Nonlocal temperature-dependent phase-field models for non-isothermal phase transitions[1]

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Abstract

We propose a model for non-isothermal phase transitions with non-conserved order parameter driven by a spatially nonlocal free energy with respect to both the temperature and the order parameter. The resulting system of equations is shown to be thermodynamically consistent and to admit a strong solution.

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1 Introduction

Phase-field models for temperature-induced phase transitions in a physical body Ω consist in deriving equations for the temperature θ (we will consider the *absolute temperature* $\theta > 0$ here) and the order parameter χ which characterizes the physical state of the material. For example, in a simple melting-solidification process, χ takes values in the interval $[0,1]$, where $\chi = 0$ corresponds to the solid, $\chi = 1$ to the liquid, and $0 < \chi < 1$ is the liquid fraction in a mixture of both phases. The mathematical model we consider below may or may not contain a restriction on the domain of admissible values of χ.

The general scheme in phase-field modeling is based on defining a suitable *free energy functional* $\mathcal{F}[\theta, \chi]$. In the spatially non-local setting, the state variables θ and χ cannot be treated as numbers any more, but as functions of the space variable $x \in \Omega$. Consequently, the symbol δ_θ in the definition of *entropy*

$$\mathcal{S}[\theta, \chi] \;=\; -\delta_\theta \mathcal{F}[\theta, \chi]\,, \tag{1.1}$$

has to be understood as the variational derivative with respect to the *function* $\theta(x)$, and the brackets in the expression for the *internal energy*

$$\mathcal{E}[\theta, \chi] \;=\; \mathcal{F}[\theta, \chi] + \langle \mathcal{S}[\theta, \chi], \theta \rangle \tag{1.2}$$

denote the duality pairing in the corresponding function spaces (e. g. $L^2(\Omega)$).

The evolution of the process is governed by the *energy conservation principle*

$$\frac{d}{dt}\mathcal{E}[\theta, \chi] \;=\; 0\,, \tag{1.3}$$

and the *order parameter evolution equation*

$$\mu(\theta)\frac{\partial \chi}{\partial t} \;\in\; -\delta_\chi \mathcal{F}[\theta, \chi]\,, \tag{1.4}$$

which expresses the tendency of the system to move towards local minima of the free energy with speed proportional to $1/\mu(\theta)$. The inclusion sign in (1.4) denotes the fact that $\mathcal{F}$ may contain components which are not Fréchet differentiable, but are convex, and the derivative can be interpreted as the subdifferential which may be multivalued.

To account for interactions between neighbouring points in Ω, the classical Ginzburg-Landau free energy (cf. [4]) contains a term $(\nu/2)|\nabla\chi(x)|^2$ with a positive parameter ν. A non-local alternative is suggested e. g. in [1, 2, 5, 6, 7, 8, 9, 10] in the form of integral $\int_\Omega k(x,y)\,|\chi(x) - \chi(y)|^2\,dy$ with a given symmetric kernel $k(x,y)$. We follow these lines and allow the kernel to depend additionally on the temperature. More specifically, we consider a total free energy given by the formula

$$\begin{aligned}\mathcal{F}[\theta, \chi] = \int_\Omega \Big(& c_V\theta(x)(1 - \ln\theta(x)) + \theta(x)\sigma(\chi(x)) + \lambda(\chi(x)) + (\beta + \theta(x))\varphi(\chi(x)) \\ & + \int_\Omega K(\theta(x) + \theta(y), x, y)\, G(\chi(x) - \chi(y))\,dy\Big)\,dx,\end{aligned} \tag{1.5}$$

where $c_V > 0$ is the specific heat, σ and λ are smooth functions describing the local dependence on χ of the entropy and of the latent heat, respectively, φ is a general proper, convex, and lower semicontinuous function, $\beta > 0$ is a constant parameter, $K : \mathbb{R}^+ \times \Omega \times \Omega \to \mathbb{R}$ is a sufficiently regular symmetric kernel describing nonlocal interactions, and G is an even smooth function having some boundedness properties on the domain of φ that we specify below.

We consider the evolution system (1.3–1.4) in a fixed time interval $t \in [0, T]$ and denote $Q_T := \Omega \times (0, T)$. The state variables θ and χ in (1.5) will therefore depend also on t. In order to keep the formulæ as simple as possible, we will not explicitly mention the dependence on t. We however preserve, in order to emphasize the non-local character of the problem, the dependence on the spatial variable. In other words, for $x, y \in \Omega$ and $t \in [0, T]$ we write $\theta(x), \theta(y)$ instead of $\theta(x,t), \theta(y,t)$, while $\theta_t(x), \theta_t(y)$ will denote the partial derivative with respect to t of $\theta(x,t), \theta(y,t)$ etc. With this convention, we derive in the next section from (1.3–1.5) the following system of equations for $(x, t) \in Q_T$,

$$
\begin{aligned}
(1.6)\quad & c_V\theta_t(x) - 2\theta(x)\int_\Omega K_{\tau\tau}(\tau, x, y)\big|_{\tau=\theta(x)+\theta(y)}(\theta_t(x)+\theta_t(y))\,G(\chi(x)-\chi(y))\,dy \\
& = \kappa\Delta\theta(x) + 2\theta(x)\int_\Omega K_\tau(\tau, x, y)\big|_{\tau=\theta(x)+\theta(y)} G'(\chi(x)-\chi(y))(\chi_t(x)-\chi_t(y))dy \\
& - (\lambda(\chi(x)) + \beta\varphi(\chi(x)))_t - 2\chi_t(x)\int_\Omega K(\theta(x)+\theta(y), x, y)G'(\chi(x)-\chi(y))dy,
\end{aligned}
$$

$$
\begin{aligned}
(1.7)\quad & \mu(\theta(x))\chi_t(x) + \theta(x)\sigma'(\chi(x)) + \lambda'(\chi(x)) \\
& + 2\int_\Omega K(\theta(x)+\theta(y), x, y)G'(\chi(x)-\chi(y))\,dy \in -(\beta+\theta(x))\partial\varphi(\chi(x)),
\end{aligned}
$$

which we couple with boundary and initial conditions

$$
(1.8)\qquad \partial_{\mathbf{n}}\theta(x,t) = 0, \quad \text{on } \partial\Omega \times (0,T),
$$

$$
(1.9)\qquad \chi(x,0) = \chi_0(x), \quad \theta(x,0) = \theta_0(x) \quad \text{in } \Omega,
$$

where Δ is the Laplace operator, the subscripts $_t$ and $_\tau$ denote partial derivatives, $\kappa > 0$ is a constant which stands for the heat conductivity, $\partial_{\mathbf{n}}$ denotes the normal derivative, and χ_0, θ_0 are given initial configurations.

Under technical assumptions on the data, we prove that the above system admits a strong solution, the absolute temperature remains positive if it is initially positive, and the Clausius-Duhem inequality (Second Principle of Thermodynamics) is satisfied in a distributional sense. The problem of uniqueness still seems to be open.

The paper is organized as follows. In Section 2 we derive in detail the equations (1.6–1.9) from the above considerations, show the thermodynamic consistency of the model, and state the main results. We postpone the proofs of existence to Section 4. Section 3 is devoted to some auxiliary results, namely an analysis of the solution operator which with each given θ associates the solution χ to (1.7), and a maximum principle for Eq. (1.6).

2 Main results

2.1 Thermodynamic consistency

We first specify formally the way from (1.3–1.4) to (1.6–1.7) with the choice (1.5) of the free energy. Let us consider a test function $h(x)$ and compute according to (1.1) the entropy as the Fréchet derivative of $\mathcal{F}$ with respect to θ, which in direction h yields

$$\begin{aligned}(2.1)\quad \langle \mathcal{S}[\theta,\chi],h\rangle = -\langle \delta_\theta \mathcal{F}[\theta,\chi],h\rangle = \int_\Omega \Big(&(c_V \ln\theta(x) - \sigma(\chi(x)) - \varphi(\chi(x)))\, h(x) \\ &- \int_\Omega K_\tau(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)} (h(x)+h(y)) G(\chi(x)-\chi(y))\, dy\Big)\, dx\,.\end{aligned}$$

The internal energy $\mathcal{E}$ is given by (1.2), that is,

$$\begin{aligned}(2.2)\quad \mathcal{E}[\theta,\chi] = \int_\Omega \Big(&c_V\theta(x) + \lambda(\chi(x)) + \beta\varphi(\chi(x)) \\ &+ \int_\Omega (K(\tau,x,y) - \tau K_\tau(\tau,x,y))\,\big|_{\tau=\theta(x)+\theta(y)} G(\chi(x)-\chi(y))\, dy\Big)\, dx\,.\end{aligned}$$

Assuming that time differentiation and space integration can be interchanged in the energy conservation law (1.3), we obtain, using the symmetry of the kernel K, that

$$(2.3)\qquad \int_\Omega \bar{E}[\theta,\chi](x)\, dx \;=\; 0\,,$$

where we set

$$\begin{aligned}(2.4)\qquad \bar{E}[\theta,\chi](x) \;:=\; &c_V\theta_t(x) + (\lambda(\chi(x)) + \beta\varphi(\chi(x)))_t \\ &- 2\theta(x)\int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)} (\theta_t(x)+\theta_t(y))\, G(\chi(x)-\chi(y))\, dy \\ &+ 2\chi_t(x)\int_\Omega K(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)} G'(\chi(x)-\chi(y))\, dy \\ &- 2\theta(x)\int_\Omega K_\tau(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)} G'(\chi(x)-\chi(y))(\chi_t(x)-\chi_t(y))\, dy\,.\end{aligned}$$

Formally, by (2.3), there exists a vector function $\mathbf{q}$ (the *heat flux*) such that $\mathbf{q}\cdot\mathbf{n}=0$ on $\partial\Omega$ ($\mathbf{n}$ is the unit outward normal) and

$$(2.5)\qquad \bar{E} + \nabla\cdot\mathbf{q} = 0.$$

Assuming now the Fourier law $\mathbf{q} := -\kappa\nabla\theta$, where $\kappa > 0$ denotes the constant heat conductivity, we obtain (1.6) and (1.8) as energy balance. Note the presence of θ_t in non-local form. The parabolicity of the equation (hence the maximum principle) can be ensured only if the product $\tau\, K_{\tau\tau} G$ is small with respect to c_V. This will follow from Hypothesis 2.1 below. By the same argument, the internal energy in (2.2) is bounded from below for $\theta > 0$ provided λ and φ are bounded from below.

The order parameter equation (1.4) is compatible with the Lagrange method which consists in choosing the multiplier $\ell(x)$ in order to maximize the augmented entropy

$$
\begin{aligned}
\mathcal{S}_\ell[\theta,\chi] &:= \langle \mathcal{S}[\theta,\chi],\ell\rangle + \mathcal{E}[\theta,\chi] \\
&= \int_\Omega c_V\left(\theta(x)+\ell(x)\ln\theta(x)\right) + \lambda(\chi(x)) - \sigma(\chi(x))\ell(x) + (\beta-\ell(x))\varphi(\chi(x))\,dx \\
&\quad + \int_\Omega\int_\Omega G(\chi(x)-\chi(y))\Big[K(\theta(x)+\theta(y),x,y) \\
&\quad - (\theta(x)+\theta(y)+\ell(x))K_\tau(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)}\Big]\,dy\,dx\,.
\end{aligned}
$$

Indeed, the first Euler-Lagrange equation for critical points reads

$$
\begin{aligned}
\langle \delta_\theta \mathcal{S}_\ell[\theta,\chi],h\rangle = \int_\Omega c_V h(x)\left(1+\frac{\ell(x)}{\theta(x)}\right)\,dx - \int_\Omega\int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)} \\
\times (\theta(x)+\theta(y)+\ell(x)+\ell(y))(h(x)+h(y))G(\chi(x)-\chi(y))\,dy\,dx = 0
\end{aligned}
$$

for every test function $h(x)$. Putting $h(x)=\theta(x)(\theta(x)+\ell(x))$ we obtain

$$
\begin{aligned}
0 = \int_\Omega c_V(\ell(x)+\theta(x))^2 dx - \int_\Omega\int_\Omega G(\chi(x)-\chi(y))K_{\tau\tau}(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)} \\
\times \big(\theta(x)+\ell(x)+\theta(y)+\ell(y)\big)\big(\theta(x)(\theta(x)+\ell(x))+\theta(y)(\theta(y)+\ell(y))\big)dy\,dx.
\end{aligned}
\tag{2.6}
$$

Assume that $\theta(x)>0$ for almost all x (this will be established in the next sections), and let $\Gamma>0$ be an upper bound for $|\tau\, K_{\tau\tau}\, G|$ for all admissible arguments. The double integral on the right hand side in (2.6), which we denote by J, can be estimated from above as

$$
\begin{aligned}
|J| &\le 2\int_\Omega\int_\Omega \Big|G(\chi(x)-\chi(y))K_{\tau\tau}(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)}\Big|\,\theta(x)\big((\theta(x)+\ell(x))^2 \\
&\qquad + |\theta(x)+\ell(x)|\,|\theta(y)+\ell(y)|\big)\,dy\,dx \\
&\le 2\Gamma\int_\Omega\int_\Omega \big((\theta(x)+\ell(x))^2 + |\theta(x)+\ell(x)|\,|\theta(y)+\ell(y)|\big)\,dy\,dx \\
&\le 4\Gamma|\Omega|\int_\Omega |\theta(x)+\ell(x)|^2\,dx.
\end{aligned}
\tag{2.7}
$$

It suffices to require that $4\Gamma|\Omega| < c_V$ (which follows from Hypothesis 2.1 below), and from (2.6) with (2.7) we get $\ell=-\theta$. With this Lagrange multiplier, we have $\mathcal{S}_\ell[\theta,\chi] = -\mathcal{F}[\theta,\chi]$, so that Eq. (1.4) can also be interpreted as the postulate that the evolution of χ runs in the direction $\delta_\chi\mathcal{S}_\ell[\theta,\chi]$ of increasing augmented entropy. Then (1.7) just follows from the definition (1.5) of $\mathcal{F}$.

To conclude this subsection, we check that the model is compatible with the Second Principle of Thermodynamics. Assuming again that $\theta>0$, let us verify that for every test function $h(x)\ge 0$ we have

$$
\frac{d}{dt}\langle \mathcal{S}[\theta,\chi],h\rangle + \int_\Omega \operatorname{div}\left(\frac{\mathbf{q}(x)}{\theta(x)}\right)\,h(x)\,dx \ge 0\,,
\tag{2.8}
$$

that is just the pointwise Clausius-Duhem inequality in the distributional sense.

We first differentiate (2.1) and obtain

$$
\begin{aligned}
(2.9)\qquad \frac{d}{dt}\langle \mathcal{S}[\theta,\chi],h\rangle &= \int_\Omega \left(c_V\frac{\theta_t(x)}{\theta(x)} - (\sigma(\chi(x)) + \varphi(\chi(x)))_t \right) h(x)\,dx \\
&- 2\int_\Omega h(x)\int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)}(\theta_t(x)+\theta_t(y))G(\chi(x)-\chi(y))\,dy\,dx \\
&- 2\int_\Omega h(x)\int_\Omega K_\tau(\tau,x,y)G'(\chi(x)-\chi(y))(\chi_t(x)-\chi_t(y))\,dy\,dx\,.
\end{aligned}
$$

Using (1.6) we have

$$
\begin{aligned}
(2.10)\quad \int_\Omega \operatorname{div}\left(\frac{\mathbf{q}(x)}{\theta(x)}\right) h(x)\,dx &= \int_\Omega \frac{\operatorname{div}\mathbf{q}(x)}{\theta(x)} h(x)\,dx - \int_\Omega \frac{\mathbf{q}(x)\cdot\nabla\theta(x)}{\theta^2(x)} h(x)\,dx \\
&= \int_\Omega \frac{\kappa\,|\nabla\theta(x)|^2}{\theta^2(x)} h(x)\,dx - \kappa\int_\Omega \frac{\Delta\theta(x)}{\theta(x)} h(x)\,dx \\
&\ge -\int_\Omega c_V\frac{\theta_t(x)}{\theta(x)} h(x)\,dx - \int_\Omega (\lambda(\chi(x)) + \beta\varphi(\chi(x)))_t \frac{h(x)}{\theta(x)}\,dx \\
&+ 2\int_\Omega h(x)\int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)}(\theta_t(x)+\theta_t(y))\,G(\chi(x)-\chi(y))dy\,dx \\
&+ 2\int_\Omega h(x)\int_\Omega K_\tau(\tau,x,y)\big|_{\tau=\theta(x)+\theta(y)}\,G'(\chi(x)-\chi(y))(\chi_t(x)-\chi_t(y))dy\,dx \\
&- 2\int_\Omega \chi_t(x)\left(\int_\Omega K(\theta(x)+\theta(y),x,y)\,G'(\chi(x)-\chi(y))\,dy\right)\frac{h(x)}{\theta(x)}\,dx.
\end{aligned}
$$

From (1.7) it follows that there exists some $\xi(x)\in\partial\varphi(\chi(x))$ such that

$$
\begin{aligned}
(2.11)\quad & \mu(\theta(x))\chi_t(x) + \theta(x)\sigma'(\chi(x)) + \lambda'(\chi(x)) \\
&\qquad + 2\int_\Omega K(\theta(x)+\theta(y),x,y)\,G'(\chi(x)-\chi(y))\,dy + (\beta+\theta(x))\xi(x) \;=\; 0\,.
\end{aligned}
$$

We have $\xi(x)\,\chi_t(x) = \varphi(\chi(x))_t$ a. e. in Q_T, and using (2.9–2.10), we obtain that

$$
\frac{d}{dt}\langle \mathcal{S}[\theta,\chi],h\rangle + \int_\Omega \operatorname{div}\left(\frac{\mathbf{q}(x)}{\theta(x)}\right) h(x)\,dx \ge \int_\Omega \frac{\mu(\theta(x))}{\theta(x)}\chi_t^2(x)\,h(x)\,dx \;\ge\; 0,
$$

which is exactly the desired inequality (2.8).

2.2 Existence of solutions

In this subsection, we state our main results on solvability conditions for the system (1.6–1.9). Throughout the paper, the following assumptions on the data are supposed to hold.

Hypothesis 2.1. *We consider a bounded domain $\Omega\subset\mathbb{R}^N$ with Lipschitzian boundary, N being an arbitrary integer, $T>0$ is a fixed final time, and for $t\in[0,T]$*

we denote $Q_t = \Omega \times (0,t)$. In addition to the fixed positive parameters c_V and β in (1.5), we assume the existence of positive constants $S, M, R, \zeta, \mu_*, \mu^*$ such that $8|\Omega|\zeta M \le c_V$, and that

(i) *$\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous function, $\mathcal{D}(\varphi)$ is its domain;*

(ii) *$\sigma,\ \lambda \in W^{2,\infty}(\mathcal{D}(\varphi))$, $\max\{|\sigma'(z)|, |\lambda'(z)|\} \le S$ for all $z \in \mathcal{D}(\varphi)$;*

(iii) *$G \in W^{2,\infty}(\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$, $G(z) = G(-z)$ for all $z \in (\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$, and the estimate $|G(z_1 - z_2)| \le M$ holds for all $z_1, z_2 \in \mathcal{D}(\varphi)$;*

(iv) *the real function K of variables $(\tau, x, y) \in \mathbb{R}^+ \times \Omega \times \Omega$ and its derivatives $K_\tau, K_{\tau\tau}$ are bounded, continuous in τ for a. e. $x, y \in \Omega$, measurable in x, y for all $\tau \ge 0$, and $K(\tau, x, y) = K(\tau, y, x)$, $|K(\tau, x, y)| \le R$, $|\tau\, K_{\tau\tau}(\tau, x, y)| \le \zeta$, for all $\tau \ge 0$ and a. e. $x, y \in \Omega$;*

(v) *μ is locally Lipschitz in $\mathbb{R}^+$, $\mu_*(1+\tau) \le \mu(\tau) \le \mu^*(1+\tau)$ for all $\tau \in \mathbb{R}^+$.*

Let us first introduce some notation. For any $C > 0$ we denote

$$\mathcal{D}_C(\varphi) \ = \ \{\chi \in \mathcal{D}(\varphi)\,;\, \partial\varphi(\chi) \cap [-C, C] \ne \emptyset\}\,. \tag{2.12}$$

By [3, Example 2.3.4], $\partial\varphi$ is maximal monotone, hence $\mathcal{D}_C(\varphi)$ is a closed (possibly unbounded or degenerate) interval for every $C > 0$.

We use, for the sake of simplicity, the same symbol H for both $L^2(\Omega)$ and $L^2(\Omega\,;\,\mathbb{R}^N)$. We further denote by V the space $H^1(\Omega)$, and identify H with a subspace of the topological dual V' to V. The symbol $\langle\cdot,\cdot\rangle$ is used for both the scalar product in H and the duality pairing between V' and V.

We are in the position of stating the existence theorem.

Theorem 2.2. *Let the Hypothesis 2.1 hold, let the initial data in (1.9) satisfy the conditions*

$$\begin{aligned} &\theta_0 \in V \cap L^\infty(\Omega)\,,\ \exists \theta_* > 0\,:\, \theta_0(x) \ge \theta_* \quad \text{a. e. in } \Omega\,,\\ &\chi_0 \in L^\infty(\Omega)\,,\ \exists\, C_0 > 0\,:\, \chi_0(x) \in \mathcal{D}_{C_0}(\varphi) \quad \text{a. e. in } \Omega\,. \end{aligned} \tag{2.13}$$

Then there exists at least one pair (θ, χ) which solves (1.6–1.9) a. e. in Q_T and such that

$$\theta \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(Q_T) \hookrightarrow C^0([0,T];V)\,, \tag{2.14}$$

$$\theta(x,t) > 0 \quad \text{a. e. in } Q_T\,, \tag{2.15}$$

$$\chi, \chi_t \in L^\infty(Q_T), \quad \exists\, C > 0\,:\, \chi(x,t) \in \mathcal{D}_C(\varphi) \quad \text{a. e. in } Q_T\,. \tag{2.16}$$

Remark 2.3. If we compare Hypothesis 2.1 (v) with Eq. (1.7), we see that the phase transition speed is controlled by the entropy at very large temperatures and by the latent heat at low temperatures. Furthermore, the global bound for χ_t is obtained by dividing Eq. (1.7) by $\beta+\theta$. The case $\varphi = I_{[0,1]}$ is somewhat special in the sense that $\partial\varphi$ remains invariant when multiplied by any positive scalar, so that the question whether $\beta > 0$ or $\beta = 0$ is irrelevant similarly as in [12]. For the same reason, only the lower bound for $\mu(\theta)$ in Hypothesis 2.1 (v) is needed in this case. A possible growth in θ of $\mu(\theta)$ can again be compensated by putting formally in front of $\partial\varphi$ a factor, say, $1+\theta+\mu(\theta)$. For general potentials φ, the situation is analogous to [15], where the assumption $\beta > 0$ also seems to be necessary.

3 Auxiliary results

In this section we provide some auxiliary results that are used in the rest of the paper. The first part of this section deals with the local Lipschitz continuity of solution operators to general differential inclusions, while the second one recalls some parabolic maximum principle results.

3.1 Solution operators to differential inclusions

Consider a functional φ as in Hypothesis 2.1 (i). For a given initial condition χ_0, and a given function $\theta \in L^1(Q_T)$, we solve the following differential inclusion

$$(3.1) \qquad \alpha(\theta)\,\chi_t + \partial\varphi(\chi) \ \ni\ f[\chi,\theta] \quad \text{a.e. in } Q_T, \quad \chi(x,0) = \chi_0(x) \quad \text{a.e. in } \Omega,$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a given function, and where $f : L^1(Q_T) \times L^1(Q_T) \to L^\infty(Q_T)$ is a given operator satisfying the following hypothesis.

Hypothesis 3.1. *There exist positive constants α_0, L, C such that*

(i) $\alpha_0 \le \alpha(\theta)$ *for all* $\theta \in \mathbb{R}$*;*

(ii) $|\alpha(\theta_1) - \alpha(\theta_2)| \ \le\ L|\theta_1 - \theta_2|$ *for all* $\theta_1,\, \theta_2 \in \mathbb{R}$*;*

(iii) $|f[\chi,\theta](x,t)| \le C$ *a. e. in* Q_T *for all* $\chi, \theta \in L^1(Q_T)$ *such that* $\chi(x,t) \in \mathcal{D}(\varphi)$ *a. e. in* Q_T*;*

(iv) $|f[\chi_1,\theta_1] - f[\chi_2,\theta_2]|_{L^1(Q_t)} \ \le\ L(|\chi_1 - \chi_2|_{L^1(Q_t)} + |\theta_1 - \theta_2|_{L^1(Q_t)})$

for all $\chi_1, \theta_1, \chi_2, \theta_2 \in L^1(Q_T)$ *and* $t \in [0,T]$.

The main result of this subsection reads as follows.

Proposition 3.2. *Let Hypothesis 3.1 hold, and let $\mathcal{D}_C(\varphi)$ be as in (2.12). Then for every $\theta \in L^1(Q_T)$ and for every $\chi_0 \in L^\infty(\Omega)$, $\chi_0(x) \in \mathcal{D}_C(\varphi)$ a. e. in Ω, there exists a unique solution $\chi \in L^\infty(Q_T)$ to Eq. (3.1) such that $\chi_t \in L^\infty(Q_T)$, and we have*

$$(3.2) \qquad \chi(x,t) \in \mathcal{D}_C(\varphi)\,, \quad |f[\chi,\theta](x,t) - \alpha(\theta(x,t))\,\chi_t(x,t)| \le C \quad \text{a.e. in } Q_T\,.$$

Moreover, there exists a positive constant M such that the solutions $\chi_1, \chi_2 \in L^\infty(Q_T)$ associated with $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$ and $\theta_1, \theta_2 \in L^1(Q_T)$ satisfy for all $t \in [0,T]$ the inequality

$$(3.3)\quad |(\chi_1)_t - (\chi_2)_t|_{L^1(Q_t)} + |(\chi_1 - \chi_2)(t)|_{L^1(\Omega)} \ \le\ M\Big(|\chi_{01} - \chi_{02}|_{L^1(\Omega)} + |\theta_1 - \theta_2|_{L^1(Q_t)}\Big)\,.$$

Remark 3.3. The L^1-Lipschitz continuity estimate (3.3) in Proposition 3.2 cannot be extended to $L^p(Q_T)$ for $p > 1$, see [11, Example 3], except in the case when φ is a C^1-function with locally Lipschitz continuous derivative. Strong continuity $L^1(Q_T) \to L^p(Q_T)$ of the solution mapping for $p < \infty$ follows however from the uniform L^∞-bound (3.2). Indeed, testing (3.1) by χ_t, we obtain the identity

$$(3.4)\qquad \varphi(\chi)_t \ =\ -\alpha(\theta)\,\chi_t^2 + f[\chi,\theta]\,\chi_t \quad \text{a.e. in } Q_T\,.$$

Let now $\theta^{(n)}, \theta$ be such that $\theta^{(n)} \to \theta$ strongly in $L^1(Q_T)$ as $n \to \infty$, and let $\chi^{(n)}, \chi$ be the corresponding solutions to Eq. (3.1). Using Proposition 3.2 and taking into account the L^∞-bound (3.2), we see that $\chi^{(n)} \to \chi$, $\chi_t^{(n)} \to \chi_t$, $\varphi(\chi^{(n)})_t \to \varphi(\chi)_t$ strongly in any $L^p(Q_T)$ for $1 \le p < \infty$ as a consequence of the Lebesgue Dominated Convergence Theorem.

Let us start with a space-independent problem. For a given initial condition $\chi_0 \in \mathcal{D}(\varphi)$ and a given function $\theta \in L^1(0,T)$ we consider the differential inclusion

$$(3.5)\qquad \alpha(\theta(t))\,\dot\chi(t) + \partial\varphi(\chi(t)) \ \ni\ g(t) \quad \text{a.e. in } (0,T), \quad \chi(0) = \chi_0,$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is as in Hypothesis 3.1 and $g \in L^\infty(0,T)$ is such that

$$(3.6)\qquad |g(t)| \le C \quad \text{a.e. in } (0,T).$$

We prove the following result.

Proposition 3.4. *Let Hypotheses 3.1 (i–ii) and (3.6) hold. Then for every $\theta \in L^1(0,T)$ and every $\chi_0 \in \mathcal{D}_C(\varphi)$, there exists a unique solution $\chi \in W^{1,\infty}(0,T)$ to Eq. (3.5), and we have*

$$(3.7)\qquad \chi(t) \in \mathcal{D}_C(\varphi)\ \ \forall t \in [0,T]\,, \quad \big|g(t) - \alpha(\theta(t))\,\dot\chi(t)\big| \ \le\ C \quad \text{a.e. in } (0,T)\,.$$

Moreover, there exists a positive constant R depending only on C, α_0, and L, such that the solutions χ_1, $\chi_2 \in W^{1,\infty}(0,T)$ associated with $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$, $\theta_1, \theta_2 \in L^1(0,T)$, and $g_1, g_2 \in L^\infty(0,T)$ with the constraint (3.6) satisfy the inequality

$$(3.8)\quad |\dot\chi_1 - \dot\chi_2|(t) + \frac{d}{dt}|\chi_1 - \chi_2|(t) \le R\Big(|\theta_1 - \theta_2|(t) + |g_1 - g_2|(t)\Big) \quad \text{a.e. in } (0,T).$$

Proof of Proposition 3.4. We first prove the existence of solutions. We fix $\theta \in L^1(0,T)$, $\chi_0 \in \mathcal{D}_C(\varphi)$ and, for $n \in \mathbb{N}$ and $k = 1, \dots, n$, define the sequences

$$(3.9)\qquad \alpha_k = \frac{n}{T}\int_{t_{k-1}}^{t_k} \alpha(\theta(t))\,dt\,, \ \ g_k = \frac{n}{T}\int_{t_{k-1}}^{t_k} g(t)\,dt\,,$$

$$\chi_k = \Big(\frac{n\alpha_k}{T} I + \partial\varphi\Big)^{-1} \Big(g_k + \frac{n\alpha_k}{T}\, \chi_{k-1}\Big) \tag{3.10}$$

corresponding to the partition $t_0 = 0$, $t_k = Tk/n$, where $I(u) = u$ is the identity mapping. Assume that for some $k \ge 1$ we have

$$\partial\varphi(\chi_k) \ \ni \ g_k - \frac{n\alpha_k}{T}(\chi_k - \chi_{k-1}) \ > \ C\,. \tag{3.11}$$

By hypothesis we have $|g_k| \le C$, hence $\max \mathcal{D}_C(\varphi) \le \chi_k < \chi_{k-1}$ by the monotonicity of $\partial\varphi$. This yields, if $k-1 > 0$, that

$$g_{k-1} - \frac{n\alpha_k}{T}(\chi_{k-1} - \chi_{k-2}) \ \ge \ g_k - \frac{n\alpha_k}{T}(\chi_k - \chi_{k-1}) \ > \ C\,, \tag{3.12}$$

and by induction we obtain $\max \mathcal{D}_C(\varphi) \le \chi_k < \chi_{k-1} < \cdots < \chi_0$ which is a contradiction. We obtain a similar contradiction by assuming that $g_k - (n\alpha_k/T)(\chi_k - \chi_{k-1}) < -C$. Using the fact that $\alpha_k \ge \alpha_0$, we thus have for all $k = 1, \dots, n$ that

$$\Big|g_k - \frac{n\alpha_k}{T}(\chi_k - \chi_{k-1})\Big| \ \le \ C\,, \quad \chi_k \in \mathcal{D}_C(\varphi)\,, \quad |\chi_k - \chi_{k-1}| \ \le \ \frac{2CT}{n\alpha_0}\,. \tag{3.13}$$

We now define the interpolates

$$\alpha^{(n)}(t) = \alpha_k\,, \quad g^{(n)}(t) = g_k\,, \quad \bar\chi^{(n)}(t) = \chi_k\,, \quad \underline{\chi}^{(n)}(t) = \chi_{k-1}\,, \tag{3.14}$$

$$\chi^{(n)}(t) \ = \ \chi_{k-1} + \frac{n}{T}(t - t_{k-1})(\chi_k - \chi_{k-1}) \tag{3.15}$$

for $t \in [t_{k-1}, t_k)$, continuously extended to $t = T$. The functions $\chi^{(n)}$ are bounded in $W^{1,\infty}(0,T)$ uniformly with respect to $n \in \mathbb{N}$. Passing to a subsequence, if necessary, we find $\chi \in W^{1,\infty}(0,T)$ such that $\chi(0) = \chi_0$, $\dot\chi^{(n)} \to \dot\chi$ in $L^\infty(0,T)$ weakly-star, and $\chi^{(n)} \to \chi$ uniformly in $[0,T]$. Using the inequalities

$$|\chi^{(n)}(t) - \bar\chi^{(n)}(t)| \ \le \ \frac{2CT}{n\alpha_0}\,, \quad |\chi^{(n)}(t) - \underline{\chi}^{(n)}(t)| \ \le \ \frac{2CT}{n\alpha_0}\,, \tag{3.16}$$

we also see that $\bar\chi^{(n)} \to \chi$, $\underline{\chi}^{(n)} \to \chi$ uniformly. Using the Mean Continuity Theorem for functions in $L^1(0,T)$, we conclude that $\alpha^{(n)}$ converge to $\alpha(\theta(\cdot))$ strongly in $L^1(0,T)$, and $g^{(n)}$ converge to g strongly in any $L^p(0,T)$ for $1 \le p < \infty$ and weakly-star in $L^\infty(0,T)$. Let now $z \in L^\infty(0,T)$ be a test function, $z(t) \ge 0$ a.e. in $(0,T)$, and let $w \in \mathcal{D}(\varphi)$, $\xi \in \partial\varphi(w)$ be arbitrary. By construction, we have

$$(g^{(n)}(t) - \alpha^{(n)}(t)\,\dot\chi^{(n)}(t) - \xi)(\bar\chi^{(n)}(t) - w) \ \ge \ 0 \quad \text{a.e. in } (0,T)\,, \tag{3.17}$$

hence

$$\int_0^T (g^{(n)}(t) - \alpha^{(n)}(t)\dot\chi^{(n)}(t) - \xi)(\bar\chi^{(n)}(t) - w)\, z(t)\, dt \ \ge \ 0\,. \tag{3.18}$$

Passing to the limit as $n \nearrow \infty$ in (3.18) we obtain

$$(g(t) - \alpha(\theta(t))\,\dot\chi(t) - \xi)(\chi(t) - w) \ \ge \ 0 \quad \text{a.e.} \tag{3.19}$$

Since $\partial\varphi$ is maximal monotone, the function χ satisfies Eq. (3.5). Estimate (3.7) follows from (3.13).

We now prove inequality (3.8) which also implies uniqueness of solutions to Eq. (3.5). Let $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$, $\theta_1, \theta_2 \in L^1(0,T)$, and g_1, $g_2 \in L^\infty(0,T)$ be functions satisfying (3.6), and let $\chi_1, \chi_2 \in W^{1,\infty}(0,T)$ be corresponding solutions of Eq. (3.5). For $i = 1,2$ and $t \in (0,T)$ put

$$\xi_i(t) \;=\; g_i(t) - \alpha(\theta_i(t))\,\dot{\chi}_i(t)\,. \tag{3.20}$$

As $\partial\varphi$ is monotone and $\xi_i(t) \in \partial\varphi(\chi_i(t))$ for $i = 1,2$ a. e. in $(0,T)$, we have

$$\big(\xi_1(t) - \xi_2(t)\big)\,\big(\chi_1(t) - \chi_2(t)\big) \;\geq\; 0 \quad \text{a. e.} \tag{3.21}$$

We test the identity

$$\begin{aligned} \xi_1(t) - \xi_2(t) \;+\; \alpha(\theta_1(t))\,\big(\dot{\chi}_1(t) - \dot{\chi}_2(t)\big) \;=\; & \dot{\chi}_2(t)\,\big(\alpha(\theta_2(t)) - \alpha(\theta_1(t))\big) \\ & + g_1(t) - g_2(t) \quad \text{a. e.} \end{aligned} \tag{3.22}$$

by the sign of $\xi_1(t) - \xi_2(t)$ if $\xi_1(t) \neq \xi_2(t)$, or otherwise by the sign of $\chi_1(t) - \chi_2(t)$. By virtue of (3.21), this yields

$$\begin{aligned} & \big|\xi_1 - \xi_2\big|(t) \;+\; \alpha(\theta_1(t))\,\frac{d}{dt}\big|\chi_1 - \chi_2\big|(t) \\ & \quad \leq\; \big|\dot{\chi}_2(t)\big|\,\big|\alpha(\theta_1(t)) - \alpha(\theta_2(t))\big| + \big|g_1 - g_2\big|(t) \quad \text{a. e.}\ , \end{aligned} \tag{3.23}$$

hence

$$\begin{aligned} & \alpha(\theta_1(t))\,\left(\big|\dot{\chi}_1 - \dot{\chi}_2\big|(t) + \frac{d}{dt}\big|\chi_1 - \chi_2\big|(t)\right) \\ & \quad \leq\; 2\big|g_1 - g_2\big|(t) + 2\,\big|\dot{\chi}_2(t)\big|\,\big|\alpha(\theta_1(t)) - \alpha(\theta_2(t))\big| \quad \text{a. e.} \end{aligned} \tag{3.24}$$

Using the estimates

$$\alpha(\theta_i(t)) \;\geq\; \alpha_0 \ \text{ for } i = 1,2, \quad \big|\dot{\chi}_2(t)\big| \;\leq\; \frac{2C}{\alpha_0} \quad \text{a. e.} \tag{3.25}$$

and Hypothesis 3.1, we obtain from (3.24) that

$$\begin{aligned} & \big|\dot{\chi}_1 - \dot{\chi}_2\big|(t) + \frac{d}{dt}\big|\chi_1 - \chi_2\big|(t) \\ & \quad \leq\; \frac{2}{\alpha_0}\big|g_1 - g_2\big|(t) + \left(\frac{4CL}{\alpha_0^2}\right)\big|\theta_1 - \theta_2\big|(t) \quad \text{a. e.}\ , \end{aligned} \tag{3.26}$$

that is exactly (3.8). ∎

We now use this result to prove Proposition 3.2.

Proof of Proposition 3.2. For given $\theta \in L^1(Q_T)$ and $\chi_0 \in L^\infty(\Omega)$, $\chi_0(x) \in \mathcal{D}_C(\varphi)$ a. e. we prove the existence of a unique solution to (3.1) by the Banach contraction argument. We define the set

$$
S := \left\{ v \in L^\infty(Q_T) : \begin{array}{l} v_t \in L^\infty(Q_T)\,,\ |v_t|_{L^\infty(Q_T)} \le 2C/\alpha_0\,, \\ v(x,0) \;=\; \chi_0(x) \ \text{ a. e. in } \Omega \end{array} \right\} \tag{3.27}
$$

as a closed subset of $L^1(Q_T)$ endowed with the weighted norm

$$
|v|_{RL} := \int_0^T \mathrm{e}^{-2RLt} \int_\Omega |v(x,t)|\,dx\,dt\,, \tag{3.28}
$$

where $R = R(C, \alpha_0, L)$ and L are as in Hypothesis 3.1 and Proposition 3.4. For an arbitrary $\tilde\chi \in S$ we put $\tilde g(x,t) = f[\tilde\chi, \theta](x,t)$, and define $\chi(x,t)$ as the solution of the differential inclusion

$$
\begin{aligned} \alpha(\theta(x,t))\,\chi_t(x,t) + \partial\varphi(\chi(x,t)) &\ni \tilde g(x,t) && \text{a. e. in } Q_T\,, \\ \chi(x,0) &= \chi_0(x) && \text{a. e. in } \Omega\,. \end{aligned} \tag{3.29}
$$

For almost every $x \in \Omega$, this inclusion is of the form (3.5) with right-hand side satisfying (3.6). By Proposition 3.4, the function χ belongs to S, and we may define the solution mapping $\mathcal{T} : S \to S : \tilde\chi \mapsto \chi$. We check that $\mathcal{T}$ is a contraction with respect to the norm (3.28). Indeed, let us use estimate (3.8) with $\theta_1 = \theta_2$, $g_i(t) = f[\tilde\chi_i, \theta_i](t)$ for $i = 1, 2$. This, thanks to Hypothesis 3.1, leads to

$$
\frac{d}{dt}\int_\Omega |\chi_1 - \chi_2|(x,t)\,dx \;\le\; RL\int_\Omega |\tilde\chi_1 - \tilde\chi_2|(x,t)\,dx\,. \tag{3.30}
$$

Now, multiplying both sides of this inequality by e^{-2RLt}, and integrating over $[0,T]$, we infer that

$$
|\chi_1 - \chi_2|_{RL} \;\le\; \frac{1}{2}|\tilde\chi_1 - \tilde\chi_2|_{RL}\,. \tag{3.31}
$$

Hence $\mathcal{T}$ is a contraction on S, and the Banach fixed point theorem yields the existence and uniqueness of a solution $\chi \in S$ of the differential inclusion (3.1). Estimate (3.2) follows directly from (3.7). Finally, in order to prove (3.3), take $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$, $\theta_1, \theta_2 \in L^1(Q_T)$, and let χ_1, χ_2 be the corresponding solutions of Eq. (3.1). For almost all x we use (3.8) with $g_i(t) = f[\chi_i, \theta_i](x,t)$, $i = 1, 2$. Integrating over Ω and over $(0,t)$ for $t \in (0,T]$, and using Hypothesis 3.1, we obtain that

$$
\begin{aligned} &\int_0^t\int_\Omega |(\chi_1)_t - (\chi_2)_t|(x,s)\,dx\,ds + \int_\Omega |\chi_1 - \chi_2|(x,t)\,dx - |\chi_{01} - \chi_{02}|_{L^1(\Omega)} \\ &\qquad \le \int_0^t\int_\Omega \left(RL|\chi_1 - \chi_2|(x,s) + R(L+1)|\theta_1 - \theta_2|(x,s)\right)dx\,ds, \end{aligned} \tag{3.32}
$$

and Gronwall's argument yields

$$
\begin{aligned} &\int_0^t\int_\Omega |(\chi_1)_t - (\chi_2)_t|(x,s)\,dx\,ds + \int_\Omega |\chi_1 - \chi_2|(x,t)\,dx \\ &\qquad \le \mathrm{e}^{RLt}\left(|\chi_{01} - \chi_{02}|_{L^1(\Omega)} + R(L+1)\int_0^t\int_\Omega |\theta_1 - \theta_2|(x,s)\,dx\,ds\right), \end{aligned} \tag{3.33}
$$

which concludes the proof. ∎

3.2 The maximum principle

We recall for the reader's convenience a maximum principle result for parabolic equations with non-constant coefficients. A substantially more general theory can be found in [13, Chapter 3], but we wish to emphasize the fact that our case can be treated in an elementary way.

Proposition 3.5. *Let u_t, $\Delta u \in L^2(Q_T)$, $u^0 \in V$, $a \in L^\infty(Q_T)$ such that $0 < a_0 \leq a(x,t) \leq a_1$ a. e. in Q_T satisfy*

$$a(x,t)\, u_t - \Delta u \leq 0 \quad \text{a. e. in } Q_T, \quad u(x,0) = u^0(x) \leq 0 \quad \text{a. e. in } \Omega\,, \tag{3.34}$$

with homogeneous Neumann boundary condition. Then $u(x,t) \leq 0$ a. e. in Q_T.

Proof. Let us assume first that $|a_t(x,t)| \leq \tilde{a}(t)$ a. e. in Q_T, with $\tilde{a} \in L^1(0,T)$. Then, testing the inequality (3.34) with the positive part u^+ of u, we get

$$\frac{d}{dt} \int_\Omega a(x,t)\,(u^+)^2\, dx \;\leq\; \tilde{a}(t) \int_\Omega (u^+)^2\, dx.$$

Now, integrating over $(0,t)$ we obtain from Gronwall's Lemma that $a_0 \int_\Omega (u^+)^2\, dx \leq 0$ a. e. in Q_T, and the assertion immediately follows.

Suppose now that $a \in L^\infty(Q_T)$. Put $g := a(x,t)u_t - \Delta u \in L^2(Q_T)$. Then $g \leq 0$ a. e. in Q_T. We choose a sequence of smooth functions a_n such that $a_0 \leq a_n(x,t) \leq a_1$, and $a_n(x,t) \to a(x,t)$ a. e. in Q_T. Now, let u^n be the solution of

$$a_n(x,t)\, u_t^n - \Delta u^n = g, \quad u^n(x,0) = u^0(x)$$

with homogeneous Neumann boundary condition for each n. By the above argument, we have that $u^n(x,t) \leq 0$ a. e. in Q_T. Passing to a subsequence, we find a function $\tilde{u}$ such that and $u_t^n \to \tilde{u}_t$, $\Delta u^n \to \Delta \tilde{u}$ weakly in $L^2(Q_T)$, $u^n \to \tilde{u}$ strongly e. g. in $C([0,T]\,;\, H)$, hence also $\tilde{u}(x,t) \leq 0$ a. e. in Q_T. Both u and $\tilde{u}$ solve the equation $a(x,t)u_t - \Delta u = g$ in Q_T with the same initial and boundary condition, hence $u = \tilde{u}$. This concludes the proof. ∎

Proposition 3.6. *Let $a_1 > a_0 > 0$, $u_* > 0$, $\gamma_0 \in L^1(0,T)$, and $u^0 \in V \cap L^\infty(\Omega)$ be given. Assume that u_t, $\Delta u \in L^2(Q_T)$, $a \in L^\infty(Q_T)$, $\gamma \in L^1(Q_T)$ such that $a_0 \leq a(x,t) \leq a_1$, $|\gamma(x,t)| \leq \gamma_0(t)$ a. e. in Q_T satisfy*

$$a(x,t)\, u_t - \Delta u \geq \gamma(x,t)\, u \quad \text{a. e. in } Q_T, \quad u(x,0) = u^0(x) \geq u_* \quad \text{a. e. in } \Omega \tag{3.35}$$

with homogeneous Neumann boundary condition. Then $u(x,t) > 0$ a. e. in Q_T. If, moreover, there exist non-negative functions $b, c \in L^1(0,T)$ such that

$$a(x,t)\, u_t - \Delta u \leq b(t)\, u + c(t) \quad \text{a. e. in } Q_T\,, \tag{3.36}$$

then there exists a positive constant C depending only on $|b|_{L^1(0,T)}$, $|c|_{L^1(0,T)}$, and $|u^0|_{L^\infty(\Omega)}$ such that $u(x,t) \leq C$ a. e. in Q_T.

Proof. For $t \in [0,T]$ we denote by $e(t)$ the solution of the differential equation

$$\dot{e}(t) \;=\; -k(t)\, e(t)\,, \quad e(0) = 1\,, \qquad \text{where} \;\; k(t) \;=\; \frac{\gamma_0(t)}{a_0}\,, \tag{3.37}$$

(that is, $e(t) = \exp(-\int_0^t k(\tau)\, d\tau))$ and define the auxiliary functions

$$p(u,t) = \; -\left(u_* e(t) - u\right)^+ \;, \tag{3.38}$$

$$P(u,t) = \; \frac{e^2(t)}{2} \left((u_* e(t) - u)^+\right)^2 \tag{3.39}$$

for $t \in [0,T]$ and $u \in \mathbb{R}$. We have

$$\frac{\partial P}{\partial u} = \; e^2(t)\, p(u,t) \;\leq\; 0\,, \tag{3.40}$$

$$\frac{\partial^2 P}{\partial u^2} = \; e^2(t)\, H(u_* e(t) - u) \;\geq\; 0\,, \quad \text{where } H \text{ is the Heaviside function}\,, \tag{3.41}$$

$$\frac{\partial P}{\partial t} = \; -k(t)\, e^2(t) \left(|p(u,t)|^2 - u_* e(t)\, p(u,t)\right)\,. \tag{3.42}$$

Put $v(x,t) = P(u(x,t),t)$. Then

$$v_t = e^2(t) \;\left(p(u,t) u_t - k(t) \left(|p(u,t)|^2 - u_* e(t)\, p(u,t)\right)\right)\,, \tag{3.43}$$

$$\nabla v = e^2(t)\, p(u,t)\, \nabla u\,, \tag{3.44}$$

$$\Delta v = e^2(t)\, H(u_* e(t) - u)\, |\nabla u|^2 + e^2(t)\, p(u,t)\, \Delta u\,, \tag{3.45}$$

hence

$$\begin{aligned} a(x,t)\, v_t - \Delta v \;&=\; e^2(t)\, p(u,t)\; (a(x,t)\, u_t - \Delta u) \\ &\quad - e^2(t)\, H(u_* e(t) - u)\, |\nabla u|^2 - a(x,t)\, k(t)\, e^2(t) \left(|p(u,t)|^2 - u_* e(t)\, p(u,t)\right) \\ &\leq\; a(x,t)\, e^2(t) \left(u\, p(u,t)\, \frac{\gamma(x,t)}{a(x,t)} - k(t)\; \left(|p(u,t)|^2 - u_* e(t)\, p(u,t)\right)\right) \\ &=\; a(x,t)\, e^2(t) \Bigg(- k(t)\, |p(u,t)|^2 + \frac{\gamma(x,t)}{a(x,t)}\, p(u,t)\, (u - u_* e(t)) \\ &\quad + \left(k(t) + \frac{\gamma(x,t)}{a(x,t)}\right) u_* e(t)\, p(u,t) \Bigg)\,. \end{aligned} \tag{3.46}$$

We have $|\gamma(x,t)|/a(x,t) \leq k(t)$, $p(u,t) \leq 0$, $|p(u,t)\, (u - u_* e(t))| = |p(u,t)|^2$ a. e., hence the right-hand side of (3.46) is non-positive. By virtue of Proposition 3.5 we have $v(x,t) \leq 0$ a. e. in Q_T, hence $u(x,t) \geq u_* e(t)$ which we wanted to prove.

Assume now that (3.36) holds. We define functions f and g as solutions to the differential equations

$$\begin{aligned} a_0 \dot{f}(t) &= -b(t)\, f(t), \quad f(0) = 1\,, \\ a_0 \dot{g}(t) &= -c(t)\, f(t), \quad g(0) \leq -\operatorname{ess\,sup}_{x \in \Omega} \{u^0(x)\}\,, \end{aligned} \tag{3.47}$$

and put $w(x,t) = f(t)\,u(x,t) + g(t)$. Then w verifies a. e. in Q_T

$$(3.48)\qquad \begin{aligned} a(x,t)\,w_t - \Delta w \ &= f(t)\,(a(x,t)\,u_t - \Delta u) + a(x,t)\,\Big(\dot f(t)\,u + \dot g(t)\Big) \\ &\le f(t)\,b(t)\,u + f(t)\,c(t) + a_0\,\dot f(t)\,u + a_0\,\dot g(t) \ = \ 0\,. \end{aligned}$$

Moreover, since $w(x,0) \le 0$, we get by (3.48) and Proposition 3.5 the bound for u

$$u(x,t) \ \le \ \frac{|g(t)|}{f(t)} \ \le \ C \quad \text{a. e. in } Q_T\,,$$

which completes the proof. ■

4 Proof of the existence result

This section is devoted to the proofs of the existence result stated in Section 2.2. We use a standard technique based on approximations, a priori estimates, and passage to the limit.

4.1 Approximation

Assuming Hypothesis 2.1 and (2.13) to hold, we proceed as follows: first solve the problem corresponding to (1.6–1.9) in which we regularize the coefficient μ and replace θ by $|\theta|$ at suitable places, and then derive bounds for θ which will allow us to conclude that the solution of the modified problem satisfies also (1.6–1.9). For some $\varrho > 0$ sufficiently large, which will be specified later, we introduce for $\theta \in \mathbb{R}$ the function

$$(4.1)\qquad \mu_\varrho(\theta) \ = \ \begin{cases} \mu(|\theta|) & \text{for } |\theta| \le \varrho\,, \\ \mu(\varrho) + \mu_*(|\theta| - \varrho) & \text{for } |\theta| > \varrho\,, \end{cases}$$

and consider the following

Problem 4.1. For $T > 0$ find a pair (θ, χ) with the regularity (2.14) and (2.16), solving a. e. in Q_T the system of equations

$$(4.2)\quad \begin{aligned} &c_V\theta_t(x) - 2\theta(x)\int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=|\theta(x)|+|\theta(y)|}(\theta_t(x) + \theta_t(y))G(\chi(x) - \chi(y))\,dy \\ &\quad = \ \kappa\Delta\theta(x) + 2\theta(x)\int_\Omega K_\tau(\tau,x,y)\big|_{\tau=|\theta(x)|+|\theta(y)|}\,G'(\chi(x) - \chi(y))(\chi_t(x) - \chi_t(y))dy \\ &\quad - (\lambda(\chi(x)) + \beta\varphi(\chi(x)))_t - 2\chi_t(x)\int_\Omega K(|\theta(x)| + |\theta(y)|, x, y)G'(\chi(x) - \chi(y))dy, \end{aligned}$$

$$(4.3)\quad \begin{aligned} &\mu_\varrho(\theta(x))\,\chi_t(x) + \theta(x)\sigma'(\chi(x)) + \lambda'(\chi(x)) \\ &\quad + 2\int_\Omega K(|\theta(x)| + |\theta(y)|, x, y)\,G'(\chi(x) - \chi(y))dy \ \in \ -(\beta + |\theta(x)|)\partial\varphi(\chi(x)) \end{aligned}$$

with boundary and initial conditions (1.8–1.9).

Equation (4.3) is of the form (3.1) with

$$\alpha(\theta) = \frac{\mu_\varrho(\theta)}{\beta+|\theta|} \geq \frac{\mu_*(1+|\theta|)}{\beta+|\theta|} \geq \mu_* \min\left\{1, \frac{1}{\beta}\right\}, \tag{4.4}$$

$$f[\chi,\theta] = -\frac{1}{\beta+|\theta|}\left(\theta\sigma'(\chi)+\lambda'(\chi)+2\int_\Omega K(|\theta(x)|+|\theta(y)|,x,y)G'(\chi(x)-\chi(y))dy\right). \tag{4.5}$$

By Hypothesis 2.1 there exists a positive constant C_f such that

$$|f[\chi,\theta]| \leq C_f \quad \text{a. e. for all } \ \chi,\, \theta \in L^1(Q_T) : \chi(x,t) \in \mathcal{D}(\varphi) \text{ a. e. } \text{ in } Q_T\,. \tag{4.6}$$

The assumptions of Proposition 3.2 are satisfied with $C = \max\{C_0, C_f\}$, where C_0 is defined in (2.13), and with L replaced by some L_ϱ dependent on ϱ. We thus may define the solution mapping

$$A_\varrho : L^1(Q_T) \to L^\infty(Q_T) : \theta \mapsto \chi\,, \tag{4.7}$$

which with each $\theta \in L^1(Q_T)$ associates the solution χ of (4.3) with fixed initial condition χ_0 according to Proposition 3.2.

Problem 4.1 is solved via Faedo-Galerkin approximations. Consider an increasing sequence $\{V_n\}$ of subspaces of $\dim V_n = n$ of V such that $\overline{\bigcup V_n} = V$. Let $(v^1,\ldots,v^n)$ be a basis for V_n. Denoting with $(\cdot)'$ the derivative with respect to time, we state the approximate Problem 4.1 in the form

Problem 4.2. For $n \in \mathbb{N}$ find $\theta_n \in H^2(0,T;V_n)$ solving for all $v \in V_n$ the following system of equations

$$\begin{aligned}
&\frac{1}{n}\langle\theta_n'', v\rangle + \langle c_V\theta_n', v\rangle - \langle\kappa\Delta\theta_n, v\rangle \\
&\quad - \langle 2\theta_n \int_\Omega G(\chi_n(x)-\chi_n(y))K_{\tau\tau}(\tau,x,y)\big|_{\tau=|\theta_n(x)|+|\theta_n(y)|}(\theta_n'(x)+\theta_n'(y))\,dy, v\rangle \\
&\quad = \langle 2\theta_n \int_\Omega K_\tau(\tau,x,y)\big|_{\tau=|\theta_n(x)|+|\theta_n(y)|}G'(\chi_n(x)-\chi_n(y))(\chi_n'(x)-\chi_n'(y))\,dy, v\rangle \\
&\quad - \langle(\lambda(\chi_n)+\beta\varphi(\chi_n))', v\rangle \\
&\quad - \langle 2\chi_n' \int_\Omega K(|\theta_n(x)|+|\theta_n(y)|,x,y)G'(\chi_n(x)-\chi_n(y))\,dy, v\rangle\,,
\end{aligned} \tag{4.8}$$

$$\chi_n = A_\varrho[\theta_n]\,, \tag{4.9}$$

$$\partial_{\mathbf{n}}\theta_n = 0 \quad \text{on } \partial\Omega\times(0,T)\,, \tag{4.10}$$

$$\theta_n(\cdot,0) = \theta_0, \quad \theta_n'(\cdot,0) = 0 \quad \text{a. e. in } \Omega\,. \tag{4.11}$$

Equation (4.8) can be written as a $2n\times 2n$ system of differential equations with a Lipschitz continuous right-hand side (note that φ is Lipschitz on $\mathcal{D}_C(\varphi)$ and the dependence $\theta_n \mapsto \chi_n'$ is Lipschitz by virtue of Proposition 3.2), hence it admits a unique solution for every $n \in \mathbb{N}$ which can be constructed, e. g., by the Banach contraction argument.

4.2 Uniform estimate and passage to the limit

In this subsection we derive some estimates (uniform in ϱ and n) for the solution (θ_n, χ_n) of Problem 4.2 constructed in the previous Subsection 4.1, and then pass to the limit Problem 4.2 with the intention to find a solution to Problem 4.1.

In the sequel, we will denote by c any positive constant which depends on the data of the problem (but neither on n nor on ϱ) and may vary from line to line.

Uniform estimate. We first notice that Proposition 3.2 and Remark 3.3 yield

$$|\chi_n|_{L^\infty(Q_T)} + |\chi_n'|_{L^\infty(Q_T)} + |\varphi(\chi_n)'|_{L^\infty(Q_T)} \le c\,. \tag{4.12}$$

We may take $v = \theta_n'(t)$ in (4.8) and integrate over $(0,t)$ with $t \in (0,T]$. Using Hypotheses 2.1 and (2.13), together with the uniform bounds (4.12), we get that

$$\frac{1}{2n}|\theta_n'(t)|_H^2 + \frac{c_V}{2}\int_0^t\int_\Omega |\theta_n'(s)|^2\,dx\,ds + \frac{\kappa}{2}|\nabla\theta_n(t)|_H^2 \le c\left(1 + \int_0^t\int_\Omega |\theta_n(s)|^2\,dx\,ds\right). \tag{4.13}$$

Now, adding to both sides in (4.13) the term

$$\frac{1}{2}\int_\Omega |\theta_n(t)|^2\,dx = \frac{1}{2}\int_\Omega |\theta_n(0)|^2 + \int_0^t\int_\Omega \theta_n'(s)\,\theta_n(s)\,ds\,,$$

and applying Young's inequality and Gronwall's lemma, we obtain the following uniform bounds

$$\frac{1}{\sqrt{n}}|\theta_n'|_{L^\infty(0,T;H)} + |\theta_n'|_{L^2(0,T;H)} + |\theta_n|_{L^\infty(0,T;V)} \le c. \tag{4.14}$$

Finally, by comparison in (4.8), we also get

$$\frac{1}{n}|\theta_n''|_{L^2(0,T;V')} \le c. \tag{4.15}$$

Passage to the limit. Thanks to the uniform estimates (4.12) and (4.14–4.15), Proposition 3.2, Remark 3.3, and standard compactness results (cf. [14, Cor. 4, p. 85]), we may deduce now that there exist functions θ, χ such that (for a subsequence of $n \nearrow \infty$) the following convergences hold true

$$\theta_n' \to \theta_t \quad \text{weakly in } L^2(0,T;H), \tag{4.16}$$

$$\theta_n \to \theta \quad \text{weakly star in } L^\infty(0,T;V), \tag{4.17}$$

$$\theta_n \to \theta \quad \text{strongly in } C^0([0,T];H), \tag{4.18}$$

$$(1/n)\theta_n' \to 0 \quad \text{strongly in } C^0([0,T];H), \tag{4.19}$$

$$\chi_n \to \chi \quad \text{weakly star in } L^\infty(Q_T) \text{ and strongly in } L^p(Q_T), \tag{4.20}$$

$$\chi_n' \to \chi_t \quad \text{weakly star in } L^\infty(Q_T) \text{ and strongly in } L^p(Q_T), \tag{4.21}$$

$$\varphi(\chi_n)' \to \varphi(\chi)_t \quad \text{weakly star in } L^\infty(Q_T) \text{ and strongly in } L^p(Q_T) \tag{4.22}$$

for all $p \in [1,+\infty)$. Our aim is now to show that the limit functions (θ,χ) solve Problem 4.1. Let us fix an arbitrary $m \in \mathbb{N}$ and, for $n \geq m$, take $z \in V_m$ and a smooth test function $\psi(t)$ with compact support in $(0,T)$. Note that $V_m \subseteq V_n$ by construction, hence we can choose $v = z\psi(t)$ in (4.8) and integrate over $(0,T)$ to obtain for all $n \geq m$ that

$$
\begin{aligned}
(4.23)\quad & \int_0^T \frac{1}{n}\langle \theta_n'(t), z\psi'(t)\rangle\, dt + \int_0^T \langle c_V\theta_n'(t), z\psi(t)\rangle\, dt - \int_0^T \langle \kappa\Delta\theta_n(t), z\psi(t)\rangle\, dt \\
&= \int_0^T \langle 2\theta_n(t) \int_\Omega G(\chi_n(t,x)-\chi_n(t,y)) K_{\tau\tau}(\tau,x,y)\big|_{\tau=|\theta_n(x)|+|\theta_n(y)|}(\theta_n'(t,x) \\
&+ \theta_n'(t,y))\, dy, z\psi(t)\rangle\, dt - \int_0^T \langle (\lambda(\chi_n(t)) + \beta\varphi(\chi_n(t)))' \\
&+ 2\chi_n'(t)\int_\Omega K(|\theta_n(t,x)|+|\theta_n(t,y)|,x,y) G'(\chi_n(t,x)-\chi_n(t,y))dy, z\psi(t)\rangle dt \\
&+ \int_0^T \langle 2\theta_n(t)\int_\Omega K_\tau(\tau,x,y)\big|_{\tau=|\theta_n(x)|+|\theta_n(y)|} G'(\chi_n(t,x)-\chi_n(t,y))(\chi_n'(t,x) \\
&- \chi_n'(t,y))\, dy, z\psi(t)\rangle\, dt\,.
\end{aligned}
$$

Using now the convergences (4.16–4.22), and passing to the limit as $n \nearrow \infty$ in (4.23), we get (with a slight abuse of notation)

$$
(4.24)\qquad \int_0^T \langle (4.2), z\psi(t)\rangle\, dt \;=\; 0\,.
$$

Since both ψ and $z \in V_m$ are arbitrary and m is arbitrarily large, we conclude that Eq. (4.2) is satisfied. Using again the convergences (4.16–4.22), we can pass to the limit also in (4.9–4.11) and get a solution to Problem 4.1. Note that now, by comparison in (4.2) it follows that $\theta \in L^2(0,T;H^2(\Omega))$.

To conclude the proof of Theorem 2.2, it remains to prove the positivity and a uniform upper bound for the θ-component of the solution to (4.2–4.3). To this aim, we multiply Eq. (4.3) by χ_t and derive the identity (we now for simplicity omit the argument (x) outside the integral)

$$
\begin{aligned}
(4.25)\qquad & -(\lambda(\chi)+\beta\varphi(\chi))_t - 2\chi_t\int_\Omega K(|\theta(x)|+|\theta(y)|,x,y)G'(\chi(x)-\chi(y))\, dy \\
&= \mu_\varrho(\theta)\,\chi_t^2 + \theta\sigma(\chi)_t + |\theta|\varphi(\chi)_t
\end{aligned}
$$

which, inserted in (4.2), yields

$$
\begin{aligned}
(4.26)\quad & c_V\theta_t - 2\theta\int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=|\theta(x)|+|\theta(y)|}(\theta_t(x)+\theta_t(y))\, G(\chi(x)-\chi(y))\, dy \\
&= \kappa\Delta\theta + 2\theta\int_\Omega K_\tau(\tau,x,y)\big|_{\tau=|\theta(x)|+|\theta(y)|} G'(\chi(x)-\chi(y))(\chi_t(x)-\chi_t(y))\, dy \\
&+ \mu_\varrho(\theta)\,\chi_t^2 + \theta\sigma(\chi)_t + |\theta|\varphi(\chi)_t\,.
\end{aligned}
$$

Putting

$$a(x,t) = c_V - 2\theta(x) \int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=|\theta(x)|+|\theta(y)|} G(\chi(x)-\chi(y))\,dy \tag{4.27}$$

$$\begin{aligned}\gamma(x,t) &= 2\int_\Omega K_{\tau\tau}(\tau,x,y)\big|_{\tau=|\theta(x)|+|\theta(y)|} G(\chi(x)-\chi(y))\theta_t(y)\,dy \\ &\quad + 2\int_\Omega K_\tau(\tau,x,y)\big|_{\tau=|\theta(x)|+|\theta(y)|} G'(\chi(x)-\chi(y)(\chi_t(x)-\chi_t(y))\,dy \\ &\quad + \sigma(\chi)_t + \operatorname{sign}(\theta(x))\varphi(\chi)_t\end{aligned} \tag{4.28}$$

we see that Eq. (4.26) for $u = \kappa\theta$ is as in Proposition 3.6, which yields that $\theta > 0$ a. e. in Q_T. From the estimates (4.12) and (4.14) it follows that the functions $b(t)$ and $c(t)$ can be chosen independently of ϱ, hence we obtain from Proposition 3.6 the L^∞-bound for θ independent of ϱ. Choosing ϱ sufficiently large we thus check that θ, χ satisfy all conditions of Theorem 2.2, and the proof is complete. ■

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