Fredholm Alternative for Periodic-Dirichlet Problems for Linear Hyperbolic Systems

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Abstract

This paper concerns hyperbolic systems of two linear first-order PDEs in one space dimension with periodicity conditions in time and reflection boundary conditions in space. The coefficients of the PDEs are supposed to be time independent, but allowed to be discontinuous with respect to the space variable. We construct two scales of Banach spaces (for the solutions and for the right hand sides of the equations, respectively) such that the problem can be modeled by means of Fredholm operators of index zero between corresponding spaces of the two scales.

1 Introduction and main results

This paper concerns linear inhomogeneous hyperbolic systems of first order PDEs in one space dimension of the type

$$\left.\begin{aligned} \partial_t u + \partial_x u + a(x)u + b(x)v = f(x,t) \\ \partial_t v - \partial_x v + c(x)u + d(x)v = g(x,t) \end{aligned}\right\} \; 0 \le x \le 1,\ t \in \mathbb{R} \tag{1.1}$$

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with time-periodicity conditions

$$\left.\begin{aligned} u\left(x, t+\tfrac{2\pi}{\omega}\right) &= u(x,t) \\ v\left(x, t+\tfrac{2\pi}{\omega}\right) &= v(x,t) \end{aligned}\right\} \; 0 \le x \le 1, \; t \in \mathbb{R} \tag{1.2}$$

and reflection boundary conditions

$$\left.\begin{aligned} u(0,t) &= r_0 v(0,t) \\ v(1,t) &= r_1 u(1,t) \end{aligned}\right\} \; t \in \mathbb{R}. \tag{1.3}$$

Together with the periodic-Dirichlet problem (1.1)–(1.3) we consider its homogeneous adjoint variant

$$\left.\begin{aligned} -\partial_t u - \partial_x u + a(x)u + c(x)v &= 0 \\ -\partial_t v + \partial_x v + b(x)u + d(x)v &= 0 \end{aligned}\right\} \; 0 \le x \le 1, \; t \in \mathbb{R}, \tag{1.4}$$

$$\left.\begin{aligned} u\left(x, t+\tfrac{2\pi}{\omega}\right) &= u(x,t) \\ v\left(x, t+\tfrac{2\pi}{\omega}\right) &= v(x,t) \end{aligned}\right\} \; 0 \le x \le 1, \; t \in \mathbb{R}, \tag{1.5}$$

$$\left.\begin{aligned} v(0,t) &= r_0 u(0,t) \\ u(1,t) &= r_1 v(1,t) \end{aligned}\right\} \; t \in \mathbb{R}. \tag{1.6}$$

Here $\omega > 0$ and $r_0, r_1 \in \mathbb{R}$ are fixed numbers, $a, b, c, d : [0,1] \to \mathbb{R}$ are fixed coefficient functions, and the right-hand sides $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are supposed to be $\frac{2\pi}{\omega}$-periodic with respect to t.

Roughly speaking, we will prove the following: Suppose

$$|r_0 r_1| \neq e^{\int_0^1 (a(x)+d(x))\,dx}. \tag{1.7}$$

Then there exists a solution to (1.1)–(1.3) if and only if the pair (f, g) of the right hand sides is orthogonal in $L^2\left((0,1) \times \left(0, \frac{2\pi}{\omega}\right); \mathbb{R}^2\right)$ to any solution to (1.4)–(1.6). Moreover, the dimension of the space of all solutions to (1.1)–(1.3) with $f = g = 0$ is finite and is equal to the dimension of the space of all solutions to (1.4)–(1.6).

The main tools of the proofs are separation of variables (cf. (3.2)-(3.3)), integral representation of the solutions of the corresponding boundary value problems of the ODE systems (cf. (3.6)) and an abstract criterion for Fredholmness which seems to be new (cf. Lemma 8).

In order to formulate our results more exactly, let us introduce function spaces: For $l = 0, 1, 2, \ldots$ and $\gamma \ge 0$ we denote by $H^{l,\gamma}$ the vector space of all measurable functions $u : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that $u(x,t) = u\left(x, t+\frac{2\pi}{\omega}\right)$ for almost all $(x,t) \in [0,1] \times \mathbb{R}$ and that

$$\|u\|^2_{H^{l,\gamma}} := \sum_{k \in \mathbb{Z}} (1+k^2)^\gamma \sum_{m=0}^{l} \int_0^1 \left| \int_0^{\frac{2\pi}{\omega}} \partial_x^m u(x,t) e^{-ik\omega t}\,dt \right|^2 dx < \infty. \tag{1.8}$$

It is well-known (see, e.g., [2], [7, Chapter 5.10], and [9, Chapter 2.4]) that $H^{l,\gamma}$ is a Banach space. In fact, it is the space of all locally quadratically Bochner integrable maps $u : \mathbb{R} \to H^l(0,1)$, which are $\frac{2\pi}{\omega}$–periodic and such that all generalized derivatives up to the (possibly noninteger) order γ are locally quadratically integrable too. Further, we denote

$$
\begin{aligned}
W^\gamma &:= H^{0,\gamma} \times H^{0,\gamma}, \\
V^\gamma &:= \{(u,v) \in W^\gamma : \ (\partial_t u + \partial_x u, \partial_t v - \partial_x v) \in W^\gamma\}.
\end{aligned}
$$

The function spaces W^γ and V^γ will be endowed with the norms

$$
\|(u,v)\|^2_{W^\gamma} := \|u\|^2_{H^{0,\gamma}} + \|v\|^2_{H^{0,\gamma}}
$$

and

$$
\|(u,v)\|^2_{V^\gamma} := \|(u,v)\|^2_{W^\gamma} + \|(\partial_t u + \partial_x u, \partial_t v - \partial_x v)\|^2_{W^\gamma}.
$$

It is easy to prove (see Lemma 4) that V^γ is a Banach space. Moreover, if $\gamma \geq 1$, then for any $(u,v) \in V^\gamma$ and any $x \in [0,1]$ there exist traces $u(x,\cdot), v(x,\cdot) \in L^2_{loc}(\mathbb{R})$ (see Remark 6). Hence, for $\gamma \geq 1$ it makes sense to consider the following closed subspaces in V^γ:

$$
\begin{aligned}
V^\gamma(r_0,r_1) &:= \{(u,v) \in V^\gamma : \ \text{(1.3) is fulfilled}\}, \\
\tilde{V}^\gamma(r_0,r_1) &:= \{(u,v) \in V^\gamma : \ \text{(1.6) is fulfilled}\}.
\end{aligned}
$$

Finally, let us introduce linear operators: For $a,b,c,d \in L^\infty(0,1)$ we define $A \in \mathcal{L}(V^\gamma(r_0,r_1); W^\gamma)$, $\tilde{A} \in \mathcal{L}(\tilde{V}^\gamma(r_0,r_1); W^\gamma)$, and $B, \tilde{B} \in \mathcal{L}(W^\gamma)$ by

$$
A \begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} \partial_t u + \partial_x u + au \\ \partial_t v - \partial_x v + dv \end{bmatrix}, \quad B \begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} bv \\ cu \end{bmatrix},
$$

$$
\tilde{A} \begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} -\partial_t u - \partial_x u + au \\ -\partial_t v + \partial_x v + dv \end{bmatrix}, \quad \tilde{B} \begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} cv \\ bu \end{bmatrix}.
$$

Now we formulate our main result:

Theorem 1 *Let $\gamma \geq 1$, $a,d \in L^\infty(0,1)$, $b,c \in BV(0,1)$, and suppose (1.7). Then we have:*

(i) The operator A is an isomorphism from $V^\gamma(r_0,r_1)$ onto W^γ.

(ii) The operator $A+B$ is Fredholm of index zero from $V^\gamma(r_0,r_1)$ into W^γ.

(iii) The image of $A+B$ is the set of all $(f,g) \in W^\gamma$ such that

$$
\int_0^{\frac{2\pi}{\omega}} \int_0^1 (f(x,t)u(x,t) + g(x,t)v(x,t))\, dx\, dt = 0 \ \textit{for all} \ (u,v) \in \ker(\tilde{A} + \tilde{B}).
$$

Remark 2 *The operators B and $\tilde{B}$ depend linearly and continuously (in the operator norm of $\mathcal{L}(W^\gamma)$) on the coefficient functions b and c (in the $L^\infty(0,1)$ norm). Hence, a perturbation argument gives the following result, which is slightly more general than assertion (ii) of Theorem 1:*

Let $\gamma \geq 1$, $a, d \in L^\infty(0,1)$, and suppose (1.7). Then there exists an open and dense set $M \subseteq (L^\infty(0,1))^2$ such that $(BV(0,1))^2 \subset M$ and that for all $(b,c) \in M$ the operator $A + B$ is Fredholm of index zero from $V^\gamma(r_0, r_1)$ into W^γ.

The present paper has been motivated by the investigation of so-called traveling wave models, which are successfully used for modeling of dynamical behavior of semiconductor lasers (see, e.g., [1, 3, 4, 5, 8]). In those models, systems of the type (1.1) appear to describe the forward and backward traveling light waves in the longitudinal laser direction. If one deals with multisection lasers consisting of several sections with different electrical and optical properties, then the coefficient functions a, b, c, and d are discontinuous (piecewise constant). Note that in the laser models the coefficient functions and the reflection coefficients r_0 and r_1 as well as the unknown functions u and v are complex valued, and the linear wave system is coupled (via the coefficient functions) to a nonlinear balance equation for the carrier distribution in the active zone of the laser. Hence, from the point of view of applications to laser dynamics, our problem (1.1)–(1.3) is only a case study. On the other hand, it is well known that the Fredholm alternative for the linearization is a key for investigation of main properties of periodic solutions to nonlinear differential equations, such as persistence via implicit function theorem, bifurcation and synchronisation via Liapunov-Schmidt procedure (see, e.g., [6]). In a forthcoming paper we will use our present results for applications to laser models.

2 Some properties of the function spaces

In this section we formulate and prove some properties of the function spaces $H^{l,\gamma}$, W^γ, and V^γ, introduced in Section 1.

It is well known that for each $u \in H^{l,\gamma}$ we have

$$u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)e^{ik\omega t} \text{ with } u_k(x) = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} u(x,t)e^{-ik\omega t}\,dt, \tag{2.1}$$

where the functions u_k belong to the classical Sobolev space $H^l((0,1);\mathbb{C})$, and the series in (2.1) converges in the complexification of $H^{l,\gamma}$. And vice versa: For any sequence $(u_k)_{k\in\mathbb{Z}}$ with

$$u_k \in H^l((0,1);\mathbb{C}), \quad \overline{u_k} = u_{-k}, \quad \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \|u_k\|^2_{H^l((0,1);\mathbb{C})} < \infty \tag{2.2}$$

there exists exactly one $u \in H^{l,\gamma}$ with (2.1). In what follows, we will identify functions $u \in H^{l,\gamma}$ and sequences $(u_k)_{k\in\mathbb{Z}}$ with (2.2) by means of (2.1), and we will keep for corresponding functions and sequences the notations u and $(u_k)_{k\in\mathbb{Z}}$, respectively.

Lemma 3 *A set $M \subset H^{0,\gamma}$ is precompact in $H^{0,\gamma}$ if and only if the following two conditions are satisfied:*

(i) There exists $C > 0$ such that for all $u \in M$ it holds

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \int_0^1 |u_k(x)|^2 \, dx \le C.$$

(ii) For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in (-\delta, \delta)$ and all $u \in M$ it holds

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \int_0^1 \left| u_k(x+h)e^{ik\omega h} - u_k(x) \right|^2 dx < \varepsilon,$$

where $u_k(x+h) := 0$ for $x+h \notin [0,1]$.

Proof. Let us use the canonical isomorphism J from $H^{0,\gamma}$ onto $H^{0,0} = L^2\Big((0,1) \times \big(0, \frac{2\pi}{\omega}\big)\Big)$, which is defined by

$$(Ju)(x,t) := \sum_{k\in\mathbb{Z}} (1+k^2)^{\gamma/2} u_k(x) e^{ik\omega t}.$$

We have to show that $J(M)$ is precompact in $L^2\Big((0,1) \times \big(0, \frac{2\pi}{\omega}\big)\Big)$, i.e. that $J(M)$ is bounded in $L^2\Big((0,1) \times \big(0, \frac{2\pi}{\omega}\big)\Big)$, and that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in J(M)$ it holds

$$\int_0^{\frac{2\pi}{\omega}} \int_0^1 |u(x+h, t+h) - u(x,t)|^2 \, dx \, dt < \varepsilon, \tag{2.3}$$

where $u(x+h, t+h) := 0$ for $x+h \notin [0,1]$. Boundedness in $L^2\Big((0,1) \times \big(0, \frac{2\pi}{\omega}\big)\Big)$ is just condition (ii) of the lemma.

Now we show that (2.3) is just condition (ii) of the lemma. This follows from

$$\int_0^{\frac{2\pi}{\omega}}\int_0^1 \Big|(Ju)(x+h,t+h)-(Ju)(x,t)\Big|^2 dx\,dt$$

$$= \int_0^{\frac{2\pi}{\omega}}\int_0^1 \Big|\sum_{k\in\mathbb{Z}}(1+k^2)^{\gamma/2}\Big[u_k(x+h)e^{ik\omega(t+h)}-u_k(x)e^{ik\omega t}\Big]\Big|^2 dx\,dt$$

$$= \frac{2\pi}{\omega}\int_0^1 \sum_{k\in\mathbb{Z}}(1+k^2)^{\gamma}\Big|u_k(x+h)e^{ik\omega h}-u_k(x)\Big|^2 dx.$$

□

Lemma 4 *The space V^γ is complete.*

Proof. Let $(u^j,v^j)_{j\in\mathbb{N}}$ be a fundamental sequence in V^γ. Then $(u^j,v^j)_{j\in\mathbb{N}}$ and $(\partial_t u^j+\partial_x u^j,\partial_t v^j-\partial_x v^j)_{j\in\mathbb{N}}$ are fundamental sequences in W^γ. Because W^γ is complete, there exist $(u,v)\in W^\gamma$ and $(\tilde u,\tilde v)\in W^\gamma$ such that

$$(u^j,v^j)\to(u,v) \text{ and } (\partial_t u^j+\partial_x u^j,\partial_t v^j-\partial_x v^j)\to(\tilde u,\tilde v)$$

in W^γ as $j\to\infty$. It remains to show that $\partial_t u+\partial_x u=\tilde u$ and $\partial_t v-\partial_x v=\tilde v$ in the sense of generalized derivatives. But this is obvious: Take a smooth function $\varphi:(0,1)\times\left(0,\frac{2\pi}{\omega}\right)\to\mathbb{R}$ with compact support. Then

$$\int_0^{\frac{2\pi}{\omega}}\int_0^1 u(\partial_t+\partial_x)\varphi\,dx\,dt = \lim_{j\to\infty}\int_0^{\frac{2\pi}{\omega}}\int_0^1 u^j(\partial_t+\partial_x)\varphi\,dx\,dt$$

$$= -\lim_{j\to\infty}\int_0^{\frac{2\pi}{\omega}}\int_0^1 (\partial_t+\partial_x)u^j\varphi\,dx\,dt = -\int_0^{\frac{2\pi}{\omega}}\int_0^1 \tilde u\varphi\,dx\,dt,$$

and similarly for v and $\tilde v$. □

Lemma 5 *If $\gamma\geq 1$, then V^γ is continuously embedded into $(H^{1,\gamma-1})^2$.*

Proof. Take $(u,v) \in V^\gamma$. Then $(u,v) \in (H^{0,\gamma})^2$, hence $(\partial_t u, \partial_t v) \in (H^{0,\gamma-1})^2$. By the definition of the space V^γ, $(\partial_x u, \partial_x v) \in (H^{0,\gamma-1})^2$. Hence $(u,v) \in (H^{1,\gamma-1})^2$. Moreover, we have

$$\begin{aligned}
\|(u,v)\|^2_{(H^{1,\gamma-1})^2} &= \|u\|^2_{H^{0,\gamma-1}} + \|v\|^2_{H^{0,\gamma-1}} + \|\partial_x u\|^2_{H^{0,\gamma-1}} + \|\partial_x v\|^2_{H^{0,\gamma-1}} \\
&\le \|u\|^2_{H^{0,\gamma-1}} + \|v\|^2_{H^{0,\gamma-1}} + \|\partial_t u + \partial_x u\|^2_{H^{0,\gamma-1}} + \|\partial_t v - \partial_x v\|^2_{H^{0,\gamma-1}} \\
&\qquad + \|\partial_t u\|^2_{H^{0,\gamma-1}} + \|\partial_t v\|^2_{H^{0,\gamma-1}} \\
&\le C\|(u,v)\|^2_{V^\gamma},
\end{aligned}$$

where the constant C does not depend on (u,v). □

Remark 6 *Suppose $\gamma \ge 1$. By Lemma 5, we have*

$$V^\gamma \hookrightarrow \left(H^{0,1}\right)^2 \approx \left(H^1\left((0,1); L^2\left(0, \frac{2\pi}{\omega}\right)\right)\right)^2 \hookrightarrow \left(C\left([0,1]; L^2\left(0, \frac{2\pi}{\omega}\right)\right)\right)^2.$$

Therefore, for any $x \in [0,1]$ there exists a continuous trace map

$$(u,v) \in V^\gamma \mapsto \Big(u(x,\cdot), v(x,\cdot)\Big) \in \left(L^2\left(0, \frac{2\pi}{\omega}\right)\right)^2.$$

Now, let us consider the dual spaces $(H^{0,\gamma})^*$.

Obviously, for any $\gamma \ge 0$ the spaces $H^{0,\gamma}$ are densely and continuously embedded into the Hilbert space $H^{0,0} = L^2\left((0,1) \times \left(0, \frac{2\pi}{\omega}\right)\right)$. Hence, there is a canonical dense continuous embedding

$$H^{0,0} \hookrightarrow (H^{0,\gamma})^* : \quad [u,v]_{H^{0,\gamma}} = \langle u,v \rangle_{H^{0,0}} \text{ for all } u \in H^{0,0} \text{ and } v \in H^{0,\gamma}. \tag{2.4}$$

Here $[\cdot,\cdot]_{H^{0,\gamma}} : (H^{0,\gamma})^* \times H^{0,\gamma} \to \mathbb{R}$ is the dual pairing, and $\langle \cdot,\cdot \rangle_{H^{0,0}} : H^{0,0} \times H^{0,0} \to \mathbb{R}$ is the scalar product in $H^{0,0}$:

$$\langle u,v \rangle_{H^{0,0}} := \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \int_0^1 u(x,t)v(x,t)\,dx\,dt = \sum_{k\in\mathbb{Z}} \int_0^1 u_k(x)\overline{v_k(x)}\,dx. \tag{2.5}$$

Let us denote

$$e_k(t) := e^{ik\omega t} \text{ for } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}. \tag{2.6}$$

If a sequence $(\varphi_k)_{k\in\mathbb{Z}}$ with $\varphi_k \in L^2((0,1);\mathbb{C})$ is given, then the pointwise products $\varphi_k e_k$ belong to $L^2\left((0,1) \times \left(0, \frac{2\pi}{\omega}\right); \mathbb{C}\right)$. Hence, they belong to the complexification of $(H^{0,\gamma})^*$ (by means of the complexification of (2.4)), and it makes sense to ask if the series

$$\sum_{k\in\mathbb{Z}} \varphi_k e_k \tag{2.7}$$

converges in the complexification of $(H^{0,\gamma})^*$.

Lemma 7 *(i) For any $\varphi \in (H^{0,\gamma})^*$ there exists a sequence $(\varphi_k)_{k\in\mathbb{Z}}$ with*

$$\varphi_k \in L^2((0,1);\mathbb{C}),\ \overline{\varphi_k} = \varphi_{-k},\ \sum_{k\in\mathbb{Z}} (1+k^2)^{-\gamma} \int_0^1 |\varphi_k(x)|^2\, dx < \infty, \tag{2.8}$$

such that the series (2.7) converges to φ in the complexification of $(H^{0,\gamma})^*$. Moreover, it holds

$$\int_0^1 \varphi_k(x)u(x)\, dx = [\varphi, ue_{-k}]_{H^{0,\gamma}} \text{ for all } u \in L^2(0,1). \tag{2.9}$$

(ii) For any sequence $(\varphi_k)_{k\in\mathbb{Z}}$ with (2.8) the series (2.7) converges in the complexification of $(H^{0,\gamma})^*$ to some $\varphi \in (H^{0,\gamma})^*$, and (2.9) is satisfied.

Proof. (i) By the Riesz representation theorem, for given $\varphi \in (H^{0,\gamma})^*$ and $k \in \mathbb{Z}$ there exists exactly one $\varphi_k \in L^2((0,1);\mathbb{C})$ with (2.9). The property $\overline{\varphi_k} = \varphi_{-k}$ follows directly from (2.9).

Now, take $u \in H^{0,\gamma}$ with its representation (2.1), (2.2) (with $l = 0$ there). We have

$$\left|[\varphi, u]_{H^{0,\gamma}}\right| = \left|\left[\varphi, \sum_{k\in\mathbb{Z}} u_k e_k\right]_{H^{0,\gamma}}\right| = \left|\sum_{k\in\mathbb{Z}} \int_0^1 (1+k^2)^{-\gamma}\varphi_k(x)(1+k^2)^{\gamma}\overline{u_k(x)}\, dx\right|.$$

Taking the supremum over all $u \in H^{0,\gamma}$ with $\|u\|^2_{H^{0,\gamma}} = \sum_{k\in\mathbb{Z}}(1+k^2)^{\gamma} \int_0^1 |u_k(x)|^2\, dx = 1$, we get

$$\|\varphi\|^2_{(H^{0,\gamma})^*} = \sum_{k\in\mathbb{Z}} (1+k^2)^{-\gamma} \int_0^1 |\varphi_k(x)|^2\, dx < \infty.$$

Similarly one shows that the series (2.7) converges in the complexification of $(H^{0,\gamma})^*$: For all $u \in H^{0,\gamma}$ we have

$$\begin{aligned}
\left|\left[\varphi - \sum_{|k|\le k_0} \varphi_k e_k, u\right]_{H^{0,\gamma}}\right| &= \left|\left[\varphi - \sum_{|k|\le k_0} \varphi_k e_k, \sum_{k\in\mathbb{Z}} u_k e_k\right]_{H^{0,\gamma}}\right| \\
&= \left|\sum_{|k|>k_0} \int_0^1 (1+k^2)^{-\gamma}\varphi_k(x)(1+k^2)^{\gamma}\overline{u_k(x)}\, dx\right| \\
&\le \left(\sum_{|k|>k_0} (1+k^2)^{-\gamma} \int_0^1 |\varphi_k(x)|^2\, dx\right)^{\frac{1}{2}} \|u\|_{H^{0,\gamma}},
\end{aligned} \tag{2.10}$$

and this tends to zero as $k_0 \to \infty$ uniformly for $\|u\|_{H^{0,\gamma}} = 1$.

(ii) As in (2.10) one shows that (2.7) converges in the complexification of $(H^{0,\gamma})^*$ to some $\varphi \in (H^{0,\gamma})^*$. Moreover, (2.9) is satisfied, because for all $u \in L^2(0,1)$ we have

$$[\varphi, ue_{-k}]_{H^{0,\gamma}} = \left[\sum_{l\in\mathbb{Z}} \varphi_l e_l, ue_{-k}\right]_{H^{0,\gamma}} = \sum_{l\in\mathbb{Z}} \langle \varphi_l e_l, ue_k\rangle_{H^{0,0}} = \int_0^1 \varphi_k(x)u(x)\,dx.$$

Here we used (2.4) and (2.5). □

3 Proof of the isomorphism property

In what follows, we suppose the assumptions of Theorem 1 to be fulfilled. In this section we prove assertion (i) of Theorem 1.

Fix $(f,g) \in W^\gamma$. Then $f(x,t) = \sum\limits_{k\in\mathbb{Z}} f_k(x)e^{ik\omega t}$ and $g(x,t) = \sum\limits_{k\in\mathbb{Z}} g_k(x)e^{ik\omega t}$ with $f_k, g_k \in \mathrm{L}^2\left((0,1);\mathbb{C}\right)$ and

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \int_0^1 \left(|f_k(x)|^2 + |g_k(x)|^2\right)\,dx < \infty. \tag{3.1}$$

We have to show that there exists exactly one $(u,v) \in V^\gamma(r_0,r_1)$ such that $\partial_t u + \partial_x u + au = f$ and $\partial_t v - \partial_x v + du = g$. Writing u and v as series according to (2.1) and (2.2), we have to show that there exists exactly one pair of sequences $(u_k)_{k\in\mathbb{Z}}$ and $(v_k)_{k\in\mathbb{Z}}$ with $u_k, v_k \in H^1(0,1)$ satisfying the boundary value problem

$$u_k' + (a(x)+ik\omega)u_k = f_k(x),\ v_k' - (d(x)+ik\omega)v_k = -g_k(x), \tag{3.2}$$

$$u_k(0) = r_0 v_k(0),\ v_k(1) = r_1 u_k(1), \tag{3.3}$$

and the estimates

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \int_0^1 \left(|u_k(x)|^2 + |v_k(x)|^2\right) dx < \infty, \tag{3.4}$$

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \int_0^1 \left(|u_k'(x) + ik\omega u_k(x)|^2 + |v_k'(x) - ik\omega v_k(x)|^2\right) dx < \infty. \tag{3.5}$$

Here we used Lemma 5 and Remark 6.

The estimate (3.5) follows from (3.1), (3.2), and (3.4). Hence, it remains to show that there exists exactly one pair of sequences $(u_k)_{k\in\mathbb{Z}}$ and $(v_k)_{k\in\mathbb{Z}}$ with $u_k, v_k \in H^1((0,1);\mathbb{C})$ satisfying (3.2), (3.3), and (3.4).

In order to simplify the formulae below, let us introduce the following notation:

$$\alpha(x) := \int_0^x a(y)\,dy, \quad \delta(x) := \int_0^x d(y)\,dy, \quad \Delta_k := e^{ik\omega+\delta(1)} - r_0 r_1 e^{-ik\omega-\alpha(1)}.$$

A straightforward calculation shows that the boundary value problem (3.2), (3.3) has a unique solution $(u_k, v_k) \in H^1((0,1);\mathbb{C}^2)$, and this solution is explicitely given by

$$\begin{aligned} u_k(x) &= e^{-ik\omega x-\alpha(x)}\left(\int_0^x e^{ik\omega y+\alpha(y)} f_k(y)\,dy + \frac{r_0}{\Delta_k} w_k(f_k, g_k)\right), \\ v_k(x) &= e^{ik\omega x+\delta(x)}\left(\int_0^x e^{-ik\omega y-\delta(y)} g_k(y)\,dy + \frac{1}{\Delta_k} w_k(f_k, g_k)\right) \end{aligned} \tag{3.6}$$

with

$$w_k(f,g) := r_1 e^{-ik\omega-\alpha(1)} \int_0^1 e^{ik\omega y+\alpha(y)} f(y)\,dy - e^{ik\omega+\delta(1)} \int_0^1 e^{-ik\omega y-\delta(y)} g(y)\,dy. \tag{3.7}$$

Here we used assumption (1.7), which implies

$$|\Delta_k| \ge \left| e^{\delta(1)} - |r_0 r_1| e^{-\alpha(1)} \right| > 0 \text{ for all } k \in \mathbb{Z}. \tag{3.8}$$

From (3.6) and (3.8) it follows that

$$|u_k(x)| + |v_k(x)| \le C \left(\int_0^1 \left(|f_k(x)|^2 + |g_k(x)|^2 \right) dx \right)^{\frac{1}{2}} \tag{3.9}$$

for all $x \in [0,1]$, where the constant C does not depend on k, f_k, g_k, and x. Finally, (3.1) and (3.9) imply (3.4).

4 Proof of Fredholmness

In this section we prove that $A + B$ is Fredholm, which is a part of assertion (*ii*) of Theorem 1.

Obviously, $A+B$ is Fredholm from $V^\gamma(r_0,r_1)$ into W^γ if and only if $I+BA^{-1}$ is Fredholm from W^γ into W^γ. Here I is the identity in W^γ.

We are going to prove that $I+BA^{-1}$ is Fredholm from W^γ into W^γ using the following

Lemma 8 *Let W be a Banach space, I the identity in W, and $C \in \mathcal{L}(W)$ such that C^2 is compact. Then $I+C$ is Fredholm.*

Proof. Since $I - C^2 = (I-C)(I+C)$ and C^2 is compact, we have

$$\dim\ker(I+C) \leq \dim\ker(I-C^2) < \infty. \tag{4.1}$$

Similarly one gets $\dim\ker(I+C)^* < \infty$, hence $\operatorname{codim}\overline{\operatorname{im}(I+C)} < \infty$. It remains to show that $\operatorname{im}(I+C)$ is closed.

Take a sequence $(w_j)_{j\in\mathbb{N}} \subset W$ and an element $w \in W$ such that

$$(I+C)w_j \to w. \tag{4.2}$$

We have to show that $w \in \operatorname{im}(I+C)$.

Because of (4.1) there exists a closed subspace V of W such that

$$W = \ker(I+C) \oplus V, \tag{4.3}$$

Using the decomposition $w_j = u_j + v_j$ with $u_j \in \ker(I+C)$ and $v_j \in V$, we get from (4.2)

$$(I+C)v_j \to w. \tag{4.4}$$

First, consider the case when the sequence $(v_j)_{j\in\mathbb{N}}$ is bounded. As the operator C^2 is compact, there exist $v \in W$ and a subsequence $(v_{j_k})_{k\in\mathbb{N}}$ such that $C^2 v_{j_k} \to v$. On the other hand, (4.4) yields $(I-C^2)v_j \to (I-C)w$. Hence (4.4) yields $\lim\limits_{k\to\infty} v_{j_k} = (I-C)w + v$, and, therefore,

$$w = \lim_{k\to\infty}(I+C)v_{j_k} = (I+C)((I-C)w+v) \in \operatorname{im}(I+C).$$

Now, suppose that the sequence $(v_j)_{j\in\mathbb{N}}$ is unbounded. Without loss of generality we can assume that

$$\lim_{j\to\infty}\|v_j\| = \infty. \tag{4.5}$$

From (4.4) and (4.5) we get

$$(I+C)\frac{v_j}{\|v_j\|} \to 0, \tag{4.6}$$

hence

$$(I - C^2)\frac{v_j}{\|v_j\|} \to 0. \tag{4.7}$$

On the other side, because C^2 is compact, there exist $v \in W$ and a subsequence $(v_{j_k})_{k\in\mathbb{N}}$ such that

$$C^2\frac{v_{j_k}}{\|v_{j_k}\|} \to v. \tag{4.8}$$

Inserting (4.8) into (4.7), we get

$$\frac{v_{j_k}}{\|v_{j_k}\|} \to v \in V. \tag{4.9}$$

Combining (4.9) with (4.6), we get $(I + C)v = 0$, i.e.

$$v \in V \cap \ker(I + C) \text{ and } \|v\| = 1.$$

But this contradicts to (4.3). In other words: The sequence $(v_j)_{j\in\mathbb{N}}$ cannot be unbounded. □

In order to use Lemma 8 with $W := W^\gamma$ and $C := BA^{-1}$, let us show that $(BA^{-1})^2$ is compact from W^γ into W^γ.

Take a bounded set $N \subset W^\gamma$, and let M be its image under $(BA^{-1})^2$. In order to show that M is precompact in $W^\gamma = H^{0,\gamma} \times H^{0,\gamma}$, we use Lemma 3 "componentwise". Condition (i) of Lemma 3 is satisfied because $(BA^{-1})^2$ is a bounded operator from W^γ into W^γ.

It remains to check condition (ii) of Lemma 3. The explicit representation (3.6) of A^{-1} yields the following: For given $(f, g) \in N$ we have

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \left(BA^{-1}\right)^2 \begin{bmatrix} f \\ g \end{bmatrix} = BA^{-1}B \begin{bmatrix} u \\ v \end{bmatrix} \text{ with } \begin{bmatrix} u \\ v \end{bmatrix} = A^{-1} \begin{bmatrix} f \\ g \end{bmatrix}$$

if and only if

$$\tilde{u}_k(x) = b(x)e^{ik\omega x+\delta(x)} \left(\int_0^x e^{-ik\omega y-\delta(y)} c(y)u_k(y)\, dy + \frac{1}{\Delta_k} w_k(bv_k, cu_k) \right),$$

$$\tilde{v}_k(x) = c(x)e^{-ik\omega x-\alpha(x)} \left(\int_0^x e^{ik\omega y+\alpha(y)} b(y)v_k(y)\, dy + \frac{r_0}{\Delta_k} w_k(bv_k, cu_k) \right),$$

where the functions w_k are defined by (3.7) and the functions $u_k, v_k \in H^1(0,1)$ as the solutions to (3.2), (3.3) are given by the formulas (3.6). Hence

$$
\begin{aligned}
&\left|\tilde{u}_k(x+h)e^{ik\omega h} - \tilde{u}_k(x)\right| \\
&\quad \le \left| b(x+h)e^{2ik\omega h+\delta(x+h)} \int\limits_x^{x+h} e^{-ik\omega y-\delta(y)}c(y)u_k(y)\,dy \right| \\
&\quad + \left| b(x+h)e^{\delta(x+h)}\left(e^{2ik\omega h}-1\right)\left(\int\limits_0^x e^{-ik\omega y-\delta(y)}c(y)u_k(y)\,dy + \frac{w_k(bv_k, cu_k)}{\Delta_k}\right)\right| \\
&\quad + \left|\left(b(x+h)e^{\delta(x+h)} - b(x)e^{\delta(x)}\right)\left(\int\limits_0^x e^{-ik\omega y-\delta(y)}c(y)u_k(y)\,dy + \frac{w_k(bv_k, cu_k)}{\Delta_k}\right)\right|.
\end{aligned}
$$

On the account of (3.9) and the boundedness of N, we get the estimate

$$
\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int\limits_0^1 \left| b(x+h)e^{2ik\omega h+\delta(x+h)} \int\limits_x^{x+h} e^{-ik\omega y-\delta(y)}c(y)u_k(y)\,dy\right|^2 dx \le Ch^2,
$$

where the constant C does not depend on $(f,g) \in N$. Similarly one gets (using (3.7)–(3.9))

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int\limits_0^1 \left|b(x+h)e^{\delta(x+h)} - b(x)e^{\delta(x)}\right|^2 \\
&\qquad \times \left| \int\limits_0^x e^{-ik\omega y-\delta(y)}c(y)u_k(y)\,dy + \frac{w_k(bv_k, cu_k)}{\Delta_k}\right|^2 dx \\
&\qquad \le C \int\limits_0^1 \left|b(x+h)e^{\delta(x+h)} - b(x)e^{\delta(x)}\right|^2 dx,
\end{aligned}
$$

which tends to zero for $h \to 0$ uniformly with respect to $(f,g) \in N$ (because of the continuity in the mean of the function $x \mapsto b(x)e^{\delta(x)}$).

It remains to show that

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \int\limits_0^1 \left|b(x+h)e^{\delta(x+h)}\left(e^{2ik\omega h}-1\right)\right|^2 \\
&\qquad \times \left| \int\limits_0^x e^{-ik\omega y-\delta(y)}c(y)u_k(y)\,dy + \frac{w_k(bv_k, cu_k)}{\Delta_k}\right|^2 dx \qquad (4.10)
\end{aligned}
$$

tends to zero for $h \to 0$ uniformly with respect to $(f, g) \in N$. Using (3.2), we have for all $k \neq 0$

$$
\begin{aligned}
&e^{-ik\omega y} c(y) u_k(y) = \\
&\quad = \frac{1}{2ik\omega} c(y) \left(e^{-2ik\omega y} \frac{d}{dy} \left(e^{ik\omega y} u_k(y) \right) - \frac{d}{dy} \left(e^{-ik\omega y} u_k(y) \right) \right) \\
&\quad = \frac{1}{2ik\omega} c(y) \left(e^{-ik\omega y} \left(f_k(y) - a(y) u_k(y) \right) - \frac{d}{dy} \left(e^{ik\omega y} u_k(y) \right) \right).
\end{aligned}
$$

Moreover, assumption $c \in BV(0,1)$ yields

$$
\left| \int_0^x e^{-\delta(y)} c(y) \frac{d}{dy} \left(e^{ik\omega y} u_k(y) \right) dy \right| \leq C \|u_k\|_{L^\infty(0,1)},
$$

the constant C being independent of x, h, k, and u_k. Using (3.9), it follows

$$
\left| \int_0^x e^{-ik\omega y - \delta(y)} c(y) u_k(y) \, dy \right| \leq \frac{C}{1 + |k|} \left(\|f_k\|_{L^2(0,1)} + \|g_k\|_{L^2(0,1)} \right)
$$

for some $C > 0$ not depending on x, h, k, f_k, and g_k. Similarly we proceed in the integrals in $w_k(bv_k, cu_k)$ (cf. (3.7)) in order to get

$$
|w_k(bv_k, cu_k)| \leq \frac{C}{1 + |k|} \left(\|f_k\|_{L^2(0,1)} + \|g_k\|_{L^2(0,1)} \right).
$$

Hence, (4.10) can be estimated by

$$
C \sum_{k \in \mathbb{Z}} (1 + k^2)^{\gamma - 1} \left| e^{2ik\omega h} - 1 \right|^2 \left(\|f_k\|^2_{L^2(0,1)} + \|g_k\|^2_{L^2(0,1)} \right)
$$

with some $C > 0$ not depending on x, h, k, f_k, and g_k. Using $|e^{2ik\omega h} - 1| \leq 2k\omega h$, we see that this tends to zero as $h \to 0$ uniformly with respect to $(f, g) \in N$.

Thus, M is precompact, i.e. $(BA^{-1})^2$ is compact. Hence $I + BA^{-1}$ is Fredholm, and therefore $A + B$ is Fredholm.

In order to finish the proof of assertion (ii) of Theorem 1, it remains to show that the index of $A + B$ is zero. This will be proved in the next section.

5 Fredholmness of index zero

Directly from the definitions of the operators A, $\tilde{A}$, B, and $\tilde{B}$ it follows

$$
\begin{aligned}
&\langle (A + B)(u, v), (\tilde{u}, \tilde{v}) \rangle = \langle (u, v), (\tilde{A} + \tilde{B})(\tilde{u}, \tilde{v}) \rangle \\
&\qquad \text{for all } (u, v) \in V^\gamma(r_0, r_1) \text{ and } (\tilde{u}, \tilde{v}) \in \tilde{V}^\gamma(r_0, r_1).
\end{aligned} \tag{5.1}
$$

Here

$$\langle (u,v),(\varphi,\psi)\rangle := \frac{\omega}{2\pi}\int\limits_0^{\frac{2\pi}{\omega}}\int\limits_0^1 (u\varphi + v\psi)\;dx\,dt = \sum_{k\in\mathbb{Z}}\int\limits_0^1 \left(u_k\overline{\varphi}_k + v_k\overline{\psi}_k\right)\;dx$$

is the usual scalar product in the Hilbert space $\left(H^{0,0}\right)^2 = L^2\left((0,1)\times\left(0,\frac{2\pi}{\omega}\right);\mathbb{R}^2\right)$.

In order to prove that the Fredholm operator $A+B$ has index zero, it sufficies to show that

$$\dim\ker(A+B) = \dim\ker(A+B)^*. \tag{5.2}$$

Here $(A+B)^*$ is the dual operator to $A+B$, i.e. a linear bounded operator from $(W^\gamma)^*$ into $(V^\gamma(r_0,r_1))^*$. Using the continuous dense embedding

$$\tilde{V}^\gamma(r_0,r_1)\hookrightarrow W^\gamma \hookrightarrow (H^{0,0})^2 \hookrightarrow (W^\gamma)^*,$$

it makes sense to compare the subspaces $\ker(A+B)^*$ of $(W^\gamma)^*$ and $\ker(\tilde{A}+\tilde{B})$ of $\tilde{V}^\gamma(r_0,r_1)$:

Lemma 9 $\ker(A+B)^* = \ker(\tilde{A}+\tilde{B})$.

Proof. Let $[\cdot,\cdot] : (W^\gamma)^*\times W^\gamma \to \mathbb{R}$ be the dual pairing on W^γ. Then for all $(u,v)\in V^\gamma(r_0,r_1)$ and $(\tilde{u},\tilde{v})\in \tilde{V}^\gamma(r_0,r_1)$ we have

$$\begin{aligned}\langle(\tilde{A}+\tilde{B})(\tilde{u},\tilde{v}),(u,v)\rangle &= \langle(\tilde{u},\tilde{v}),(A+B)(u,v)\rangle\\ &= [(\tilde{u},\tilde{v}),(A+B)(u,v)] = [(A+B)^*(\tilde{u},\tilde{v}),(u,v)]\,.\end{aligned} \tag{5.3}$$

Here we used (2.4) and (5.1). Obviously, (5.3) implies $\ker(\tilde{A}+\tilde{B})\subseteq\ker(A+B)^*$.

Now, take an arbitrary $(\varphi,\psi)\in\ker(A+B)^*$, and let us show that $(\varphi,\psi)\in\ker(\tilde{A}+\tilde{B})$. By Lemma 7, we have (using notation (2.6)) $\varphi = \sum_{k\in\mathbb{Z}}\varphi_k e_k$ and $\psi = \sum_{k\in\mathbb{Z}}\psi_k e_k$ with $\varphi_k,\psi_k\in L^2\left((0,1);\mathbb{C}\right)$ and

$$\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma\int_0^1\left(|\varphi_k(x)|^2+|\psi_k(x)|^2\right)dx<\infty.$$

It follows that for all $(u,v)\in V^\gamma(r_0,r_1)$

$$\begin{aligned}0 &= [(A+B)^*(\varphi,\psi),(u,v)] = [(\varphi,\psi),(A+B)(u,v)]\\ &= \sum_{k\in\mathbb{Z}}\int_0^1(\varphi_k(u'_{-k}+(a-ik\omega)u_{-k}+bv_{-k})+\psi_k(v'_{-k}-(d-ik\omega)v_{-k}+cu_{-k}))dx.\end{aligned}$$

Therefore

$$\varphi_k\left(u'_{-k}+(a-ik\omega)u_{-k}+bv_{-k}\right)+\psi_k\left(v'_{-k}-(d-ik\omega)v_{-k}+cu_{-k}\right)=0$$

for all $u_k, v_k \in H^1((0,1);\mathbb{C})$ with (3.3). By a standard argument, we conclude that $\varphi_k, \psi_k \in H^1((0,1);\mathbb{C})$ and that they satisfy the differential equations

$$-\varphi'_k+(a-ik\omega)\varphi_k+c\psi_k=0,\ \ \psi'_k+(d-ik\omega)\psi_k+b\varphi_k=0 \tag{5.4}$$

and the boundary conditions

$$\psi_k(0)=r_0\varphi_k(0),\ \varphi_k(1)=r_1\psi_k(1). \tag{5.5}$$

This yields, as in Section 3, that $(\varphi,\psi)\in\tilde{V}^\gamma(r_0,r_1)$ and $(\tilde{A}+\tilde{B})(\varphi,\psi)=0$. □

Lemma 9 implies that assertion (iii) of Theorem 1 is true. Hence, it remains to prove (5.2), i.e.

$$\dim\ker(A+B)=\dim\ker(\tilde{A}+\tilde{B}). \tag{5.6}$$

Lemma 10 *There exists $k_0\in\mathbb{N}$ such that for all $(u,v)\in\ker(A+B)$ and all $(\tilde{u},\tilde{v})\in\ker(\tilde{A}+\tilde{B})$ and all $k\in\mathbb{Z}$ with $|k|>k_0$ we have $u_k=v_k=\tilde{u}_k=\tilde{v}_k=0$.*

Proof. Suppose, contrary to our claim. Then there exists, for example, a sequence $(u^j,v^j)_{j\in\mathbb{N}}\in\ker(A+B)$ such that for all $j\in\mathbb{N}$ there is $k_j\in\mathbb{Z}$ with $|k_j|\ge j$ and $u^j_{k_j}\neq 0$ or $v^j_{k_j}\neq 0$. Without loss of generality we can assume that $k_j\neq k_l$ for $j\neq l$. Using the notation (2.6) again, we see that the functions $(u^j_{k_j}e_{k_j}, v^j_{k_j}e_{k_j})$ belong to $\ker(A+B)$ and are linearly independent. On the other side we know that $\ker(A+B)<\infty$, and this is a contradiction. □

Lemma 10 implies that

$$\ker(A+B)=\left\{\sum_{|k|\le k_0}(u_ke_k,v_ke_k):\ (u_k,v_k)\text{ solves (3.2), (3.3) with } f_k=g_k=0\right\},$$

$$\ker(\tilde{A}+\tilde{B})=\left\{\sum_{|k|\le k_0}(\varphi_ke_k,\psi_ke_k):\ (\varphi_k,\psi_k)\text{ solves (5.4), (5.5)}\right\}.$$

It is known that, given $k\in\mathbb{Z}$, the number of linearly independent solutions to (3.2), (3.3) with $f_k=g_k=0$ (this number is zero, one or two) equals to the number of linearly independent solutions to (5.4), (5.5). Hence, (5.6) is proved.

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