Elastic-ideally plastic beams and Prandtl–Ishlinskii hysteresis operators[1]

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Abstract

In this paper, the one-dimensional equation for the transversal vibrations of an elastoplastic beam is derived from a general three-dimensional system. The plastic behavior is modeled using the classical three-dimensional von Mises plasticity model. It turns out that this single-yield model leads after a dimensional reduction to a multi-yield one-dimensional hysteresis model, given by a hysteresis operator of Prandtl–Ishlinskii type whose density function can be determined explicitly. This result indicates that the use of Prandtl–Ishlinskii hysteresis operators in the modeling of elastoplasticity is not just a questionable phenomenological approach, but in fact quite natural. In addition to the derivation of the model, it is shown that the resulting partial differential equation with hysteresis can be transformed into an equivalent system for which the existence and uniqueness of a strong solution is proved. The proof employs techniques from the mathematical theory of hysteresis operators.

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1 Introduction

The use of hysteresis operators in the modeling of the hysteretic stress-strain relations that are commonplace in nonlinear elastoplasticity, dates back to at least the early 20th century. Back in 1928, Prandtl in his pioneering work [9] introduced the input-output relation that was independently studied by Ishlinskii in [3] in the 1940's and nowadays is called the *Prandtl–Ishlinskii operator*. It describes the time-evolution of the relation between strain ε (input) and stress σ (output) in one-dimensional elastoplasticity in the form

$$\sigma(t) = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[\varepsilon](t)\, dq\,. \tag{1.1}$$

Here, t denotes the time variable, φ is some nonnegative weight function that satisfies the growth condition

$$\int_0^\infty (1+q)\, \varphi(q)\, dq < +\infty\,, \tag{1.2}$$

and $\mathfrak{s}_q$ denotes the one-dimensional *stop operator* or *Prandtl's elastic-perfectly plastic element* with thresholds $\pm q$, which is a basic hysteresis operator whose dynamic input-output behavior is described in Fig. 1.

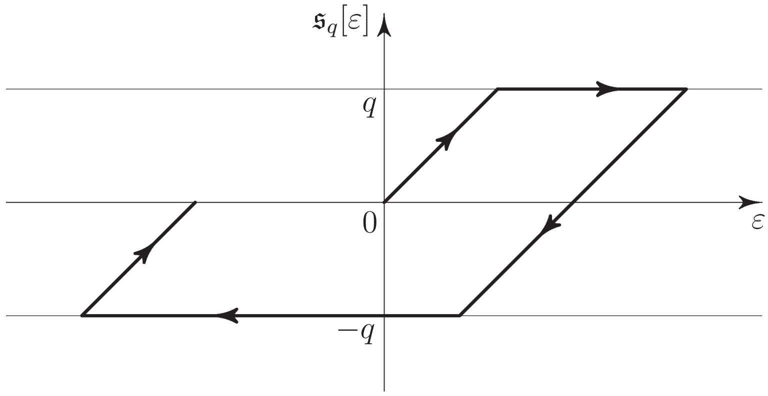

Figure 1: Hysteretic input-output behavior of $\mathfrak{s}_q$.

Between the thresholds $\pm q$, the behavior is linear elastic (with elasticity modulus 1), while along the upper (lower) threshold $+q$ $(-q)$ we have irreversible plastic yielding and can only move to the right (left). The operator $\mathfrak{s}_q$ is a special one-dimensional case of the abstract stop operator $\mathfrak{S}_Z$ in a separable Hilbert space X associated with a closed and convex set $Z \subset X$ containing 0. This operator is defined in the following way: for a given input function $v \in W^{1,1}(0,T;X)$, consider the variational inequality

$$\begin{aligned} &\chi(t) \in Z \quad \forall\, t \in [0,T]\,, \quad \chi(0) = \chi_0\,, \\ &(\dot\chi(t) - \dot v(t), z - \chi(t)) \,\ge\, 0 \quad \forall\, z \in Z\,, \quad \text{for a. e. } t \in (0,T)\,. \end{aligned} \tag{1.3}$$

Here, and throughout the paper, the superimposed dot stands for differentiation with respect to time, and $(\cdot,\cdot)$ is a scalar product in X. The investigation of such problems goes back to [8], and the existence and uniqueness of a solution $\chi \in W^{1,1}(0,T;X)$ for any given initial value $\chi_0 \in Z$ is obtained as a special case of the general theory. This allows us to define the corresponding solution operator $\mathfrak{S}_Z$ as

$$\mathfrak{S}_Z : Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)\,, \quad [\chi_0, v] \mapsto \chi\,. \tag{1.4}$$

It is proved in [5, Section I.3] that this operator is continuous and, if Z has non-empty interior, admits a continuous extension to

$$\mathfrak{S}_Z : Z \times C([0,T];X) \to C([0,T];X)\,.$$

In the case $X = \mathbb{R}^1$, we set $\mathfrak{s}_q = \mathfrak{S}_{[-q,q]}$. Notice that since $z = 0 \in Z$, we obtain from (1.3) the fundamental *energy dissipation inequality*

$$\frac{1}{2}\frac{d}{dt}\,|\mathfrak{S}_Z[\chi_0, v](t)|^2 \,\le\, (\mathfrak{S}_Z[\chi_0, v](t), v_t(t))\,, \quad \text{a.e. in } (0,T)\,. \tag{1.5}$$

In this paper, we restrict ourselves to the canonical choice of initial conditions

$$\chi_0 \;=\; \mathrm{Proj}_Z(v(0))\,, \tag{1.6}$$

where $\mathrm{Proj}_Z : X \to Z$ is the orthogonal projection onto Z. We then simply write $\chi = \mathfrak{S}_Z[v]$ instead of $\chi = \mathfrak{S}_Z[\chi_0, v]$. The operator

$$\mathfrak{P}_Z = I - \mathfrak{S}_Z\,, \tag{1.7}$$

where I denotes the identity mapping, is called the *vector play operator* associated with Z. We similarly denote $\mathfrak{p}_q = \mathfrak{P}_{[-q,q]}$. The stop and play operators form the corner stones of the mathematical theory of hysteresis operators. In the 1D case in particular, every hysteresis relation with the so-called "return point memory" (which is a common property of hysteresis relations in plasticity, ferromagnetism, piezoelectricity, etc.) can be represented by some functional on the one-parametric play system $\{\mathfrak{p}_q\,;\, q > 0\}$, see [1, Theorem 2.7.7]. The Prandtl–Ishlinskii operators (1.1) correspond in this respect to *linear functionals*. For a thorough treatment of their analytical and geometrical properties, we refer the reader to the monographs [1, 4, 5, 10]. Some important facts concerning $\mathfrak{s}_q$, which will be needed in the analysis below, are collected in Propositions 3.4, 3.5 in Section 3.

Although the Prandtl–Ishlinskii operator is easily understood and rather intuitive, its use in the physical and engineering literature is still nonstandard. The main reasons are the following: on the one hand, the operator appears to be entirely phenomenological, and its weight function φ is a priori unknown and must be identified; on the other hand, other well-established three-dimensional plasticity models like those by von Mises or Tresca are available.

The aim of this paper is twofold: first, we demonstrate that in the modeling of the (one-dimensional) transversal vibrations of an elastoplastic beam the use of the three-dimensional von Mises model leads (after normalizing all physical constants to unity) to the following beam equation for the transversal displacement:

$$w_{tt} \,-\, w_{xxtt} \,+\, \mathcal{P}\,[w_{xx}]_{xx} = g\,. \tag{1.8}$$

Here, $\mathcal{P}$ is a Prandtl–Ishlinskii operator *whose weight function* φ *can be determined explicitly*, and g is given. Observe that the Prandtl–Ishlinskii operator $\mathcal{P}$ is (as most nontrivial hysteresis operators) non-differentiable, so that (1.8) has to be given a proper meaning.

The existence and uniqueness analysis of the problem is carried out by transforming (1.8) into a system, in which no differentiation of the hysteresis operator occurs. The strong solution of this system is then interpreted as a weak solution to (1.8). The proof employs techniques from the mathematical theory of hysteresis operators; in particular, the properties of the stop operators $\mathfrak{s}_q$ will play a crucial role in the analysis.

The paper is organized as follows: in Section 2, will derive our model equation from a three-dimensional model using dimensional reduction. In Section 3, we will state the main existence and uniqueness result, which will be proved in the last two sections.

2 Derivation of the model

In this section, we derive our model from a general three-dimensional system. We restrict ourselves to *rectangular beams*, that is, to sets $\Omega \subset \mathbb{R}^3$ of the form $\Omega = (0,L) \times \omega$, where $L > 0$ is the *length* of the beam, and where, with some $h > 0$ and $b > 0$, the set $\omega = (-b,b) \times (-h,h)$ represents its (rectangular) *cross section*. We denote by $x \in (0,L)$ the longitudinal coordinate, by $(y,z) \in \omega$ the transversal coordinates, and by $t \in [0,T]$ the time, where $T > 0$ is given.

In order to compare the resulting equations, we start with the linear elastic isotropic case (Subsection 2.1), and then pass to the elastoplastic model under further simplifying assumptions (Subsection 2.2). We follow the scaling technique of [2, Sect. 5.4] in terms of a small parameter $\alpha > 0$ with the intention to keep only lowest order terms in α in the resulting equations. In particular, we assume that

$$h, b = \mathcal{O}(\alpha), \ \ L = \mathcal{O}(1).$$

Let us consider smooth displacements $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^3$ decomposed into

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1^L \\ u_2^L \\ u_3^L \end{pmatrix} + \begin{pmatrix} u_1^H \\ u_2^H \\ u_3^H \end{pmatrix} = \mathbf{u}^L + \mathbf{u}^H \,,$$

where the superscripts L and H stand for low order and high order components with respect to α, respectively. We make the following assumptions.

(A1) The low order deformation of the midsurface $\mathcal{C} = \{(x,y) \in \mathbb{R}^2;\ (x,y,0) \in \Omega\}$ is independent of y, that is,

$$\mathbf{u}^L(x,y,0,t) = \begin{pmatrix} v(x,t) \\ 0 \\ w(x,t) \end{pmatrix} \quad \forall\,(x,y) \in \mathcal{C}\,, \quad \forall\, t \in (0,T)\,, \tag{2.1}$$

with given functions $v, w : (0,L) \times (0,T) \to \mathbb{R}$.

(A2) The low order deformation leaves the cross sections $\{x\} \times \omega$ perpendicular to the midsurface, and their deformation is proportional to their distance to it; that is,

$$\mathbf{u}^L(x,y,z,t) = \mathbf{u}^L(x,y,0,t) + z\,\mathbf{n}(x,y,t) \quad \forall\,(x,y,z,t) \in \Omega \times (0,T)\,, \tag{2.2}$$

where $\mathbf{n}(x,y,t)$ is the unit "upward" normal to the deformed midsurface $\mathcal{C}(t) = \mathcal{C} + \mathbf{u}^L(\mathcal{C},0,t)$ at time t.

(A3) $v_x = \mathcal{O}(\alpha^2)\,,\ w_{xx} = \mathcal{O}(\alpha)\,.$

Under the hypothesis (A3), we can linearize the problem by replacing

$$\mathbf{n}(x,y,t) = \frac{1}{\sqrt{(1+v_x(x,t))^2 + w_x^2(x,t)}} \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1+v_x(x,t) \end{pmatrix}$$

with

$$\tilde{\mathbf{n}}(x,y,t) := \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1 \end{pmatrix}. \tag{2.3}$$

This is justified, since an elementary computation yields that

$$|\,\tilde{\mathbf{n}}(x,y,t) - \mathbf{n}(x,y,t)| \;<\; \left(|v_x(x,t)| + |w_x(x,t)|\right)^2$$

whenever $|v_x(x,t)| < 1\,,\ |w_x(x,t)| < 1\,.$ This enables us to write for every $(x,y,z,t) \in \Omega \times (0,T)$ the low order displacement $\mathbf{u}^L(x,y,z,t)$ as

$$\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} v(x,t) - z\,w_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}. \tag{2.4}$$

The smallness assumptions ensure in particular that the deformation

$$\mathbf{F}(x,y,z,t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathbf{u}(x,y,z,t) \tag{2.5}$$

is a local homeomorphism. We further compute

$$\nabla\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} v_x(x,t) - z\,w_{xx}(x,t) & 0 & -w_x(x,t) \\ 0 & 0 & 0 \\ w_x(x,t) & 0 & 0 \end{pmatrix}, \tag{2.6}$$

and the low order strain tensor $\boldsymbol{\varepsilon}^L = (\nabla\mathbf{u}^L + (\nabla\mathbf{u}^L)^T)/2$ becomes

$$\boldsymbol{\varepsilon}^L(x,y,z,t) = \begin{pmatrix} v_x(x,t) - z\,w_{xx}(x,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

2.1 Small elastic deformations

We denote by " $:$ " the canonical scalar product in the space of (3×3) - tensors, i. e.,

$$\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{i,j=1}^{3} \xi_{ij}\, \eta_{ij} \,, \quad \forall \boldsymbol{\xi} = (\xi_{ij}) \,, \ \boldsymbol{\eta} = (\eta_{ij}) \,, \quad i,j = 1,2,3 \,. \tag{2.8}$$

Moreover, we define for any given (3×3) - tensor $\boldsymbol{\xi}$ its (trace-free) *deviator* $\mathbf{d}(\boldsymbol{\xi})$ by

$$\mathbf{d}(\boldsymbol{\xi}) = \boldsymbol{\xi} - \frac{1}{3}\,(\boldsymbol{\xi} : \boldsymbol{\delta})\,\boldsymbol{\delta} \,, \tag{2.9}$$

where $\boldsymbol{\delta} = (\delta_{ij})$ denotes the Kronecker tensor.

To motivate the elastoplastic case treated below, we first study the case of linear isotropic elasticity, in which the strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ are related to each other through the formula

$$\boldsymbol{\sigma} \ = \ 2\mu\,\boldsymbol{\varepsilon} + \lambda\,(\boldsymbol{\varepsilon} : \boldsymbol{\delta})\,\boldsymbol{\delta} \,, \tag{2.10}$$

where μ, λ are the Lamé constants. The main issue is to choose a proper scaling of $\boldsymbol{\sigma}$. The component σ_{11} is of the lowest order, which is $\mathcal{O}(\alpha^2)$ due to (2.7) and (2.10). Assuming that the motion is "sufficiently slow" and no volume forces act on the body, we may for scaling purposes refer to the elastostatic equilibrium conditions $\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$ which, according to the natural scaling of the variables $y, z = \mathcal{O}(\alpha)$, $x = \mathcal{O}(1)$ and due to the symmetry of $\boldsymbol{\sigma}$, justify the scaling hypothesis

(A4) $\sigma_{12}, \sigma_{13} = \mathcal{O}(\alpha^3)$, $\sigma_{22}, \sigma_{33}, \sigma_{23} = \mathcal{O}(\alpha^4)$.

According to (2.10) and Hypothesis (A4), the high order strain tensor $\boldsymbol{\varepsilon}^H$ is scaled as

(A5) $\varepsilon^H_{12}, \varepsilon^H_{13} = \mathcal{O}(\alpha^3)$, $\varepsilon^H_{22}, \varepsilon^H_{33} = \mathcal{O}(\alpha^2)$, $\varepsilon^H_{11}, \varepsilon^H_{23} = \mathcal{O}(\alpha^4)$.

In terms of the high order displacements $\mathbf{u}^H$, this corresponds to the scaling $u^H_1 = \mathcal{O}(\alpha^4)$, $u^H_2, u^H_3 = \mathcal{O}(\alpha^3)$, with a vanishing $\mathcal{O}(\alpha^2)$ component of ε^H_{23}.

Let $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\varepsilon}}$ denote the stress and strain components of the order $\mathcal{O}(\alpha^2)$ at most. Then

$$\bar{\boldsymbol{\sigma}} : \boldsymbol{\delta} = \sigma_{11} = (2\mu + 3\lambda)\, \bar{\boldsymbol{\varepsilon}} : \boldsymbol{\delta} = (2\mu + 3\lambda)(\varepsilon^L_{11} + \varepsilon^H_{22} + \varepsilon^H_{33}) \,,$$

hence, by (2.10), $\sigma_{11} = 2\mu\varepsilon^L_{11} + \lambda/(2\mu + 3\lambda)\sigma_{11}$. In terms of the Young modulus $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$ and the Poisson ratio $\nu = \lambda/(2(\mu + \lambda))$ we thus obtain

$$\sigma_{11} = E\varepsilon^L_{11} \,, \ \bar{\varepsilon}_{11} = \varepsilon^L_{11} = v_x - z w_{xx} \,,$$

and

$$\bar{\mathbf{u}} = \mathbf{u}^L \,, \quad \bar{\boldsymbol{\sigma}} \ = \ \begin{pmatrix} E\varepsilon^L_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \bar{\boldsymbol{\varepsilon}} \ = \ \begin{pmatrix} \varepsilon^L_{11} & 0 & 0 \\ 0 & -\nu\varepsilon^L_{11} & 0 \\ 0 & 0 & -\nu\varepsilon^L_{11} \end{pmatrix} . \tag{2.11}$$

On the upper boundary, we prescribe the boundary condition $\boldsymbol{\sigma}(x,y,h,t)\cdot\boldsymbol{\nu}_3 = \mathbf{f}(x,t)$, where $\boldsymbol{\nu}_3 = (0,0,1)^T$ is the upward normal vector, and $\mathbf{f} = (f_1, 0, f_3)^T$ is a given external surface load. In component form, this boundary condition reads $\sigma_{13} = f_1$, $\sigma_{23} = 0$, $\sigma_{33} = f_3$. In agreement with the scaling hypothesis (A4), we require $f_1 = \mathcal{O}(\alpha^3), f_3 = \mathcal{O}(\alpha^4)$. On the rest of the boundary, we assume the *vanishing normal stress* boundary conditions $\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = 0$, where $\boldsymbol{\nu}$ is the unit outward normal vector. On $\{0\}\times\omega$, this means in particular

$$w_{xx}(0,t) = v_x(0,t) = 0\,, \quad w(0,t) = 0\,, \tag{2.12}$$

where the latter boundary condition is added in order to eliminate possible rigid body displacements and corresponds to a *simply supported beam*. An analogous choice of the boundary conditions is made at the right surface $\{L\}\times\omega$. In accordance with these boundary conditions, we consider the Sobolev space

$$V = \big\{(v,w) \in H^1(0,L)\times H^2(0,L);\ w(0) = w(L) = 0\big\}\,. \tag{2.13}$$

Finally, suppose that the initial conditions

$$v(x,0) = v^0(x)\,,\ \ v_t(x,0) = v^1(x)\,,\ \ w(x,0) = w^0(x)\,,\ \ w_t(x,0) = w^1(x)\,, \tag{2.14}$$

are given. As in [7], we write the *momentum balance equation* in variational form

$$\int_\Omega \rho\,\mathbf{u}_{tt}\cdot\hat{\mathbf{u}}\,dx\,dy\,dz + \int_\Omega \boldsymbol{\sigma} : \hat{\boldsymbol{\varepsilon}}\,dx\,dy\,dz = \int_{\partial\Omega}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot\hat{\mathbf{u}}\,ds\,, \tag{2.15}$$

with the unknown vector $\mathbf{u}$ and tensor $\boldsymbol{\sigma}$, for all admissible displacements $\hat{\mathbf{u}}$ and strains $\hat{\boldsymbol{\varepsilon}}$ of the form (2.11); i. e., we have

$$\hat{\mathbf{u}}(x,y,z) = \begin{pmatrix} \hat{v}(x) - z\,\hat{w}_x(x) \\ 0 \\ \hat{w}(x) \end{pmatrix}, \quad \hat{\boldsymbol{\varepsilon}}(x,y,z) = \begin{pmatrix} \hat{\varepsilon}_{11}(x) & 0 & 0 \\ 0 & -\nu\hat{\varepsilon}_{11}(x) & 0 \\ 0 & 0 & -\nu\hat{\varepsilon}_{11}(x) \end{pmatrix}, \tag{2.16}$$

with $\hat{\varepsilon}_{11} = \hat{v}_x(x) - z\,\hat{w}_{xx}(x)$, where $(\hat{v},\hat{w})$ varies over the space V. It follows from the choice of the boundary conditions that

$$\begin{aligned} \int_{\partial\Omega}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot\hat{\mathbf{u}}\,ds &= 2b\int_0^L \left(f_1\,(\hat{v} - h\,\hat{w}_x) + f_3\,\hat{w}\right)\,dx \\ &= 2b\left(\int_0^L f_1\,\hat{v}\,dx + \int_0^L (h(f_1)_x + f_3)\,\hat{w}\,dx\right). \end{aligned}$$

Keeping on the left-hand side of (2.15) only terms of the lowest order in α, we may replace $(\mathbf{u},\boldsymbol{\sigma})$ by $(\bar{\mathbf{u}},\bar{\boldsymbol{\sigma}})$ from (2.11). The test functions $\hat{v},\hat{w}$ are independent of each other, and a straightforward calculation shows that (2.15) decouples into the system

$$\rho\int_0^L v_{tt}(x,t)\,\hat{v}(x)\,dx + E\int_0^L v_x(x,t)\,\hat{v}_x(x)\,dx = \int_0^L g_1(x,t)\,\hat{v}(x)\,dx\,, \tag{2.17}$$

$$\begin{aligned} \rho\int_0^L\left(w_{tt}(x,t)\,\hat{w}(x) + \frac{h^2}{3}\,w_{xtt}(x,t)\,\hat{w}_x(x)\right)dx + \frac{E\,h^2}{3}\int_0^L w_{xx}(x,t)\,\hat{w}_{xx}(x)\,dx \\ = \int_0^L g_2(x,t)\,\hat{w}(x)\,dx\,, \end{aligned} \tag{2.18}$$

where we have set

$$g_1(x,t) = \frac{1}{2h} f_1(x,t)\,, \quad g_2(x,t) = \frac{1}{2h}\left(f_3(x,t) + h\,(f_1)_x(x,t)\right)\,. \tag{2.19}$$

The variational system (2.17), (2.18) leads formally to the partial differential equations

$$\rho\, v_{tt} - E\, v_{xx} = g_1\,, \tag{2.20}$$

$$\rho\, w_{tt} - \frac{\rho\, h^2}{3} w_{xxtt} + \frac{E\, h^2}{3} w_{xxxx} = g_2\,, \tag{2.21}$$

which describe the longitudinal (Eq. (2.20)) and transversal (Eq. (2.21)) vibrations of a straight elastic beam.

2.2 Transversal elastoplastic oscillations

We now turn our interest to elastoplasticity. We make further hypotheses.

(B1) The strain tensor $\boldsymbol{\varepsilon}$ is decomposed in elastic and plastic components $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$.

(B2) The elastic constitutive law is as in (2.11), that is,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}^e = \begin{pmatrix} \varepsilon^e_{11} & 0 & 0 \\ 0 & -\nu\varepsilon^e_{11} & 0 \\ 0 & 0 & -\nu\varepsilon^e_{11} \end{pmatrix}, \tag{2.22}$$

with

$$\sigma_{11} = E\,\varepsilon^e_{11}\,, \tag{2.23}$$

where E, ν are the Young modulus and the Poisson ratio, respectively.

(B3) The plastic deformations are *volume preserving* in the sense that

$$\boldsymbol{\varepsilon}^p : \boldsymbol{\delta} = 0\,. \tag{2.24}$$

The plastic yield condition is stated in terms of the stress deviator

$$\mathbf{d}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} - \frac{1}{3}(\boldsymbol{\sigma} : \boldsymbol{\delta})\,\boldsymbol{\delta} = \sigma_{11}\,\boldsymbol{\eta}\,,$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

The von Mises yield condition reads

(B4) $\mathbf{d}(\boldsymbol{\sigma}) : \mathbf{d}(\boldsymbol{\sigma}) \leq \frac{2}{3} R^2$, or equivalently

$$|\sigma_{11}| \;\leq\; R\,, \tag{2.25}$$

where $R > 0$ is a given *yield limit*. For the plastic strain, we prescribe the *normality flow rule*

(B5) $\boldsymbol{\varepsilon}^p_t \,\boldsymbol{:}\, (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) \;\geq\; 0 \quad \forall\, \tilde{\boldsymbol{\sigma}} \in \mathbb{R}^{(3\times 3)} \;:\quad \mathbf{d}(\tilde{\boldsymbol{\sigma}}) \,\boldsymbol{:}\, \mathbf{d}(\tilde{\boldsymbol{\sigma}}) \leq \frac{2}{3}R^2\,,$

and assume for simplicity that

(B6) the motion is only transversal,

that is, the component f_1 of the external surface load vanishes, and

$$\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} -z\, w_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}, \quad \varepsilon_{11}(x,y,z,t) = \varepsilon^L_{11}(x,y,z,t) \;=\; -z\, w_{xx}(x,t)\,.$$

From (B3) it follows that (B5) can be equivalently written as

$$\boldsymbol{\varepsilon}^p_t \,\boldsymbol{:}\, (\sigma_{11}\, \boldsymbol{\eta} - \tilde{\boldsymbol{\sigma}}) \;\geq\; 0 \quad \forall\, \tilde{\boldsymbol{\sigma}} \in \mathbb{R}^{(3\times 3)} \;:\quad \mathbf{d}(\tilde{\boldsymbol{\sigma}}) \,\boldsymbol{:}\, \mathbf{d}(\tilde{\boldsymbol{\sigma}}) \leq \frac{2}{3}R^2. \tag{2.26}$$

Whenever $\boldsymbol{\varepsilon}^p_t \neq 0$, we may choose

$$\tilde{\boldsymbol{\sigma}} \;=\; \sqrt{\frac{2}{3}}\, R\, \frac{\boldsymbol{\varepsilon}^p_t}{\sqrt{\boldsymbol{\varepsilon}^p_t \,\boldsymbol{:}\, \boldsymbol{\varepsilon}^p_t}}$$

and obtain from (2.26) that

$$\sigma_{11}\, (\boldsymbol{\varepsilon}^p_t \,\boldsymbol{:}\, \boldsymbol{\eta}) \geq R\, \sqrt{\boldsymbol{\varepsilon}^p_t \,\boldsymbol{:}\, \boldsymbol{\varepsilon}^p_t}\, \sqrt{\boldsymbol{\eta} \,\boldsymbol{:}\, \boldsymbol{\eta}}\,,$$

whence $\boldsymbol{\varepsilon}^p_t \;=\; \frac{3}{2}(\varepsilon^p_{11})_t\, \boldsymbol{\eta}$. Assuming that we have $\boldsymbol{\varepsilon}^p(x,y,z,0) \;=\; \frac{3}{2}\varepsilon^p_{11}(x,y,z,0)\, \boldsymbol{\eta}$ at initial time $t = 0$, we thus obtain

$$\boldsymbol{\varepsilon}^p(x,y,z,t) \;=\; \frac{3}{2}\varepsilon^p_{11}(x,y,z,t)\, \boldsymbol{\eta} \tag{2.27}$$

for all admissible values of the arguments. This enables us to rewrite (2.26) in the form

$$(\varepsilon^p_{11})_t\; (E\,(\varepsilon_{11} - \varepsilon^p_{11}) - \tilde{\sigma}) \;\geq\; 0 \quad \forall\, \tilde{\sigma} \in [-R, R]\,, \tag{2.28}$$

under the constraint $|\varepsilon_{11} - \varepsilon^p_{11}| \leq R/E$. At this point, the notion of hysteresis operators comes into play. Suppose that the initial condition $\varepsilon^e_{11}(x,y,z,0)$ is related to $\varepsilon_{11}(x,y,z,0)$ by (1.6), that is,

$$\varepsilon^e_{11}(x,y,z,0) = Q_{R/E}(-z\, w_{xx}(x,0))\,, \quad \text{where } Q_r(s) = \max\{-r, \min\{s, r\}\} \quad \forall\, s \in \mathbb{R},\; r > 0\,. \tag{2.29}$$

We then arrive at the conclusion that

$$\varepsilon^e_{11}(x,y,z,t) = \mathfrak{s}_{R/E}\left[-z\, w_{xx}(x,\cdot)\right](t)\,, \quad \varepsilon^p_{11}(x,y,z,t) = \mathfrak{p}_{R/E}\left[-z\, w_{xx}(x,\cdot)\right](t)\,, \tag{2.30}$$

where $\mathfrak{p}_r, \mathfrak{s}_r$ are the scalar play and stop operators with threshold r. Using the simple identity

$$\mathfrak{s}_r[\alpha\, u] = \alpha\ \mathfrak{s}_{r/|\alpha|}[u]\,,$$

which, with the convention $\mathfrak{s}_\infty[u] := u$, holds for all $\alpha \in \mathbb{R}$ and every input function u, we rewrite (2.30) as

$$\varepsilon^e_{11}(x,y,z,t) = -z\,\mathfrak{s}_{R/(E|z|)}\left[w_{xx}(x,\cdot)\right](t)\,, \quad \varepsilon^p_{11}(x,y,z,t) = -z\,\mathfrak{p}_{R/(E|z|)}\left[w_{xx}(x,\cdot)\right](t)\,. \tag{2.31}$$

We now aim to derive the momentum balance in the same way as in (2.15) to (2.19). To this end, we again make the test functions independent of $\hat{v}$, so that

$$\boldsymbol{\sigma} : \hat{\boldsymbol{\varepsilon}} \;=\; E\, z^2\, \mathfrak{s}_{R/(E|z|)}\left[w_{xx}\right]\, \hat{w}_{xx}\,.$$

Integrating over ω we obtain

$$\begin{aligned}\int_\omega z^2\ \mathfrak{s}_{R/(E|z|)}\left[w_{xx}\right]\, dy\, dz \;&=\; 2b\int_{-h}^{h} z^2\, \mathfrak{s}_{R/(E|z|)}\left[w_{xx}\right]\, dz\\ =\; 4b\int_0^h z^2\, \mathfrak{s}_{R/(Ez)}\left[w_{xx}\right]\, dz \;&=\; 4b\left(\frac{R}{E}\right)^3 \int_{R/(Eh)}^{\infty} q^{-4}\, \mathfrak{s}_q\left[w_{xx}\right]\, dq\,,\end{aligned}$$

where

$$\mathcal{P}[u] := \int_{R/(Eh)}^{\infty} q^{-4}\ \mathfrak{s}_q[u]\, dq \tag{2.32}$$

is a Prandtl–Ishlinskii operator with the weight function

$$\varphi(q) = \begin{cases} 0\,, & \text{if} \quad 0 \le q \le \frac{R}{Eh}\,,\\ q^{-4}\,, & \text{if} \quad q > \frac{R}{Eh}\,. \end{cases} \tag{2.33}$$

The counterpart of (2.21) then reads formally

$$\rho\, w_{tt} \;-\; \frac{\rho\, h^2}{3}\, w_{xxtt} \;+\; \frac{R^3}{E^2\, h}\, \mathcal{P}\left[w_{xx}\right]_{xx} \;=\; g_2\,. \tag{2.34}$$

Here, we have used the abbreviation

$$\mathcal{P}\left[w_{xx}\right]_{xx}(x,t) \;=\; \frac{\partial^2}{\partial x^2}\, \mathcal{P}\left[w_{xx}(x,\cdot)\right](t)\,. \tag{2.35}$$

Remark 2.1. Note that the Prandtl–Ishlinskii *initial loading curve* $\sigma = \Phi(\varepsilon)$ is bounded and saturation occurs. Indeed, Φ is given by the formula (see [1], [5])

$$\Phi(\varepsilon) \;=\; \frac{R^3}{E^2\, h}\int_{R/(Eh)}^{\infty} q^{-4}\, \min\{q,\varepsilon\}\, dq\,, \quad \text{for } \varepsilon \ge 0\,, \tag{2.36}$$

so that

$$\Phi(\varepsilon) = \begin{cases} \frac{E h^2}{3} \varepsilon \,, & \text{if} \quad \varepsilon \le \frac{R}{Eh} \,, \\ \frac{Rh}{2} \left(1 - \frac{R^2}{3 E^2 h^2 \varepsilon^2}\right) \,, & \text{if} \quad \varepsilon > \frac{R}{Eh} \,, \end{cases} \tag{2.37}$$

hence $Rh/2$ is the saturation limit. More general Prandtl–Ishlinskii initial loading curves describe the cases that the shape of ω is no longer a rectangle, but a domain of the form

$$\omega = \{(y,z) \in \mathbb{R}^2 \,;\; -h < z < h \,,\; -b(z) < y < b(z)\}$$

with a positive measurable function b. The same equation with a different Prandtl–Ishlinskii operator also comes out if we let the Young modulus E depend on z as a model for a layered beam.

Remark 2.2. Note that (2.44) reduces to (2.21) if we replace $\mathfrak{s}_q[u]$ by u in the expression (2.42) for $\mathcal{P}[u]$ (no plasticity). Also, if we pass to the elastic limit as $r \to \infty$ in (2.44), we recover (2.21) in agreement with natural expectations.

3 Statement of the mathematical results

In what follows, we use the usual notations for the spaces of continuous functions and for the standard Lebesgue and Sobolev spaces. The L^2-norm is always denoted by $\|\cdot\|$.

We now formulate the main mathematical results of this paper. To this end, we normalize all physical constants in (2.44) to unity, which has no bearing on the mathematical analysis. We thus study the following initial-boundary value problem in Q_T, where $Q_t := (0,1) \times (0,t)$ for any $t > 0$:

$$w_{tt} - w_{xxtt} + \mathcal{P}[w_{xx}]_{xx} = g \quad \text{in } Q_T \,, \tag{3.1}$$

$$w(0,t) = \mathcal{P}[w_{xx}](0,t) = w(1,t) = \mathcal{P}[w_{xx}](1,t) = 0 \,, \quad 0 \le t \le T \,, \tag{3.2}$$

$$w(x,0) = z_0(x) \,, \quad w_t(x,0) = z_1(x) \,, \qquad 0 \le x \le 1 \,. \tag{3.3}$$

We make the following general assumptions on the data of the system:

(H1) $g \in L^2(Q_T)$.

(H2) $z_0 \in H^3(0,1), z_1 \in H^2(0,1)$, and the following compatibility conditions are satisfied:

$$z_0(0) = z_{0,xx}(0) = z_0(1) = z_{0,xx}(1) = 0 \,,\; z_1(0) = z_1(1) = 0 \,. \tag{3.4}$$

(H3) The weight function $\varphi : (0,\infty) \to [0,\infty)$ of the Prandtl–Ishlinskii operator

$$\mathcal{P}[u] = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[u]\, dq$$

is measurable and satisfies the growth condition

$$\int_0^\infty \left(1 + q^2\right) \varphi(q)\, dq < +\infty \,. \tag{3.5}$$

Remark 3.1. Under condition (3.5) the so-called *clockwise admissible potential* of $\mathcal{P}$, given by the hysteresis operator

$$\mathcal{Q}[u] = \frac{1}{2} \int_0^\infty \varphi(q)\, \mathfrak{s}_q^2[u]\, dq\,, \tag{3.6}$$

is well defined. It then follows from the dissipation inequality (1.5) for the stop operator that for any input function $u \in W^{1,1}(0,T)$ it holds

$$(\mathcal{Q}[u])_t\,(t) \;=\; \int_0^\infty \varphi(q)\, \mathfrak{s}_q[u](t)\; (\mathfrak{s}_q[u])_t\,(t)\, dq \;\le\; \mathcal{P}[u](t)\, u_t(t)\,, \quad \textit{for a. e. } \; t \in (0,T). \tag{3.7}$$

We now associate with problem (3.1)–(3.3) the following system of initial-boundary value problems

$$\begin{aligned}
u_t \;&=\; \mathcal{P}\,[w_{xx}] && \text{in } Q_T\,, && (3.8)\\
w_t \,-\, w_{xxt} \;&=\; -u_{xx} + f(x,t) && \text{in } Q_T\,, && (3.9)\\
u(0,t) = u(1,t) \;&=\; 0\,, && 0 \le t \le T\,, && (3.10)\\
w(0,t) = w(1,t) \;&=\; 0\,, && 0 \le t \le T\,, && (3.11)\\
u(x,0) \;&=\; z_1(x)\,, && 0 \le x \le 1\,, && (3.12)\\
w(x,0) \;&=\; z_0(x)\,, && 0 \le x \le 1\,, && (3.13)
\end{aligned}$$

which arises from (3.1)–(3.3) if we put

$$u(x,t) \;=\; z_1(x) + \int_0^t \mathcal{P}\,[w_{xx}]\,(x,s)\, ds\,, \quad f(x,t) \;=\; z_1(x) + \int_0^t g(x,s)\, ds\,. \tag{3.14}$$

Conversely, one should expect that a sufficiently smooth solution (u,w) to the system (3.8)–(3.13) induces a solution to (3.1)–(3.3). We will therefore in the following examine the solvability of (3.8)–(3.13). It will turn out, however, that we will not be able to extract enough regularity from the system (3.8)–(3.13) so that the existence of a strong solution to (3.1)–(3.3) can be guaranteed. Instead, we will show the following weaker result.

Theorem 3.2. *Suppose that the conditions (H1)–(H3) are satisfied. Then the system (3.8)–(3.13) has a unique solution pair (u,w) having the following properties:*

(i) $u \in W^{2,\infty}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^2(0,1)) \cap H^1(0,T;H^1(0,1))$.

(ii) $w \in W^{1,\infty}(0,T;H^2(0,1)) \cap H^2(0,T;H^1(0,1))$.

(iii) *Eq. (3.8) is fulfilled pointwise in $\overline{Q_T}$, and Eq. (3.9) holds almost everywhere in Q_T.*

(iv) *The initial and boundary conditions (3.10)–(3.13) are satisfied pointwise, and it holds*

$$\mathcal{P}[w_{xx}](0,t) = \mathcal{P}[w_{xx}](1,t) = 0 \quad \forall\, t \in [0,T]\,.$$

Remark 3.3. We call (u, w) a strong solution to (3.8)–(3.13), and w a weak solution to (3.1)–(3.3). The meaning of conditions (i), (ii) in Theorem 3.2 is that

$$\left.\begin{aligned} u_{tt}, u_{xx}, w_{xxt} &\in L^\infty(0,T;L^2(0,1)), \\ u_{xt}, w_{xtt} &\in L^2(Q_T). \end{aligned}\right\} \tag{3.15}$$

By virtue of the boundary conditions and embedding theorems, we then have

$$u, u_x, u_t, w, w_x, w_t, w_{xt} \in C(\overline{Q_T}). \tag{3.16}$$

Before proving Theorem 3.2 in the next sections, we now collect some well-known properties of the one-dimensional stop operator that can be found in a more general form in the monographs [1] or [5], and in the paper [6]. For the reader's convenience, we give a brief outline of the proofs.

Proposition 3.4. *Let* $v_1, v_2 \in W^{1,1}(0,T)$ *be given,* $\chi_i = \mathfrak{s}_q[v_i]$, $p_i = v_i - \chi_i = \mathfrak{p}_q[v_i]$, $i = 1, 2$. *Then*

(i) $(\chi_1(t) - \chi_2(t))(\dot v_1(t) - \dot v_2(t)) \ \ge\ \dfrac{1}{2}\dfrac{d}{dt}(\chi_1(t) - \chi_2(t))^2$ *a. e.;*

(ii) $|\dot p_1(t) - \dot p_2(t)| + \dfrac{d}{dt}|\chi_1(t) - \chi_2(t)| \ \le\ |\dot v_1(t) - \dot v_2(t)|$ *a. e.;*

(iii) $|\chi_1(t) - \chi_2(t)| \ \le\ 2 \max\limits_{0\le\tau\le t} |v_1(\tau) - v_2(\tau)| \quad \forall t \in [0,T]$*;*

(iv) $|\dot\chi_i(t)| \ \le\ |\dot v_i(t)|$ *a. e.*

Sketch of the proof. We have by (1.3) that $\dot p_1(\chi_1 - \chi_2) \ge 0$, $\dot p_2(\chi_2 - \chi_1) \ge 0$ a. e., hence

$$(\dot p_1(t) - \dot p_2(t))(\chi_1(t) - \chi_2(t)) \ \ge\ 0 \quad \text{a. e.}, \tag{3.17}$$

which is nothing but (i). We obtain (ii) from (3.17) whenever $\chi_1(t) \ne \chi_2(t)$. If $\chi_1(t) = \chi_2(t) \in (-q, q)$, then $\dot p_1(t) = \dot p_2(t) = 0$, while on the set of all t such that $\chi_1(t) = \chi_2(t) = \pm q$, we have

$$\dot\chi_1(t) = \dot\chi_2(t) = \frac{d}{dt}|\chi_1(t) - \chi_2(t)| = 0 \quad \text{a. e.},$$

and (ii) follows. To prove (iii), we fix any $t \in (0,T]$, assume for instance that $\chi_1(t) > \chi_2(t)$, and find a smallest $t_0 < t$ such that $\chi_1(\tau) > \chi_2(\tau)$ for all $\tau \in (t_0, t]$. Then, by (3.17), $\dot p_1(\tau) \ge \dot p_2(\tau)$ for a. e. $\tau \in (t_0, t)$, hence

$$p_1(t_0) - p_2(t_0) \ \le\ p_1(t) - p_2(t) \ \le\ v_1(t) - v_2(t)$$

(note that $p_i + \chi_i = v_i$). Then either $t_0 > 0$ with $\chi_1(t_0) = \chi_2(t_0)$, or $t_0 = 0$ with $|p_1(t_0) - p_2(t_0)| \le |v_1(t_0) - v_2(t_0)|$. In both cases we have

$$|p_1(t) - p_2(t)| \ \le\ \max\{|v_1(t_0) - v_2(t_0)|, |v_1(t) - v_2(t)|\},$$

hence (iii). Part (iv) follows from the obvious identity $\dot{p}_i(t)\,\dot{\chi}_i(t) = 0$ a. e. $\square$

As a consequence of Proposition 3.4 (i), we obtain for the Prandtl–Ishlinskii operator $\mathcal{P}$ from Hypothesis (H3) the inequality

$$(\mathcal{P}[v_1](t) - \mathcal{P}[v_2](t))(\dot{v}_1(t) - \dot{v}_2(t)) \;\geq\; \frac{1}{2}\frac{d}{dt}\int_0^\infty \varphi(q)(\mathfrak{s}_q[v_1] - \mathfrak{s}_q[v_2])^2(t)\,dq \tag{3.18}$$

for every $v_1, v_2 \in W^{1,1}(0,T)$ and a. e. $t \in (0,T)$.

Proposition 3.5. *Let $v \in C(\overline{Q_T})$ be such that $v_{xt} \in L^1(Q_T)$. For $(x,t) \in Q_T$ set $\chi(x,t) = \mathfrak{s}[v(x,\cdot)](t)$. Then $\chi_{xt} \in L^1(Q_T)$, and*

$$\left.\begin{array}{rcl} |\chi_x(x,t)| & \leq & 2\,\max_{0\leq\tau\leq t}|v_x(x,\tau)| \\ & & \qquad\textit{for a. e. } \ x \in (0,1) \ \textit{ and all } \ t \in [0,T]\,, \\ |\chi_{xt}(x,t)| + \frac{\partial}{\partial t}|\chi_x(x,t)| & \leq & 2\,|v_{xt}(x,t)| \quad \textit{a. e. in } \ Q_T\,. \end{array}\right\} \tag{3.19}$$

If moreover $v_{xt} \in L^2(Q_T)$, then for all $t \in [0,T]$ we have

$$\int_0^t\int_0^1 \chi_x(x,\tau)\,v_{xt}(x,\tau)\,dx\,d\tau \;\geq\; \frac{1}{2}\int_0^1 (\chi_x^2(x,t) - \chi_x^2(x,0))\,dx\,. \tag{3.20}$$

Sketch of the proof. By Proposition 3.4 (ii),(iii), we have for all $0 < x_1 < x_2 < 1$ and $t > 0$ that

$$\begin{array}{rcl} |\chi(x_1,t) - \chi(x_2,t)| \;\leq\; 2\displaystyle\max_{0\leq\tau\leq t}|v(x_1,\tau) - v(x_2,\tau)| & \leq & 2\int_{x_1}^{x_2}\displaystyle\max_{0\leq\tau\leq t}|v_x(x,\tau)|\,dx\,, \\ |\chi_t(x_1,t) - \chi_t(x_2,t)| + \frac{\partial}{\partial t}|\chi(x_1,t) - \chi(x_2,t)| & \leq & 2\,|v_t(x_1,t) - v_t(x_2,t)|\,, \end{array}$$

hence (3.19) holds. To prove (3.20), we first notice that by Proposition 3.4 (i), we have for each $h \in (0,1)$ and $t \in (0,T]$ that

$$\int_0^t\int_h^1 \frac{\chi(x,\tau) - \chi(x-h,\tau)}{h}\cdot\frac{v_t(x,\tau) - v_t(x-h,\tau)}{h}\,dx\,d\tau$$
$$\geq\; \frac{1}{2}\int_h^1 \left(\left(\frac{\chi(x,t) - \chi(x-h,t)}{h}\right)^2 - \left(\frac{\chi(x,0) - \chi(x-h,0)}{h}\right)^2\right)dx\,.$$

Using e. g. the Mean Continuity Theorem, we pass to the limit as $h \searrow 0+$ and obtain the assertion. $\square$

4 Proof of existence

In this section, we will prove the existence result of Theorem 3.2. To this end, we use Faedo-Galerkin approximations. Let $\{\psi_k\}_{k\in\mathbb{N}}$ denote the system of eigenfunctions to the eigenvalue problem

$$-\,\psi_k'' = \lambda_k\,\psi_k\,, \quad \text{in } [0,1]\,, \quad \psi_k(0) = \psi_k(1) = 0\,, \quad k \in \mathbb{N}\,,$$

normalized with respect to the standard scalar product $\langle\cdot,\cdot\rangle$ in $L^2(0,1)$. Clearly, $\lambda_k = k^2\pi^2$ and $\psi_k(x) = \sqrt{2}\,\sin(k\pi x)$, for $k \in \mathbb{N}$. We set $V_m = \operatorname{span}\{\psi_1,\ldots,\psi_m\}$. Then $V_m \subset V_{m+1}$, $m \in \mathbb{N}$, and $\bigcup_{m\in\mathbb{N}} V_m$ is dense in any of the spaces $L^2(0,1)$, $H_0^1(0,1)$, and $\tilde{H}_0^3(0,1) := \{v \in H^3(0,1);\, v(0) = v''(0) = v(1) = v''(1) = 0\}$.

For given $m \in \mathbb{N}$, we consider approximations for u, w of the form

$$u^m(x,t) = \sum_{j=1}^{m} \mu_j(t)\,\psi_j(x)\,, \quad w^m(x,t) = \sum_{j=1}^{m} \eta_j(t)\,\psi_j(x)\,. \tag{4.1}$$

Denoting by Q_m the $L^2(0,1)$-orthogonal projection onto V_m, and using the standard notation $u(t)(x) = u(x,t)$ for functions of space and time, we consider the system of Faedo-Galerkin equations

$$\langle u_t^m(t),\psi\rangle \;=\; \langle \mathcal{P}\,[w_{xx}^m]\,(t),\psi\rangle \qquad \forall\,\psi \in V_m\,, \quad 0 \le t \le T\,, \tag{4.2}$$

$$\langle w_t^m(t) - w_{xxt}^m(t),\psi\rangle \;=\; \langle -u_{xx}^m(t) + f(t),\psi\rangle \qquad \forall\,\psi \in V_m\,, \quad 0 \le t \le T\,, \tag{4.3}$$

$$u^m(0) \;=\; Q_m(z_1)\,, \quad w^m(0) \;=\; Q_m(z_0)\,, \tag{4.4}$$

which is equivalent to the system

$$\dot{\mu}_k(t) \;=\; \langle \mathcal{P}\,[w_{xx}^m]\,(t),\psi_k\rangle\,, \quad 0 \le t \le T\,, \tag{4.5}$$

$$\dot{\eta}_k(t) \;=\; \frac{k^2\pi^2}{1+k^2\pi^2}\,\mu_k(t) \;+\; \frac{1}{1+k^2\pi^2}\,\langle f(t),\psi_k\rangle\,, \quad 0 \le t \le T\,, \tag{4.6}$$

$$\mu_k(0) \;=\; \langle z_1,\psi_k\rangle\,, \quad \eta_k(0) \;=\; \langle z_0,\psi_k\rangle\,, \tag{4.7}$$

for $k = 1,\ldots,m$. Here, we have used the abbreviation

$$\mathcal{P}\,[w_{xx}^m]\,(x,t) \;=\; \mathcal{P}\,[w_{xx}^m(x,\cdot)]\,(t) \;=\; \mathcal{P}\Big[-\sum_{j=1}^{m} j^2\pi^2\,\eta_j\,\psi_j(x)\Big](t)\,. \tag{4.8}$$

Obviously, (4.5)–(4.7) is an initial value problem for a system of $2m$ ordinary differential equations whose right-hand side is globally Lipschitz continuous on $C\big([0,T];\mathbb{R}^{2m}\big)$. Indeed, owing to Proposition 3.4 (iii) and Eq. (3.5), we have for any $u_1, u_2 \in C[0,T]$ the estimate

$$\begin{aligned} |\mathcal{P}\,[u_1]\,(t) - \mathcal{P}\,[u_2]\,(t)| \;&\le\; \int_0^\infty \varphi(q)\;|\mathfrak{s}_q\,[u_1]\,(t) - \mathfrak{s}_q\,[u_2]\,(t)|\;dq \\ &\le\; 2\max_{0\le s\le t}|u_1(s) - u_2(s)|\int_0^\infty \varphi(q)\,dq\,, \quad \forall\,t \in [0,T]\,, \end{aligned}$$

from which the claim follows. Consequently, the system (4.5)–(4.7) has a unique (global) solution $(\mu_1,\ldots,\mu_m,\eta_1,\ldots,\eta_m) \in C^1\left([0,T];\mathbb{R}^{2m}\right)$ that defines the solution (u^m, w^m) of (4.2)–(4.4) through Eq. (4.1). We have in fact $(\mu_1,\ldots,\mu_m,\eta_1,\ldots,\eta_m) \in H^2\left(0,T;\mathbb{R}^{2m}\right)$ as a consequence of Proposition 3.4 (iv). In the following, we derive a series of a priori estimates to pave the way for the passage to the limit as $m \to \infty$. To this end, we differentiate Eq. (4.3) with respect to t to obtain

$$\langle w_{tt}^m(t) - w_{xxtt}^m(t), \psi \rangle \;=\; -\, \langle u_{xxt}^m, \psi \rangle \,+\, \langle g(t), \psi \rangle \quad \forall\, \psi \in V_m \quad \text{for a. e.} \;\; t \in (0,T)\,. \tag{4.9}$$

Inserting $\psi = w_t^m(t) \in V_m$ in (4.9), integrating by parts, and employing Young's inequality, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\|w_t^m(t)\|^2 + \|w_{xt}^m(t)\|^2\right) + \langle u_t^m(t), w_{xxt}^m(t)\rangle \le \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|w_t^m(t)\|^2 \quad \text{a. e.} \tag{4.10}$$

Now observe that $w_{xxt}^m(t) \in V_m$, so that it follows from (4.2) that

$$\langle u_t^m(t), w_{xxt}^m(t)\rangle \;=\; \langle \mathcal{P}\left[w_{xx}^m\right](t), w_{xxt}^m(t)\rangle \;\; \text{a. e.} \tag{4.11}$$

Recalling (3.6) and (3.7), we can infer that

$$\langle \mathcal{P}\left[w_{xx}^m\right](t), w_{xxt}^m(t)\rangle \;\ge\; \frac{d}{dt}\int_0^1 \mathcal{Q}\left[w_{xx}^m\right](t)\, dx \quad \text{a. e.}\,, \tag{4.12}$$

where $\mathcal{Q}\left[w_{xx}^m\right](t) \ge 0$. Hence, integrating (4.10) over $[0,t]$ for any $t \ge 0$, we arrive at the estimate

$$\begin{aligned} \|w_t^m(t)\|^2 + \|w_{xt}^m(t)\|^2 \;\le\; & \|w_t^m(0)\|^2 + \|w_{xt}^m(0)\|^2 \\ & + 2\int_0^1 \mathcal{Q}\left[w_{xx}^m\right](0)\, dx + \int_0^t \|g(s)\|^2\, ds + \int_0^t \|w_t^m(s)\|^2\, ds\,. \end{aligned} \tag{4.13}$$

In the following, we denote by C_ℓ, $\ell \in \mathbb{N}$, positive constants that may depend on the data of the system, but not on $m \in \mathbb{N}$. First notice that we have $|\mathfrak{s}_q\left[w_{xx}^m\right]| \le q$ for $q \ge 0$. Hence, by (3.5),

$$\int_0^1 \mathcal{Q}\left[w_{xx}^m\right](0)\, dx \;\le\; \int_0^1\int_0^\infty \varphi(q)\, q^2\, dq\, dx \;\le\; C_1\,. \tag{4.14}$$

Next, observe that

$$\|w_t^m(0)\|^2 \;=\; \sum_{k=1}^m \dot\eta_k^2(0)\,. \tag{4.15}$$

Now, in view of (3.14), (4.6), and (4.7),

$$\dot\eta_k(0) \;=\; \langle z_1, \psi_k\rangle\,, \tag{4.16}$$

and it follows from Bessel's inequality and (H2) that

$$\|w_t^m(0)\|^2 \;\le\; \sum_{k=1}^m \langle z_1, \psi_k\rangle^2 \;\le\; \|z_1\|^2 \;\le\; C_2\,. \tag{4.17}$$

Likewise,

$$\|w_{xxt}^m(0)\|^2 \;=\; \sum_{k=1}^m k^4\pi^4\, \dot\eta_k^2(0) \;=\; \sum_{k=1}^m k^4\pi^4\, \langle z_1, \psi_k\rangle^2 \;\le\; \|z_{1,xx}\|^2 \;\le\; C_3\,, \tag{4.18}$$

hence

$$\|w_{xt}^m(0)\|^2 \;=\; -\,\langle w_t^m(0), w_{xxt}^m(0)\rangle \;\le\; C_4 \;=\; \sqrt{C_2 C_3}\,. \tag{4.19}$$

Combining the above estimates with (4.13), and invoking Gronwall's lemma, we have proved the a priori estimate

$$\max_{0\le t\le T}\left(\|w_t^m(t)\|^2 \,+\, \|w_{xt}^m(t)\|^2\right) \;\le\; C_5\,. \tag{4.20}$$

As second step in the proof, we insert $\psi = -w_{xxt}^m \in V_m$ in (4.9). Integrating by parts, and invoking (4.2) and Young's inequality, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\|w_{xt}^m(t)\|^2 \,+\, \|w_{xxt}^m(t)\|^2\right) \,+\, \langle(\mathcal{P}[w_{xx}^m])_x(t), w_{xxxt}^m(t)\rangle \;\le\; \frac{1}{2}\,\|g(t)\|^2 \,+\, \frac{1}{2}\,\|w_{xxt}^m(t)\|^2 \tag{4.21}$$

holds for a. e. $t\in(0,T)$, where

$$\langle(\mathcal{P}[w_{xx}^m])_x(t), w_{xxxt}^m(t)\rangle \;=\; \int_0^1\int_0^\infty \varphi(q)\,\left(\mathfrak{s}_q[w_{xx}^m(x,\cdot)]\right)_x(t)\, w_{xxxt}^m(t)\,dq\,dx\,. \tag{4.22}$$

Recalling Proposition 3.5, and integrating (4.21) over $[0,t]$ for any $t\in[0,T]$, we arrive at the estimate

$$\begin{aligned}
&\|w_{xt}^m(t)\|^2 \,+\, \|w_{xxt}^m(t)\|^2 \,+\, \int_0^\infty \varphi(q)\,\left\|\left(\mathfrak{s}_q[w_{xx}^m]\right)_x(t)\right\|^2\,dq\\
&\le\; \|w_{xt}^m(0)\|^2 \,+\, \|w_{xxt}^m(0)\|^2 \,+\, \int_0^\infty \varphi(q)\,\left\|\left(\mathfrak{s}_q[w_{xx}^m]\right)_x(0)\right\|^2\,dq\\
&\quad +\frac{1}{2}\int_0^t\left(\|g(s)\|^2 \,+\, \|w_{xxt}^m\|^2\right)\,ds\,.
\end{aligned} \tag{4.23}$$

Since z_0 satisfies the compatibility conditions (3.4), we have

$$\langle z_0,\psi_k\rangle = \frac{1}{k^3\pi^3}\,\langle z_{0,xxx},\hat{\psi}_k\rangle \quad \forall\, k\in\mathbb{N}\,, \tag{4.24}$$

where $\hat{\psi}_k(x) = -\sqrt{2}\cos(k\pi x)$. Thus,

$$\|w_{xxx}^m(0)\|^2 \;=\; \sum_{k=1}^m |\langle z_0,\psi_k\rangle|^2\,k^6\pi^6 \;\le\; \|z_{0,xxx}\|^2\,. \tag{4.25}$$

Finally, we employ the property (3.19) of the stop operator $\mathfrak{s}_q$ to deduce that

$$\left|\left(\mathfrak{s}_q[w_{xx}^m]\right)_x(0)\right| \;\le\; 2\,|w_{xxx}^m(0)| \quad \text{a. e.}\,, \tag{4.26}$$

whence it follows that

$$\int_0^\infty \varphi(q)\,\left\|\left(\mathfrak{s}_q[w_{xx}^m]\right)_x(0)\right\|^2\,dq \;\le\; C_6\,. \tag{4.27}$$

On the other hand, we have

$$
\begin{aligned}
\|\mathcal{P}[w^m_{xx}]_x(t)\|^2 &= \int_0^1 \left| \int_0^\infty \varphi(q) \mathfrak{s}_q[w^m_{xx}]_x(x,t)\, dq \right|^2 dx \\
&\le \int_0^\infty \varphi(q)\, dq \int_0^1 \int_0^\infty \varphi(q) (\mathfrak{s}_q[w^m_{xx}]_x(x,t))^2\, dq\, dx
\end{aligned}
$$

In conclusion, using also (4.18) and (4.19), we have shown in (4.23) the estimate

$$
\max_{0\le t\le T} \left(\|w^m_{xt}(t)\|^2 + \|w^m_{xxt}(t)\|^2 + \|\mathcal{P}[w^m_{xx}]_x(t)\|^2 \right) \le C_7 . \tag{4.28}
$$

Now observe that Proposition 3.4 (iv) shows that

$$
|(\mathcal{P}[w^m_{xx}])_t\,(x,t)| \le \int_0^\infty \varphi(q) \left| (\mathfrak{s}_q[w^m_{xx}])_t\,(x,t) \right| dq \le C_8\, |w^m_{xxt}(x,t)| \quad \text{a. e. in } Q_T .
$$

Hence, differentiating (4.2) with respect to t, inserting $\psi = u^m_{tt}(t) \in V_m$, and invoking (4.28), we can infer that

$$
\max_{0\le t\le T} (\|\mathcal{P}[w^m_{xx}]_t(t)\| + \|u^m_{tt}\|) \le C_9 . \tag{4.29}
$$

Moreover, by inserting $\psi = u^m_{xx}(t) \in V_m$ in (4.3), we directly find that

$$
\max_{0\le t\le T} \|u^m_{xx}(t)\| \le C_{10} . \tag{4.30}
$$

We now use the elementary formula

$$
\int_0^T \left(\dot\mu_k^2 + \ddot\mu_k\, \mu_k\right)(t)\, dt = \dot\mu_k(T)\, \mu_k(T) - \dot\mu_k(0)\, \mu_k(0)
$$

to estimate u^m_{xt} as follows.

$$
\begin{aligned}
\int_0^T \|u^m_{xt}(t)\|^2 dt &= \pi^2 \sum_{k=1}^m \int_0^T k^2 \dot\mu_k^2(t)\, dt \\
&\le \pi^2 \sum_{k=1}^m \int_0^T k^2\, |\ddot\mu_k(t)\, \mu_k(t)|\, dt + 2\pi^2 \max_{0\le t\le T} \sum_{k=1}^m k^2\, |\dot\mu_k(t)\, \mu_k(t)| \\
&\le \pi^2 \int_0^T \left(\sum_{k=1}^m |\ddot\mu_k(t)|^2 \right)^{1/2} \left(\sum_{k=1}^m k^4 |\mu_k(t)|^2 \right)^{1/2} dt \\
&\quad + 2\pi^2 \max_{0\le t\le T} \left(\sum_{k=1}^m |\dot\mu_k(t)|^2 \right)^{1/2} \left(\sum_{k=1}^m k^4 |\mu_k(t)|^2 \right)^{1/2} \\
&= \int_0^T \|u^m_{tt}(t)\| \|u^m_{xx}(t)\|\, dt + 2 \max_{0\le t\le T} \|u^m_t(t)\| \|u^m_{xx}(t)\| \\
&\le C_{11} ,
\end{aligned} \tag{4.31}
$$

by virtue of (4.29)–(4.30). To estimate w^m_{xtt}, we refer to (4.6), which yields for almost every $t \in [0,T]$ and every $k = 1, \dots, m$ that

$$
\begin{aligned}
|k\pi \ddot{\eta}_k(t)| &\le \frac{k^3\pi^3}{1+k^2\pi^2}\, |\dot{\mu}_k(t)| + \frac{k\pi}{1+k^2\pi^2}\, |\langle g(t), \psi_k\rangle| \\
&\le k\pi\, |\dot{\mu}_k(t)| + \frac{1}{2}\, |\langle g(t), \psi_k\rangle|\,, \qquad (4.32)
\end{aligned}
$$

hence

$$
\begin{aligned}
\int_0^T \|w^m_{xtt}(t)\|^2 dt &= \pi^2 \sum_{k=1}^m \int_0^T k^2 \ddot{\eta}_k^2(t)\, dt \\
&\le 2\pi^2 \sum_{k=1}^m \int_0^T k^2 \dot{\mu}_k^2(t)\, dt + \frac{1}{2} \sum_{k=1}^m \int_0^T \langle g(t), \psi_k\rangle^2 dt \\
&\le 2\int_0^T \|u^m_{xt}(t)\|^2 dt + \frac{1}{2}\int_0^T \|g(t)\|^2 dt \\
&\le C_{12}\,. \qquad (4.33)
\end{aligned}
$$

Combining the above estimates, and possibly selecting a suitable subsequence again indexed by m, we find that there exist functions u, w in the appropriate Sobolev spaces such that the following convergences take place:

$$
\left.
\begin{array}{rl}
\left.\begin{array}{r} w^m_{xxt} \to w_{xxt},\ u^m_{tt} \to u_{tt},\ u^m_{xx} \to u_{xx}, \\ \mathcal{P}[w^m_{xx}]_t \to u_{tt},\ \mathcal{P}[w^m_{xx}]_x \to u_{xt}, \end{array}\right\} & \text{weakly-* in } L^\infty(0,T;L^2(0,1))\,, \\
u^m_{xt} \to u_{xt},\ w^m_{xtt} \to w_{xtt}, & \text{weakly in } L^2(Q_T)\,.
\end{array}
\right\} \qquad (4.34)
$$

Then, by compact embedding,

$$
\left.
\begin{array}{l}
u^m \to u,\ u^m_x \to u_x,\ u^m_t \to u_t,\ \mathcal{P}[w^m_{xx}] \to u_t, \\
w^m \to w,\ w^m_x \to w_x,\ w^m_t \to w_t,\ w^m_{xt} \to w_{xt},
\end{array}
\right\} \quad \text{strongly in } C(\overline{Q_T})\,. \qquad (4.35)
$$

The convergences (4.34)–(4.35) entail that the functions u, v have the regularity stated in Theorem 3.1, and Eqs. (3.9)–(3.13) are satisfied in appropriate sense. It remains to prove that (3.8) holds, that is, $u_t = \mathcal{P}[w_{xx}]$. To this end, we apply a variant of Minty's trick based on the monotonicity (3.18) of the Prandtl–Ishlinskii operator. We fix an arbitrary function $z \in C(\overline{Q_T})$, and set $Z(x,t) = \int_0^t z(x,\tau)\, d\tau$. For all $\delta > 0$ and $m \in \mathbb{N}$ we have, by virtue of (3.18) and of the Lipschitz continuity of the projection mapping $\mathrm{Proj}_{[-q,q]}$ (see (1.6)), that

$$
\begin{aligned}
&\int_0^T \int_0^1 \left(\mathcal{P}[w^m_{xx}] - \mathcal{P}[w_{xx} + \delta Z]\right) \left(w^m_{xxt} - w_{xxt} - \delta z\right)(x,t)\, dx\, dt \\
&\quad \ge -\frac{1}{2} \int_0^\infty \int_0^1 \varphi(q) \left(\mathfrak{s}_q[w^m_{xx}] - \mathfrak{s}_q[w_{xx} + \delta Z]\right)^2 (x,0)\, dx\, dq \\
&\quad \ge -\frac{1}{2} \int_0^\infty \varphi(q)\, dq\, \|Q_m(z_{0,xx}) - z_{0,xx}\|^2\,. \qquad (4.36)
\end{aligned}
$$

Letting $m \to \infty$, this yields

$$\int_0^T \int_0^1 \left(u_t - \mathcal{P}[w_{xx} + \delta Z]\right)\, z(x,t)\, dx\, dt \;\leq\; 0\,. \tag{4.37}$$

We now let δ tend to 0 and obtain the desired result. The existence part of Theorem 3.1 is thus proved. $\square$

5 Proof of uniqueness and concluding remarks

Let us consider two solutions u_1, w_1, u_2, w_2 to (3.8)–(3.13), with the regularity stated in Theorem 3.2, and set $u = u_1 - u_2$, $w = w_1 - w_2$. We then have

$$\begin{aligned}
u_t &= \mathcal{P}\left[w_{1,xx}\right] - \mathcal{P}\left[w_{2,xx}\right] && \text{in } Q_T\,, && (5.1)\\
w_t - w_{xxt} &= -u_{xx} && \text{in } Q_T\,, && (5.2)\\
u(0,t) = u(1,t) &= 0\,, && 0 \leq t \leq T\,, && (5.3)\\
w(0,t) = w(1,t) &= 0\,, && 0 \leq t \leq T\,, && (5.4)\\
u(x,0) &= 0\,, && 0 \leq x \leq 1\,, && (5.5)\\
w(x,0) &= 0\,, && 0 \leq x \leq 1\,. && (5.6)
\end{aligned}$$

By Proposition 3.4 (i), we have a. e. in Q_T that

$$\left(\mathcal{P}\left[w_{1,xx}\right] - \mathcal{P}\left[w_{2,xx}\right]\right)\, w_{xxt} \;\geq\; \mathcal{R}_t\,,$$

where

$$\mathcal{R}(x,t) \;=\; \frac{1}{2}\int_0^\infty \varphi(q)\left(\mathfrak{s}_q[w_{1,xx}](x,t) - \mathfrak{s}_q[w_{2,xx}](x,t)\right)^2 dq \;\geq\; 0\,.$$

We now test Eq. (5.1) by w_{xxt}, Eq. (5.2) by $-w_{tt}$, and sum them up. The regularity (3.15)–(3.16) enables us to obtain for almost all $t \in (0,T)$ that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \left(\mathcal{R} - \frac{1}{2}w_t^2 - \frac{1}{2}w_{xt}^2\right)(x,t)\, dx\\
&\leq \int_0^1 \left(u_t\, w_{xxt} + u_{xx}\, w_{tt}\right)(x,t)\, dx\\
&= -\int_0^1 \left(u_{xt}\, w_{xt} + u_x\, w_{xtt}\right)(x,t)\, dx\\
&= -\frac{d}{dt}\int_0^1 u_x(x,t)\, w_{xt}(x,t)\, dx\\
&= \frac{d}{dt}\int_0^1 u_{xx}(x,t)\, w_t(x,t)\, dx\\
&= -\frac{d}{dt}\int_0^1 \left(w_t^2 + w_{xt}^2\right)(x,t)\, dx\,, && (5.7)
\end{aligned}$$

hence

$$\frac{d}{dt}\int_0^1 \left(\mathcal{R} + \frac{1}{2}w_t^2 + \frac{1}{2}w_{xt}^2\right)(x,t)\,dx \;\leq\; 0 \quad \text{a.e.} \tag{5.8}$$

The initial conditions for w_1 and w_2 coincide, hence $w_1 = w_2$ in Q_T, and consequently also $u_1 = u_2$. This completes the proof of Theorem 3.2. $\square$

Remark 5.1. The uniqueness of the limit pair (u, w) entails that the convergences (4.34)–(4.35) hold for the entire sequence $\{(u^m, w^m)\}$ and not only for a subsequence. Hence the Faedo-Galerkin scheme (4.2)–(4.7) constitutes a convergent method to approximate the solution numerically.

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