac{w^* A' v}{w^* v}.$$

Here, w and v are the left and right eigenvectors of A corresponding to the eigenvalue λ.

A matrix $M \in \mathbb{R}^{2n,2n}$ is called *Hamiltonian* if it is symmetric in the J–inner product, i.e.,

$$JM = (JM)^T \quad \text{with} \quad J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{3}$$

Any Hamiltonian matrix M can be represented as

$$M = \begin{bmatrix} F & G \\ Q & -F^T \end{bmatrix},$$

where F, G, $Q \in \mathbb{R}^{n,n}$ and $G = G^T$, $Q = Q^T$. Let v be a right eigenvector of M corresponding to an eigenvalue λ. It follows from (3) that

$$\lambda Jv = JMv = (JM)^T v = M^T J^T v = -M^T Jv.$$

Since M is real, we have

$$(Jv)^* M = -\overline{\lambda}(Jv)^*.$$

One can see that together with λ also $-\overline{\lambda}$ is an eigenvalue of M with a left eigenvector Jv. In particular, if $\lambda = j\omega$ is an imaginary eigenvalue of M with a right eigenvector v, then Jv is a left eigenvector of M corresponding to the eigenvalue $-\overline{\lambda} = j\omega$. Thus, the left eigenvectors corresponding to purely imaginary eigenvalues of a Hamiltonian matrix can be immediately obtained from the right eigenvectors without additional computations.

If $j\omega$ is a simple imaginary eigenvalue of a Hamiltonian matrix M, then the perturbation result (2) for a perturbed matrix $M + \epsilon M'$ with $M' \in \mathbb{R}^{2n,2n}$ takes the form

$$\lambda_\epsilon = j\omega + \epsilon \frac{v^* J M' v}{v^* J v} + \mathcal{O}(\epsilon^2),$$

where v is a right eigenvector of M corresponding to the eigenvalue $j\omega$ and λ_ϵ is an eigenvalue of $M + \epsilon M'$. If the perturbation matrix M' is also Hamiltonian, then JM' is real symmetric and, hence, $v^* J M' v$ is real. Since J is real skew-symmetric, the term $v^* J v$ is purely imaginary. In this case λ_ϵ stays imaginary, at least to the first order perturbation. It can be shown that a simple imaginary eigenvalue of a real Hamiltonian matrix cannot leave the imaginary axis under Hamiltonian perturbations, as long as it stays simple. Thus, we have $\lambda_\epsilon = j\omega_\epsilon$ with

$$\omega_\epsilon = \omega + \epsilon \frac{v^* J M' v}{j v^* J v} + \mathcal{O}(\epsilon^2). \tag{4}$$

3 Passivity check

In this section, a method to check the γ_*-passivity of the LTI system (1) is discussed. For $\gamma_* = 1$, we get the classic notion of passivity. The value $\gamma_* < 1$ can be used in order to decide whether the singular values of $H(j\omega)$ are far away from 1. Since $H(j\omega)$ approaches D as $\omega \to \infty$, we will assume that $\sigma_{\max}(D) < \gamma_*$.

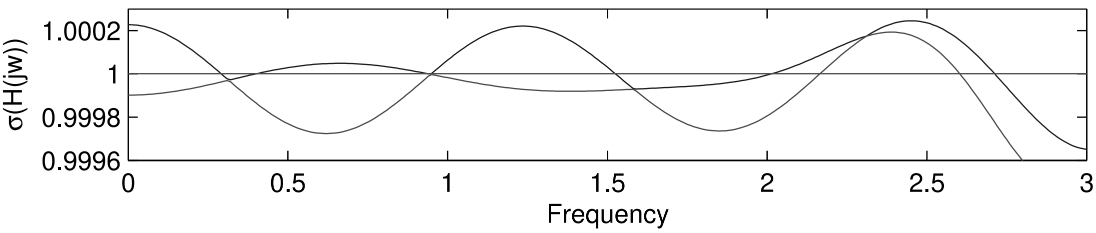


Figure 1: The singular values of $H(j\omega)$.

For some example system, we plotted in Figure 1 the singular values of $H(j\omega)$ for the frequency range $\omega \in [0, 3]$. One can see that this system has passivity violation in the regions ca. $[0, 0.3]$, $[0.4, 0.94]$, $[0.95, 1.5]$ and $[2, 2.7]$. These regions are exactly the kind of information we seek. To determine them more precisely, we can use the following theorem.

Theorem 1 *Consider an LTI system* (1)*, where* A *has no purely imaginary eigenvalues. Let* $\gamma > 0$ *be a scalar such that* γ *is not a singular value of* D*. Then* γ *is a singular value of* $H(j\omega)$ *if and only if* $j\omega$ *is an eigenvalue of the Hamiltonian matrix*

$$M_\gamma = \begin{bmatrix} A - BR_\gamma^{-1} D^T C & -\gamma B R_\gamma^{-1} B^T \\ \gamma C^T S_\gamma^{-1} C & -A^T + C^T D R_\gamma^{-1} B^T \end{bmatrix}, \tag{5}$$

where $R_\gamma = D^T D - \gamma^2 I$ *and* $S_\gamma = DD^T - \gamma^2 I$.

Proof: See, e.g., [7]. ■

Thus, by calculating the imaginary eigenvalues of M_{γ_*} we find the frequencies, where the singular value curves of $H(j\omega)$ cross the level γ_*. The imaginary eigenvalues of M_1 for the above example are given by

$$0.2912j,\ 0.4010j,\ 0.9386j,\ 0.9487j,\ 1.5247j,\ 2.0144j,\ 2.1652j,\ 2.6029j,\ 2.7120j.$$

They coincide with the frequencies, where the singular value curves cross the level $\gamma_* = 1$, see Figure 1. But this is not enough to determine the regions of passivity violation. The next result provides the missing information.

Theorem 2 *Consider an LTI system* (1). *Let* $0 < \gamma_* \leq 1$ *be a scalar such that* $\sigma_{\max}(D) < \gamma_*$. *Further, let* v *be a right eigenvector of* M_{γ_*} *corresponding to a simple imaginary eigenvalue* $j\omega_{\gamma_*}$ *and let* $\sigma(\omega)$ *be a singular value curve of* $H(j\omega)$ *that crosses the level* γ_* *at* ω_{γ_*}, *i.e.,* $\sigma(\omega_{\gamma_*}) = \gamma_*$. *Then the slope of* $\sigma(\omega)$ *is positive (negative) at* ω_{γ_*} *if* $jv^*Jv > 0$ ($jv^*Jv < 0$).

Proof: We want to motivate the technique of the proof by the following idea. To decide whether the curve increases or decreases at the point ω_{γ_*}, we could compute the point $\omega_{\gamma_*} + \delta$, where the curve crosses the level $\gamma_* + \epsilon$ with $\epsilon > 0$ and then check whether δ is positive or negative. Thus, the actual proof starts by analyzing the eigenvalues of the perturbed matrix

$$M_{\gamma_*+\epsilon} = M_{\gamma_*} + \epsilon M'_{\gamma_*} + \mathcal{O}(\epsilon^2),$$

where

$$M'_{\gamma_*} = \left.\frac{dM_\gamma}{d\gamma}\right|_{\gamma=\gamma_*} = \begin{bmatrix} -2\gamma_* B R_{\gamma_*}^{-2} D^T C & -B(2\gamma_* R_{\gamma_*}^{-2} + R_{\gamma_*}^{-1})B^T \\ C^T(2\gamma_* S_{\gamma_*}^{-2} + S_{\gamma_*}^{-1})C & 2\gamma_* C^T D R_{\gamma_*}^{-2} B^T \end{bmatrix}.$$

Note that the matrix M'_{γ_*} is Hamiltonian. Let $j\omega_{\gamma_*}$ be an eigenvalue of M_{γ_*} and let $j\omega_{\gamma_*+\epsilon}$ be the corresponding perturbed eigenvalue of $M_{\gamma_*+\epsilon}$. Then, using (4) we have

$$\omega_{\gamma_*+\epsilon} = \omega_{\gamma_*} + \epsilon\,\kappa + \mathcal{O}(\epsilon^2),$$

where

$$\kappa = \left.\frac{d\omega_\gamma}{d\gamma}\right|_{\gamma=\gamma_*} = \frac{v^* J M'_{\gamma_*} v}{jv^*Jv}.$$

Since σ and γ are interchangeable, it follows that the slope of the singular value curve crossing the level γ_* at ω_* can be written as

$$\xi := \left.\frac{d\sigma}{d\omega}\right|_{\omega=\omega_{\gamma_*}} = \frac{1}{\left.\frac{d\omega}{d\gamma}\right|_{\gamma=\gamma_*}} = \frac{1}{\kappa} = \frac{jv^*Jv}{v^*JM'_{\gamma_*}v}.$$

We only need to know the sign of ξ to decide whether the singular value curve increases or decreases. The computation can be simplified by noting that JM'_{γ_*} is positive semidefinite. Indeed, we have

$$JM'_{\gamma_*} = \begin{bmatrix} C^T & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} S_{\gamma_*}^{-1} & 0 \\ 0 & R_{\gamma_*}^{-1} \end{bmatrix} \widetilde{M} \begin{bmatrix} S_{\gamma_*}^{-1} & 0 \\ 0 & R_{\gamma_*}^{-1} \end{bmatrix}^T \begin{bmatrix} C^T & 0 \\ 0 & B \end{bmatrix}^T,$$

where

$$\widetilde{M} = \begin{bmatrix} D & \gamma_* I \\ \gamma_* I & D^T \end{bmatrix} \begin{bmatrix} D & \gamma_* I \\ \gamma_* I & D^T \end{bmatrix}^T + \begin{bmatrix} 2\gamma_*(1-\gamma_*)I & 0 \\ 0 & 2\gamma_*(1-\gamma_*)I \end{bmatrix}$$

is positive semidefinite. Thus, $\operatorname{sign}(\xi) = \operatorname{sign}(jv^*Jv)$. ■

The frequencies, where the singular value curves cross the level γ_*, together with the information whether the curves are increasing or decreasing admit to decide for γ_*-passivity.

Corollary 3 (Passivity check) *Assume that an LTI system* (1) *is asymptotically stable. Let* $0 < \gamma_* \leq 1$ *be a scalar such that* $\sigma_{\max}(D) < \gamma_*$. *Let the matrix* M_{γ_*} *as in* (5) *have the simple imaginary eigenvalues* $\pm j\omega_1, \ldots, \pm j\omega_k$ *with the corresponding right eigenvectors* $v_1, \ldots, v_k$. *Then*

1. *system* (1) *is* γ_**-passive if* $k = 0$, *i.e., if* M_{γ_*} *does not have purely imaginary eigenvalues;*
2. *system* (1) *is* γ_**-passive at frequency* ω *if* $\sum_{i\,:\,\omega_i \geq \omega} \operatorname{sign}(jv_i^* J v_i) = 0$.

Proof: See, [10]. ■

This result has been applied to above example. In Figure 2 the down aiming and up aiming arrowheads mark the location of the imaginary eigenvalues whose corresponding eigenvectors v fulfill $jv^*Jv < 0$ and $jv^*Jv > 0$, respectively. The marks coincide with the locations, where the singular value curves cross the level $\gamma_* = 1$. The slopes are also predicted correctly. By just looking at the marks, it is possible to decide whether the system is passive at a given frequency ω (the numbers of arrow heads to the right of ω pointing upwards and downwards must be equal).

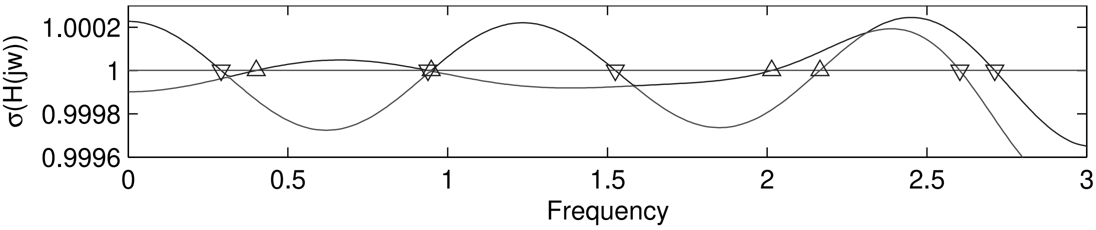


Figure 2: The singular values of $H(j\omega)$ with the crossing points.

Note that a rigorous treatment of the theory would involve the consideration of multiple imaginary eigenvalues of M_{γ_*}. Such eigenvalues correspond to the singular value curves that are touching the level γ_*, but do not cross it, or to several singular value curves crossing the level γ_* at the same frequency. However, multiple eigenvalues never occurred during our tests. Nevertheless, this will be a topic of further research.

3.1 Computing eigenvalues and eigenvectors of Hamiltonian matrices

As described above, in order to check the passivity of system (1), one has to compute eigenvalues and eigenvectors of a Hamiltonian matrix. Of course, one could use the standard solver tool for the non-symmetric eigenvalue problem based on the computation of the Schur form [9]. But because of rounding errors all computed eigenvalues will be slightly perturbed. This raises the difficulty to decide whether a computed eigenvalue is purely imaginary or not. On the other hand, Hamiltonian matrices provide a lot of structure that can be used to compute the eigenvalues and eigenvectors more precisely [6, 15].

An alternative approach is provided by the symplectic URV decomposition [5, 6]. A matrix $S \in \mathbb{R}^{2n,2n}$ is called *symplectic* if it is orthogonal in the J–inner product, i.e.,

$$S^T J S = J.$$

A matrix $U \in \mathbb{R}^{2n,2n}$ is called *orthogonal symplectic*, if it is orthogonal and symplectic, i.e., $U^T U = I$ and $U^T J U = J$. Any orthogonal symplectic matrix U can be written as

$$U = \begin{bmatrix} U_1 & U_2 \\ -U_2 & U_1 \end{bmatrix}, \qquad U_1, U_2 \in \mathbb{R}^{n,n}.$$

For every matrix $M \in \mathbb{R}^{2n,2n}$, there exists a *symplectic URV decomposition*

$$M = U \begin{bmatrix} R_1 & R_3 \\ 0 & R_2^T \end{bmatrix} V^T, \tag{6}$$

where $U, V \in \mathbb{R}^{2n,2n}$ are orthogonal symplectic, $R_1 \in \mathbb{R}^{n,n}$ is upper triangular and $R_2 \in \mathbb{R}^{n,n}$ is quasi upper triangular with 1×1 and 2×2 blocks on the diagonal. If M is Hamiltonian, then we also have

$$M = V \begin{bmatrix} -R_2 & R_3^T \\ 0 & -R_1^T \end{bmatrix} U^T. \tag{7}$$

It follows from (6) and (7) that

$$M^2 = U \begin{bmatrix} -R_1R_2 & R_1R_3^T - R_3^TR_1 \\ 0 & (-R_1R_2)^T \end{bmatrix} U^T.$$

Since R_1 is upper triangular and R_2 is quasi-upper triangular, also the product R_1R_2 is quasi-upper triangular. Thus, all eigenvalues of M^2 can be read off the diagonal blocks of $-R_1R_2$. Since the eigenvalues of Hamiltonian matrices appear in pairs $(\lambda, -\lambda)$, the eigenvalues of M are given as the square root of the eigenvalues of M^2, i.e., λ^2 is an eigenvalue of M^2 if and only if $\pm\lambda$ are the eigenvalues of M. In particular, if $-R_1R_2$ has a 1×1 diagonal block containing a negative number $-\omega^2$, then $\pm j\omega$ are imaginary eigenvalues of M. Thus, the computation of the URV decomposition of the Hamiltonian matrix will provide the eigenvalues on the imaginary axis but not close by as when using an unstructured eigenvalue solver. The computation of the URV decomposition of Hamiltonian matrices was implemented in the HAPACK Library [2, 3].

Now we will discuss the computation of eigenvectors of a Hamiltonian matrix M corresponding to the imaginary eigenvalues. Let z be a right eigenvector of $-R_1R_2$ corresponding to an eigenvalue λ^2, i.e., $-R_1R_2z = \lambda^2 z$. Then

$$q = U \begin{bmatrix} z \\ 0 \end{bmatrix}$$

is an eigenvector of M^2 corresponding to the eigenvalue λ^2, because

$$M^2 q = U \begin{bmatrix} -R_1R_2 & R_1R_3^T - R_3^TR_1 \\ 0 & (-R_1R_2)^T \end{bmatrix} \begin{bmatrix} z \\ 0 \end{bmatrix} = \lambda^2 U \begin{bmatrix} z \\ 0 \end{bmatrix} = \lambda^2 q.$$

Therefore, $M[\,Mq,\ q\,] = [\,Mq,\ q\,]\Lambda$, where the matrix

$$\Lambda = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix}$$

has the eigenvectors $[\,1,\ \pm\lambda]^T$ corresponding to the eigenvalues $\pm\lambda$. Thus, $v_\pm = Mq \pm \lambda q$ are the eigenvectors of M corresponding to the eigenvalues $\pm\lambda$. Indeed, we have

$$Mv_\pm = M\,[\,Mq,\ q\,] \begin{bmatrix} 1 \\ \pm\lambda \end{bmatrix} = [\,Mq,\ q\,] \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \pm\lambda \end{bmatrix} = \pm\lambda\,[\,Mq,\ q\,] \begin{bmatrix} 1 \\ \pm\lambda \end{bmatrix} = \pm\lambda v_\pm.$$

Note that Mq can be written as

$$Mq = V \begin{bmatrix} -R_2 & R_3^T \\ 0 & -R_1^T \end{bmatrix} U^T U \begin{bmatrix} z \\ 0 \end{bmatrix} = -V \begin{bmatrix} R_2 z \\ 0 \end{bmatrix}$$

which is faster to compute than by the usual matrix vector multiplication.

We finish this section with an algorithm for computing the eigenvectors of a real Hamiltonian matrix corresponding to the purely imaginary eigenvalues with positive imaginary part. We use above formalism, but do everything blockwise.

Algorithm 1
Input: a Hamiltonian matrix $M \in \mathbb{R}^{2n,2n}$ that has simple purely imaginary eigenvalues.
Output: a matrix $X \in \mathbb{R}^{2n,k}$ whose columns are the eigenvectors of M corresponding to the imaginary eigenvalues with a positive imaginary part.

1. Compute the URV decomposition $M = \begin{bmatrix} U_1 & U_2 \\ -U_2 & U_1 \end{bmatrix} \begin{bmatrix} R_1 & R_3 \\ 0 & R_2^T \end{bmatrix} \begin{bmatrix} V_1 & V_2 \\ -V_2 & V_1 \end{bmatrix}^T$.

2. Set $\hat{R} = -R_1R_2$.

3. Compute the real Schur form $[W_1,\, W_2]^T \hat{R}[W_1,\, W_2] = \begin{bmatrix} \hat{R}_1 & \hat{R}_3 \\ 0 & \hat{R}_2 \end{bmatrix}$, where $[W_1,\, W_2] \in \mathbb{R}^{n,n}$ is orthogonal, $\hat{R}_1 \in \mathbb{R}^{k,k}$ is upper triangular with negative numbers on the diagonal and $\hat{R}_2$ does not have real negative eigenvalues.

4. Compute a nonsingular matrix $Q \in \mathbb{R}^{k,k}$ such that $\hat{R}_1 Q = Q\,\mathrm{diag}(-\omega_1^2, ..., -\omega_k^2)$.

5. Compute $X = \begin{bmatrix} -V_1 \\ V_2 \end{bmatrix} R_2 W_1 Q + \begin{bmatrix} U_1 \\ -U_2 \end{bmatrix} W_1 Q\,\mathrm{diag}(j\omega_1, \ldots, j\omega_k)$.

Remark 4 Balancing is a way to increase the accuracy of the computations, see [15]. Instead of computing eigenvalues and eigenvectors of the Hamiltonian matrix M, these are computed for $\widetilde{M} = S^{-1}MS$, where S is a symplectic matrix that can be written as a permuted diagonal matrix. Since S is symplectic, $\widetilde{M}$ is Hamiltonian and, hence, the techniques of this section can be applied. Further, M and $\widetilde{M}$ have the same eigenvalues and if $\widetilde{v}$ is an eigenvector of $\widetilde{M}$ then $v = S\widetilde{v}$ is an eigenvector of M.

4 Passivation

Assume that the passivity check discussed in the last section has been applied to a given system $H = [\, A,\, B,\, C,\, D\,]$ declaring it to be non-passive. Often the matrices A, B, C and D are determined from a real world system by some sort of approximation scheme like vector fitting, modal or balanced truncation [1, 12, 17]. However, the underlying physical problem is known to be passive, i.e., the non-passivity of $H = [\, A,\, B,\, C,\, D\,]$ is an approximation error. This motivates the desire for a passivation method, i.e., a method to find a passive system $\widetilde{H} = [\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, \widetilde{D}\,]$ such that $\|\widetilde{H} - H\|$ is small in some system norm. Such a passivation method will be discussed in this section. We will keep the matrix A in order to preserve the poles of the transfer matrix. Hence, stability is not at risk. Further, the feedthrough matrix D is preserved in order to keep the behavior of the system at large frequencies. That leaves B and C. For convenience, we choose to perturb C only. Further, we assume that the system is nearly passive, i.e., one can find a small perturbation Δ such that the perturbed system $\widetilde{H} = [\, A,\, B,\, C + \Delta,\, D\,]$ is γ_*-passive.

We start by analyzing the effect of the perturbation of C on the imaginary eigenvalues $j\omega_i$ of M_{γ_*} given in (5). Let $\widetilde{M}_{\gamma_*}$ be a perturbed Hamiltonian matrix as in (5) with C replaced by $C + \Delta$. Then $\widetilde{M}_{\gamma_*} = M_{\gamma_*} + \hat{M}_{\gamma_*} + \mathcal{O}(\|\Delta\|^2)$ with

$$\hat{M}_{\gamma_*} = \begin{bmatrix} -BR_{\gamma_*}^{-1}D^T\Delta & 0 \\ \gamma_* C^T S_{\gamma_*}^{-1}\Delta + \gamma_*\Delta^T S_{\gamma_*}^{-1} C & \Delta^T D R_{\gamma_*}^{-1} B^T \end{bmatrix}.$$

Let v_i be a right eigenvector of M_{γ_*} corresponding to an eigenvalue $j\omega_i$. Then by (4) the imaginary eigenvalues $j\widetilde{\omega}_i$ of $\widetilde{M}_{\gamma_*}$ and those of M_{γ_*} are related up to first order perturbation via

$$\widetilde{\omega}_i - \omega_i = \frac{v_i^* J \hat{M}_{\gamma_*} v_i}{j v_i^* J v_i}. \tag{8}$$

The numerator of the right-hand side in (8) can be expressed as

$$v_i^* J \hat{M}_{\gamma_*} v_i = 2\,\Re e(z_i^* \Delta v_{i1}), \tag{9}$$

where $v_i = [\, v_{i1}^T,\, v_{i2}^T\,]^T$ with $v_{i1}, v_{i2} \in \mathbb{C}^n$ and $z_i = DR_{\gamma_*}^{-1}B^T v_{i2} + \gamma_* S_{\gamma_*}^{-1} C v_{i1}$. Applying the vectorization operator to (9) we obtain

$$v_i^* J M'_{\gamma_*} v_i = 2\,\Re e(v_{i1}^T \otimes z_i^*)\,\mathrm{vec}(\Delta).$$

Inserting this into (8) gives

$$\frac{2}{j v_i^* J v_i} \Re e(v_{i1}^T \otimes z_i^*)\,\mathrm{vec}(\Delta) = \widetilde{\omega}_i - \omega_i. \tag{10}$$

This is a linear relation between $\mathrm{vec}(\Delta)$ and $\widetilde{\omega}_i - \omega_i$. Collecting these relations for every imaginary eigenvalue we obtain the linear system

$$Z\,\mathrm{vec}(\Delta) = \widetilde{w} - w, \tag{11}$$

where $\widetilde{w} = [\,\widetilde{\omega}_1, \ldots, \widetilde{\omega}_k\,]^T$, $w = [\,\omega_1, \ldots, \omega_k\,]^T$ and the i-th row of $Z \in \mathbb{R}^{k,np}$ has the form

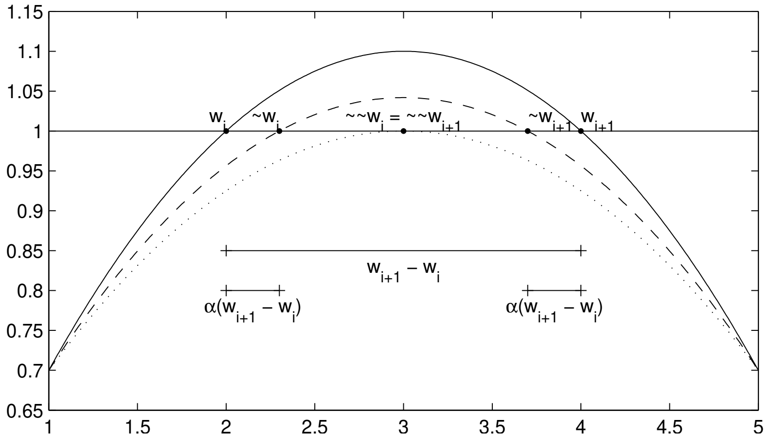
$$e_i^T Z = \frac{2}{jv_i^* J v_i} \Re e(v_{i1}^T \otimes z_i^*).$$

Here n is the state space dimension, p is the number of outputs and k is the number of the imaginary eigenvalues of M_{γ_*}. So, given a perturbation Δ, the new (first order approximate) location of the imaginary eigenvalues can be computed. We are going to use this relation the other way round: after choosing 'better' values for the imaginary eigenvalues, we use relation (11) to get a perturbation that achieves these 'better' eigenvalues. Since $k < np$ (usually $k \ll np$), the linear system (11) is underdetermined. In this case among the possible solutions we want to choose the one that minimizes the error $\|\widetilde{H} - H\|$ in some system norm.

There are three questions left: how to choose the new imaginary eigenvalues, which system norm should be used, and how to compute the minimal solution?

- *The choice of* $\widetilde{\omega}_i$

  Let the frequencies ω_i, where the singular value curves cross the level γ_*, be ordered in increasing order, i.e., $0 \le \omega_1 < \omega_2 < \ldots < \omega_k$. Assume that a single singular value curve has positive and negative slopes at ω_i and ω_{i+1}, respectively. For $\gamma_* = 1$, this situation is sketched in the following figure by the solid line.



  It is reasonable to assume that the level of non-passivity will decrease when the crossing points are moved to each other, i.e., ω_i is moved to the left and ω_{i+1} is moved to the right. Choosing the displacement proportionally to the distance between ω_{i+1} and ω_i, we obtain the following equations

$$\widetilde{\omega}_i = \begin{cases} \omega_i + \alpha(\omega_{i+1} - \omega_i) & \text{if} \quad jv_i^* J v_i > 0, \\ \omega_i - \alpha(\omega_i - \omega_{i-1}) & \text{if} \quad jv_i^* J v_i < 0,\ i \neq 1, \\ (1 - 2\alpha)\omega_1 & \text{if} \quad jv_i^* J v_i < 0,\ i = 1. \end{cases} \tag{12}$$

  Here, $\alpha \in (0,\, 0.5]$ is a tuning parameter. The dashed and the dotted lines in above figure correspond to the values $\alpha = 0.15$ and $\alpha = 0.5$, respectively. It is tempting to use $\alpha = 0.5$ since then the system is passive. But this corresponds to a rather large perturbation. This is dangerous, because it might take us out of the region, where the first order perturbation theory holds. Therefore, we suggest to use a value around $\alpha = 0.25$ and to apply the whole method iteratively, until the system is passive. Other choices of $\widetilde{\omega}_i$ are also possible, see [10] for details.

- *The choice of system norm*

  As proposed in [10], we compute the perturbation Δ that minimizes the $\mathbb{H}_2$-norm of the error $E(s) = \widetilde{H}(s) - H(s) = \Delta(sI - A)^{-1}B$ given by

$$\|E\|_{\mathbb{H}_2} = \left(\int_{-\infty}^{\infty} \|E(j\omega)\|_F^2 \, d\omega \right)^{1/2}.$$

  Consider a controllability Gramian $\mathcal{P}$ of the system $E = [\,A,\, B,\, \Delta,\, 0\,]$ that is defined as the unique symmetric positive semidefinite solution of the Lyapunov equation

$$A\mathcal{P} + \mathcal{P}A^T = -BB^T. \tag{13}$$

  Note that $\mathcal{P}$ is also the controllability Gramian of the system $H = [\,A,\, B,\, C,\, D\,]$. Since $\mathcal{P}$ can be represented as

$$\mathcal{P} = \int_{-\infty}^{\infty} (i\omega I - A)^{-1} BB^T (-i\omega I - A^T)^{-1} \, d\omega,$$

  we have $\|E\|_{\mathbb{H}_2} = \|\Delta L^T\|_F$, where L is a Cholesky factor of the controllability Gramian $\mathcal{P} = L^T L$.

- *The computation of the perturbation Δ*

  To compute the perturbation Δ that satisfies (11) and minimizes $\|E\|_{\mathbb{H}_2}$ we have to solve the following minimization problem

$$\min_{\Delta} \|\Delta L^T\|_F \quad \text{subject to} \quad Z \operatorname{vec}(\Delta) = \widetilde{w} - w.$$

  If the system $H = [\,A,\, B,\, C,\, D\,]$ is controllable, then the Gramian $\mathcal{P}$ is positive definite, and, hence, its Cholesky factor L is nonsingular. Changing the basis $\Delta_L = \Delta L^T$ we obtain the problem

$$\min_{\Delta_L} \|\Delta_L\|_F \quad \text{subject to} \quad Z_L \operatorname{vec}(\Delta_L) = \widetilde{w} - w, \tag{14}$$

  where $Z_L = Z(L^{-1} \otimes I)$. Note that there is no need to build the matrix $L^{-1} \otimes I$ explicitly, since the i-th row of Z_L can be computed as

$$e_i^T Z_L = e_i^T Z(L^{-1} \otimes I) = \frac{2}{j v_i^* J v_i} \Re e(v_{i1}^T L^{-1} \otimes z_i^*). \tag{15}$$

  In this case the minimization problem (14) reduces to the standard least squares problem

$$\min_{\Delta_L} \|\operatorname{vec}(\Delta_L)\|_2 \quad \text{subject to} \quad Z_L \operatorname{vec}(\Delta_L) = \widetilde{w} - w.$$

  The solution of this problem is given by

$$\operatorname{vec}(\Delta_L) = Z_L^+(\widetilde{w} - w),$$

  where Z_L^+ denotes the Moore-Penrose inverse of Z_L. The computation of Z_L^+ involves a singular value decomposition of the $k \times np$ matrix Z_L that costs $\mathcal{O}(npk^2)$ operations. The required perturbation is then computed as

$$\Delta = \Delta_L L^{-T}. \tag{16}$$

  Note that the upper triangular Cholesky factor L of the controllability Gramian $\mathcal{P}$ can be determined directly from the Lyapunov equation (13) without forming the product BB^T and computing $\mathcal{P}$ explicitly. This can be done using the Hammarling method [13] implemented in the SLICOT routine SB03OD [4] that is used in the MATLAB function `lyapchol.m`.

Now we are ready to state the passivation algorithm.

Algorithm 2. Passivation
Input: an asymptotically stable system $H = [\,A,\, B,\, C,\, D\,]$ such that $\sigma_{\max}(D) < \gamma_* \leq 1$, $0 < \alpha \leq 0.5$.
Output: a γ_*-passive system.

1. Compute the Cholesky factor L of the controllability Gramian $\mathcal{P} = L^T L$ by solving the Lyapunov equation (13).
2. Compute the imaginary eigenvalues and the corresponding eigenvectors of M_{γ_*} as in (5) using Algorithm 1.
3. **While** M_{γ_*} has imaginary eigenvalues **do**
   (a) choose new eigenvalues as in (12);
   (b) solve $\min_{\Delta_L} \|\operatorname{vec}(\Delta_L)\|_2$ subject to $Z_L \operatorname{vec}(\Delta_L) = \widetilde{w} - w$ with Z_L as in (15);
   (c) update $C \leftarrow C + \Delta_L L^{-T}$;
   (d) update M_{γ_*};
   (e) compute the imaginary eigenvalues and the corresponding eigenvectors of M_{γ_*};

   End

5 Numerical results

Algorithms 1 and 2 have been implemented and applied to several systems obtained from discrete S-parameters using the vector fitting followed by modal and balanced truncation, see [8] for details. The S-parameters were provided by CST GmbH Darmstadt. All systems are slightly non-passive. The following table gives an overview. Here, $\|H\|_{\mathbb{H}_\infty} = \sup_{\omega \in \mathbb{R}} \|H(j\omega)\|_2$ is given as a level of non-passivity.

Problem	n	$m = p$	$\|H\|_{\mathbb{H}_\infty}$
coax1	320	2	1.002
coax2	35	2	1.002
twolines	49	4	1.485
branch	100	4	1.133
rj45	160	8	1.176

The problem 'coax' appears twice. The larger version 'coax1' is the result of vector fitting applied to the discrete S-parameters. Applying modal and balanced truncation yields the small version 'coax2'. We chose a value of $\gamma_* = 1$ in all tests.

All computations have been performed on a PC equipped with a Pentium 4 CPU running at 3.2GHz and 1GB memory. The algorithms were tested in MATLAB 7.1 (R14) under Suse Linux 9.3. The symplectic URV decomposition used in Algorithm 1 was computed by the precompiled version of HAPACK [2].

5.1 URV vs Schur, reliability, accuracy, speed

Here we compare Algorithm 1 with an unstructured solver to compute the eigenvectors corresponding to the imaginary eigenvalues of a Hamiltonian matrix M. Recall, that Algorithm 1 will return eigenvalues that are exactly imaginary. This is not the case for unstructured algorithms since due to rounding errors the eigenvalues may leave the imaginary axis. This results in the problem of deciding whether a computed eigenvalue λ is imaginary or not. In our tests, we declared λ to be imaginary if

$$|\Re e(\lambda)| < 10^{-8}. \tag{17}$$

The whole unstructured eigenvalue solver is given in the following algorithm.

Algorithm 3
Input: a Hamiltonian matrix $M \in \mathbb{R}^{2n,2n}$ that has simple purely imaginary eigenvalues.
Output: a matrix $X \in \mathbb{C}^{2n,k}$ whose columns are the eigenvectors of M corresponding to the imaginary eigenvalues with a positive imaginary part.

1. Compute the real Schur form $M = URU^T$ using the MATLAB function `schur.m`.
2. Reorder the real Schur form $[W_1,\, W_2]^T R [W_1,\, W_2] = \begin{bmatrix} \widetilde{R}_1 & \widetilde{R}_3 \\ 0 & \widetilde{R}_2 \end{bmatrix}$, where $[W_1,\, W_2] \in \mathbb{R}^{2n,2n}$ is orthogonal, $\widetilde{R}_1 \in \mathbb{R}^{2k,2k}$ is quasi-upper triangular with only eigenvalues satisfying (17) and $\widetilde{R}_2$ does not have eigenvalues satisfying (17).
3. Compute a nonsingular matrix $Q = [Q_1,\, Q_2]$ with $Q_1, Q_2 \in \mathbb{R}^{2k,k}$ such that $\widetilde{R}_1 Q = Q\,\mathrm{diag}(\lambda_1, \ldots, \lambda_k, \overline{\lambda}_1, \ldots, \overline{\lambda}_k)$.
4. Compute $X = UW_1Q_1$.

In the following table the numerical results are presented. Next to the problem name and size we list the number of computed imaginary eigenvalues, the eigenvector accuracy defined as

$$\max_i \|Mv_i - j\omega_i v_i\|_2,$$

the computation time and the resulting speedup. The first values correspond to the structured Algorithm 1 and the values in parenthesis correspond to the unstructured Algorithm 3.

Problem	n	im. ev found	vector accuracy	time	speedup
coax1	320	21 (20)	2e-10 (1e- 8)	5 (9)	1.9
coax2	35	21 (21)	7e-14 (6e-12)	0.02 (0.03)	1.6
twolines	49	25 (25)	3e-11 (1e-12)	0.03 (0.06)	1.8
branch	100	54 (54)	2e-12 (8e-13)	0.2 (0.3)	1.4
rj45	160	11 (11)	1e-13 (7e-12)	0.7 (1.0)	1.6

It can be concluded that compared to Algorithm 3, the structured Algorithm 1 is

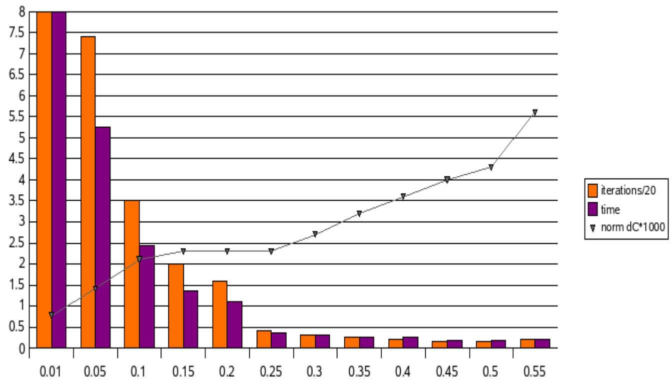
- more reliable
  (For the problem 'coax1', Algorithm 3 missed an eigenvalue),
- as accurate as
  (the computed eigenvector residuals are of about the same order for both algorithms. It should be mentioned that in theory, Algorithm 1 has problems with eigenvectors to eigenvalues that are small in magnitude),
- faster
  (both algorithms share a complexity of $\mathcal{O}(n^3)$ flops, but Algorithm 1 is between 30% and 50% faster than Algorithm 3.

Thus, the use of a structured eigenvalue solver is strongly advisable.

5.2 Passivation algorithm

In this subsection, we discuss the performance of the passivation scheme presented in Section 4. A first test addresses the choice of the tuning parameter α used in (12) for the 'coax2' problem. The following table and figure give the number of iterations, computation time and norms of the resulting perturbation $\|E\|_{\mathbb{H}_2} = \|\Delta L^T\|_F$ for various values of α.

α	# iter.	time	$\|\Delta L^T\|_F$
0.01	901	31.1	0.00077
0.05	148	5.26	0.0014
0.1	70	2.43	0.0021
0.15	40	1.37	0.0023
0.2	32	1.11	0.0023
0.25	8	0.35	0.0023
0.3	6	0.3	0.0027
0.35	5	0.25	0.0032
0.4	4	0.25	0.0036
0.45	3	0.19	0.004
0.5	3	0.19	0.0043
0.55	4	0.22	0.0056



One can see that higher values of α result in fewer iterations, and hence, less computation time, but also in larger perturbations. For the following tests, we used a value $\alpha = 0.25$ providing a good compromise between speed and small perturbation.

The passivation scheme was then applied to all the test problems. The results are listed in the following table.

Problem	n	p	$\frac{\|\Delta L^T\|_F}{\|CL^T\|_F}$	# iter.	time (sec)
coax1	320	2	2.3e-3	21	115
coax2	35	2	2.3e-3	8	0.35
twolines	49	4	3.8e-2	18	0.9
branch	100	4	7.5e-2	29	6.3
rj45	160	8	5.5e-2	35	28

It should be noted that the algorithm converged for every problem. Further, the perturbations are in the order of the approximation errors.

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