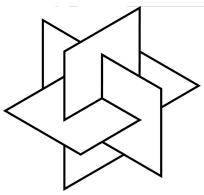

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AN INTERIOR POINT METHOD FOR A PARABOLIC OPTIMAL CONTROL PROBLEM WITH REGULARIZED POINTWISE STATE CONSTRAINTS

UWE PRÜFERT AND FREDI TRÖLTZSCH*

ABSTRACT. A primal-dual interior point method for state-constrained parabolic optimal control problems is considered. By a Lavrentiev type regularization, the state constraints are transformed to mixed control-state constraints which, after a simple transformation, can be handled as control constraints. Existence and convergence of the central path are shown. Moreover, the convergence of a short step interior point algorithm is proven in a function space setting. The theoretical properties of the algorithm are confirmed by numerical examples.

AMS SUBJECT CLASSIFICATION. 49M15, 49M37

KEY WORDS AND PHRASES. Parabolic optimal control, pointwise state constraints, Lavrentiev type regularization, interior point method

1. INTRODUCTION

In this paper, we extend our investigations on interior point methods for elliptic state-constrained optimal control problems in [18] and [14] to the parabolic case.

The main difficulty of the numerical analysis of interior point methods for such problems is the lack of regularity of Lagrange multipliers associated with the state constraints. Therefore, it is helpful to improve the properties of the multipliers have to be improved by suitable regularization techniques.

For instance, this task can be accomplished by discretization and subsequent application of interior point methods. We mention the work by Bergounioux et al. [2], who carefully compare the performance of primal-dual active set strategies and interior point methods for elliptic problems, Grund and Rösch [7], who handle such problems with maximum norm functional,

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and Maurer and Mittelmann [17], who solve several state-constrained elliptic control problems by standard interior point codes.

To consider the interior point algorithm in function space, we suggested in [18], [14] a Lavrentiev type regularization. The Lavrentiev regularization of elliptic problems was introduced in [15]. This method ensures regular Lagrange multipliers and preserves, in some sense, the structure of a state-constrained control problem. Moreover, compared with a direct application of interior point methods to state-constrained problems, the regularization improves the performance of the algorithm, [14].

In [24, 25], primal-dual interior point methods are analyzed for ODE problems in an infinite dimensional function space setting, and their computational realization by inexact path-following methods has been suggested. In [18], this method is extended to the optimal control of linear elliptic PDEs with regularized pointwise state constraints, where the analysis is performed in L^∞-spaces. Nonlinear equations are considered in the recent paper [22]. In particular, the convergence of primal-dual interior point methods is shown in L^p-spaces with $p < \infty$ for the control-constrained case.

Today, there exist also several papers on the numerical analysis of interior point methods for parabolic optimal control problems. For instance, trust-region interior point techniques were considered by M. Ulbrich, S. Ulbrich, and Heinkenschloss [23] for the optimal control of semilinear parabolic equations in a function space setting. Affine-scaling interior-point methods are presented for semilinear parabolic boundary control in [21]. Sachs and Leibfritz [12, 11, 10] considered interior point methods in the context of SQP-methods for parabolic optimization problems.

In our paper, we are able to prove the convergence of a conceptual primal interior point method in function space. We confine ourselves to a problem with linear equation and an objective functional with observation at the final time. This seems to be more challenging in the analysis than functionals of tracking type.

The analysis is very similar to the one for the elliptic case that was discussed in [18]. Therefore, we concentrate on those parts of the proofs that need essential modifications for parabolic problems. For parts of the theory that are completely analogous to elliptic problems, we refer to [18].

In the parabolic case, the presence of pointwise state constraints causes stronger restrictions on the dimension of the spatial domain than for elliptic equations. We do not impose control constraints. Therefore, the natural control space is of type L^2. To derive first-order necessary optimality conditions of Karush-Kuhn-Tucker type, the state functions should be continuous. This restricts the theory to distributed problems in one-dimensional domains.

This obstacle is completely overcome by our Lavrentiev regularization, which is crucial for the analysis. After regularization, we obtain Lagrange multipliers for any dimension of the domain. Moreover, we do not need constraint-qualifications. This remarkable advantage of our regularization is worth mentioning.

The paper is organized as follows: After defining our problem and introducing our main assumptions in Section 2, Section 3 is devoted to recall known results concerning the parabolic equation. In particular, we regard the properties of the control-to-state mapping.

In Section 4, we introduce the Lavrentiev type regularization. We motivate why the Lagrange multipliers are regular and show that the optimal control of the regularized problem converges towards the optimal control of the original problem. Section 5 is devoted to existence and convergence of the central path defined by the interior point method. In Section 6, we discuss the convergence of a simple interior-point algorithm in function space and finally, in Section 7, we confirm our theory by some numerical examples.

2. Problem setting

We consider the optimal control problem

$$\min J(y,u) = \frac{1}{2}\|y(T) - y_d\|_\Omega^2 + \frac{\kappa}{2}\|u\|_Q^2 \tag{1}$$

subject to the parabolic initial boundary value problem

$$\begin{aligned} y_t - \nabla\cdot(A\,\nabla y) + c_0 y &= u && \text{in } Q,\\ \partial_n y + \alpha y &= 0 && \text{in } \Sigma,\\ y(0) &= 0 && \text{in } \Omega, \end{aligned} \tag{2}$$

and to the pointwise state constraints

$$y_a(x,t) \le y(x,t) \le y_b(x,t) \quad \text{for all } (x,t)\in Q. \tag{3}$$

In this setting, $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with $C^{1,1}$-boundary Γ, and $(0,T)$ is a fixed time interval. We define $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$.

$A = (a_{ij}(x))$, $i,j = 1,...,N$, is a symmetric matrix with $a_{ij} \in C^{1,\gamma}(\Omega)$, $\gamma \in (0,1)$. It is assumed to satisfy the following condition of uniform ellipticity: There is an $m > 0$ such that

$$\lambda^\top A(x)\lambda \;\geq\; m|\lambda|^2 \quad \text{for all } \lambda \in \mathbb{R}^N \text{ and all } x \in \bar{\Omega}.$$

Moreover, functions $c_0 \in L^\infty(Q)$, $y_d \in L^\infty(\Omega)$ and y_a, y_b from $C(\bar{Q})$ are given that satisfy $y_a(x,t) < y_b(x,t)$ for all $(x,t) \in \bar{Q}$.

By the continuity of y_a and y_b, there is some $c_Q > 0$, such that it holds

$$y_b(x,t) - y_a(x,t) =\geq c_Q \qquad \forall (x,t) \in \bar{C}. \tag{4}$$

Notations: By $\|\cdot\|_{L^p(M)}$, $M \in \{Q, \Sigma, \Omega\}$ we denote the standard norm of $L^p(M)$. By $(\cdot,\cdot)_{L^2(M)}$ inner product of $L^2(M)$ is denoted. In $L^2(Q)$, the norm and the inner product are written without subscript, i.e. $\|\cdot\| := \|\cdot\|_{L^2(Q)}$ and $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(Q)}$ the associated inner product of $L^2(Q)$. We use $\|B\|_{V\to W}$ for the norm of a linear continuous operator $B : V \to W$. If $V = W = L^2(Q)$ we just write $\|B\|$. Throughout the paper, c is a generic positive constant. To shorten the notation, we write e.g. $B + \frac{\mu}{w-y_a}$ instead of $B + \frac{\mu}{w-y_a} I$, although B is an operator and $\frac{\mu}{w-y_a}$ is a function.

3. Some facts about the parabolic equation

In this section, we recall some known facts about the parabolic equation defined in (2). For the proof, we refer to [4] and [9], or to the survey in [20].

By $W(0,T)$, we denote the Hilbert space of functions $y \in L^2(0,T;V)$ with time derivative y' in $L^2(0,T;V^*)$, endowed with its standard norm, c.f. [13]. For the notion of a weak solution to (2) we refer to [9] or [13].

Theorem 3.1. *The control-to-state mapping $u \mapsto y$ is linear and continuous from $L^2(Q)$ to $W(0,T)$.*

With the linearity of the parabolic PDE, we can write $y = G_Q u$, where the control-to-state mapping $G_Q : L^2(Q) \to W(0,T)$ is continuous in view of Theorem 3.1. The mapping

$u \mapsto y(T)$, considered from $L^2(Q)$ to $L^2(\Omega)$, the "observation" of y at T, is denoted by S. Define $E_T : W(0,T) \to L^2(\Omega)$ by $E_T : y \mapsto y(T)$. Then S is given by $S = E_T G_Q$.

If we consider G_Q with range in $L^2(Q)$, then we denote this operator by G, i.e. $G = EG_Q$, where E is the embedding operator from $W(0,T)$ to $L^2(Q)$.

Corollary 3.2. *The mapping $S : u \mapsto y(T)$ is continuous from $L^2(Q)$ to $L^2(\Omega)$.*

Summarizing up, we have introduced the mappings

$$\begin{aligned} G_Q &: L^2(Q) \to W(0,T) \\ G &: L^2(Q) \to L^2(Q) \\ S &: L^2(Q) \to L^2(\Omega). \end{aligned}$$

Remark 3.3. Although we have fixed the spaces of L^2-type, where G and S are defined, we shall consider them also in other spaces without changing their notation, as in the next theorem.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$-domain and assume $f \in L^r(Q)$ with $r > N/2+1$, $g \in L^s(\Sigma)$ for $s > N+1$ and $y_0 \in C(\bar{\Omega})$. Then the weak solution y of*

$$\begin{aligned} y_t - \nabla(A\nabla y) + c_0 y &= f \qquad \text{in } Q \\ \partial_n y + \alpha y &= g \qquad \text{on } \Sigma \\ y(0) &= y_0 \qquad \text{in } \Omega \end{aligned}$$

belongs to $C(\bar{Q})$ and there is a constant independent of u, such that

$$\|y\|_{C(\bar{Q})} \leq c\left(\|f\|_{L^r(Q)} + \|g\|_{L^s(\Sigma)} + \|y_0\|_{C(\bar{\Omega})}\right).$$

Proof. We refer to [3], or [19], cf. also [20], Lemma 7.10. □

For a spatial dimension of $N = 2$, we need $r > 2$ and for $N = 3$ we need $r > 5/2$ to satisfy the assumptions.

Remark 3.5. We present the theory for homogeneous boundary data and zero initial value. Problems with fixed inhomogeneous data in the parabolic equation,

$$\begin{aligned} y_t - \nabla \cdot (A \nabla y) + c_0 y &= u + f && \text{in } Q \\ \partial_n y + \alpha y &= g && \text{in } \Sigma \\ y(0) &= y_0 && \text{in } \Omega, \end{aligned} \tag{5}$$

where $f \in L^r(Q)$, $r > N/2 + 1$, $g \in L^s(\Sigma)$, $s > N + 1$, and $y_0 \in C(\bar{\Omega})$ are given, can be easily transformed to a problem of type (1)–(3). This problem can be simplified in the following standard way:

With the linearity of the parabolic PDE, we can write $y = G_Q(u + f) + G_\Sigma g + G_0 y_0$. In view of Theorem 3.1, the linear operators G_Σ, G_0 are continuous from $L^2(\Sigma)$ to $W(0,T)$, and $L^2(\Omega)$ to $W(0,T)$, respectively. We obtain

$$y(T) = S\, u + \hat{y}(T),$$

where $\hat{y} = G_Q f + G_\Sigma g + G_0 y_0$.

Since $y = G_Q u + \hat{y}$, we have $y(T) - y_\Omega = Su - (y_\Omega - \hat{y}(T)) =: Su - y_d$. In the same way, the constraints can be transformed to $\alpha \le G_Q u \le \beta$ with $\alpha = y_a - \hat{y}$ and $\beta = y_b - \hat{y}$. If f and g satisfy the assumptions of the theorem above, then y_d remains bounded, while α, β remain continuous. Therefore, we can assume without loss of generality that y_0, f, and g are zero. On the other hand, by transforming back, the theory remains valid also for non-vanishing y_0, f, and g.

4. Mixed control-state constraints

In this section we consider the regularized optimal control problem

$$\min J(y,u) = \frac{1}{2}\|y(T) - y_d\|_\Omega^2 + \frac{\kappa}{2}\|u\|_Q^2 \tag{P}$$

subject to

$$\begin{aligned} y_t - \nabla \cdot (A \nabla y) + c_0 y &= u && \text{in } Q \\ \partial_n y + \alpha y &= 0 && \text{on } \Sigma \\ y(0) &= 0 && \text{in } \Omega \end{aligned}$$

and to the mixed (ε-regularized) control-state constraints

$$y_a \le y + \varepsilon u \le y_b \quad \text{a.e. in } Q. \tag{6}$$

We are able to show that the optimal control u_ε of this problem tends in $L^2(Q)$ to the solution $\bar{u}$ of the original problem, provided that a Slater type condition is satisfied for the original one. The method of proof is analogous to the one in Hintermüller et al. [8]. We do not prove this result, since we aim at concentrating on the interior point method for problem (P) rather than to discuss the relation with the unregularized problem. Following [15], we transform the mixed control-state constraints into control constraints. Using the operators introduced in Section 3, we can write

$$y + \varepsilon u = Gu + \varepsilon u = (G + \varepsilon\, I)u.$$

We introduce $w := y + \varepsilon u$ as a new auxiliary control. Then $u = D\, w$, where $D : L^2(Q) \to L^2(Q)$ is defined by

$$D = (G + \varepsilon\, I)^{-1}. \tag{7}$$

D is well defined, as the next result shows:

Lemma 4.1. *For all $\varepsilon \neq 0$, the operator D exists and is continuous in $L^2(Q)$.*

Proof. First we show that the kernel of $G + \varepsilon I$ is trivial. To see this, consider the equation

$$Gu + \varepsilon u = 0.$$

This is equivalent to $u = G(-\varepsilon^{-1}u)$. By the definition of G, u solves the system

$$\begin{aligned} u_t - \Delta u - c_0 u &= -\frac{1}{\varepsilon} u \\ \partial_n u + \alpha u &= 0 \\ u(0) &= 0. \end{aligned}$$

By taking $(-1/\varepsilon)u$ to the other side of the equation we see that u solves a homogeneous initial-boundary value problem that has only the trivial solution.

It remains to show that $\varepsilon u + Gu$ is surjective. Then the Banach theorem on the inverse operator ensures the continuity of $D = (\varepsilon u + Gu)^{-1}$. Let $w \in L^2(Q)$ be given arbitrarily and consider the equation

$$\varepsilon u + Gu = w.$$

To solve it, we consider the equation

$$\begin{aligned} y_t - \Delta y - c_0 y &= \frac{1}{\varepsilon}(w - y) \\ \partial_n y + \alpha y &= 0 \\ y(0) &= 0. \end{aligned} \tag{8}$$

Taking $-\frac{1}{\varepsilon}y$ to the other side, we see that this equation has a unique solution $y \in W(0,T)$. Now we define

$$u := \frac{1}{\varepsilon}(w - y). \tag{9}$$

Then we have $y = Gu$ and hence

$$u = \frac{1}{\varepsilon}(w - Gu).$$

Obviously, this u solves the equation $\varepsilon u + Gu = w$ and we have shown the surjectivity. □

4.1. **Regular Lagrange multipliers.** By the technique used in [14] for an elliptic problem, we will show the existence of regular multipliers. We do not directly need this result for our convergence analysis. However, it shows how the regularization helps to construct a problem with better properties. In particular, this explains why our numerical method does not have to deal with measures as multipliers. First of all, we transform problem (P) with mixed control-state constraints (6) in a control-constrained problem with new control $w := D^{-1}u$. With G, S, and D, we transform problem (P) to one depending on the control w as

$$\min F(w) = \frac{1}{2}\|SDw - y_d\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|Dw\|^2 \tag{10}$$

subject to

$$y_a \le w \le y_b \text{ a.e. in } Q. \tag{11}$$

This transformation of our control problem (P) will be used for the analysis of the interior point algorithm, while all computations are performed with the original form of (P).

The functional F is continuously Fréchet-differentiable on $L^2(Q)$. Its Fréchet derivative is represented by

$$F'(w)v = ((SD)^*(SDw - y_d), v) + \kappa\,(D^*Dw, v)\,.$$

We can identify it with the function

$$g := (SD)^*(SDw - y_d) + \kappa D^*Dw \in L^2(Q),$$

the Riesz representation of the derivative. Using the same arguments as in [14, 15], we define Lagrange multipliers by η_a and $\eta_b \in L^2(Q)$ by

$$\begin{aligned} \eta_a(x,t) &= g(x,t)_+ \\ \eta_b(x,t) &= g(x,t)_- \end{aligned}$$

so that $g = g_+ - g_- = \eta_a - \eta_b$.

Remark 4.2. In all what follows, a bar as in $\bar{u}$, $\bar{y}$, or $\bar{w}$ etc. indicates optimality.

The optimal solution $\bar{w}$ fulfills, together with η_a and η_b, the following necessary and (by convexity) sufficient optimality conditions:

$$S^*(SD\bar{w} - y_d) + \kappa D\bar{w} - D^{-*}\eta_a + D^{-*}\eta_b = 0, \tag{12}$$

together with the complementary conditions

$$\begin{cases} (\eta_a, \bar{w} - y_a) = (\eta_b, y_b - \bar{w}) = 0, \\ \eta_a(x,t) \geq 0,\ \eta_b(x,t) \geq 0, \ \text{ a.e. in } Q, \\ \bar{w}(x,t) - y_a(x,t) \geq 0,\ y_b(x,t) - \bar{w}(x,t) \geq 0 \text{ a.e. in } Q. \end{cases} \tag{13}$$

Following the same steps as in [16], η_a, η_b are verified to be the Lagrange multipliers associated with the mixed constraints (6).

4.2. **Transformation in terms of PDEs.** By $D^{-1} = \varepsilon I + G$ we can write (12) as follows

$$S^*(SD\bar{w} - y_d) + \kappa D\bar{w} + \varepsilon(\eta_b - \eta_a) + G^*(\eta_b - \eta_a) = 0.$$

Re-substituting $D\bar{w} = \bar{u}$, and defining an adjoint state p by

$$p = G^*(\eta_b - \eta_a) + S^*(S\bar{u} - y_d), \tag{14}$$

we obtain the optimality conditions

$$\bar{y} = G\bar{u}\,, \tag{15}$$

$$p + \kappa\bar{u} = \varepsilon(\eta_a - \eta_b), \tag{16}$$

together with the complementary conditions (13), where we re-substitute $\bar{w} := \varepsilon\bar{u} + \bar{y}$. The adjoint state p defined by (14) is the unique solution of the following adjoint equation:

$$\begin{aligned} -p_t - \nabla\cdot(A\,\nabla p) &= \eta_b - \eta_a && \text{in } Q, \\ \partial_n p + \alpha p &= 0 && \text{on } \Sigma, \\ p(T) &= \bar{y}(T) - y_d && \text{in } \Omega. \end{aligned} \tag{17}$$

This can be easily confirmed, cf. [20], Section 3.6.4. The adjoint equation has, for every pair a unique solution $p \in W(0,T)$. It holds

$$\|p\|_{W(0,T)} \le c_W \left(\|\eta_b - \eta_a\|_{L^2(Q)} + \|y(\bar{T}) - y_d\|_{L^2(\Omega)} \right)$$

with some c_w not depending on the given data.

This follows from Theorem 3.1 after the transformation of time $\tau := T - t$.

Remark 4.3. The case $\varepsilon = 0$ is formally covered by the optimality system (13)–(16), too. Here, possibly, $\eta_a(0)$, $\eta_b(0)$ belong to $\mathcal{M}(\bar{Q})$, the space of regular Borel measures defined at $\bar{Q}$. Then equation (17) is a parabolic PDE with measures on the right-hand-side, which may even appear in the boundary condition, we refer to Casas [4]. In this case, our theory does not work, since the operator D is unbounded and not defined on the whole space $L^2(Q)$.

In summary, we have derived the following theorem:

Theorem 4.4. *For all $\varepsilon \neq 0$, problem (P) has a unique optimal control $\bar{u}_\varepsilon$ with associated state $\bar{y}_\varepsilon$. There exist non-negative Lagrange multipliers $\eta_a \in L^2(Q)$ and $\eta_b \in L^2(Q)$ and an associated adjoint state $p \in W(0,T)$, such that the optimality system* (13)–(16) *is satisfied.*

Remark 4.5. The existence of the optimal control follows in particular from the fact that the equation $\varepsilon u + Gu = y_a$ is solvable for all nonzero ε. Therefore, the admissible set is never empty.

Due to the convexity of the objective functional F, the necessary optimality conditions are also sufficient for optimality.

5. Interior-point method in function space

By the interior point method, the constrained problem (10)–(11) is transformed into a formally unconstrained problem by adding a logarithmic penalty term to the objective functional F. In this section, we show that the transformed problems are solvable and that the associated *central path* exists.

In terms of PDE, the problem (P) is converted to the following one:

$$\begin{aligned}\min J_\mu(y,u) := &\frac{1}{2}\|y(T)-y_d\|_\Omega^2 + \frac{\kappa}{2}\|u\|_Q^2 \\ &- \mu \iint_Q \ln(y+\varepsilon u - y_a) + \ln(y_b - \varepsilon u - y)\, dxdt\end{aligned}$$

subject to the equation (2).

Let us first state the associated necessary optimality conditions. In a standard, but slightly formal way, we obtain the adjoint equation

$$\begin{aligned} -p_t - \nabla\cdot(A\nabla p) + c_0 p &= -\frac{\mu}{y+\varepsilon u - y_a} + \frac{\mu}{y_b - \varepsilon u - y} && \text{in } Q, \\ \partial_n p + \alpha p &= 0 && \text{on } \Sigma, \\ p(T) &= y(T) - y_d && \text{in } \Omega, \end{aligned} \tag{18}$$

and the gradient equation

$$p + \kappa u - \frac{\varepsilon\mu}{y+\varepsilon u - y_a} + \frac{\varepsilon\mu}{y_b - \varepsilon u - y} = 0 \quad \text{a.e in } Q. \tag{19}$$

The solution (u_μ, y_μ, p_μ), if it exists, is expected to converge to the solution of Problem (P). We prove the existence of the solution and the optimality conditions by considering the problem

(10)–(11), i.e.,

$$\text{(Q)} \qquad \min F_\mu(w) = \frac{1}{2}\|SDw - y_d\|_\Omega^2 + \frac{\kappa}{2}\|Dw\|^2 \\ -\mu \iint_Q \ln(w - y_a) + \ln(y_b - w)\, dx\, dt,$$

where $\mu > 0$ is a path parameter that will tend to zero. This is a formally unconstrained problem, but the logarithmic penalty term can only be well defined for $w \in L^2(Q)$ with $y_a < w < y_b$ a.e. in Q. Therefore, the admissible set of (Q) is open in some sense. Notice that $F_\mu(w)$ is a convex functional.

To prove the existence of a solution of problem (Q), we apply a method that has been introduced in [18]. It considers the minimization of F_μ in a closed subset and, at the same time, finally permits to show that the solution w_μ has some positive distance to the bounds: We have $y_a + \tau \le w_\mu \le y_b - \tau$ for some sufficiently small $\tau > 0$ that depends on μ.

5.1. **Existence.** For fixed $\tau > 0$, we consider the auxiliary problem

$$\text{(Aux)} \qquad \min_{y_a+\tau \le w \le y_b-\tau} F_\mu(w).$$

In contrast to (Q), the admissible set of this problem is closed.

We define the following admissible sets:

$$W := \{w \in L^2(Q) \,|\, y_a \le w \le y_b \text{ a.e. in } Q\}, \tag{20}$$

$$W_\tau := \{w \in L^2(Q) \,|\, y_a + \tau \le w \le y_b - \tau \text{ a.e. in } Q\}. \tag{21}$$

Theorem 5.1. *For every $0 < \tau < c_Q/3$, and for all $\mu > 0$, problem (Aux) has a unique solution $w_{\tau,\mu}$. There is a bound c not depending on τ and μ such that it holds $\|w_{\tau,\mu}\|_{L^\infty(Q)} \le c$.*

Proof. It is clear that W_τ is non-empty, convex, closed and bounded. F_μ is strictly convex and continuous on W_τ, and hence weakly lower semicontinuous. Therefore, standard arguments show the existence of a unique solution of (Aux). The uniform boundedness of the solution is an obvious consequence of the boundedness of $W_\tau \subset W$ in $L^\infty(Q)$. □

In the case of one-sided constraints $y + \varepsilon u \leq y_c$ or $y_c \leq y + \varepsilon u$, Theorem 5.1 cannot be shown in this way, since the associated W_τ is not bounded. Here, the following Lemma applies that can be shown completely analogous as Lemma 3.2 in [18].

Lemma 5.2. *For all $\mu \geq 0$, it holds that $F_\mu(w) \to \infty$ if $\|w\| \to \infty$ and $w \leq y_c$ or $w \geq y_c$, respectively.*

The function F_μ is directionally differentiable at $w_{\tau,\mu}$ in all directions $w - w_\tau$ with $w \in W_\tau$. The optimality of $w_{\tau,\mu}$ gives

$$F'_\mu(w_{\tau,\mu})(w - w_{\tau,\mu}) \quad \geq \quad 0 \quad \forall\, w \in W_\tau,$$

where F'_μ denotes the directional derivative of F_μ. According to the definition of F_μ, we obtain the variational inequality

$$(g_{\tau,\mu}, w - w_{\tau,\mu})_Q \quad \geq \quad 0 \tag{22}$$

for all $w \in W_\tau$, where the function $g_{\varepsilon,\mu} \in L^2(Q)$ is defined by

$$g_{\tau,\mu} := (SD)^*(SDw_{\tau,\mu} - y_d) + \kappa D^* D w_{\tau,\mu} - \frac{\mu}{w_{\tau,\mu} - y_a} + \frac{\mu}{y_b - w_{\tau,\mu}}. \tag{23}$$

Next, we define two auxiliary functions, namely

$$p_{\tau,\mu} := (SD)^*(SDw_{\tau,\mu} - y_d) \quad \text{and} \quad \theta_{\tau,\mu} := \kappa D^* D w_{\tau,\mu}.$$

We show that they are bounded in $L^\infty(Q)$, uniformly with respect to τ and μ. To this aim, we need the following result.

Lemma 5.3. *The operators D and D^* are continuous in $L^\infty(Q)$ for any dimension N.*

Proof. To find $u = Dw$, we have to solve the equation $\varepsilon u + Gu = w$. In view of (8) and (9), this is equivalent to the following two steps: We solve first (8) to find y. Next, we obtain u by formula (9). Thanks to Theorem 3.4, the mapping $w \mapsto y$ is linear and continuous in $L^\infty(Q)$. Therefore, the same holds true for the mapping $u \mapsto \varepsilon^{-1}(w - y(w))$. This shows the continuity of D.

The proof for D^* is analogous, since G^* is related to an adjoint parabolic equation that has the same properties as equation (8). □

The following Lemma asserts the L^∞-boundedness of p and θ.

Lemma 5.4. *There is a positive constant $c_{p,\theta}$ such that*

$$\|p_{\tau,\mu}\|_{L^\infty} + \|\theta_{\tau,\mu}\|_{L^\infty} \leq c_{p,\theta}$$

holds true for all $0 < \tau < c_Q/3$ and all $\mu > 0$.

Proof. We have

$$\|p_{\tau,\mu}\|_{L^\infty(Q)} + \|\theta_{\tau,\mu}\|_{L^\infty(Q)} \leq \|(SD)^*(SDw - y_d)\|_{L^\infty(Q)} + \kappa\|D^*Dw\|_{L^\infty(Q)}.$$

In view of Theorem 3.4 and Lemma 5.3, all operators appearing in this formula are continuous in L^∞-spaces on associated domains. Moreover, we have assumed that $y_d \in L^\infty(\Omega)$. Therefore, the result of the Lemma is an immediate conclusion. □

The main result of this section, the existence of the central path, can be shown completely analogous to the elliptic case discussed in [18]. Nevertheless, we briefly sketch the proof for convenience of the reader. To this aim, we define the sets

$$\begin{aligned} M_+(\tau,\mu) &:= \{(x,t) \in Q \,|\, g_{\tau,\mu}(x,t) > 0\}, \\ M_-(\tau,\mu) &:= \{(x,t) \in Q \,|\, g_{\tau,\mu}(x,t) < 0\}, \\ M_0(\tau,\mu) &:= \{(x,t) \in Q \,|\, g_{\tau,\mu}(x,t) = 0\}. \end{aligned}$$

Lemma 5.5. *For all $\mu > 0$, there are positive numbers $\tau_+(\mu)$ and $\tau_-(\mu)$ such that, for all $0 < \tau < \tau(\mu) := \min\{\tau_+(\mu), \tau_-(\mu)\}$, the sets $M_+(\tau)$ and $M_-(\tau)$ have measure zero.*

Proof. A standard evaluation of (22) yields for almost all $(x,t) \in Q$ that

$$w_{\tau,\mu}(x,t) = \begin{cases} y_a(x,t) + \tau, & (x,t) \in M_+(\tau,\mu) \\ y_b(x,t) - \tau, & (x,t) \in M_-(\tau,\mu). \end{cases}$$

Almost everywhere on $M_+(\tau,\mu)$, Lemma 5.4 implies

$$\begin{aligned} 0 \;<\; g_{\tau,\mu}(x,t) \;&=\; p_{\tau,\mu}(x,t) + \theta_{\tau,\mu}(x,t) - \frac{\mu}{\tau} + \frac{\mu}{y_b(x,t) - y_a(x,t) - \tau} \\ &\leq\; c_{p,\theta} - \frac{\mu}{\tau} + 2\frac{\mu}{c_Q}. \end{aligned}$$

For $\tau \downarrow 0$, the right hand side tends to $-\infty$, a contradiction for all sufficiently small $\tau > 0$, say $\tau < \tau_+(\mu)$. Consequently, $M_+(\tau, \mu)$ is of measure zero for these τ. Analogously, $M_-(\tau, \mu)$ can be handled. □

Now we can formulate the main result of this section.

Theorem 5.6. *For all $\mu > 0$ and all $0 < \tau < \tau(\mu) = \min\{\tau_+(\mu), \tau_-(\mu)\}$, the solution $w_{\tau,\mu}$ of* (Aux) *is the unique solution w_μ of problem* (Q).

Proof. Since $Q = M_0(\tau) \cup M_+(\tau) \cup M_-(\tau)$ and the set $M_+(\tau) \cup M_-(\tau)$ has measure zero for $0 < \tau < \min\{\tau_+(\mu), \tau_-(\mu)\}$, we have

$$g_{\tau,\mu}(x,t) \;=\; 0 \quad \text{a.e. in } Q.$$

Therefore, it holds that

$$F'_\mu(w_{\tau,\mu})h \;=\; \iint_Q g_{\tau,\mu}(x,t)\, h(x,t)\, dxdt = 0 \quad \forall h \in L^2(Q),$$

and hence $w_{\tau,\mu}$ satisfies the necessary optimality condition for problem (Q). By convexity, the necessary conditions are sufficient for optimality. Strong convexity yields uniqueness (notice that $\kappa > 0$). In view of this, $w_{\tau,\mu}$ is the unique solution w_μ of (Q). □

Corollary 5.7. *For all $\mu > 0$, the solution w_μ of (Q) satisfies*

$$y_a(x,t) + \tau(\mu) \le w_\mu(x,t) \le y_b(x,t) - \tau(\mu), \qquad \textit{a.e. in } Q. \tag{24}$$

where $\tau(\mu) > 0$ is given by Theorem 5.6.

Proof. By Theorem 5.6, we have $w_{\tau,\mu} = w_\mu$ for all $0 < \tau < \tau(\mu)$. By the definition of (Aux), $w_{\tau,\mu}$ satisfies

$$y_a(x,t) + \tau(\mu) \le w_\mu(x,t) \le y_b(x,t) - \tau(\mu)$$

for all $\tau < \tau(\mu)$, i.e., inequality (24) is satisfied. □

After having solved the problem of existence, let us verify and re-formulate the optimality conditions (18)–(19). We denote by u_μ the optimal control with state y_μ given by $\varepsilon u_\mu + y_\mu = w_\mu$.

The associated adjoint state is p_μ. Define $\eta_{a,\mu}$ and $\eta_{b,\mu}$ by

$$\eta_{a,\mu} = \frac{\mu}{y_\mu + \varepsilon u_\mu - y_a}, \quad \eta_{b,\mu} = \frac{\mu}{y_b - \varepsilon u_\mu - y_\mu}. \tag{25}$$

Multiplying (23) by $(D^*)^{-1} = (\varepsilon I + G^*)$, we obtain in view of $Dw_\mu = u_\mu$ and $Su_\mu = y_\mu$ that

$$S^*(y_\mu(T) - y_d) + \kappa u + \varepsilon(\eta_{b,\mu} - \eta_{a,\mu}) + G^*(\eta_{b,\mu} - \eta_{a,\mu}) = 0. \tag{26}$$

We set

$$p_\mu := S^*(y_\mu(T) - y_d) + G^*(\eta_{b\mu} - \eta_{a,\mu}). \tag{27}$$

Then, analogous to (17), p_μ, y_μ, u_μ solve the adjoint equation (18). Moreover, (26) becomes

$$p_\mu + \kappa u + \varepsilon(\eta_{b,\mu} - \eta_{a,\mu}) = 0.$$

This is equivalent to (19). Summarizing up, we get the optimality system

$$\begin{aligned} y_t - \nabla(A\,\nabla y) + c_0 y &= u \text{ in } Q, \\ \partial_n y + \alpha y &= 0 \text{ on } \Sigma, \\ y(0) &= 0 \text{ in } \Omega, \end{aligned} \tag{28}$$

$$\begin{aligned} -p_t - \nabla(A\,\nabla p) + c_0 y &= -\eta_{a,\mu} + \eta_{b,\mu} && \text{in } Q, \\ \partial_n p + \alpha p &= 0 && \text{on } \Sigma, \\ p(T) &= y - y_d && \text{in } \Omega, \end{aligned} \tag{29}$$

$$p + \kappa u + \varepsilon(-\eta_{a,\mu} + \eta_{b,\mu}) = 0 \text{ a.e. in } Q, \tag{30}$$

$$\begin{aligned} \eta_{a,\mu} \geq 0,\ y + \varepsilon u - y_a \geq 0,\ \eta_{a,\mu}(y + \varepsilon u - y_a) &= \mu \quad \text{a.e. in } Q, \\ \eta_{b,\mu} \geq 0,\ y_b - \varepsilon u - y \geq 0,\ \eta_{b\mu}(y_b - \varepsilon u - y) &= \mu \quad \text{a.e. in } Q. \end{aligned} \tag{31}$$

Notice that (25) can be rewritten as $\mu = \eta_{a,\mu}(y_\mu + \varepsilon u_\mu - y_a)$, $\mu = \eta_{b,\mu}(y_b - y_\mu - \varepsilon u_\mu)$.

5.2. **Convergence.** In Section 5.1 we established the existence of the central path $\mu \mapsto w_\mu$ for all fixed $\mu > 0$. Now we proceed with proving the continuity of the mapping $\mu \mapsto w_\mu$ and the

convergence towards a solution $\bar{w}$ of (10)–(11). We implicitly differentiate the function F with respect to μ and estimate the arc length of the central path in the L^∞-Norm.

The unique minimizer w_μ of (Q) is the solution of

$$\begin{aligned} H(w_\mu;\mu) &:= (SD)^*(SDw_\mu - y_d) + \kappa D^*Dw_\mu - \frac{\mu}{w_\mu - y_a} + \frac{\mu}{y_b - w_\mu} \\ &= (D^*S^*SD + \kappa D^*D)w_\mu - D^*S^*y_d - \frac{\mu}{w_\mu - y_a} + \frac{\mu}{y_b - w_\mu} \\ &= 0. \end{aligned}$$

By Corollary 5.7, we have $w_\mu - y_a \geq \tau(\mu)$ and $y_b - w_\mu \geq \tau(\mu)$ for all sufficiently small $\mu > 0$. H is Fréchet-differentiable in all directions $w \in L^\infty(Q)$ for all $\mu > 0$. Let $\partial_\mu H$ denote the derivative of H with respect to μ and let $\partial_w H$ be the derivative of H with respect to w. The derivative $\partial_w H$ is

$$\partial_w H(w;\mu) = D^*S^*SD + \kappa D^*D + \frac{\mu}{(w - y_a)^2} + \frac{\mu}{(y_b - w)^2}. \tag{32}$$

It satisfies the estimate

$$\begin{aligned} (v, \partial_w H(w;\mu)v) &= (SDv, SDv)_\Omega + \kappa\,(Dv, Dv) \\ &\quad + \mu\left(\frac{v}{(w - y_a)}, \frac{v}{(w - y_a)}\right) \\ &\quad + \mu\left(\frac{v}{(y_b - w)}, \frac{v}{(y_b - w)}\right) \geq \kappa \frac{1}{\|D^{-1}\|^2}\|v\|^2. \end{aligned}$$

By Lemma 5.3, $\partial_w H$ is continuous in $L^\infty(Q)$ for all $w \in L^\infty(Q)$ with $y_a \leq w \leq y_b$ a.e. in Q. We show the boundedness of the inverse $(\partial_w H)^{-1}$ in $L^\infty(Q)$.

Theorem 5.8. *For all $\mu > 0$, the mapping $\partial_w H(w;\mu) : L^\infty(Q) \to L^\infty(Q)$ is a bijection. Its inverse is uniformly bounded for all $\mu > 0$; i.e. there exists a $c_{inv} > 0$ such that*

$$\|\partial_w H(w;\mu)\|_{L^\infty \to L^\infty} \leq c_{inv} \text{ for all } \mu > 0 \text{ and for all } w \in W.$$

The proof is the same as the one for Lemma 4.1 in [18], cf. the argumentation there.

Theorem 5.9. *For $\mu \to 0$, w_μ converges towards the solution $\bar{w}$ of (10)–(11). There is a constant $c > 0$, such that*

$$\|w_\mu - \bar{w}\|_{L^\infty(Q)} \quad \leq \quad c\sqrt{\mu}$$

holds for all sufficiently small μ.

The proof is analogous to the one of Theorem 4.3 in [18] for the elliptic case with a bilateral constraint. However it is more technical in view of our bilateral constraints. Therefore, we present the full proof for convenience of the reader.

Proof. First, we establish an L^2- estimate of $\partial_\mu H$, then we infer an L^∞-bound from that. In a third step, we estimate a bound for the distance between two points on the central path. Finally, we check the first order optimality conditions.

The derivative of w_μ with respect to μ is given by

$$w_\mu^{'} \quad = \quad -\partial_w H(w;\mu)^{-1}\partial_\mu H(w;\mu), \tag{33}$$

with

$$\partial_w H(w;\mu)^{-1} = \left(D^*S^*SD + \kappa D^*D + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1}$$

and

$$\partial_\mu H(w;\mu) = -\frac{1}{w - y_a} + \frac{1}{y_b - w}. \tag{34}$$

In preparation of the proof, we introduce the following operator splitting. From $D = \varepsilon^{-1}(I - DG)$ and $D^* = \varepsilon^{-1}(I - G^*D^*)$ we get

$$D^*D \quad = \quad \frac{1}{\varepsilon^2}(1 + G^*D^*DG - (G^*D^* + DG)). \tag{35}$$

Let us define the operator K as

$$K \quad := \quad D^*S^*SD + \frac{\kappa}{\varepsilon^2}(G^*D^*DG - (G^*D^* + DG)). \tag{36}$$

Now we can write (32) as

$$\partial_w H(w_\mu;\mu) \quad = \quad K + \frac{\kappa}{\varepsilon^2}I + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}. \tag{37}$$

(i) *L^2-bound.* We introduce the diagonal preconditioning operator

$$\phi(\mu) = \sqrt{\frac{\kappa}{\varepsilon^2} + \mu\frac{(w_\mu - y_a)^2 + (y_b - w_\mu)^2}{((w_\mu - y_a)(y_b - w_\mu))^2}}. \tag{38}$$

It is clear, that $\phi(\mu) > \underline{c} = \frac{\sqrt{\kappa}}{\varepsilon} > 0$. From Corollary 5.7 we know, that there is some constant $\bar{c} = \bar{c}(\mu) = \sqrt{\frac{\kappa}{\varepsilon^2} + c\frac{\mu}{\tau(\mu)^4}} > 0$, $c = c(y_a, y_b)$, such that for fixed μ the function $\phi(\mu)$ is bounded: $0 < \underline{c} \le \phi(\mu) \le \bar{c} < \infty$. Pre-conditioning w'_μ by (38), we get

$$\begin{aligned} \phi(\mu)w'_\mu &= \left(\phi(\mu)^{-1}\left(K + \frac{\kappa}{\varepsilon^2} + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)\phi(\mu)^{-1}\right)^{-1} \\ &\quad \cdot \phi(\mu)^{-1}\left(\frac{1}{w - y_a} - \frac{1}{(y_b - w}\right) \\ &= (\phi(\mu)^{-1}K\phi(\mu)^{-1} + I)^{-1}\phi(\mu)^{-1}\left(\frac{1}{w - y_a} - \frac{1}{y_b - w}\right). \end{aligned} \tag{39}$$

Obviously, $\phi(\mu)$ satisfies

$$0 < \frac{\sqrt{\kappa}}{\varepsilon} \le \phi(\mu) \quad \text{and} \quad 0 < \sqrt{\mu}\frac{\sqrt{(w - y_a)^2 + (y_b - w)^2}}{(w - y_a)(y_b - w)} \le \phi(\mu),$$

hence $\phi(\mu)^{-1} < \dfrac{\varepsilon}{\sqrt{\kappa}}$ and $\phi(\mu)^{-1} \le \dfrac{1}{\sqrt{\mu}}\dfrac{(w - y_a)(y_b - w)}{\sqrt{(w - y_a)^2 + (y_b - w)^2}}$. We estimate

$$\begin{aligned} \left\|\phi(\mu)^{-1}\frac{\mu}{w - y_a} - \frac{\mu}{y_b - w}\right\| &= \left\|\phi(\mu)^{-1}\frac{(y_b - w) - (w - y_a)}{(w - y_a)(y_b - w)}\right\| \\ &\le \left\|\frac{1}{\sqrt{\mu}}\sqrt{\frac{(w - y_a)^2(y_b - w)^2}{(w - y_a)^2 + (y_b - w)^2}} \cdot \frac{(y_b - w) - (w - y_a)}{(w - y_a)(y_b - w)}\right\| \\ &\le \frac{1}{\sqrt{\mu}}\left\|\sqrt{\frac{(w - y_a)^2(y_b - w)^2}{(w - y_a)^2 + (y_b - w)^2}} \cdot \frac{(w - y_a) + (y_b - w)}{(w - y_a)(y_b - w)}\right\| \\ &= \frac{1}{\sqrt{\mu}}\left\|\frac{(w - y_a) + (y_b - w)}{\sqrt{(w - y_a)^2 + (y_b - w)^2}}\right\|. \end{aligned}$$

Since $\phi^{-1}K\phi^{-1} + I$ is a positive definite operator (in fact, it is nothing else then $\phi^{-1}\partial_w H(w;\mu)\phi^{-1}$), we have $\|(\phi^{-1}K\phi^{-1} + 1)^{-1}\| \le c_K|Q|$ with some $c_K > 0$ and

$$\frac{(w - y_a) + (y_b - w)}{\sqrt{(w - y_a)^2 + (y_b - w)^2}} < \sqrt{2}$$

holds for all $w - y_a > 0$, $y_b - w > 0$ a.e. in Q , we may estimate

$$\|\phi(\mu)w_\mu^{'}\| \leq c_K\sqrt{2}|Q|\frac{1}{\mu}$$

and finally

$$\begin{aligned} \|w_\mu^{'}\| &= \|\phi^{-1}(\mu)\phi(\mu)w_\mu^{'}\| \leq \|\phi^{-1}(\mu)\|\|\phi(\mu)w_\mu^{'}\| \\ &\leq \frac{\varepsilon c_K\sqrt{2}|\,Q\,|}{\sqrt{\kappa}}\frac{1}{\sqrt{\mu}} = c\frac{1}{\sqrt{\mu}}. \end{aligned} \tag{40}$$

(ii) *L^∞-bound.* Using the splitting (36), we get from (33) the representation

$$\begin{aligned} w_\mu^{'} &= \left(\frac{\kappa}{\varepsilon^2}I + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1}\left(\frac{1}{w_\mu - y_a} - \frac{1}{y_b - w_\mu}\right) \\ &\quad - \left(\frac{\kappa}{\varepsilon^2}I + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1} Kw_\mu^{'}. \end{aligned} \tag{41}$$

We define the set

$$\hat{Q} := \{(x,t) \in Q \,|\, y_a(x,t) + \tau(\mu) \leq w_\mu(x,t) \leq y_b(x,t) - \tau(\mu)\}.$$

Obviously, $\hat{Q} \subset Q$ and by Corollary 5.7 $\{Q\backslash\hat{Q}\}$ has measure zero. In the following, we use the Young inequality in the form $ax + \frac{b}{x} \geq 2\sqrt{ab}$.

Let

$$\lambda = \lambda(x,t) := \begin{cases} \max\left\{\dfrac{1}{w_\mu(x,t) - y_a}, \dfrac{1}{y_b - w_\mu(x,t)}\right\} & (x,t) \in \hat{Q} \\ 0 & \text{else .} \end{cases}$$

By its definition, it holds $\lambda \in L^\infty(Q)$ and

$$\lambda(x,t) \geq \frac{1}{2}\left(\frac{1}{w_\mu(x,t) - y_a} + \frac{1}{y_b - w_\mu(x,t)}\right) \text{ a.e. in } Q.$$

We define a splitting of Q by

$$Q_a := \left\{(x,t) \in Q \,|\, \lambda(x,t) = \frac{1}{w_\mu(x,t) - y_a}\right\}, \tag{42}$$

$$Q_b := \hat{Q}\backslash Q_a, \tag{43}$$

e.a $\lambda = \frac{1}{y_b - w}$ on Q_b. Furthermore, we have

$$\left(\frac{\kappa}{\varepsilon^2} I + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1} \leq \left(\frac{\kappa}{\varepsilon^2} I + \frac{\mu}{(w_\mu - y_a)^2}\right)^{-1}$$

and

$$\left(\frac{\kappa}{\varepsilon^2} I + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1} \leq \left(\frac{\kappa}{\varepsilon^2} I + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1}$$

a.e in Q. We will estimate $\|w_\mu^{'}\|_{L^\infty(Q)}$ on Q_a and Q_b separately. On Q_a we have by its definition $\lambda = \frac{1}{w - y_a}$. We obtain from (41)

$$\begin{aligned}
\|w_\mu^{'}\|_{L^\infty(Q)} &= \left\| \left(\frac{\kappa}{\varepsilon^2} I + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1} \right. \\
&\qquad \left. \left(\frac{1}{w_\mu - y_a} - \frac{1}{y_b - w_\mu} - K w_\mu^{'}\right) \right\|_{L^\infty(Q)} \\
&\leq \left\| \left(\frac{\kappa}{\varepsilon^2} I + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}\right)^{-1} \right. \\
&\qquad \left. \left(\frac{1}{w_\mu - y_a} + \frac{1}{y_b - w_\mu} - K w_\mu^{'}\right) \right\|_{L^\infty(Q)} \\
&\leq \left\| \left(\frac{\kappa}{\varepsilon^2} I + \frac{\mu}{(w_\mu - y_a)^2}\right)^{-1} \left(2\lambda - K w_\mu^{'}\right) \right\|_{L^\infty(Q)} \\
&\leq \left\| \left(\left(\frac{\kappa}{2\varepsilon^2} I + \frac{\mu}{2}\lambda^2\right)^{-1} \lambda \right) - \left(\frac{\kappa}{\varepsilon^2} I + \mu\lambda^2\right)^{-1} K w_\mu^{'} \right\|_{L^\infty(Q)} \\
&\leq \left\| \left(\frac{\kappa}{\varepsilon^2 \lambda} + \mu\lambda\right)^{-1} \right\|_{L^\infty(Q)} + \left\| \frac{\varepsilon^2}{\kappa} K w_\mu^{'} \right\|_{L^\infty(Q)}.
\end{aligned}$$

By Young's inequality, we estimate $\frac{\kappa}{2\varepsilon^2\lambda} + \frac{\mu}{2}\lambda \geq \frac{\sqrt{\kappa\mu}}{\varepsilon}$. Finally, we obtain

$$\begin{aligned}
\|w_\mu^{'}\|_{L^\infty(Q)} &\leq \left\| \left(\frac{\kappa}{\varepsilon^2 \lambda} + \mu\lambda\right)^{-1} \right\|_{L^\infty(Q)} + \frac{\varepsilon^2}{\kappa} \|K\|_{2,\infty} \|w_\mu^{'}\|_2 \\
&\leq \frac{\varepsilon}{\sqrt{\kappa\mu}} + \frac{\varepsilon^2}{\kappa} \|K\|_{2,\infty} \frac{c_K \sqrt{2} |\, Q \,|}{\sqrt{\mu}}.
\end{aligned}$$

Analogously, we get the same result on Q_b. Both inequalities give us

$$\|w_\mu^{'}\|_{L^\infty(Q)} \leq \frac{c}{\sqrt{\mu}} \quad \text{a.e. in } Q. \tag{44}$$

(iii) *Distance to the limit point.* Having now estimate (44) for $\|w_\mu'\|$, the distance between two points on the central path is bounded by

$$\|w_{\mu_1} - w_{\mu_2}\|_{L^\infty(Q)} \;\leq\; \int_{\mu_1}^{\mu_2} \|w_\mu'\|_{L^\infty(Q)} d\mu \;\leq\; \frac{c}{2}(\sqrt{\mu_2} - \sqrt{\mu_1})$$

for $0 < \mu_1 < \mu_2$. Therefore, for any sequence $(\mu_k)_{k\in\mathbb{N}}$ with $\mu_k \to 0$, the corresponding sequence $(w_{\mu_k})_{k\in\mathbb{N}}$ of points on the central path forms a Cauchy-sequence and hence, the central path converges towards some limit point $\bar{w}$. Pass to the limit $\mu_1 \downarrow 0$ verifies the error bound

$$\|w_\mu - w_0\|_{L^\infty(Q)} \;\leq\; c\sqrt{\mu}.$$

(iv) *First order necessary conditions.* Using the Lagrange multiplier approximations $\eta_{a,\mu} = \frac{\mu}{w - y_a}$ and $\eta_{b,\mu} = \frac{\mu}{y_b - w}$, we write $F'(w_\mu) = \eta_{a,\mu} - \eta_{b,\mu}$. From Lemma 5.4 we know,

$$\|\eta_{a,\mu} - \eta_{b,\mu}\|_{L^\infty(Q)} \leq \|p_\mu\|_{L^\infty(Q)} + \|\theta_\mu\|_{L^\infty(Q)} \leq c_{p,\theta}.$$

Defining the sets

$$\begin{aligned} M_a &:= \left\{ (x,t) \,|\, y_a(x,t) \leq w_\mu(x,t) \leq \frac{y_a(x,t) + y_b(x,t)}{2} \right\} \\ M_b &:= \left\{ (x,t) \;\middle|\; \frac{y_a(x,t) + y_b(x,t)}{2} < w_\mu(x,t) \leq y_b(x,t) \right\} = Q \backslash M_a, \end{aligned}$$

we observe $\eta_{a,\mu}(x,t) - \eta_{b,\mu}(x,t) \leq c_{p,\theta}$ on M_a, hence

$$0 \leq \eta_{a,\mu}(x,t) \leq c_{p,\theta} + \eta_{b,\mu}(x,t) = c_{p,\theta} + \frac{\mu}{y_b(x,t) - w_\mu(x,t)} \qquad \text{a.e. in } Q.$$

By $y_b(x,t) - y_a(x,t) \geq c_Q$ a.e. in Q, we can estimate

$$y_b(x,t) - w_\mu(x,t) \geq y_b(x,t) - \frac{y_a(x,t) + y(x,t)}{2} = \frac{1}{2}(y_b - y_a) \geq \frac{c_Q}{2},$$

hence

$$\eta_{b,\mu}(x,t) = \frac{\mu}{y_b(x,t) - w_\mu(x,t)} \leq \frac{2\mu}{c_Q}.$$

On the other hand, $\eta_{a,\mu}$ is bounded by

$$\eta_{a,\mu}(x,t) \le c_{p,\theta} + \frac{2\mu}{c_Q} \le 2c_{p,\theta} = c \qquad \text{a.e. in } Q$$

for sufficiently small μ. In summary, we have on M_a that $\eta_{b,\mu} \to 0$ if $\mu \to 0$ and $\eta_{a,\mu}$ is bounded by some constant c. By the same argumentation, we get on M_b that $\eta_{a,\mu} \to 0$ if $\mu \to 0$ and $\eta_{b,\mu}$ is bounded by some constant, hence $\|\eta_{a,\mu}\|_{L^\infty(Q)} \le c$, $\|\eta_{b,\mu}\|_{L^\infty(Q)} \le c$.

From $\eta_{a,\mu}(w_\mu - y_a) = \mu$ and $\eta_{b\mu}(y_b - w_\mu) = \mu$ a.e. in Q, we observe that

$$\mu|Q| = (\eta_{a,\mu}, w_\mu - y_a) = (\eta_{b,\mu}, y_b - w_\mu).$$

By the boundedness of $\eta_{a,\mu}$ and $\eta_{b,\mu}$, we can assume that subsequences are weakly converging in $L^2(Q)$. Without loss of generality, we can assume $\eta_{a,\mu} \rightharpoonup \eta_{a,0}$, $\eta_{b,\mu} \rightharpoonup \eta_{b,0}$ in $L^2(Q)$. Along with strong convergence $w_\mu \to w_0$ in $L^2(Q)$, we obtain

$$0 = (\eta_{a,0}, w_0 - y_a) = (\eta_{b,0}, y_b - w_0).$$

Obviously, $\eta_{a,0} \ge 0$ and $\eta_{b,0} \ge 0$, and hence the necessary optimality conditions hold. By uniqueness of the optimal control, it holds $w_0 = \bar{w}$.

□

6. An interior point algorithm

A conceptual interior point algorithm in function space can be described by the following steps.

Algorithm IP

Choose $0 < \sigma < 1$, $0 < eps$, *and an initial function* $w^0 \in L^\infty$ *such that* $y_a + \tau \le w^0 \le y_b - \tau$ *holds for some* $\tau > 0$ *and take* $\mu^0 > 0$. *Set* $k = 0$.

while $\mu^k > eps$ do {
$\mu^{k+1} = \sigma\mu^k$,
$d^{k+1} = -\partial H_w(w^k;\mu^{k+1})^{-1} H(w^k;\mu^{k+1})$
$w^{k+1} = w^k + d^{k+1}$
k = k+1
}

The code-sequence in the while-loop performs one classical Newton step for solving the equation $H(w^{k+1};\mu^{k+1}) = 0$ for fixed μ_{k+1}.

In the following, we denote the solutions of (Q) associated with the parameter μ_k by subscripts, i.e. w_{μ_k} is a point on the central path and solves $H(w_{\mu_k}; \mu_k) = 0$. On the other hand, let w^k , $k = 1, 2, ...$ denote the iterates of Algorithm IP associated with the parameter μ_k. Figure 1 illustrates the situation.

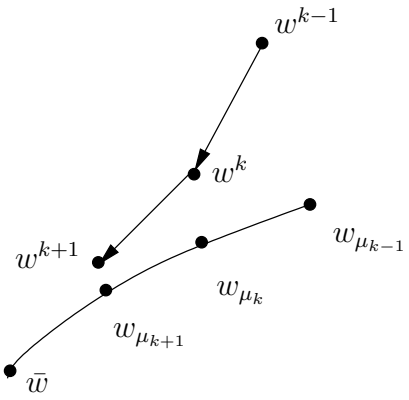

FIGURE 1. Some iterates of Algorithm IP and the associated points on the central path.

Under our assumptions, the Newton method provides for fixed μ_k a unique solution w_{μ_k}. It converges quadratically, if the starting point (in the Figure, we choose w^{k-1}) is sufficiently close to w_{μ_k}.

To prove the convergence of our method in function space, we show that

$$\|w^k - w_{\mu_k}\| \leq c\sqrt{\mu^k} \quad \text{and} \quad \|w^k - \bar{w}\| \quad \leq \quad c\sigma^k$$

holds for some constant $c > 0$. Clearly, it holds that

$$w^{k+1} - w^k = d^{k+1}.$$

In contrast to the Algorithm 5.1 in [18], the Newton corrector is assumed to be exact for simplicity, i.e. we assume to compute d^{k+1} exactly. Certainly, this is not realistic for a practical implementation. However, we do not aim at estimating here all errors that occur in a real computation.

In Section 5, we have estimated the distance of two points on the central path in the L^2- and L^∞-norm. In this section, we make use of the Newton-Mysovskikh theorem to estimate the distance between w^k and w_{μ_k}.

6.1. **Scaled norms.** Local norms are a valuable tool in the context of interior point methods. Here we will use the scaled norm

$$\|w\|_\mu \;:=\; \|\phi(\mu)w\|_{L^\infty(Q)}\,,$$

where $\phi(\mu)$ is defined in (38). For the theory of affine scaled norms, we refer to [5]. First, we provide some results on this scaled norm.

Lemma 6.1. *For all $w \in L^\infty(Q)$, the norm $\|\cdot\|_\mu$ satisfies the estimate*

$$\|w\|_{L^\infty(Q)} \;\leq\; \frac{\varepsilon}{\sqrt{\kappa}}\|w\|_\mu.$$

Proof. It holds by the definition of $\|\cdot\|_\mu$ and $\phi(\mu)$ that

$$\|w\|_\mu \;=\; \|\phi(\mu)w\|_{L^\infty(Q)} \;\geq\; \|\sqrt{\frac{\kappa}{\varepsilon^2}}w\|_{L^\infty(Q)} \;=\; \sqrt{\frac{\kappa}{\varepsilon^2}}\|w\|_{L^\infty(Q)}.$$

□

The following counterpart of Theorem 5.8 for the scaled norm can be shown completely analogous to Lemma 5.3 in [18].

Lemma 6.2. *There is some constant c_ϕ, $1 \leq c_\phi < \infty$, independent of μ, such that*

$$\|\partial_w H(w;\mu)\eta_b\|_\mu \;\leq\; c_\phi\|\phi(\mu)^{-1}\eta_b\|_{L^\infty(Q)} \quad \forall\, \eta_b \in L^\infty(Q)$$

holds true for all $w \in B(w_\mu;\theta\sqrt{\mu}) := \{w \in L^\infty(Q) \,:\, \|w - w_\mu\|_\mu \leq \theta\sqrt{\mu}\}$ with some $\theta < 1$.

Proof. The proof is analogous to the one of Lemma 5.3 in [18]. We get the estimate

$$\begin{aligned}\|\partial_w H(w;\mu)^{-1}\eta_b\|_\mu \;&=\; \|\phi_\mu\partial_w H(w;\mu)^{-1}\eta_b\|_{L^\infty(Q)}\\ &\leq\; \left(1 + \frac{\varepsilon^2}{\kappa\sqrt{|Q|}}\|K\|_{L^2(Q)\to L^\infty(Q)}\right)\|\phi_\mu^{-1}\eta_b\|_{L^\infty(Q)},\end{aligned}$$

which gives us the constant c_ϕ. □

Lemma 6.3. *For all $w \in B_\mu(w_\mu;\theta\sqrt{\mu})$ and all $0 < \theta < 1$, the following estimates hold true*

$$w - y_a \geq (1-\theta)(w_\mu - y_a)$$

and

$$y_b - w \geq (1-\theta)(y_b - w_\mu).$$

Proof. By its definition, the diagonal preconditioner $\phi(\mu)$ satisfies

$$\sqrt{\frac{\mu}{(w_\mu - y_a)^2}} \leq \phi(\mu)$$

and

$$\sqrt{\frac{\mu}{(y_b - w_\mu)^2}} \leq \phi(\mu)$$

for all $y_a < w_\mu < y_b$.

For all $w \in B_\mu(w_\mu; \theta\sqrt{\mu})$ we obtain

$$\begin{aligned} \left\| \frac{w - w_\mu}{w_\mu - y_a} \right\|_{L^\infty(Q)} &= \left\| \frac{w - w_\mu}{\sqrt{\mu}} \sqrt{\frac{\mu}{(w_\mu - y_a)^2}} \right\|_{L^\infty(Q)} \\ &\leq \frac{1}{\sqrt{\mu}} \left\| \phi(\mu)(w - w_\mu) \right\|_{L^\infty(Q)} \\ &= \frac{1}{\sqrt{\mu}} \|w - w_\mu\|_\mu < \theta < 1. \end{aligned}$$

From $w_\mu - y_a > 0$ a.e. in Q we get therefore

$$\pm(w - w_\mu) \leq \theta(w_\mu - y_a) \text{ a.e. in } Q,$$

hence, by multiplying the minus-version by (-1) and adding on both sides $(w_\mu - y_a)$,

$$w - y_a \geq (1-\theta)(w_\mu - y_a).$$

By the same argumentation we can estimate

$$y_b - w \geq (1-\theta)(y_b - w_\mu).$$

□

Lemma 6.4. *(Lipschitz-Condition) For all $0 < \theta < 1$, θ sufficiently small, and all w, $\hat{w} \in B_\mu(w_\mu, \theta\sqrt{\mu})$, the following Lipschitz condition holds:*

$$\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w - \hat{w})\|_\mu \leq \frac{4c_\phi}{(1-\theta)^3\sqrt{\mu}} \|w - \hat{w}\|_\mu^2. \tag{45}$$

Proof. The main idea of the proof is analogous to the proof of Lemma 5.5 in [18] for unilateral constraints. The difficulty here is the more complicated structure of $\phi(\mu)$, what results in a more technical proof. For convenience of the reader, we perform it here in detail.

Using Lemma 6.2, we obtain

$$\begin{aligned}
&\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w-\hat{w})\|_\mu \\
&\qquad \leq\ c_\phi \|\phi_\mu^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w-\hat{w})\|_{L^\infty(Q)} \\
&=\ c_\phi \left\| \phi_\mu^{-1} \left(\frac{\mu}{(w-y_a)^2} + \frac{\mu}{(y_b-w)^2} - \frac{\mu}{(\hat{w}-y_a)^2} - \frac{\mu}{(y_b-\hat{w})^2} \right) \cdot (w-\hat{w}) \right\|_{L^\infty(Q)},
\end{aligned}$$

where the constant parts of $\partial_w H(w;\mu)$ are compensated by the constant parts of $\partial_w H(\hat{w};\mu)$, cf. the definition of $\partial_w H(w;\mu)$ in (32) or (37), respectively. By Lemma 6.3 we obtain $w - y_a \geq (1-\theta)(w_\mu - y_a)$ and $y_b - w \geq (1-\theta)(y_b - w_\mu)$. Using the fact, that the Lipschitz constant of x^{-2} for $x \geq a > 0$ is given by $2a^{-3}$, we can estimate

$$\begin{aligned}
&\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w-\hat{w})\|_\mu \\
&\quad \leq c_\phi \left\| \phi(\mu)^{-1} \left(\frac{2\mu}{(1-\theta)^3(w_\mu - y_a)^3} + \frac{2\mu}{(1-\theta)^3(y_b - w_\mu)^3} \right) (w-\hat{w})^2 \right\|_{L^\infty(Q)} \\
&\quad = \frac{2c_\phi}{(1-\theta)^3} \left\| \left(\frac{\mu}{\phi(\mu)^3(w_\mu - y_a)^3} + \frac{\mu}{\phi(\mu)^3(y_b - w_\mu)^3} \right) \phi(\mu)^2 (w-\hat{w})^2 \right\|_{L^\infty(Q)} \\
&\quad \leq \frac{2c_\phi}{(1-\theta)^3} \left\| \frac{\mu}{\phi(\mu)^3(w_\mu - y_a)^3} + \frac{\mu}{\phi(\mu)^3(y_b - w_\mu)^3} \right\|_{L^\infty(Q)} \|w-\hat{w}\|_\mu^2 \\
&\quad \leq \frac{2c_\phi}{(1-\theta)^3} \left(\left\| \frac{\mu}{\phi(\mu)^3(w_\mu - y_a)^3} \right\|_{L^\infty(Q)} + \left\| \frac{\mu}{\phi(\mu)^3(y_b - w_\mu)^3} \right\|_{L^\infty(Q)} \right) \|w-\hat{w}\|_\mu^2
\end{aligned}$$

We show that $\left\| \frac{\mu}{\phi(\mu)^3(w_\mu - y_a)^3} \right\|_{L^\infty(Q)}$ and $\left\| \frac{\mu}{(\phi(\mu)^3(y_b - w_\mu)^3} \right\|_{L^\infty(Q)}$ are bounded by $c\sqrt{\mu}$. First, we have

$$\begin{aligned}
\phi(\mu)^3(w_\mu - y_a)^3 &= \left(\sqrt{\frac{\kappa}{\varepsilon^2}(w_\mu - y_a)^2 + \frac{\mu(w_\mu - y_a)^2}{(w_\mu - y_a)^2} + \frac{\mu(w_\mu - y_a)^2}{(y_b - w_\mu)^2}} \right)^3 \\
&\geq (\sqrt{\mu})^3,
\end{aligned}$$

and from that we get $\left\| \frac{\mu}{\phi(\mu)^3 (w_\mu - y_a)^3} \right\|_{L^\infty(Q)} \leq \mu/\mu^{3/2} = 1/\sqrt{\mu}$. With the same argumentation, we get $\left\| \frac{\mu}{(\phi(\mu)^3 (y_b - w_\mu)^3} \right\| \leq 1/\sqrt{\mu}$. Altogether, this yields the Lipschitz condition

$$\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w - \hat{w})\|_\mu \leq \frac{4c_\phi}{(1-\theta)^3\sqrt{\mu}} \|w - \hat{w}\|_\mu^2. \tag{46}$$

□

With this Lipschitz constant at hand, we are able to prove the convergence of Algorithm IP.

6.2. **Estimation of the error $\|w^k - \bar{w}\|_{L^\infty(Q)}$.** To prove the convergence of the interior point method defined by Algorithm IP, we have to show, that $\|w^k - \bar{w}\|_{L^\infty(Q)} \to 0$ for $k \to \infty$.

Theorem 6.5. *Let $\|w^0 - w_{\mu_1}\|_{\mu_1} \leq \theta\sqrt{\mu_0\sigma}$ and $0 < \sigma < 1$. Then the iterates w^k of Algorithm IP converge linearly towards the solution $\bar{w}$ of problem (P):*

$$\|w^k - \bar{w}\|_{L^\infty(Q)} \leq c\sigma^k.$$

Proof. We have

$$\begin{aligned} \|w^k - \bar{w}\|_{L^\infty(Q)} &= \|w^k - w_{\mu_k} + w_{\mu_k} - \bar{w}\|_{L^\infty(Q)} \\ &\leq \|w^k - w_{\mu_k}\|_{L^\infty(Q)} + \|w_{\mu_k} - \bar{w}\|_{L^\infty(Q)} \\ &\leq \frac{\varepsilon}{\sqrt{\kappa}} \|w^k - w_{\mu_k}\|_{\mu_k} + \|w_{\mu_k} - \bar{w}\|_{L^\infty(Q)}. \end{aligned} \tag{47}$$

The second part of (47) can be estimated by the length of a segment of the central path: using Theorem 5.9 yields $\|w_{\mu_k} - \bar{w}\|_{L^\infty(Q)} \leq c\sqrt{\mu_k} = c\sqrt{\mu_0}\sigma^{k/2}$. Now we have to show, that the norm of the first part of (47) can be bounded by $c\sqrt{\mu^k}$ with a constant $c < \infty$. We interpret w^k as the first iterate of the Newton method for the problem

$$H(w;\mu_k) = 0 \tag{48}$$

with some starting point w^{k-1} close enough to w_{μ_k}, where w_{μ_k} is the solution of (48). To show the bound, we use the refined Newton-Mysovskikh theorem, provided in [6], Theorem 1.5. First, we have by Lemma 6.4 the affine covariant Lipschitz condition

$$\|H'(w)^{-1}(F'(w) - H'(\hat{w}))(w - \hat{w})\|_\mu \leq \omega\|w - \hat{w}\|_\mu$$

that holds for all w, $\hat{w}$ with $\omega = \frac{4c_z}{(1-\theta)^3\sqrt{\mu}}$. From Theorem 1.5 in [6], we get

$$\|w^k - w_{\mu_k}\|_{\mu_k} \leq \frac{4c_\phi}{(1-\theta)^3\sqrt{\mu_k}}\|w^{k-1} - w_{\mu_k}\|^2_{\mu_k}.$$

From our assumption we deduce by induction that

$$\|w^{k-1} - w_{\mu_k}\|_{\mu_k} \leq c_k\sqrt{\mu^k} = c_k\sqrt{\mu_0}\sigma^{k/2}.$$

This gives us

$$\|w^k - w_{\mu_k}\|^2_{\mu_k} \leq \frac{4c_\phi}{(1-\theta)^3\sqrt{\mu_k}}c_k^2\mu_k = \frac{4c_\phi}{(1-\theta)^3}c_k^2\sqrt{\mu_0}\sigma^{k/2}.$$

Together with (47), we arrive at

$$\|w^k - \bar{w}\|_{L^\infty(Q)} \leq \left(\frac{4c_\phi}{(1-\theta)^3}c_k^2\sqrt{\mu_0} + c\sqrt{\mu_0}\right)\sigma^{k/2}.$$

This proves the convergence $w^k \to \bar{w}$. □

Corollary 6.6. *For every $\varepsilon > 0$, the sequences of states $y^k := SDw^k$ and of controls $u^k := \frac{1}{\varepsilon}(w^k - y^k)$ converge linearly to the optimal state $\bar{y}$ and the optimal control $\bar{u}$, respectively.*

7. Numerical examples

7.1. **Discretization of the optimality system.** In Section 5.1, we have introduced the optimality system (28)–(31) for our problem. In view of our test examples, we will use the extended form (5) of our problem, for which the theory works as well, cf. Remark 3 in Section 3. Using (31), we write $\eta_a = \frac{\mu}{\varepsilon u + y - y_a}$ and $\eta_b = \frac{\mu}{y_b - \varepsilon u - y}$ and we have to solve the optimality system

$$\begin{aligned} y_t - \nabla\cdot(A\nabla y) + c_0 y &= u + f && \text{in } Q,\\ \partial_n y + \alpha y &= g && \text{on } \Sigma, \\ y(0) &= y_0 && \text{in } \Omega, \end{aligned} \tag{49}$$

$$\begin{aligned} -p_t - \nabla\cdot(A\nabla p) + c_0 p &= -\frac{\mu}{\varepsilon u + y - y_a} + \frac{\mu}{y_b - \varepsilon u - y} && \text{in } Q,\\ \partial_n y + \alpha y &= 0 && \text{on } \Sigma, \\ y(0) &= y(T) - y_d && \text{in } \Omega, \end{aligned} \tag{50}$$

$$\kappa u + p - \frac{\varepsilon\mu}{\varepsilon u + y - y_a} + \frac{\varepsilon\mu}{y_b - \varepsilon u - y} = 0 \qquad \text{a.e. in } Q. \tag{51}$$

Our test examples are defined in one-dimensional domains $\Omega = (a, b)$. Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of $[0, T]$, and denote by $\delta_k = t_k - t_{k-1}$ the time step sizes. Define $y_k = y(\cdot, t_k)$, $u_k = u(\cdot, t_k)$, $p_k = p(\cdot, t_k)$, $(y_a)_k = y_a(\cdot, t_k)$, $(y_b)_k = y_b(\cdot, t_k)$, $k = 0, 1, ..., n$. Using an implicit Euler scheme for discretizing (49) and (50) in time, we have to solve a sequence of elliptic problems

$$\begin{aligned} -\nabla \cdot (A\nabla y_{k+1}) + \frac{1 + \delta_{k+1} c_0}{\delta_{k+1}} y_{k+1} &= \frac{1}{\delta_{k+1}} y_k + u_{k+1} + f_{k+1}, \\ \partial_n y_{k+1} + \alpha y_{k+1} &= g_{k+1} \end{aligned} \tag{52}$$

for $k = 0, ..., n-1$, starting at

$$y(\cdot, 0) = y_0.$$

To get a fully discrete system, we use linear finite elements to discretize the elliptic subproblems. Let $a = x_0 < x_1 < ... < x_n = b$ be a partition of $(a, b) = \Omega \subset \mathbb{R}$ with mesh sizes $h_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. By using standard hat functions with $\varphi_i(x_j) = \delta_{ij}$ $i, j \in I$, where $I \subset \mathbb{N}$ is the set of indices of the nodes x_i, we can identify the coefficients of the FEM approximation of a function by the values of the function f in the nodes $f(x) \approx \sum_{i \in I} f(x_i)\varphi_i(x)$. In all what follows, we identify the functions f, y, u, etc. by their coefficient vectors $(f(x_i))$, $(y(x_i))$, $(u(x_i))$ and denote them by the same notifiers, i.e., we will write f instead $(f(x_i))$ etc.

By the stiffness matrix

$$\mathbf{K} = (K_{ij}), \quad K_{ij} = \int_\Omega (a_{ij}\nabla\varphi_j) \cdot (\nabla\varphi_i)\, dx,$$

mass matrices

$$\mathbf{M}_{k+1} = (M_{ij})_{k+1}, \quad M_{ij,k+1} = \int_\Omega \frac{1 + \delta_{k+1} c_0}{\delta_{k+1}} \varphi_j \varphi_i \, dx,$$

$$\bar{\mathbf{M}} = (\bar{M}_{ij}), \quad \bar{M}_{ij} = \int_\Omega \varphi_j \varphi_i \, dx,$$

and the matrices associated with the boundary Γ,

$$\mathbf{Q} = (Q_{ij}), \quad Q_{ij} = \int_\Gamma \alpha\varphi_j \varphi i \, ds,$$

$$\mathbf{G} = (G_i), \quad G_i = \int_\Gamma g\varphi_i \, ds,$$

the FEM representation of the elliptic subproblems is given by

$$\begin{aligned} y_0 &= y(0) \\ (\mathbf{K} + \mathbf{M}_{k+1} + \mathbf{Q})y_{k+1} &= \frac{1}{\delta_{k+1}}\bar{\mathbf{M}}y_k + \mathbf{G}_{k+1} \\ &\quad + \bar{\mathbf{M}}(u_{k+1} + f_{k+1}), \end{aligned} \tag{53}$$

for $k = 0, 1, ..., n-1$.

Analogously, the adjoint equation is discretized by

$$\begin{aligned} (\mathbf{K} + \mathbf{M}_k + \mathbf{Q})p_k &= \bar{\mathbf{M}}\left(\frac{\mu}{y_k + \varepsilon u_k - (y_a)_k}\right) \\ &\quad - \bar{\mathbf{M}}\left(\frac{\mu}{(y_b)_k - y_k - \varepsilon u_k}\right) + \frac{1}{\delta_k}\bar{\mathbf{M}}p_{k+1} \end{aligned} \tag{54}$$

for $k = n-1, ..., 0$ and

$$p_n = y_n - y_d.$$

The vectors $\frac{\mu}{y_k + \varepsilon u_k - (y_a)_k}$ and $\frac{\mu}{(y_b)_k - y_k - \varepsilon u_k}$ are defined by

$$\left(\frac{\mu}{y_k + \varepsilon u_k - (y_a)_k}\right)_i = \frac{\mu}{(y_k)_i + \varepsilon(u_k)_i - ((y_a)_k)_i}$$

and

$$\left(\frac{\mu}{(y_b)_k - y_k - \varepsilon u_k}\right)_i = \frac{\mu}{((y_b)_k)_i - (y_k)_i - \varepsilon(u_k)_i},$$

respectively.

These equations are coupled through the discrete version of the gradient equation

$$\kappa u_k + p_k + \frac{\varepsilon\mu}{\beta_k - y_k - \varepsilon u_k} - \frac{\varepsilon\mu}{y_k + \varepsilon u_k - \alpha_k} = 0, \tag{55}$$

for $k = 0, ..., n$.

We arrange the coefficient vectors as follows:

$$z = \left[y_0^T, y_1^T, \ldots, y_n^T, u_0^T, u_1^T, \ldots, u_n^T, p_0^T, p_1^T, \ldots, p_n^T\right]^T$$

Algorithm PCAG

(i) Define equidistant initial partitions $\mathbf{T}_0 = \{t_0, t_0 + \delta_t, ..., T\}$ of $[0, T]$ and $\mathbf{\Omega}_0 = \{a = x_0, x_0 + h, ..., x_n = b\}$ of $\Omega = (a, b)$, where δ_t and h are the fixed initial stepsizes in time and space, respectively.

(ii) Choose $z_0 = (y_0^T, u_0^T, p_0^T)^T$ feasible, e.a. $y_a \leq y_0 + \varepsilon u_0 \leq y_b$ and p_0 can be taken arbitrarily.

(iii) Assemble the matrices K, M_k, $\bar{M}$, Q, and the vector G.

(iv) Choose $\mu_0 > 0$. Compute a solution of

$$F(z; \mu_0) = 0$$

by the Newton Method.

(v) Refine the space and time discretization by suitable methods.

(vi) Reassemble all matrices and compose the associated system matrix Ξ. Interpolate z onto the new grids.

and write the optimality conditions as a nonlinear system

$$F(z; \mu) := \Xi z + \Psi(z) + \Phi = 0,$$

where Ξ is a large, sparse matrix, essentially build of blocks $\mathbf{K} + \mathbf{M}_k + \mathbf{Q}$ on the diagonal and $\bar{\mathbf{M}}$ on the subdiagonal. Ψ is a function that covers the nonlinearity and Φ is a vector that contains the constant parts of the equations (53)–(55).

One difficulty in the Algorithm IP is how to find a suitable initial function w^0. Algorithm PCAG provide a feasible initial function that can be expected sufficiently close to w_{μ_0}. Moreover, the time and space discretizations can be adapted during the computations.

The spatial grids may change between the different time steps. After Algorithm PCAG is finished, the joint refinement of all spatial grids is taken as the fixed spatial grid for Algorithm IP. The discretized version of Algorithm IP is started with $z_0 = z$. For all computations, we used Matlab 7.1.0 R14 on a Pentium IV machine with 1GB memory. The linear subproblems are solved by direct methods. For refining meshes in Algorithm PCAG, we used for time refinement ode15s with the setting RelTol = 1e-3, MaxOrder = 1, and BDF=on. For the grid refinement in space, we used an error indicator function similar to the one described in [1], which detects the maxima of $\|f\|$, where f is the right hand side of a elliptic PDE. The spatial grid is fixed in all time steps.

7.2. **Examples.**

Example 7.1. We tested our method by the problem

$$\min J(y,u) := \frac{1}{2}\|y(T) - y_d\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(Q)}$$

subject to

$$\begin{aligned} y_t - \Delta y &= u \qquad \text{in } Q, \\ \partial_n y + 10y &= 0 \qquad \text{on } \Sigma, \\ y(0) &= y_0 \qquad \text{in } \Omega, \end{aligned}$$

and to the mixed control-state constraints

$$y + \varepsilon u \geq y_a := \max\{-100(t(t-1) + x(x-1)) - 49.0, 0.5\} \qquad \text{a.e. in } Q.$$

We choose $\Omega = (0,1) \subset \mathbb{R}^1$, $T = 1$. Further, let be $y_d \equiv 0$ and $y_0 = \sin(\pi x)$ given. Obviously, this problem fits in our general setting with $\alpha = 10$.

In our examples, there is no upper bound y_b, but it is clear that our method covers the onesided case as well, cf. our comments before Lemma 5.2. In contrast to the next example, here the exact optimal control $\bar{u}$ and the associated functions $\bar{y}$, p and η_a are unknown.

The initial function for Algorithm PCAG was $w_0 \equiv 2$. The initial stepsizes were $h = 0.01$ and $\delta_t = 0.005$. In Algorithm IP, we choose $\sigma = 0.8$, $\mu_0 = 10^{-3}$, and $eps = 10^{-5}$. Figure 2 shows the computed optimal solutions $\bar{u}_\mu$, $\bar{y}_\mu$, $\bar{p}_\mu$, and $\eta_a = \frac{\mu}{\varepsilon\bar{u}_\mu + \bar{y}_\mu - y_a}$ for the regularized problem with $\varepsilon = 10^{-3}$ and $\kappa = 10^{-3}$.

Example 7.2. In our second example, we consider the slightly modified problem

$$\min J(y,u) := \frac{1}{2}\|y(T) - y_\Omega\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(Q)} + \iint_Q y_Q y \, dxdt,$$

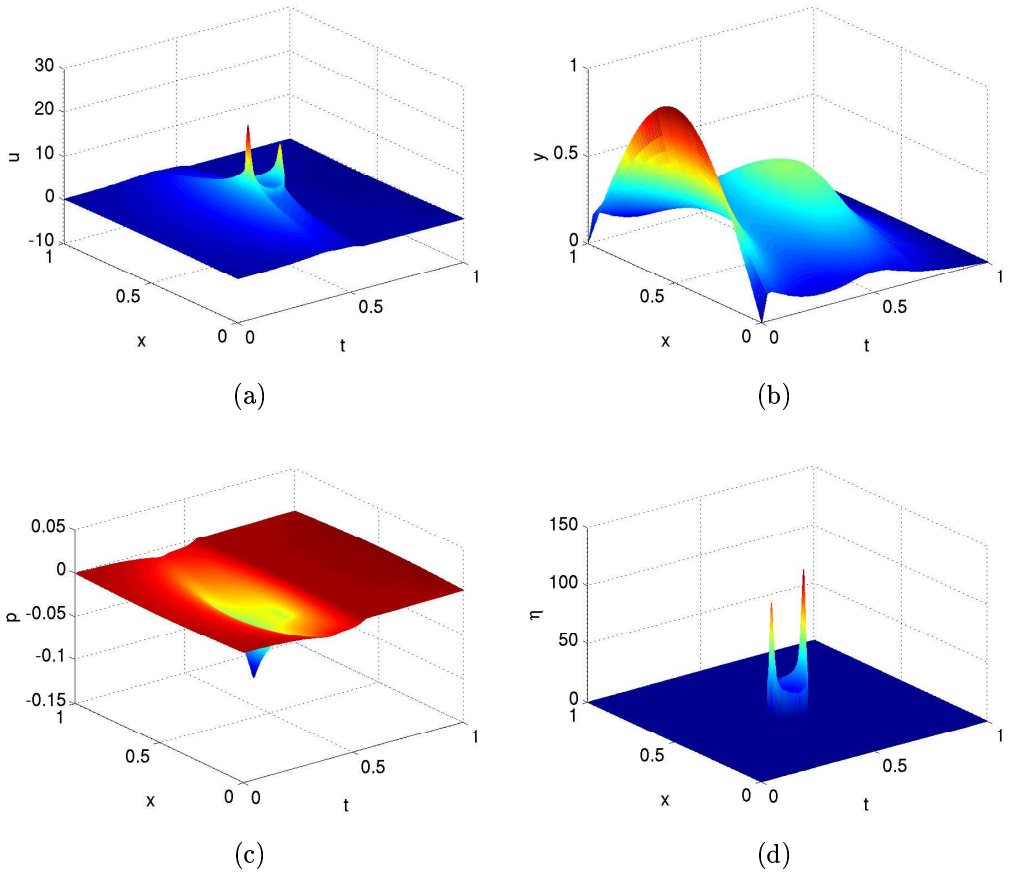

FIGURE 2. Computed solutions to Example 1: (a) control $\bar{u}_\mu$, (b) state $\bar{y}_\mu$, (c) adjoint state $\bar{p}_\mu$ and (d) Lagrange multiplier η_a.

subject to

$$\begin{aligned} y_t - \Delta y &= u + f \quad \text{in } Q \\ y &= 0 \quad \text{on } \Sigma \\ y(0) &= y_0 \quad \text{in } \Omega \end{aligned}$$

and to the mixed control-state constraints

$$y_a \leq y + \varepsilon(u + f) \qquad \text{a.e. in } Q.$$

The last term in the objective function was added to construct an example with explicitly known optimal solution. This term does not change our theory. We simply have to add its derivative y_Q to the right hand side of the adjoint equation.

We construct an optimal solution which fulfils the optimality conditions (49)–(51) for the unregularized problem, i.e., for $\varepsilon = 0$.

The integral $\iint_Q y_Q y \, dxdt$ in the objective function leads to the adjoint equation

$$\begin{aligned} -p_t + \Delta p &= y_Q - \frac{\mu}{y - y_a} && \text{in } Q \\ p &= 0 && \text{on } \Sigma \\ p(T) &= y(T) - y_d && \text{in } \Omega \end{aligned}$$

instead of (50).

Construction of the optimal solution. We choose $\Omega = (0, \pi)$ and $T = 1$, and we just define the optimal state by $\bar{y}(x,t) := e^{-t}\sin(x)$. Together with $y(\cdot, 0) = \sin(x)$ and $y(\cdot, T) = \mathrm{e}^{-1}\sin(x)$ we obtain from (49) and $y_t - \Delta y = 0$ the condition $\bar{u} + f = 0$.

From the gradient equation (51) and $\varepsilon = 0$ we get $f = \frac{1}{\kappa} p$. Next, we construct the state constraint such that $\bar{y}$ touches the bound y_a only on a set $(t_1, t_2) \times \left\{\frac{\pi}{2}\right\}$. This set has measure zero, so that we construct a Lagrange-multiplier as a regular Borel measure. We choose $t_1 = 0.3$ and $t_2 = 0.6$. The bound y_a is fixed by $y_a(x,t) = \eta(t)\eta(x)$ with

$$\eta(t) = \begin{cases} \frac{1}{2}\frac{t - t_1}{t_0 - t_1} + \mathrm{e}^{-t_1}\frac{t}{t_1} & t \in (0, t_1) \\ \mathrm{e}^{-t} & t \in (t_1, t_2) \\ \mathrm{e}^{-t_2}\frac{t-1}{t_2 - 1} + \frac{1}{8}\frac{t - t_2}{1 - t_2} & t \in (t_2, 1) \end{cases}$$

and

$$\eta(x) = \begin{cases} \frac{3}{\pi} - 0.5 & x \in (0, \pi/2) \\ 2.5 - \frac{3}{\pi} & x \in (\pi/2, \pi). \end{cases}$$

The adjoint state is constructed by the ansatz $p = \varphi(t)v(x)$. To this aim, let

$$\varphi(t) = \begin{cases} -\sin^2\left(\frac{\pi}{t_2 - t_1}(t - t_1)\right) & t \in (t_1, t_2) \\ 0 & \text{else.} \end{cases}$$

The derivative of φ is given by the continuous function

$$\varphi'(t) = \begin{cases} -\frac{2\pi}{t_2-t_1}\cos\left(\frac{\pi}{t_2-t_1}\right)(t-t_1)\sin\left(\frac{\pi}{t_2-t_1}(t-t_1)\right) & t \in (t_1,t_2) \\ 0 & \text{else,} \end{cases}$$

which is continuous. Next, we introduce the continuous piecewise linear function

$$v(x) = \begin{cases} \frac{2}{\pi}x & x \in [0,\frac{\pi}{2}] \\ 2-\frac{2}{\pi}x & x \in [\frac{\pi}{2},\pi]. \end{cases}$$

The second derivative of $v(x)$ with respect to x is a multiple of the δ-distribution: $v_{xx} = -\frac{4}{\pi}\delta_{\frac{\pi}{2}}(x)$. The adjoint equation gives us

$$-p_t - p_{xx} = -\mu + y_Q,$$

so we can set

$$\mu = \varphi(t)v_{xx} = -\varphi(t)\frac{4}{\pi}\delta_{\frac{\pi}{2}} \geq 0$$

and

$$y_Q = -\varphi'(t)v(x).$$

Obviously, μ and $y - y_c$ fulfill the complementary slackness conditions

$$\begin{aligned} \iint_Q (y-y_a)\,d\mu(x,t) &= 0 \\ y - y_a &\geq 0 \text{ a.e. in } Q \\ \mu &\geq 0 \text{ in } Q. \end{aligned}$$

Giving the optimal solutions, we are able to show convergence rates for u_μ by $\mu \to 0$. We choose $\kappa = 10^{-2}$, $\varepsilon = 10^{-6}$, $\sigma = 0.8$, $\mu_0 = 10^{-3}$, and $eps = 10^{-5}$ fixed. Figure 3 shows the numerical solutions.

With the given exact solutions for the unregularized problem and our choice of small ε, we can show linear convergence in u and y. Figure 4(c) shows the value of the objective function J_μ.

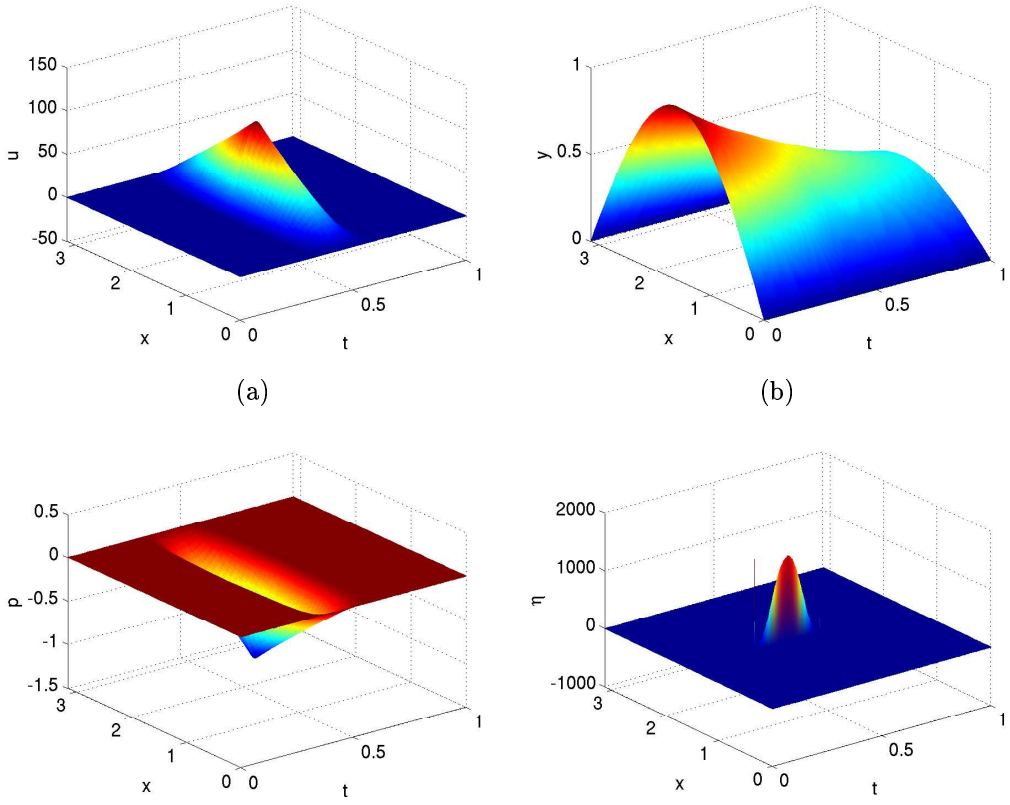

FIGURE 3. Computed solutions to Example 2: (a) control $\bar{u}_\mu$, (b) state $\bar{y}_\mu$, (c) adjoint state $\bar{p}_\mu$ and (d) Lagrange multiplier η_a.

In Table 1, we present the errors of the solutions and the value of the objective function to Example 2 for selected values of μ.

μ	$\|y_\mu - \bar{y}\|/\|\bar{y}\|$	$\|u_\mu - \bar{u}\|/\|\bar{u}\|$	$\|p_\mu - \bar{p}\|/\|\bar{p}\|$	$J(y,u;\mu)$
8.0^{-2}	2.2954	4.3332^{-1}	4.3332^{-1}	7.3130
4.3980^{-3}	1.7467^{-2}	2.9738^{-2}	2.9738^{-2}	6.1299
7.3787^{-4}	3.8415^{-2}	6.6231^{-3}	6.6234^{-3}	6.1211
3.0223^{-4}	2.0936^{-2}	4.0684^{-3}	4.0685^{-3}	6.1204
9.9035^{-5}	1.3354^{-2}	3.1801^{-3}	3.1799^{-3}	6.1202

TABLE 1. Relative errors of the computed solutions y_μ, u_μ, and p_μ as well as values of $J_\mu(y,u)$ depending on μ.

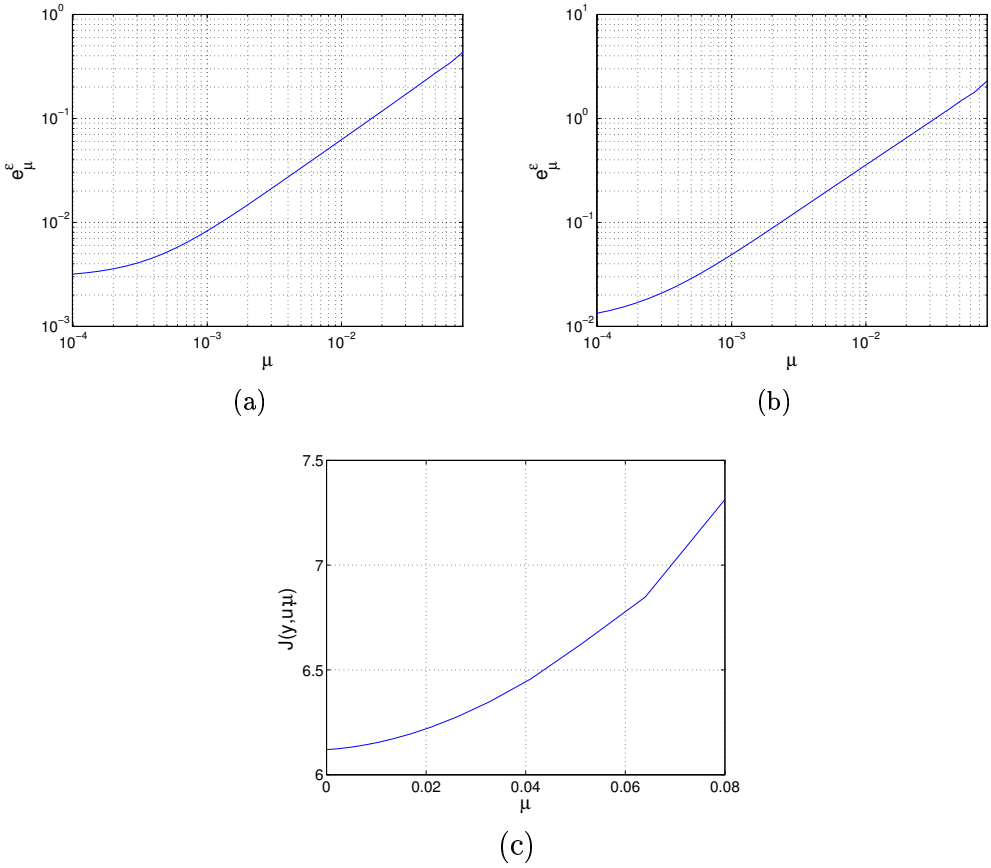

FIGURE 4. Relative errors of the computed solutions (a) control u_μ and (b) state u_μ, as well as (c) value of the objective function J_μ depending on μ.

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