Nonlinear evolution inclusions arising from phase change models[1]

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Abstract

This paper is devoted to the analysis of an abstract evolution inclusion with a non-invertible operator, motivated by problems arising in nonlocal phase separation modeling. Existence, uniqueness, and long-time behaviour of the solution to the related Cauchy problem are discussed in detail.

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1 Introduction

In this paper we study the evolution inclusion

$$\frac{du}{dt}(t) + A[\partial\Phi(u(t)) + Bu(t)] \ni g(t) \quad \text{in } V', \tag{1.1}$$

for t varying in a time interval $(0,T)$, where V' is the dual of a reflexive Banach space V, and Φ is a proper, convex, and lower semicontinuous functional on a Hilbert space H (in which V is compactly and densely embedded) with values in $\mathbb{R} \cup \{+\infty\}$; hence its subdifferential $\partial\Phi$ is maximal monotone on H. The symbol B stands for a continuous possibly nonlinear operator from H to V. Finally, A is a linear continuous symmetric operator from V to V' with a non-trivial null-space, and $g : (0,T) \to V'$ is a given function. We will see in Section 3 that the structure of equation (1.1) guarantees that $\partial\Phi(u) \cap V$ is non-empty and so $A[\partial\Phi(u) + B(u)]$ is well-defined. For other types of doubly nonlinear evolution equations the reader may refer to [9, 10] and the references therein.

The abstract problem (1.1) was inspired by a model of Cahn-Hilliard type for phase separation in a two-phase system involving nonlocal interactions presented by Gajewski and Zacharias in [14]. The Cahn-Hilliard model itself goes back to [7] and a fairly complete review on the recent related literature can be found e. g. in [20].

The authors of [14] consider the system

$$\frac{\partial u}{\partial t} - \mathrm{div}(\mu \nabla v) = 0, \quad \mu = \mu(x, \nabla v, u) = \frac{a(x, \nabla v)}{f''(u)}, \tag{1.2}$$

$$v = f'(u) + w, \quad w(x,t) = \int_\Omega \mathcal{K}(|x-y|)(1 - 2u(y,t))\, dy \tag{1.3}$$

in $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a Lipschitzian domain. The equations are coupled with the boundary condition

$$\mu \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T).$$

Here, the variable u represents the local relative concentration of one of the two phases, that is, $u(x,t) \in [0,1]$ for all admissible $x \in \Omega$ and $t \in (0,T)$, v is the chemical potential, and f', f'' are the first and the second derivatives, respectively, of a given convex function f (in fact, only the case $f(u) = u \log u + (1-u)\log(1-u)$ is considered). The function a in the formula for the mobility μ and the kernel $\mathcal{K}$ are assumed to satisfy appropriate natural technical hypotheses.

The model is compatible with the general scheme of [6, 8, 15] that consists in choosing the free energy of the form

$$F(u) = \int_\Omega \Big\{ f(u)(x) + k_1(x) u(x)(1-u(x)) \; + \; \frac{1}{2}\int_\Omega \mathcal{K}(x,y)|u(x) - u(y)|^2\, dy \Big\}\, dx, \tag{1.4}$$

with a more general symmetric kernel $\mathcal{K}$ of two variables and $k_1(x) = \int_\Omega \mathcal{K}(x,y)\, dy$. Under the hypothesis that the mass flux is proportional to the negative gradient of the external thermodynamic force, we write the mass balance in the form

$$\frac{\partial u}{\partial t} = -\,\mathrm{div}\Big\{ -\mu \nabla\Big(\frac{\delta F}{\delta u}\Big)[u]\Big\} + g. \tag{1.5}$$

Here, $\delta F/\delta u$ stands for the variational derivative of F with respect to u, and g represents an external source. The results of [14] include the existence and uniqueness of solutions and a proof that stationary solutions exist in the ω-limit sets of global solutions.

The aim of the present paper is to establish a general Hilbert-space framework for such situations, where the mobility coefficient μ can be assumed to be constant (and, in particular, independent of $f''(u)$). This choice has however some justification and it was followed in a number of contributions for the standard Cahn-Hilliard equation (let us refer again to [20]), and in particular, it has been recently considered by the authors of the paper [4] in which a nonlocal Cahn-Hilliard equation is investigatedfor a rather general class of kernels $\mathcal{K}$. Note that (1.1) fits with the above model, provided we interpret $-A$ as the Laplacian with Neumann boundary data, B as the nonlocal integral term in (1.3) or, more precisely, the integral operator

$$u \mapsto \int_\Omega -2\mathcal{K}(\cdot, y)u(y)\,dy,$$

and Φ stands for the convex potential

$$u \mapsto f(u) + k_1(x)u.$$

In our abstract setting, the null-space of A is allowed to have an arbitrary finite dimension, while in [4, 14] or, e. g., [19], it is one-dimensional. We state sufficient conditions on A, B and Φ which ensure the existence and/or uniqueness of solutions to (1.1) for a suitable class of data. We also study the long-time behaviour of solutions to (1.1) under more restrictive assumptions on B and $\partial\Phi$. Note that our analysis covers the vectorial case in which u is replaced by $\vec{u} : Q \to \mathbb{R}^N$, with $N \geq 1$, cf. Subsection 2.2. We only point out that in this case the term $\int_\Omega [-\int_\Omega \mathcal{K}(x, y)u(y)\,dy]u(x)\,dx$ in the nonlocal free energy potential (1.4) can be generalized as

$$\int_\Omega -\Big(\int_\Omega \mathcal{K}(x, y)\vec{u}(y)\,dy\Big) \cdot \vec{u}(x)\,dx,$$

where $\mathcal{K}$ is a $N \times N$ symmetric and positive definite matrix and $\cdot$ denotes the scalar product in $\mathbb{R}^N$.

Also other applications of our theory seem to be relevant. A recent investigation [12] has been devoted to the Czochralski crystal growth process in a simplified framework, namely with a constant radius of the crystal and a known fluid velocity in the liquid. The model consists of heat equations in the domains of liquid, solid and gas phases, a Stefan condition at the liquid-solid interface and transmission conditions at the liquid-gas and solid-gas interfaces. By an enthalpy formulation the problem can be reformulated as a degenerate parabolic differential equation, which in a very simplified version (reminiscent of the problem studied in [21]) reads

$$u_t - \Delta\beta(u) + \mathbf{v} \cdot \nabla u = f$$

for the enthalpy u in the fixed domain Ω with a monotone function $\beta(u)$ and the given fluid velocity $\mathbf{v}$, supplemented by boundary conditions $\frac{\partial\beta(u)}{\partial\nu} + n_0\beta(u) = p$ and $\mathbf{v} \cdot \boldsymbol{\nu} = 0$

on $\partial\Omega$, and initial conditions for u. A weak formulation of the model is presented in [12]. Of course, this model fits into our framework with obvious definition for A (which is now invertible) and Φ, while Bu is defined as the solution of $\langle A(Bu), z\rangle = \int_\Omega u\,\mathbf{v}\cdot\nabla z$ for all $z \in V$ $(= H^1(\Omega)$ in this example).

Let us briefly outline a detailed plan of the paper. Section 2 summarizes the necessary background related to the operators A and $\partial\Phi$. We mainly focus on the technique of estimating the component of the solution in the null-space of A using special properties of Φ. In Section 3 we give a precise formulation of the initial value problem for equation (1.1) and present two existence results which require either the strong monotonicity of $\partial\Phi$ or the linearity of B. Uniqueness and continuous dependence on the data are obtained under a general condition which is satisfied if e. g. B is Lipschitz continuous and $\partial\Phi$ is strongly monotone. Section 4 is devoted to the proofs of the above statements. Finally, in the last Section 5, we present some results on the long-time behaviour of solutions to this problem provided B is the Fréchet derivative of a potential Ψ satisfying a suitable growth condition.

2 Preliminaries

In what follows, the symbol H denotes a real Hilbert space endowed with a scalar product $\langle\cdot,\cdot\rangle_H$. Let V be a reflexive Banach space densely and compactly embedded into H. Assuming that H is identified with its dual, we obtain for the dual space V' of V that $V \subset H \subset V'$ with dense and compact injections. By $\langle\cdot,\cdot\rangle$ we denote the duality pairing between V' and V, and $\|\cdot\|_E$ stands for the norm in a generic Banach space E. In particular, we set $\|u\|_H = \sqrt{\langle u,u\rangle_H}$ for $u \in H$, and fix a constant κ such that

$$\|v\|_H \le \kappa\,\|v\|_V \quad \forall v \in V. \tag{2.1}$$

Note that the injection $H \subset V'$ can be defined in such a way that

$$\langle u, v\rangle \;=\; \langle u, v\rangle_H \quad \forall u \in H, \quad \forall v \in V. \tag{2.2}$$

2.1 A linear operator with non-trivial kernel

We start with basic hypotheses on the operator A.

Hypothesis 2.1. *The map $A : V \to V'$ is linear and has the following properties.*

(i) *There exists $a_0 > 0$ such that $\|Av\|_{V'} \le a_0\,\|v\|_V \quad \forall v \in V$;*

(ii) *$\langle Av, w\rangle = \langle Aw, v\rangle \quad \forall v,\, w \in V$;*

(iii) *$V_0 := \mathcal{N}(A) = \{v \in V \;:\; Av = 0\}$ is closed in H.*

For the sake of completeness, we now state and prove a series of easy auxiliary results.

Lemma 2.2. *Under Hypothesis 2.1, we have that* $\dim V_0 < +\infty$.

Proof. By continuity of A, V_0 is a closed in V. Thanks to Hypothesis 2.1 (iii), both $W_0 = \big(V_0, \|\cdot\|_V\big)$ and $\widetilde{W_0} = \big(V_0, \|\cdot\|_H\big)$ are Banach spaces, and the identity mapping $I : W_0 \to \widetilde{W_0}$, $Iu = u$ for $u \in V_0$ is a bounded linear operator of W_0 onto $\widetilde{W_0}$ with trivial null-space. Whence, by the inverse mapping theorem (cf. [22, Thm. 4.1, p. 63]), $I^{-1} : \widetilde{W_0} \to W_0$ is continuous, hence the two norms $\|\cdot\|_V$, $\|\cdot\|_H$ are equivalent on V_0. Since V is compactly embedded into H, we conclude that the unit ball in $\widetilde{W_0}$ is compact, hence $\dim V_0 < +\infty$. ∎

We define in a standard way the orthogonal projection P_0 of H onto V_0 for $u \in H$ by the formula

$$\begin{aligned} w_0 = P_0 u \iff & w_0 \in V_0,\ \|u - w_0\|_H = \min\{\|u - w\|_H\ ;\ w \in V_0\} \\ \iff & \langle u - w_0, w\rangle_H = 0 \quad \forall w \in V_0. \end{aligned} \tag{2.3}$$

Set now $H_1 := V_0^{\perp} = \{u \in H : \langle u, w_0\rangle_H = 0 \quad \forall w_0 \in V_0\} = (I - P_0)H$. Then $V_1 = V \cap H_1$ is closed in V and every element $v \in V$ (as element of H) can be decomposed in a unique way into the sum $v = v_0 + v_1$ with $v_0 \in V_0$ and $v_1 \in V_1$. In view of Hypothesis 2.1 (ii), for all $v, w \in V$ with $v = v_0 + v_1$, $w = w_0 + w_1$, we have

$$\langle Av, w\rangle = \langle Av_1, w_1\rangle,$$

hence A maps V into the space

$$V'_* := \{y \in V' : \langle y, w_0\rangle = 0 \quad \forall w_0 \in V_0\}. \tag{2.4}$$

Moreover, we have the following

Lemma 2.3. *The space* V'_* *defined by* (2.4) *is isomorphic to the dual space* V'_1 *of* V_1.

Proof. For $y \in V'_1$ and $v \in V$, we define $y_* \in V'_*$ by the formula $\langle y_*, v\rangle = y(v_1)$ referring to the decomposition $v = v_0 + v_1$ with $v_0 \in V_0$ and $v_1 \in V_1$. The correspondence between y and y_* is one-to-one and, by definition of the dual norm $\|\cdot\|_{V'_1}$, we have the inequality

$$\|y\|_{V'_1} \le \|y_*\|_{V'}.$$

To prove a reverse inequality, we notice that for all $v \in V$ we have $\|v\|_H^2 = \|v_0\|_H^2 + \|v_1\|_H^2$, hence $\|v_0\|_H \le \|v\|_H$. But, due to Lemma 2.2, all the norms in V_0 are equivalent, hence there exists a positive constant ρ such that

$$\|v_0\|_V \le \rho \|v_0\|_H \quad \forall v_0 \in V_0, \tag{2.5}$$

consequently (cf. also (2.1))

$$\|v_1\|_V \le \|v\|_V + \|v_0\|_V \le (1 + \kappa\rho)\|v\|_V \quad \forall v \in V.$$

Then, for $y_* \in V'_*$ and $v_1 \in V_1$, we define $y \in V'_1$ by the formula $y(v_1) = \langle y_*, v_1\rangle$ which yields

$$\|y_*\|_{V'} = \sup_{\|v\|_V \le 1} |\langle y_*, v_1\rangle| \le \sup_{\|v_1\|_V \le 1+\kappa\rho} |y(v_1)| = (1 + \kappa\rho)\|y\|_{V'_1},$$

hence V'_* and V'_1 are isomorphic. ∎

The next lemma explores the structure of V'.

Lemma 2.4. *The space V' is isomorphic to the direct sum $V'_* \oplus V_0$.*

Proof. For $v \in V$, $y_* \in V'_*$, and $w_0 \in V_0$, we define $y \in V'$ by the formula

$$\langle y, v \rangle = \langle y_*, v \rangle + \langle w_0, v \rangle_H,$$

which yields

$$\|y\|_{V'} \leq \|y_*\|_{V'} + \kappa \|w_0\|_H.$$

Conversely, for $y \in V'$ we use the Riesz representation theorem to find $w_0 \in V_0$ such that

$$\langle y, v_0 \rangle = \langle w_0, v_0 \rangle_H \quad \forall v_0 \in V_0.$$

Then $\|w_0\|_H^2 = \langle y, w_0 \rangle \leq \|y\|_{V'} \|w_0\|_V$, and (2.5) implies that $\|w_0\|_H \leq \rho \|y\|_{V'}$. Putting, for $v \in V$,

$$\langle y_*, v \rangle = \langle y, v \rangle - \langle w_0, v \rangle_H,$$

we obtain $y_* \in V'_*$ and $\|y_*\|_{V'} \leq (1 + \kappa\rho)\|y\|_{V'}$. Thus, the proof is complete. ∎

Let us observe that the restriction of the operator A to V_1 is continuous from V_1 to V'_1 and its null-space is trivial. We now make an additional coercivity hypothesis on the operator A, namely

Hypothesis 2.5. *There exists $\gamma_A > 0$ such that $\langle Av_1, v_1 \rangle \geq \gamma_A \|v_1\|_V^2$ for all $v_1 \in V_1$.*

Under Hypothesis 2.5, we can define the scalar product

$$\langle v, w \rangle_A := \langle Av_1, w_1 \rangle + \langle v_0, w_0 \rangle_H \tag{2.6}$$

referring to the decomposition $v = v_0 + v_1$, $w = w_0 + w_1$ with $v_0, w_0 \in V_0$ and $v_1, w_1 \in V_1$. Note that (2.6) generates in V a norm equivalent to $\|\cdot\|_V$, which will be used from now on. Moreover, (2.6) transforms V into a Hilbert space with V_1 as the orthogonal complement of V_0.

The following lemma immediately follows from the Riesz representation theorem and the inverse mapping theorem.

Lemma 2.6. *For every $y_1 \in V'_1$ there exists a unique $v_1 \in V_1$ such that $y_1 = Av_1$ and the mapping $A^{-1} : V'_1 \to V_1$ is continuous.*

Similarly as in (2.6), the scalar product

$$\langle v', w' \rangle_{A^{-1}} := \left\langle v'_1, A^{-1} w'_1 \right\rangle + \langle v_0, w_0 \rangle_H \tag{2.7}$$

referring to the decomposition $v' = v_0 + v'_1$, $w' = w_0 + w'_1$ with $v_0, w_0 \in V_0$ and $v_1, w_1 \in V'_1$ generates in V' a norm equivalent to $\| \cdot \|_{V'}$ which will be used in the sequel. With this choice of the norms in V and V', we have

$$\left\{ \begin{array}{rclcll} \|Av\|_{V'}^2 & = & \langle Av, (I - P_0)v \rangle & = & \|(I - P_0)v\|_V^2 & \forall v \in V, \\ \|A^{-1}w'\|_V^2 & = & \langle A^{-1}w', w' \rangle & = & \|w'\|_{V'}^2 & \forall w' \in V'_1. \end{array} \right. \tag{2.8}$$

2.2 The functional Φ

The symbol $\partial\Phi$ in (1.1) represents the subdifferential of a proper convex lower semicontinuous mapping $\Phi : H \to \mathbb{R} \cup \{+\infty\}$. By $\mathrm{Dom}(\Phi)$, $\mathrm{Dom}(\partial\Phi)$ we denote the domains of Φ and $\partial\Phi$, respectively. If $\dim H < \infty$, then $\mathrm{Dom}(\Phi) = \mathrm{Dom}(\partial\Phi)$, otherwise $\mathrm{Dom}(\partial\Phi)$ is in general only a dense subset of $\mathrm{Dom}(\Phi)$, see [2, Ch. 4, Thm. 3.11, p. 192] (actually, to check that the two domains do not necessarily coincide, it suffices to consider $H = \ell^2$ and $\Phi(x) = \sum_{k=1}^{\infty} k x_k^2$). For every $u \in \mathrm{Dom}(\partial\Phi)$, the set $\partial\Phi(u)$ is convex and closed, and we denote by $m(\partial\Phi(u))$ its element with minimal norm.

Before stating precise hypotheses on Φ, we briefly recall the notion of the *Yosida approximation*, see [2, 3, 5] for proofs.

Proposition 2.7. *For $\varepsilon > 0$ and $u \in H$ define*

$$\Phi_\varepsilon(u) = \min_{z\in H} \left\{ \frac{1}{2\varepsilon} \|u - z\|_H^2 + \Phi(z) \right\}. \tag{2.9}$$

Then Φ_ε is convex, Fréchet-differentiable in H, and its subdifferential $\partial\Phi_\varepsilon(u)$ contains a unique element $D\Phi_\varepsilon(u)$ for every $u \in H$, where D denotes the Fréchet derivative. Moreover, the so-called resolvent *J_ε of $\partial\Phi$, defined as*

$$J_\varepsilon = (I + \varepsilon\, \partial\Phi)^{-1}, \tag{2.10}$$

where $I : H \to H$ is the identity, is non-expansive in H; the mapping $D\Phi_\varepsilon : H \to H$ is monotone and Lipschitz continuous, and has for every $u \in H$ the properties

$$D\Phi_\varepsilon(u) = \frac{1}{\varepsilon}(u - J_\varepsilon u) \in \partial\Phi(J_\varepsilon u) \quad \forall \varepsilon > 0, \tag{2.11}$$

$$u \in \mathrm{Dom}(\partial\Phi) \ \Rightarrow \ \left\{ \begin{array}{l} \|D\Phi_\varepsilon(u) - m(\partial\Phi(u))\|_H \to 0 \\ \|D\Phi_\varepsilon(u)\|_H \nearrow \|m(\partial\Phi(u))\|_H \end{array} \right. \quad \text{as } \varepsilon \searrow 0, \tag{2.12}$$

$$\Phi_\varepsilon(u) = \frac{\varepsilon}{2} \|D\Phi_\varepsilon u\|_H^2 + \Phi(J_\varepsilon u) \quad \forall \varepsilon > 0, \tag{2.13}$$

$$\Phi_\varepsilon(u) \nearrow \Phi(u) \quad \text{as } \varepsilon \searrow 0. \tag{2.14}$$

In the sequel we require the following hypothesis, which in particular implies that $0 \in \mathrm{Dom}(\partial\Phi)$.

Hypothesis 2.8. *There exist two constants $C_\Phi > 0$, $C'_\Phi \ge 0$ and two Banach spaces X, Y such that*

(i) *the inequality*
$$\Phi(u) \ge C_\Phi \|u\|_H^2 - C'_\Phi \quad \text{holds for all } u \in H; \tag{2.15}$$

(ii) *$X \supset H \supset Y \supset V_0$ with continuous injections; moreover, for some constants a, b, c, $r > 0$ we have*

$$w \in Y,\ \|w\|_Y \le a \ \Rightarrow \ \left\{ \begin{array}{l} w \in \mathrm{Dom}(\partial\Phi), \\ \|\xi\|_Y \le b \ \ \forall \xi \in \partial\Phi(w), \\ \|D\Phi_\varepsilon(w)\|_Y \le b \ \ \forall \varepsilon > 0, \end{array} \right. \tag{2.16}$$

as well as

$$r\|\xi - \eta\|_X \ \leq \ \langle \xi - \eta, w - u\rangle_H + c \tag{2.17}$$

for every $w \in Y$ such that $\|w\|_Y \leq a$ and every $u \in \mathrm{Dom}(\partial\Phi)$, for all selections $\xi \in \partial\Phi(w)$ and $\eta \in \partial\Phi(u)$.

Hypothesis 2.8 looks rather technical and we illustrate now its meaning by considering a special case which occurs frequently in PDE's, namely $H = L^2(\Omega;\mathbb{R}^N)$, $X = L^1(\Omega;\mathbb{R}^N)$, $V_0 \subset Y = L^\infty(\Omega;\mathbb{R}^N)$, where $\Omega \subset \mathbb{R}^n$ is an open bounded domain, and n, N are integers. Let $\varphi : \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ be a proper convex lower semicontinuous mapping, and for $u \in H$ set

$$\Phi(u) \ = \begin{cases} \displaystyle\int_\Omega \varphi(u(x))\,dx & \text{if } \varphi(u) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.18}$$

We will systematically use the easy relation stated below (see, e.g., [5, Ex. 2.3.3, p. 25] and [3, Ex. 3, p. 61]). Let us report the proof for the reader's convenience.

Lemma 2.9. *For $u \in \mathrm{Dom}(\partial\Phi)$ and $\xi \in \partial\Phi(u)$ we have $\xi(x) \in \partial\varphi(x)$ for a.e. $x \in \Omega$. Conversely, let $u \in H$ be such that $u(x) \in \mathrm{Dom}(\varphi) = \mathrm{Dom}(\partial\varphi)$ for a.e. $x \in \Omega$, and for each admissible $x \in \Omega$ put $\xi(x) = m(\partial\varphi(u(x)))$. Then ξ is measurable, and if $\xi \in H$, then $u \in \mathrm{Dom}(\partial\Phi)$ and $\xi = m(\partial\Phi(u))$.*

Proof. Every $\xi \in \partial\Phi(u)$ satisfies the inequality

$$\langle \xi, v - u\rangle_H \ \leq \ \Phi(v) - \Phi(u) \qquad \forall v \in H. \tag{2.19}$$

Choosing any $v_0 \in \mathrm{Dom}(\varphi)$ and any measurable set $\Omega' \subset \Omega$, we may put $v(x) = v_0$ for $x \in \Omega'$, $v(x) = u(x)$ for $x \in \Omega \setminus \Omega'$. Then, it is not difficult to obtain from (2.19) that

$$\xi(x)\cdot(v_0 - u(x)) \ \leq \ \varphi(v_0) - \varphi(u(x)) \qquad \text{for a.e. } x \in \Omega,\ \forall v_0 \in \mathbb{R}^N, \tag{2.20}$$

where "$\cdot$" denotes the scalar product in $\mathbb{R}^N$, and the first assertion follows. Conversely, if $u \in H$, $x \in \Omega$, and $\xi(x) = m(\partial\varphi(u(x)))$, then $\xi(x)$ is the pointwise limit of the Yosida approximations $D\varphi_\varepsilon(u(x))$ as $\varepsilon \searrow 0$. By Proposition 2.7, the functions $D\varphi_\varepsilon$ are Lipschitz continuous, hence $D\varphi_\varepsilon(u(\cdot)) \in H$ for all $\varepsilon > 0$, and we conclude that ξ is measurable. If moreover $\xi \in H$, then (2.20) and the fact that φ is bounded from below by an affine function entail that $u \in \mathrm{Dom}(\partial\Phi)$. Finally, every $\eta \in \partial\Phi(u)$ satisfies $|\xi(x)| \leq |\eta(x)|$ for a.e. $x \in \Omega$, hence $\xi = m(\partial\Phi(u))$ and the proof is complete. ∎

Proposition 2.10. *Assume that there exist positive constants $c_\varphi, c'_\varphi, a', b', c', d', r'$ such that*

$$\varphi(z) \ \geq \ c_\varphi|z|^2 - c'_\varphi \qquad \forall z \in \mathbb{R}^N; \tag{2.21}$$

$$|z| \leq a' + d' \ \Longrightarrow \ z \in \mathrm{Dom}(\varphi),\ |\xi| \leq b' \qquad \forall \xi \in \partial\varphi(z); \tag{2.22}$$

$$\begin{aligned} &|y| \leq a',\ |y - z| \geq d',\ z \in \mathrm{Dom}(\varphi) \ \Longrightarrow \\ &\qquad r'|\xi - \eta| \leq (\xi - \eta)\cdot(y - z) + c' \qquad \forall \xi \in \partial\varphi(y),\ \forall \eta \in \partial\varphi(z). \end{aligned} \tag{2.23}$$

Then the functional Φ defined by (2.18) satisfies Hypothesis 2.8.

Proof. Inequality (2.15) follows immediately from (2.21). To prove (2.16), set $a = a'$ and consider $w \in Y$ such that $|w(x)| \leq a$ for a.e. $x \in \Omega$. By Lemma 2.9 we have $\partial\Phi(w) \neq \emptyset$, and each $\xi \in \partial\Phi(w)$ satisfies $\xi(x) \in \partial\varphi(w(x))$, hence $|\xi(x)| \leq b'$ for a.e. $x \in \Omega$. For $\varepsilon > 0$ set $\xi_\varepsilon = D\Phi_\varepsilon(w)$, $w_\varepsilon = J_\varepsilon(w)$. Then $\xi_\varepsilon(x) \in \partial\varphi(w_\varepsilon(x))$ and $w_\varepsilon(x) + \varepsilon\xi_\varepsilon(x) = w(x)$ for a.e. $x \in \Omega$, hence

$$-\varepsilon\xi_\varepsilon(x) \cdot (\xi_\varepsilon(x) - \xi(x)) \;=\; (\xi_\varepsilon(x) - \xi(x)) \cdot (w_\varepsilon(x) - w(x)) \;\geq\; 0$$

and we easily conclude that $|\xi_\varepsilon(x)| \leq |\xi(x)| \leq b'$ for a.e. $x \in \Omega$. We thus checked (2.16) for $b = b'$.

It remains to prove (2.17). Keeping $a = a'$, $b = b'$, consider $w \in Y$, $\|w\|_Y \leq a$ and $u \in \mathrm{Dom}(\partial\Phi)$, and let $\xi \in \partial\Phi(w)$, $\eta \in \partial\Phi(u)$ be arbitrary. We have, as before, $\xi(x) \in \partial\varphi(w(x))$, $\eta(x) \in \partial\varphi(u(x))$ for a.e. $x \in \Omega$. Set

$$\Omega_+ = \{x \in \Omega\,;\, |u(x) - w(x)| \geq d'\}, \quad \Omega_- = \Omega \setminus \Omega_+. \tag{2.24}$$

By (2.23) we infer

$$r'|\xi(x) - \eta(x)| \leq (\xi(x) - \eta(x)) \cdot (w(x) - u(x)) + c' \qquad \text{for a.e. } x \in \Omega_+. \tag{2.25}$$

Using the fact that $(\xi(x) - \eta(x)) \cdot (w(x) - u(x)) \geq 0$ for a.e. $x \in \Omega$, we obtain that

$$r' \int_{\Omega_+} |\xi(x) - \eta(x)|\, dx \leq \langle \xi - \eta, w - u\rangle_H + c'|\Omega_+|.$$

On the other hand, for $x \in \Omega_-$ we have $|w(x)| \leq a'$, $|u(x)| \leq a' + d'$, hence $|\xi(x)| \leq b'$, $|\eta(x)| \leq b'$ by virtue of (2.22). This yields that

$$\int_{\Omega_-} |\xi(x) - \eta(x)|\, dx \leq 2b'|\Omega_-|. \tag{2.26}$$

Combining the above inequalities, we obtain

$$r'\, \|\xi - \eta\|_X \;\leq\; \langle \xi - \eta, u - w\rangle_H + c'|\Omega_+| + 2r'b'|\Omega_-|,$$

which is precisely (2.17) with $r = r'$ and $c = |\Omega| \max\left\{c', 2r'b'\right\}$. ∎

We now give a hint how to check conditions (2.22)–(2.23) in concrete situations. If M stands for a symmetric positive definite matrix, then the function $\varphi_M(z) = Mz \cdot z$, $z \in \mathbb{R}^N$, as well as its small and smooth perturbations, provide the most canonical example. Furthermore, if φ_1, φ_2 fulfil the above conditions, then so does any combination $k_1\varphi_1 + k_2\varphi_2$ with $k_1, k_2 \geq 0$. The case $N = 1$ is particularly easy: then every convex lower semicontinuous function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with $[-a' - d', a' + d'] \subset \mathrm{Dom}(\varphi)$ satisfies (2.22)–(2.23). Another example which typically arises in applications is the subject of the following statement.

Proposition 2.11. *Let $Z \subset \mathbb{R}^N$ be a convex closed set containing in its interior the ball $\{z \in \mathbb{R}^N\,;\, |z| \leq a' + d'\}$, and let I_Z be the indicator function of Z, that is, $I_Z(z) = 0$ for $z \in Z$, $I_Z(z) = +\infty$ if $z \notin Z$. Then $\varphi = I_Z$ satisfies conditions* (2.22)–(2.23).

Proof. Condition (2.22) is automatically fulfilled with $b' = 0$. Consider now $y, z \in Z$, $|y| \le a'$, $|y - z| \ge d'$. Then $\partial\varphi(y) = \{0\}$ and $\eta \cdot (z - v) \ge 0$ for each $\eta \in \partial\varphi(z)$ and $v \in Z$. There is nothing to prove if $\eta = 0$; otherwise we put

$$v \;=\; y + \frac{d'}{|\eta|}\eta \;\in\; Z,$$

and obtain $d'|\eta| \le \eta \cdot (z - y)$. This corresponds to (2.23) with $r' = d'$ and $c' = 0$. ∎

Remark 2.12. Conditions (2.22)–(2.23) formalize and generalize the special case $N = 1$ and $Z = [-1, 1]$ considered by Kenmochi, Niezgódka, and Pawlow in [19]. There, the authors devise an argument leading to an a priori estimate for the V_0-component of the solution $u(t)$ to the Cahn-Hilliard equation with constraint (see [19, Lemma 5.2]). In Theorem 2.14 below we show a counterpart of this technique adapted to our situation. Before, we prove that conditions (2.15)–(2.17) are stable with respect to Yosida approximations.

Proposition 2.13. *Let Φ satisfy Hypothesis 2.8. Then there exists $\bar\varepsilon > 0$ such that the Yosida approximations Φ_ε of Φ for $\varepsilon \in (0, \bar\varepsilon)$ have the following properties.*

(i) *There exist two constants $\hat C_\Phi > 0$ and $\hat C'_\Phi \ge 0$ such that*

$$\Phi_\varepsilon(u) \ge \hat C_\Phi \|u\|_H^2 - \hat C'_\Phi \quad \forall u \in H \text{ and } \varepsilon \in (0, \bar\varepsilon); \tag{2.27}$$

(ii) *There exists a constant $\hat a > 0$ such that*

$$w \in Y,\ \|w\|_Y \le \hat a \;\Rightarrow\; \|D\Phi_\varepsilon(w)\|_Y \le b \quad \forall \varepsilon \in (0, \bar\varepsilon), \tag{2.28}$$

and for every $w \in Y$ such that $\|w\|_Y \le \hat a$ and every $u \in H$ we have

$$r\|D\Phi_\varepsilon(w) - D\Phi_\varepsilon(u)\|_X \;\le\; \langle D\Phi_\varepsilon(w) - D\Phi_\varepsilon(u), w - u\rangle_H + c. \tag{2.29}$$

Proof. By (2.13) and (2.15) we have

$$\Phi_\varepsilon(z) \;\ge\; \frac{1}{2\varepsilon}\|z - J_\varepsilon(z)\|_H^2 + C_\Phi \|J_\varepsilon(z)\|_H^2 - C'_\Phi \;\ge\; \frac{C_\Phi}{1 + 2\varepsilon C_\Phi}\|z\|_H^2 - C'_\Phi,$$

hence (2.27) is verified for $\hat C_\Phi = C_\Phi/(1 + 2\bar\varepsilon C_\Phi)$ and $\hat C'_\Phi = C'_\Phi$. Indeed, (2.28) is a particular case of (2.16) with any $\hat a \le a$. To prove (2.29), set $\hat a = a/2$, $\bar\varepsilon = a/(2b)$, and consider arbitrary elements $u \in H$ and $w \in Y$, with $\|w\|_Y \le \hat a$. For $\varepsilon \in (0, \bar\varepsilon)$ put $w_\varepsilon = J_\varepsilon(w)$, $u_\varepsilon = J_\varepsilon(u)$. Then, by (2.11) we have $D\Phi_\varepsilon(w) \in \partial\Phi(w_\varepsilon)$, $D\Phi_\varepsilon(u) \in \partial\Phi(u_\varepsilon)$, $w_\varepsilon + \varepsilon D\Phi_\varepsilon(w) = w$, hence $\|w_\varepsilon\|_Y \le \|w\|_Y + \varepsilon\|D\Phi_\varepsilon(w)\|_Y \le a$. Using (2.17) for w_ε and u_ε, we immediately obtain (2.29). ∎

We are now ready to state and prove the main result of this section.

Theorem 2.14. *Let Φ satisfy Hypothesis 2.8, and let $\bar\varepsilon$ be as in Proposition 2.13. Then there exist positive constants a^*, b^*, r^*, m^* such that for every $u \in H$ such that $\|P_0 u\|_H \le a^*$ we have*

$$r^*\|P_0\xi\|_H \;\le\; (\|(I - P_0)\xi\|_H + b^*)\,(\|(I - P_0)u\|_H + m^*) + c, \tag{2.30}$$

$$r^*\|P_0 D\Phi_\varepsilon(u)\|_H \;\le\; (\|(I - P_0)D\Phi_\varepsilon(u)\|_H + b^*)\,(\|(I - P_0)u\|_H + m^*) + c \tag{2.31}$$

for all $\xi \in \partial\Phi(u)$ and $\varepsilon \in (0, \bar\varepsilon)$.

Proof. We fix positive constants γ_i, $i = 1, \ldots, 4$, such that

$$\gamma_1 \|v\|_X \le \|v\|_H \quad \forall v \in H, \qquad \|v\|_H \le \gamma_2 \|v\|_Y \quad \forall v \in Y,$$
$$\gamma_3 \|w\|_X \ge \|w\|_H \ge \gamma_4 \|w\|_Y \quad \forall w \in V_0.$$

Consider $\hat{a}$ as in Proposition 2.13, and set $a^* = \gamma_4 \hat{a}$. Let $u \in H$ satisfy $\|P_0 u\|_H \le a^*$. We have $P_0 u \in V_0$, hence $\|P_0 u\|_Y \le \hat{a} \le a$. Since the proof is essentially the same for both inequalities (2.30) and (2.31), we restrict ourselves to show the latter. From (2.29) it results that

$$r \|D\Phi_\varepsilon(u) - D\Phi_\varepsilon(P_0 u)\|_X \ \le \ \langle D\Phi_\varepsilon(u) - D\Phi_\varepsilon(P_0 u), (I - P_0)u\rangle_H + c, \tag{2.32}$$

where (cf. (2.28))

$$\begin{aligned}
&|\langle D\Phi_\varepsilon(u) - D\Phi_\varepsilon(P_0 u), (I - P_0)u\rangle_H| \\
&\quad = \ |\langle (I - P_0)(D\Phi_\varepsilon(u) - D\Phi_\varepsilon(P_0 u)), (I - P_0)u\rangle_H| \\
&\quad \le \ (\|(I - P_0)D\Phi_\varepsilon(u)\|_H + \|D\Phi_\varepsilon(P_0 u)\|_H) \ \|(I - P_0)u\|_H \\
&\quad \le \ (\|(I - P_0)D\Phi_\varepsilon(u)\|_H + \gamma_2 b) \ \|(I - P_0)u\|_H \quad \forall \varepsilon \in (0, \bar{\varepsilon}).
\end{aligned}$$

In addition, observe that

$$\begin{aligned}
&\frac{1}{\gamma_3} \|P_0 D\Phi_\varepsilon(u)\|_H \ \le \ \|P_0 D\Phi_\varepsilon(u)\|_X \\
&\quad \le \ \|(I - P_0)D\Phi_\varepsilon(u)\|_X + \|D\Phi_\varepsilon(u) - D\Phi_\varepsilon(P_0 u)\|_X + \|D\Phi_\varepsilon(P_0 u)\|_X \\
&\quad \le \ \|D\Phi_\varepsilon(u) - D\Phi_\varepsilon(P_0 u)\|_X + \frac{1}{\gamma_1} (\|(I - P_0)D\Phi_\varepsilon(u)\|_H + \gamma_2 b)
\end{aligned} \tag{2.33}$$

for all $\varepsilon \in (0, \bar{\varepsilon})$. Combining (2.33) with (2.32), we thus obtain (2.31) for $r^* = r/\gamma_3$, $b^* = \gamma_2 b$, and $m^* = r/\gamma_1$. ∎

3 Main results

In this section, the main results of the paper are stated under the following hypotheses on the data.

Hypothesis 3.1. *Let Hypotheses 2.1, 2.5, 2.8 hold and assume that*

(i) *the operator B maps continuously H into V and there exists a constant $b_0 > 0$ such that*
$$\|Bz\|_V \le b_0 \big(1 + \|z\|_H\big) \quad \forall z \in H; \tag{3.1}$$

(ii) *elements $g \in L^2(0, T; V_*')$ and $u_0 \in \mathrm{Dom}(\Phi)$ are given such that $\|P_0 u_0\|_H \le a^*$, where a^* is as in Theorem 2.14.*

We now state our initial value problem.

PROBLEM (P). For every fixed $T > 0$, find $u \in H^1(0,T;V') \cap L^\infty(0,T;H)$ such that $\Phi(u) \in W^{1,1}(0,T)$ and there exist $v,\ \xi \in L^2(0,T;V)$ satisfying

$$u'(t) + Av(t) = g(t) \quad \text{in } V', \quad \text{for a.e. } t \in (0,T), \tag{3.2}$$

$$v(t) = \xi(t) + Bu(t) \quad \text{in } V, \quad \text{for a.e. } t \in (0,T), \tag{3.3}$$

$$u(t) \in \mathrm{Dom}(\partial\Phi), \quad \xi(t) \in \partial\Phi(u(t)) \quad \text{for a.e. } t \in (0,T), \tag{3.4}$$

$$u(0) = u_0 \quad \text{in } H. \tag{3.5}$$

In (3.2) we use the symbol $(\cdot)'$ to denote the time derivative $d(\cdot)/dt$.

Remark 3.2. As $u \in H^1(0,T;V') \cap L^\infty(0,T;H)$, it turns out that u is weakly continuous from $[0,T]$ to H, hence the initial condition (3.5) makes sense. Furthermore, the argument below (see Proposition 4.2) shows that our notion of solution automatically yields the additional smoothness property $\Phi(u) \in W^{1,1}(0,T)$.

The existence results read as follows.

Theorem 3.3. *Under Hypothesis 3.1, let moreover $\partial\Phi$ be strongly monotone, i.e., there is a positive constant C''_Φ such that*

$$\langle w_1 - w_2, z_1 - z_2\rangle_H \ \geq\ C''_\Phi \|z_1 - z_2\|_H^2 \quad \forall z_i \in \mathrm{Dom}(\partial\Phi),\ w_i \in \partial\Phi(z_i),\ i = 1,2. \tag{3.6}$$

Then there exists at least one solution u of PROBLEM (P).

Theorem 3.4. *Let Hypothesis 3.1 hold and assume that the operator B defined by (3.1) satisfies the further condition*

$$B \text{ is linear.} \tag{3.7}$$

Then there exists at least one solution u of PROBLEM (P).

Remark 3.5. Note that $u'(t) \in V_1' \cong V_*'$ for a.e. $t \in (0,T)$ (cf. Lemma 2.3). In fact, as a consequence of Hypotheses 2.1 and 3.1 (ii), if we take $w_0 \in V_0$, we have

$$\langle u'(t), w_0\rangle = -\langle Av(t), w_0\rangle + \langle g(t), w_0\rangle \ =\ -\langle Aw_0, v(t)\rangle = 0$$

for a.e. $t \in (0,T)$. In particular, it follows that every solution of (3.2)–(3.5) satisfies $P_0 u(t) = P_0 u_0$ for all $t \in [0,T]$.

With an additional assumption on the sum of B and $\partial\Phi$, we prove a continuous dependence result in following form.

Theorem 3.6. *Let Hypothesis 3.1 hold and assume that there is a positive constant γ such that*

$$\langle z_1 - z_2, w_1 - w_2\rangle_H + \langle z_1 - z_2, Bz_1 - Bz_2\rangle \geq -\gamma\|z_1 - z_2\|_{V'}^2 \tag{3.8}$$

for all $z_i \in \mathrm{Dom}(\partial\Phi)$ and $w_i \in \partial\Phi(z_i)$, $i = 1,2$. Take two sets of data $\{u_{0i}, g_i\}$, $i = 1,2$, satisfying Hypothesis 3.1 (ii) and suppose that u_1 and u_2 are two respective solutions to PROBLEM (P). *Then, there exists a positive constant C_{cd}, depending in particular on γ, T, $\|g_i\|_{L^2(0,T;V')}$, $\Phi(u_{0i})$, and $\|u_{0i}\|_{V'}$ for $i = 1,2$, such that*

$$\|u_1 - u_2\|_{C^0([0,T];V')} \leq C_{\mathrm{cd}}\big(\|u_{01} - u_{02}\|_{V'} + \|g_1 - g_2\|_{L^1(0,T;V')} + \|P_0(u_{01} - u_{02})\|_H^{1/2}\big). \tag{3.9}$$

In particular, PROBLEM (P) *has at most one solution for each admissible set of data.*

Remark 3.7. Note that in the case where (3.6) holds and the operator $B : H \to V$ is Lipschitz continuous for some positive constant L, that is,

$$\|Bu_1 - Bu_2\|_V \ \leq\ L\,\|u_1 - u_2\|_H \qquad \forall u_1, u_2 \in H, \tag{3.10}$$

then the solution of Problem (P) ensured by Theorem 3.3 is unique. Indeed, (3.10) implies

$$-\langle z_1 - z_2, Bz_1 - Bz_2\rangle \leq \frac{C''_\Phi}{2}\|z_1 - z_2\|_H^2 + \frac{L^2}{2C''_\Phi}\|z_1 - z_2\|_{V'}^2 \qquad \forall z_1, z_2 \in H,$$

so that (3.8) follows from (3.6). Besides, let us point out that (3.8) holds true also when the mapping $\partial\Phi$ is only monotone (and not *strongly monotone* as in (3.6)), and B is the restriction to H of a Lipschitz continuous operator from V' to V (think, for instance, to some linear mapping B which regularizes its argument). Hence (cf. (3.7)), the last framework could be partly combined with Theorem 3.4 to investigate existence and uniqueness of the solution in some situations.

Remark 3.8. In view of Hypothesis 3.1 (i), note that (3.7) entails (3.10). Anyhow, we point out that linear integral operators mentioned in the Introduction are natural prototypes of operators B satisfying the various conditions.

The proofs of the above results are contained in Section 4. We conclude this section by showing a simple example of non-existence for Problem (P), in the case where Hypothesis 3.1 (ii) is violated.

Example 3.9. Non-existence of solutions.
We show here that the existence result for Problem (P) does not hold if Hypothesis 3.1 (ii) on the initial size of P_0u_0 is removed. Consider the problem in $\mathbb{R}^2$

$$\begin{pmatrix} \dot u_1(t) \\ \dot u_2(t) \end{pmatrix} + A \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \dot f(t) \\ 0 \end{pmatrix}, \tag{3.11}$$

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \in \partial\Phi \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \tag{3.12}$$

where Φ is the indicator function of the bounded closed convex set $K \subset \mathbb{R}^2$ defined below, $f \in W^{1,2}(0,T)$ is a given function, and the data are

$$\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} \ \in\ K, \tag{3.13}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \operatorname{conv}\left(B_1\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cup B_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \tag{3.14}$$

where $B_r(x)$ denotes the ball centered at $x \in \mathbb{R}^2$ with radius $r > 0$ and conv$\,(S)$ denotes the convex hull of the set S, see Figure 1.

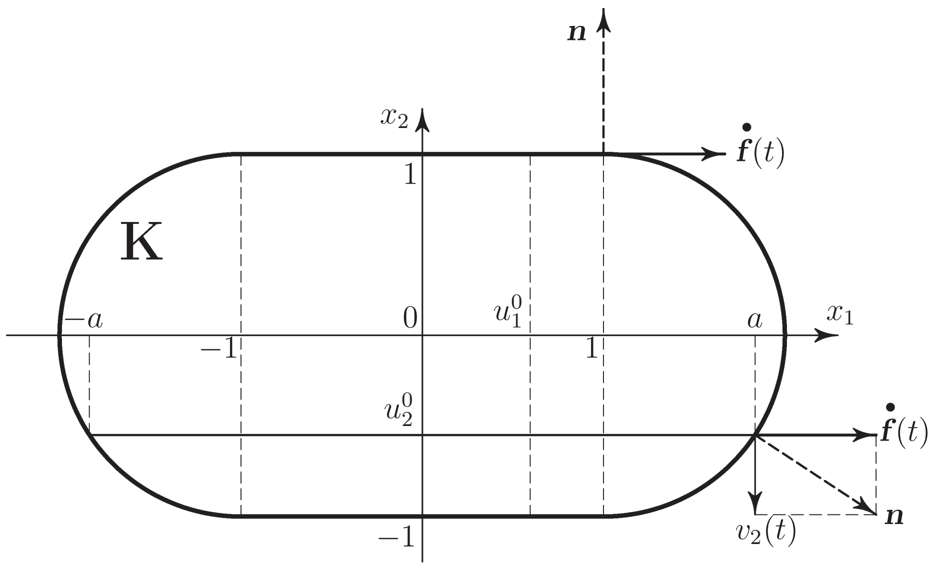

Figure 1: An illustration to Example 3.9.

System (3.11)–(3.12) can be written in the form

$$\dot{u}_1(t) + v_1(t) = \dot{f}(t), \tag{3.15}$$

$$u_2(t) = u_2^0, \tag{3.16}$$

$$\begin{pmatrix} u_1(t) \\ u_2^0 \end{pmatrix} \in K, \tag{3.17}$$

$$\left\langle \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}, \begin{pmatrix} u_1(t) \\ u_2^0 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \geq 0 \qquad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K\,. \tag{3.18}$$

Set

$$K_1 = \left\{ z \in \mathbb{R}\,;\ \begin{pmatrix} z \\ u_2^0 \end{pmatrix} \in K \right\}.$$

We may choose $x_2 = u_2^0$ in (3.18) and obtain

$$u_1(t) \in K_1, \tag{3.19}$$

$$\left(\dot{f}(t) - \dot{u}_1(t)\right)(u_1(t) - x_1) \geq 0 \qquad \forall x_1 \in K_1\,. \tag{3.20}$$

Relation (3.19)–(3.20) is nothing but the definition of the *stop operator* with input f, output u_1, and characteristic K_1. Let us consider now the special case

$$u_2^0 = 1, \quad \dot{f}(t) = 1 \quad \text{for } t \in [0,T], \quad T \geq 2\,. \tag{3.21}$$

Then, it results that $K_1 = [-1,1]$ and $u_1(t) = \min\{u_1^0 + t, 1\}$ for $t \in [0,T]$. In particular, for $t > 1 - u_1^0$ we have $u_1(t) = 1$, $\dot{u}_1(t) = 0$ and consequently $v_1(t) = 1$. According to (3.18), we have to find $v_2(t)$ in such a way that $\begin{pmatrix} 1 \\ v_2(t) \end{pmatrix}$ belongs to the outward normal cone $N_K\begin{pmatrix}1\\1\end{pmatrix}$ to K at the point $\begin{pmatrix}1\\1\end{pmatrix}$. However, this is not possible, since $N_K\begin{pmatrix}1\\1\end{pmatrix}$ contains only non-negative multiples of the vector $\begin{pmatrix}0\\1\end{pmatrix}$. Hence, we see that Problem (3.11)–(3.12) with data (3.21) does not even have a local solution if $u_1^0 = 1$. On the other hand, such a function $v_2(t)$ can always be found if $|u_2^0| < 1$ in agreement with Theorem 3.4, see Figure 1.

4 Proofs

This section is devoted to the proofs of the existence and uniqueness results stated in Section 3. We use a standard technique based on approximations, a priori estimates, and passage to the limit.

In the sequel, we will denote by C any positive constant which depends on the data of the problem and may vary from line to line; the dependence on T will be accounted for by writing $C(T)$.

4.1 Approximation

Keeping the notation from Proposition 2.13 and Theorem 2.14, we state for $\varepsilon \in (0,\bar{\varepsilon})$ the following problem.

Problem $(\mathrm{P})_\varepsilon$. For fixed $T > 0$ and $\varepsilon \in (0,\bar{\varepsilon})$, find a function $u_\varepsilon \in H^1(-\bar{\varepsilon}, T; H)$ such that for $t \in (0,T)$ we have

$$u_\varepsilon'(t) + A(\varepsilon u_\varepsilon'(t) + v_\varepsilon(t)) = g(t), \tag{4.1}$$

$$v_\varepsilon(t) = \xi_\varepsilon(t) + Bu_\varepsilon(t-\varepsilon), \tag{4.2}$$

$$\xi_\varepsilon(t) = D\Phi_\varepsilon(u_\varepsilon(t)), \tag{4.3}$$

and for $t \in [-\bar{\varepsilon}, 0]$ the function u_ε satisfies the (initial) condition

$$u_\varepsilon(t) = u_0. \tag{4.4}$$

Lemma 4.1. *Under Hypothesis 3.1, for each $\varepsilon \in (0,\bar{\varepsilon})$ Problem $(\mathrm{P})_\varepsilon$ has a unique solution u_ε with the prescribed regularity and such that*

$$P_0 u_\varepsilon(t) = P_0 u_0 \quad \forall t \in [-\bar{\varepsilon}, T]. \tag{4.5}$$

Proof. Arguing as in Remark 3.5, we see that every solution u_ε of Problem $(\mathrm{P})_\varepsilon$ satisfies $\langle u_\varepsilon'(t), w_0\rangle = 0$ for all $w_0 \in V_0$, hence $P_0 u_\varepsilon'(t) = 0$ for a.e. $t \in (0,T)$, and consequently (4.5) holds. Equation (4.1) is therefore equivalent to

$$u_\varepsilon'(t) = (A^{-1} + \varepsilon I)^{-1}\left(A^{-1}g(t) - (I - P_0)v_\varepsilon(t)\right) \quad \text{for a.e. } t \in (0,T). \tag{4.6}$$

Note that the mapping $G_\varepsilon = -(A^{-1}+\varepsilon I)^{-1}(I-P_0)D\Phi_\varepsilon : H \to H$ is Lipschitz continuous. Indeed, $D\Phi_\varepsilon$ is Lipschitz continuous by Proposition 2.7; moreover, by virtue of (2.2) and (2.8) we have

$$\|u\|_{V'}^2 + \varepsilon\,\|u\|_H^2 \;=\; \langle (A^{-1}+\varepsilon I)u, u\rangle_H \quad \forall u \in H_1. \tag{4.7}$$

As the left hand side of (4.7) is the square of an equivalent norm in H_1, it turns out that $(A^{-1}+\varepsilon I)^{-1}$ is a linear continuous operator on H_1. Let us consider equation (4.6) coupled with (4.2)–(4.4) consecutively on intervals $[(k-1)\varepsilon, k\varepsilon]$ for $k = 1, 2, \ldots$ until $k\varepsilon \geq T$. For each fixed k, it can be written as a fixed point problem of the form

$$u_\varepsilon(t) \;=\; u_\varepsilon((k-1)\varepsilon) + \int_{(k-1)\varepsilon}^t \big(G_\varepsilon(u_\varepsilon(s)) + h_\varepsilon^k(s)\big)\, ds \quad \text{ for } t \in [(k-1)\varepsilon, k\varepsilon] \tag{4.8}$$

with a given $h_\varepsilon^k \in L^2((k-1)\varepsilon, k\varepsilon\,; H)$, namely

$$h_\varepsilon^k(t) := (A^{-1}+\varepsilon I)^{-1}\big(A^{-1}g(t) - (I-P_0)Bu_\varepsilon(t-\varepsilon)\big), \quad t \in [0,T].$$

Then, the integral equation (4.8) admits a unique solution in $C^0([(k-1)\varepsilon, k\varepsilon]; H)$ by virtue of, e.g., the Contraction Mapping Principle. After a finite number of steps we construct a unique solution on $[-\bar\varepsilon, T]$. ∎

4.2 A priori estimates

We test (4.1) by $A^{-1}(u_\varepsilon'(\cdot))$ and integrate over $(0,t)$ for some $t \in (0,T)$. Using (3.1) and (2.8) we obtain

$$\begin{aligned}
&\int_0^t \|u_\varepsilon'(s)\|_{V'}^2\, ds + \varepsilon \int_0^t \|u_\varepsilon'(s)\|_H^2\, ds + \Phi_\varepsilon(u_\varepsilon(t)) - \Phi_\varepsilon(u_0)\\
&\quad = \; -\int_0^t \langle u_\varepsilon'(s), Bu_\varepsilon(s-\varepsilon)\rangle ds + \int_0^t \langle g(s), A^{-1}u_\varepsilon'(s)\rangle\, ds\\
&\quad \leq \; b_0^2 \int_{-\varepsilon}^{t-\varepsilon} \big(1 + \|u_\varepsilon(s)\|_H\big)^2\, ds + \frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_{V'}^2\, ds + \int_0^t \|g(s)\|_{V'}^2\, ds.
\end{aligned} \tag{4.9}$$

Inequality (2.14) yields

$$\Phi_\varepsilon(u_0) \;\leq\; \Phi(u_0), \tag{4.10}$$

and from (4.9) combined with (2.27) it follows that

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_{V'}^2\, ds + \varepsilon \int_0^t \|u_\varepsilon'(s)\|_H^2\, ds + \hat{C}_\Phi\,\|u_\varepsilon(t)\|_H^2\\
&\qquad\qquad \leq\; C(T)\left(1 + \int_0^t \|u_\varepsilon(s)\|_H^2\, ds\right).
\end{aligned} \tag{4.11}$$

Applying the Gronwall lemma to (4.11) leads to the estimate

$$\|u_\varepsilon'\|_{L^2(0,T;V')} + \|u_\varepsilon\|_{L^\infty(0,T;H)} + \sqrt{\varepsilon}\,\|u_\varepsilon'\|_{L^2(0,T;H)} \;\leq\; C(T). \tag{4.12}$$

Hence, from (4.9) and (2.27) we further deduce that

$$|\Phi_\varepsilon(u_\varepsilon(t))| \ \le \ C(T) \quad \forall t \in [0,T]. \tag{4.13}$$

Finally, thanks to (2.8), by a comparison in (4.1) we infer that

$$\|(I-P_0)(\varepsilon u_\varepsilon' + v_\varepsilon)\|^2_{L^2(0,T;V)} \ \le \ 2\left(\|u_\varepsilon'\|^2_{L^2(0,T;V')} + \|g\|^2_{L^2(0,T;V_*')}\right) \ \le \ C(T). \tag{4.14}$$

We now use Theorem 2.14 to estimate the quantity $\|P_0(\varepsilon u_\varepsilon' + v_\varepsilon)\|_{L^2(0,T;V)}$. Set $p_\varepsilon(t) := \varepsilon u_\varepsilon'(t) + Bu_\varepsilon(t-\varepsilon)$ and observe that (3.1) and (4.12) enable us to check that

$$\|p_\varepsilon\|_{L^2(0,T;H)} \le \varepsilon\|u_\varepsilon'\|_{L^2(0,T;H)} + T^{1/2}C(1+\|u_\varepsilon\|_{L^\infty(0,T;H)}) \le C(T). \tag{4.15}$$

From (4.2) and (4.14)–(4.15) it follows in particular that

$$\|(I-P_0)\xi_\varepsilon\|_{L^2(0,T;H)} \ \le \ C(T). \tag{4.16}$$

Hypothesis 3.1 (ii) and equation (4.5) entail $\|P_0 u_\varepsilon(t)\|_H \le a^*$ for all $t \in [0,T]$, hence we may use (2.31) and (4.12) to derive the bounds

$$\begin{aligned} r^* \|P_0\xi_\varepsilon(t)\|_H &\le (\|(I-P_0)\xi_\varepsilon(t)\|_H + b^*)\,(\|(I-P_0)u_\varepsilon(t)\|_H + m^*) + c \\ &\le C(T)\ (1 + \|(I-P_0)\xi_\varepsilon(t)\|_H) \end{aligned} \tag{4.17}$$

for all $t \in [0,T]$. In addition, we have

$$\|P_0(\varepsilon u_\varepsilon' + v_\varepsilon)\|_{L^2(0,T;V)} \ \le \ \rho\,\|P_0(\xi_\varepsilon + p_\varepsilon)\|_{L^2(0,T;H)} \ \le \ C(T) \tag{4.18}$$

as a direct consequence of (2.5) and (4.15)–(4.17). Thus, in view of (4.14)–(4.18), we obtain the estimate

$$\|\varepsilon u_\varepsilon' + v_\varepsilon\|_{L^2(0,T;V)} + \|\xi_\varepsilon\|_{L^2(0,T;H)} \le C(T). \tag{4.19}$$

We finally exploit (4.13) which, in combination with (2.13) and (2.15), yields

$$\|J_\varepsilon u_\varepsilon\|_{L^\infty(0,T;H)} + \sqrt{\varepsilon}\|\xi_\varepsilon\|_{L^\infty(0,T;H)} \le C(T). \tag{4.20}$$

4.3 Passage to the limit

Our aim now is to obtain a solution to Problem (P) by passing to the limit in Problem $(\mathrm{P})_\varepsilon$ as $\varepsilon \searrow 0$. We start with convergences which are independent of the special assumptions (3.6) and (3.7), and then distinguish the two cases corresponding to Theorems 3.3 and 3.4.

From (4.12) and (4.19)–(4.20) it follows that, up to the extraction of some subsequence of ε as $\varepsilon \searrow 0$, there exist four functions $u, v, \xi, w : (0,T) \to H$ such that, putting

$u(t) = u_0$ for $t \in [-\bar\varepsilon, 0)$, we have

$$
\begin{aligned}
u_\varepsilon \to u \quad & \text{weakly star in } H^1(-\bar\varepsilon, T; V') \cap L^\infty(-\bar\varepsilon, T; H) \\
& \qquad \text{and strongly in } C^0([-\bar\varepsilon, T]; V'), && (4.21)\\
\varepsilon u_\varepsilon \to 0 \quad & \text{strongly in } H^1(0,T;H), && (4.22)\\
\varepsilon \xi_\varepsilon \to 0 \quad & \text{strongly in } C^0([0,T];H), && (4.23)\\
\xi_\varepsilon \to \xi \quad & \text{weakly in } L^2(0,T;H), && (4.24)\\
J_\varepsilon u_\varepsilon \to u \quad & \text{weakly star in } L^\infty(0,T;H) \\
& \qquad \text{and strongly in } C^0([0,T];V'), && (4.25)\\
\varepsilon u_\varepsilon' + v_\varepsilon \to v \quad & \text{weakly in } L^2(0,T;V), && (4.26)\\
B u_\varepsilon(\cdot - \varepsilon) \to w \quad & \text{weakly star in } L^\infty(0,T;V), && (4.27)\\
u_\varepsilon(\cdot - \varepsilon) \to u \quad & \text{strongly in } C^0([0,T];V') \\
& \qquad \text{and weakly star in } L^\infty(0,T;H) && (4.28)
\end{aligned}
$$

as $\varepsilon \searrow 0$. Note that the strong convergence in (4.21) is a consequence of the generalized Ascoli theorem (see, e.g., [23, Cor. 8, p. 90]). We also point out that (4.23) follows from (4.20) and the fact that $\xi_\varepsilon \in C^0([0,T];H)$, while (4.25) results from (4.21), (4.23), and the formula (cf. (2.11)) $u_\varepsilon = J_\varepsilon u_\varepsilon + \varepsilon \xi_\varepsilon$, for $\varepsilon \in (0, \bar\varepsilon)$, We obtain (4.27) directly from (3.1) and (4.12). Since $u(\cdot - \varepsilon) \to u$ strongly in $C^0([0,T];V')$, (4.28) follows from (4.21).

Passing to the weak limit in $L^2(0,T;V')$ in (4.1) and in $L^2(0,T;H)$ in (4.2), we obtain from the above convergences that (3.2) holds and $v = \xi + w$. As a consequence of (4.26)–(4.27) we deduce $\xi \in L^2(0,T;V)$. Furthermore, thanks to (4.3) and (2.11), for every measurable subset $E \subset (0,T)$, every $z \in \mathrm{Dom}(\partial\Phi)$ and every $\eta \in \partial\Phi(z)$ we have that

$$
\int_0^T \langle \xi_\varepsilon(t) - \eta, J_\varepsilon u_\varepsilon(t) - z \rangle_H \, \chi_E(t)\, dt \ \geq \ 0, \tag{4.29}
$$

where χ_E is the characteristic function of E. Using (2.2), the identity

$$
\begin{aligned}
&\langle \xi_\varepsilon(t), J_\varepsilon u_\varepsilon(t)) \rangle_H \\
&\quad = \langle J_\varepsilon u_\varepsilon(t), \varepsilon u_\varepsilon'(t) + v_\varepsilon(t) \rangle - \langle J_\varepsilon u_\varepsilon(t), \varepsilon u_\varepsilon'(t) \rangle_H - \langle J_\varepsilon u_\varepsilon(t), B u_\varepsilon(t - \varepsilon) \rangle \,,
\end{aligned} \tag{4.30}
$$

and the convergences (4.22), (4.25)–(4.27), we can pass to the limit in (4.29) and conclude that there exists a set $M \subset (0,T)$ of zero measure such that

$$
\langle \xi(t) - \eta, u(t) - z \rangle_H \geq 0 \quad \forall t \in (0,T) \setminus M, \quad \forall z \in \mathrm{Dom}(\partial\Phi), \quad \forall \eta \in \partial\Phi(z). \tag{4.31}
$$

As the multivalued mapping $z \mapsto \partial\Phi(z)$ is maximal monotone (cf. [2, Ch. 6, Sec. 7]), it turns out that (3.4) holds.

The absolute continuity of $\Phi(u)$ is a consequence of the following chain rule formula.

Proposition 4.2. *Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous mapping, and let $u \in L^2(0,T;H)$ be such that $u' \in L^2(0,T;V')$, $\xi \in L^2(0,T;V)$, and $\xi(t) \in \partial\Phi(u(t))$ for a.e. $t \in (0,T)$. Then the function $\psi = \Phi(u(\cdot))$ is absolutely continuous in $[0,T]$ and $\psi'(t) = \langle u'(t), \xi(t) \rangle$ for a.e. $t \in (0,T)$.*

Proof. For each $v \in H$ we have

$$\langle v - u(t), \xi(t)\rangle \ \le \ \Phi(v) - \Phi(u(t)), \qquad \text{for a.e. } t \in (0,T).$$

Since Φ is bounded from below by an affine function, we conclude that $\psi \in L^1(0,T)$. Let now $w \in W^{1,\infty}(0,T)$ be a non-negative function with compact support in $(0,T)$. We choose $h > 0$ such that $\mathrm{supp}\,(w) \subset [h, T-h]$. For a.e. $t \in [h,T]$ there holds

$$\langle u(t) - u(t-h), \xi(t-h)\rangle \ \le \ \psi(t) - \psi(t-h) \ \le \ \langle u(t) - u(t-h), \xi(t)\rangle\,.$$

Observe that we can extend w outside of $(0,T)$ with the 0 value. Hence, multiplying by $w(t)$, integrating with respect to t, and letting $h \searrow 0$, we obtain

$$\begin{aligned}
\frac{1}{h}\int_h^T \langle u(t) - u(t-h), \xi(t-h)\rangle\, w(t)dt &= \frac{1}{h}\int_0^{T-h} \langle u(t+h) - u(t), \xi(t)\rangle\, w(t+h)dt \\
&\to \int_0^T \langle u'(t), \xi(t)\rangle\, w(t)dt, \\
\frac{1}{h}\int_h^T (\psi(t) - \psi(t-h))w(t)dt &= \frac{1}{h}\int_0^T \psi(t)(w(t) - w(t+h))dt \\
&\to -\int_0^T \psi(t)w'(t)dt, \\
\frac{1}{h}\int_h^T \langle u(t) - u(t-h), \xi(t)\rangle\, w(t)dt &\to \int_0^T \langle u'(t), \xi(t)\rangle\, w(t)dt.
\end{aligned}$$

Therefore, we conclude that

$$-\int_0^T \psi(t)w'(t)dt \ = \ \int_0^T \langle \xi(t), u'(t)\rangle\, w(t)dt$$

for all non-negative Lipschitz continuous test functions w with compact support. Since both the positive and the negative part of a Lipschitz continuous function are Lipschitz continuous, we obtain the assertion. ∎

In order to establish the existence of solutions to Problem (P), it remains to prove that $w = Bu$. The argument is different in each of the two cases corresponding to Theorems 3.3 and 3.4.

Proof of Theorem 3.3. Let (3.6) hold. We test the difference of the equations (4.1) written for two different indices ε, ε' by $A^{-1}(u_\varepsilon - u_{\varepsilon'})(t) \in V_1$ and integrate over $(0,T)$. With the help of (2.2) and (2.8), we find

$$\begin{aligned}
&\frac{1}{2}\|(u_\varepsilon - u_{\varepsilon'})(T)\|^2_{V'} + \int_0^T \langle \xi_\varepsilon(s) - \xi_{\varepsilon'}(s), u_\varepsilon(s) - u_{\varepsilon'}(s)\rangle_H\, ds \\
&\quad \le -\int_0^T \langle \varepsilon u'_\varepsilon(s) - \varepsilon' u'_{\varepsilon'}(s), u_\varepsilon(s) - u_{\varepsilon'}(s)\rangle_H\, ds \\
&\qquad - \int_0^T \langle (u_\varepsilon - u_{\varepsilon'})(s), Bu_\varepsilon(s-\varepsilon) - Bu_{\varepsilon'}(s-\varepsilon')\rangle\, ds.
\end{aligned} \tag{4.32}$$

Note that the right hand side of (4.32) tends to 0 as ε, $\varepsilon' \to 0$ because of the weak star (or strong) against strong convergence in (4.21)–(4.22) and (4.27). The term which has to be estimated from below is (we omit the arguments (s) for simplicity)

$$\begin{aligned}
&\int_0^T \langle \xi_\varepsilon - \xi_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'} \rangle_H \, ds \\
&\quad = \int_0^T \langle \xi_\varepsilon - \xi_{\varepsilon'}, J_\varepsilon u_\varepsilon - J_{\varepsilon'} u_{\varepsilon'} \rangle_H \, ds + \int_0^T \langle \xi_\varepsilon - \xi_{\varepsilon'}, \varepsilon\xi_\varepsilon - \varepsilon'\xi_{\varepsilon'} \rangle_H \, ds \\
&\quad \geq \; C''_\Phi \, \| J_\varepsilon u_\varepsilon - J_{\varepsilon'} u_{\varepsilon'} \|^2_{L^2(0,T;H)} + \int_0^T \langle \xi_\varepsilon - \xi_{\varepsilon'}, \varepsilon\xi_\varepsilon - \varepsilon'\xi_{\varepsilon'} \rangle_H \, ds.
\end{aligned}$$

Here, we have used (4.3), (2.11) and hypothesis (3.6). Note that the last integral tends to 0 again due to weak against strong convergences in (4.23)–(4.24). Then, in view of (4.25), we conclude that

$$J_\varepsilon u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;H), \tag{4.33}$$

from which we also deduce

$$u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;H). \tag{4.34}$$

It is known that (see, e.g., [11, Thm. III.3.6]) convergence (4.34) is equivalent to the convergence in measure of u_ε to u plus the 2-uniform integrability of u_ε. Then, it is not difficult to check that continuity of B and (3.1) imply the same properties for the sequence Bu_ε, referring now to the space $L^2(0,T;V)$. Hence, we have that

$$Bu_\varepsilon \to Bu \quad \text{strongly in } L^2(0,T;V), \tag{4.35}$$

from which, by the continuity of the translation operator in $L^2(0,T;V)$, it follows

$$Bu_\varepsilon(\cdot - \varepsilon) \to Bu = w \quad \text{strongly in } L^2(0,T;V). \tag{4.36}$$

This concludes the proof of Theorem 3.3. ∎

Proof of Theorem 3.4. Suppose now the validity of (3.7). As $B : H \to V$ is linear and bounded, it is clear that B generates a linear bounded operator from $L^\infty(0,T;H)$ to $L^\infty(0,T;V)$, so that

$$Bu_\varepsilon(\cdot - \varepsilon) \to Bu \quad \text{weakly star in } L^\infty(0,T;V), \tag{4.37}$$

and the proof is complete. ∎

4.4 Continuous dependence

This subsection is devoted to the proof of Theorem 3.6. We start with an auxiliary boundedness result for the solutions of Problem (P).

Lemma 4.3. *There exists a function $S : \mathbb{R}^3 \to (0,+\infty)$, non-decreasing with respect to each of the variables, such that every solution to* Problem (P) *satisfies the estimate*

$$\int_0^t \big(\|u'(s)\|_{V'}^2 + \|\xi(s)\|_V^2\big)\, ds + \Phi(u(t)) \\ \leq S\big(T, \|g\|_{L^2(0,T;V')}, \Phi(u_0)\big) \qquad \forall t \in [0,T]\,. \tag{4.38}$$

Proof. We argue as in Subsection 4.2. The estimate for $\|u'\|_{V'}$ and $\Phi(u)$ is obtained directly by testing equation (3.2) by $A^{-1}u'$ and using Proposition 4.2. The estimate for $\|(I-P_0)\xi\|_V$ follows from (3.3), and inequality (2.30) yields the assertion. ∎

Proof of Theorem 3.6. Let u_1, u_2 be two solutions to Problem (P) corresponding to the sets of data $\{u_{0i}, g_i\}$, with $\xi_i \in \partial\Phi(u_i)$, $i = 1,2$. Set $\bar{u} = u_1 - u_2$, $\bar{\xi} = \xi_1 - \xi_2$, $\bar{g} = g_1 - g_2$, $\bar{u}_0 = u_{01} - u_{02}$. We then have

$$\bar{u}'(t) + A(\bar{\xi}(t) + Bu_1(t) - Bu_2(t)) \;=\; \bar{g}(t)\,. \tag{4.39}$$

We test equation (4.39) by $A^{-1}(\bar{u}(t) - P_0(\bar{u}_0))$. After integration over $(0,t)$ we obtain the estimate

$$\begin{aligned}\frac{1}{2}\|\bar{u}(t) - P_0(\bar{u}_0)\|_{V'}^2 \leq \; & -\int_0^t \big(\langle \bar{u}, \bar{\xi}\rangle_H + \langle \bar{u}, Bu_1 - Bu_2\rangle\big)\, ds \\ & + \|P_0(\bar{u}_0)\|_H \int_0^t (\|\xi_1\|_H + \|\xi_2\|_H + \|Bu_1\|_H + \|Bu_2\|_H)\, ds \\ & + \frac{1}{2}\|(I-P_0)\bar{u}_0\|_{V'}^2 + \int_0^t \|\bar{g}\|_{V'}\|\bar{u}\|_{V'}\, ds\,.\end{aligned}$$

As $\|\bar{u}(t)\|_{V'}^2 - 2\|P_0(\bar{u}_0)\|_{V'}^2 \leq 2\|\bar{u}(t) - P_0(\bar{u}_0)\|_{V'}^2$, it turns out that inequality (3.9) follows from (3.8), (3.1), (2.15), Lemma 4.3, if one applies a generalized Gronwall lemma (combine, for instance, the two versions reported in [5, pp. 156–157]). ∎

5 Long-time behaviour

If $g \in L^2(0,T;V_*')$ for all $T > 0$, then the above existence theorems allow us to construct a solution $u : [0,+\infty) \to H$, i.e., to build up trajectories of solutions on the halfline $[0,+\infty)$. Indeed, having a solution on $[0,T]$ for some $T > 0$, we can use Proposition 4.2 and Remark 3.5 to conclude that $u(T) \in \mathrm{Dom}(\Phi)$ and $\|P_0 u(T)\|_H = \|P_0 u_0\|_H \leq a^*$. This enables us to start again with the new initial datum $u(T)$ to solve the problem in the interval $[T, 2T]$, and so on. It thus makes sense to investigate the long-time behaviour of the solutions u to Problem (P) given by Theorems 3.3–3.4.

In this framework, we have to make an additional assumption about the operator B in (3.1): we ask B to be the Fréchet derivative of a potential Ψ with growth controlled by Φ.

Hypothesis 5.1. *Let Hypothesis 3.1 hold and assume that there exists a functional* $\Psi : H \to \mathbb{R}$ *and two constants* $\vartheta \in [0,1)$ *and* $C_\Psi \geq 0$ *such that*

$$
\begin{aligned}
Bz &= D\Psi(z)\,, && (5.1)\\
\Psi(z) &\geq -\vartheta\Phi(z) - C_\Psi && (5.2)
\end{aligned}
$$

for all $z \in H$*, where* $D\Psi$ *denotes again the Fréchet derivative of* Ψ*.*

First, we derive uniform bounds with respect to time for solutions to Problem (P).

Theorem 5.2. *Assume that Hypothesis 5.1 and either* (3.6) *or* (3.7) *hold. Moreover, let the datum* g *in Hypothesis 3.1 (ii) be defined on* $(0,+\infty)$ *and fulfil*

$$
g \in L^\infty(0,+\infty;V_*'), \quad g' \in L^1(0,+\infty;V_*'). \tag{5.3}
$$

Then, there exist a solution $u : (0,+\infty) \to H$ *to* Problem (P) *and a positive constant* $C_{\rm s}$ *such that*

$$
\mathcal{E}(t) := \int_0^t \|u'(s)\|_{V'}^2\,ds + \|u(t)\|_H^2 + \Phi(u(t)) + C_\Phi' \leq C_{\rm s} \quad \forall t > 0, \tag{5.4}
$$

with $C_{\rm s}$ *depending only on* C_Φ, C_Φ', ϑ, C_Ψ, $\|u_0\|_H$, $\Phi(u_0)$, $\Psi(u_0)$, $\|A^{-1}g\|_{L^\infty(0,+\infty;H)}$, *and* $\|A^{-1}g'\|_{L^1(0,+\infty;H)}$. *Moreover, for every* $T > 0$ *there holds*

$$
\|v\|_{L^2(t,t+T;V)} \leq C(T) \quad \text{for all } t \geq 0, \tag{5.5}
$$

for some constant $C(T)$ *which depends in particular on* $C_{\rm s}$, $\|g\|_{L^\infty(0,+\infty;V_*')}$, *and* T.

Remark 5.3. The constant C_Φ' has been included into the left-hand side of (5.4) in order that $\mathcal{E}$ be non-negative by virtue of (2.15). Equivalently, in view of (5.1)–(5.2) we could have considered the natural non-negative Lyapunov functional

$$
\mathcal{E}_0(t) := \int_0^t \|u'(s)\|_{V'}^2\,ds + \Phi(u(t)) + \Psi(u(t)) + (1-\vartheta)C_\Phi' + C_\Psi
$$

associated with the autonomous case ($g \equiv 0$) of Problem (P).

Proof of Theorem 5.2. As noticed at the beginning of this section, Theorems 3.3 – 3.4 ensure that a global solution exists on the halfline $[0,+\infty)$. We test (3.2) by $A^{-1}u'(t)$, exploit (3.3)–(3.4) and Proposition 4.2, integrate with respect to t, and find out that

$$
\begin{aligned}
\int_0^t \|u'(s)\|_{V'}^2\,ds + \Phi(u(t)) - \Phi(u_0) + \int_0^t \langle u'(s), Bu(s)\rangle\,ds \\
= \int_0^t \langle g(s), A^{-1}u'(s)\rangle\,ds. \qquad (5.6)
\end{aligned}
$$

Now, let us make use of the chain rule formula

$$
\int_0^t \langle u'(s), Bu(s)\rangle\,ds = \Psi(u(t)) - \Psi(u_0) \tag{5.7}
$$

which is obvious if $u' \in L^2(0,T;H)$ and $t < T$ (see, e.g., [1, pp. 9–12] for definitions and basic properties of Fréchet derivatives). Since in the general case we just know that $u' \in L^2(0,T;V')$ for all $T > 0$, we can proceed as follows. Let $J_A : V \to V'$ be the Riesz operator defined by the scalar product in (2.6), i.e., $\langle J_A v, w\rangle := \langle v, w\rangle_A$ for all $v, w \in V$, and for $\varepsilon \in (0,1)$ consider the singular perturbations u_ε of u defined as the solutions to the equation

$$u_\varepsilon(t) + \varepsilon J_A u_\varepsilon(t) \;=\; u(t)\,. \qquad t \in (0,T).$$

Formula (5.7) is valid for u_ε instead of u. Moreover, one can check that $\|u_\varepsilon(t)\|_H \le \|u(t)\|_H$ for all $\varepsilon \in (0,1)$ and $u_\varepsilon(t) \to u(t)$ in H as $\varepsilon \searrow 0$, for every $t \in [0,T]$ (cf. Remark 3.2). On the other hand, we also have $\|u'_\varepsilon(t)\|_{V'} \le \|u'(t)\|_{V'}$ for a.e. $t \in (0,T)$ and $u'_\varepsilon \to u'$ strongly in $L^2(0,T;V')$. Hence, passing to the limit and using the continuity of $B : H \to V$ and $\Psi : H \to \mathbb{R}$, we obtain (5.7).

In the subsequent calculation we use the assumptions (2.15) on Φ and (5.1)–(5.2) on Ψ to obtain

$$\begin{aligned}\Phi(u(t)) - \Phi(u_0) + \Psi(u(t)) - \Psi(u_0) &\ge (1-\vartheta)\Phi(u(t)) - C_\Psi - \Phi(u_0) - \Psi(u_0)\\ &\ge \frac{1-\vartheta}{2} C_\Phi \|u(t)\|_H^2 + \frac{1-\vartheta}{2}\Phi(u(t)) - C'_\Phi - C_\Psi - \Phi(u_0) - \Psi(u_0).\end{aligned} \tag{5.8}$$

Next, we estimate the integral on the right hand side of (5.6). By C_1, C_2 ... we denote suitable positive constants depending only on C_Φ, C'_Φ, ϑ, C_Ψ, $\Phi(u_0)$, $\Psi(u_0)$, $\|u_0\|_H$, and $\|A^{-1}g\|_{L^\infty(0,+\infty;H)}$, at most. Using the symmetry properties of A^{-1} and integrating by parts in time, we deduce

$$\begin{aligned}\int_0^t \langle g(s), A^{-1}u'(s)\rangle ds &= -\int_0^t \langle u(s), A^{-1}g'(s)\rangle_H ds + \langle u(t), A^{-1}g(t)\rangle_H - \langle u_0, A^{-1}g(0)\rangle_H\\ &\le \int_0^t \|u(s)\|_H \|A^{-1}g'(s)\|_H\, ds + \|u(t)\|_H\|A^{-1}g(t)\|_H + \|u_0\|_H \|A^{-1}g(0)\|_H\\ &\le \int_0^t \|u(s)\|_H \|A^{-1}g'(s)\|_H\, ds + \frac{1-\vartheta}{4} C_\Phi \|u(t)\|_H^2 + C_1\,.\end{aligned} \tag{5.9}$$

Then, combining (5.6) and (5.8)–(5.9), we get the inequality

$$\begin{aligned}&\int_0^t \|u'(s)\|_{V'}^2\, ds + \frac{1-\vartheta}{4} C_\Phi \|u(t)\|_H^2 + \frac{1-\vartheta}{2}\Phi(u(t))\\ &\qquad \le C_2 + \int_0^t \|(A^{-1}g)'(s)\|_H \|u(s)\|_H\, ds\,.\end{aligned} \tag{5.10}$$

Now, recalling the definition of $\mathcal{E}$ in (5.4), we can rewrite (5.10) as

$$\mathcal{E}(t) \le C_3 + C_4 \int_0^t \|(A^{-1}g)'(s)\|_H \sqrt{\mathcal{E}(s)}\, ds$$

Finally, by applying a variation of the Gronwall lemma (cf., e.g., [5, Lemme A5, p. 157]), we obtain

$$\sqrt{\mathcal{E}(t)} \;\le\; \sqrt{C_3} + \frac{C_4}{2}\int_0^t \|(A^{-1}g)'(s)\|_H\, ds \le\; C_5\,, \tag{5.11}$$

whence the estimate in (5.4) follows immediately. At this point, it remains to prove (5.5). From (2.8), (3.2), (5.3), Remark 3.5, and (5.11) it follows that

$$\begin{aligned}
&\int_t^{t+T} \|(I-P_0)v(s)\|_V^2\,ds \;=\; \int_t^{t+T} \|Av(s)\|_{V'}^2\,ds \\
&\quad\le\; 2\int_0^{t+T} \|u'(s)\|_{V'}^2\,ds + \int_t^{t+T} 2\,\|g(s)\|_{V'}^2\,ds \;\le\; 2\left(C_{\mathrm{s}} + T\|g\|^2_{L^\infty(0,\infty;V'_*)}\right). \qquad (5.12)
\end{aligned}$$

Similarly as in Subsection 4.2, we use (2.30) and (5.3)–(5.4) to obtain

$$\|P_0\xi(s)\|_H \;\le\; C\,(1+\|(I-P_0)\xi(s)\|_H) \qquad \text{for } s\in[t,t+T]\,. \qquad (5.13)$$

Hence, owing to (2.5), (3.3), and (5.4), for $s\in[t,t+T]$ we deduce that

$$\begin{aligned}
&\|P_0v(s)\|_V \;\le\; C\,(\|P_0\xi(s)\|_H + \|Bu(s)\|_H) \\
&\quad\le\; C\,(1+\|(I-P_0)\xi(s)\|_H + \|Bu(s)\|_V) \\
&\quad\le\; C\,(1+\|(I-P_0)v(s)\|_H + \|Bu(s)\|_V) \;\le\; C\,(1+\|(I-P_0)v(s)\|_V)\,, \qquad (5.14)
\end{aligned}$$

and consequently (5.5) follows from (5.12). This concludes the proof of Theorem 5.2. ∎

With the intention of investigating the long-time behaviour of solutions to Problem (P), we define the ω-limit set $\omega(u)$ of the single trajectory u in V' by

$$\omega(u) = \left\{ \begin{array}{c} u_\infty\in V':\ \text{there exists a sequence of times } t_n\nearrow+\infty \\ \text{such that } u(t_n) \text{ converges to } u_\infty \text{ strongly in } V' \end{array} \right\}. \qquad (5.15)$$

Remark 5.4. Note that in the case where Problem (P) has a unique solution (cf. Theorem 3.6 and Remark 3.7), the trajectory $u:(0,+\infty)\to H$ is uniquely determined by the initial datum u_0 so that, in this case, $\omega(u)$ can be replaced by $\omega(u_0)$.

The main result of this section can be stated as follows.

Theorem 5.5. *Under the same assumptions as in Theorem 5.2, let $u:(0,+\infty)\to H$ be a solution to Problem (P). Then, the ω-limit set $\omega(u)$ is a nonempty, compact, and connected subset of V'. Moreover, if $u_\infty\in\omega(u)$, then*

$$u_\infty\in H, \qquad P_0u_\infty = P_0u_0, \qquad (5.16)$$

and there exists a selection $\xi_\infty\in\partial\Phi(u_\infty)\cap V$ such that

$$A(\xi_\infty + Bu_\infty) = g_\infty \quad \text{in } V', \qquad (5.17)$$

where g_∞ denotes the limit, as $t\nearrow+\infty$, of $g(t)$ in V'_*, existing by virtue of (5.3).

Proof. We first note that, thanks to the estimate (5.4) (cf. also Remark 3.2) the set $\{u(t),\,t\ge 0\}$ is bounded in H and relatively compact in V'. Therefore, the set $\omega(u)$ is a nonempty compact subset of V'. Actually, $\omega(u)$ is also connected, due to the continuity of u from $[0,+\infty)$ to V' and to a standard argument from the theory of dynamical systems (see, e.g., [17, p. 12]). Then, let $u_\infty \in \omega(u)$ and take a strictly increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $t_n \nearrow +\infty$ as $n \nearrow +\infty$ and

$$u(t_n) \to u_\infty \quad \text{weakly in } H \text{ and strongly in } V'. \tag{5.18}$$

In addition, for every integer $n \geq 1$, we define the functions $u_n(t) := u(t_n + t)$, $v_n(t) := v(t_n + t)$, $\xi_n(t) := \xi(t_n + t)$, and $g_n(t) := g(t_n + t)$, $t \geq 0$. We are interested in studying the limiting behaviour of the above sequences as $n \nearrow +\infty$ in some finite time interval $[0, T]$. Hence, for a fixed $T > 0$ let us rewrite here PROBLEM (P) at the time $(t + t_n)$ in terms of the new unknowns u_n, v_n, ξ_n and data g_n, i.e.,

$$u_n'(t) + Av_n(t) = g_n(t) \quad \text{in } V', \text{ for a.e. } t \in (0,T), \tag{5.19}$$

$$v_n(t) = \xi_n(t) + Bu_n(t) \quad \text{in } V, \text{ for a.e. } t \in (0,T), \tag{5.20}$$

$$u_n(t) \in \mathrm{Dom}(\partial\Phi), \quad \xi_n(t) \in \partial\Phi(u_n(t)) \quad \text{for a.e. } t \in (0,T), \tag{5.21}$$

$$u_n(0) = u(t_n) \quad \text{in } H\,. \tag{5.22}$$

In view of (5.3), let us point out that

$$g_n \to g_\infty = g(0) + \int_0^{+\infty} g(t)dt \quad \text{strongly in } L^1(0,T;V') \text{ as } n \nearrow +\infty \tag{5.23}$$

because of

$$\|g_n - g_\infty\|_{L^1(0,T;V')} \leq \int_0^T \int_{t_n+t}^{+\infty} \|g'(s)\|_{V'}\, ds\, dt \leq T\|g'\|_{L^1(t_n,+\infty;V')} \searrow 0. \tag{5.24}$$

As a consequence of Theorem 5.2, we derive some estimates for u_n, v_n, and ξ_n, uniform with respect to $n \geq 1$. Since $u' \in L^2(0,+\infty;V')$ by (5.4), we infer that

$$\|u_n'\|_{L^2(0,T;V')} \leq \|u'\|_{L^2(t_n,+\infty;V')} \searrow 0 \quad \text{as } n \nearrow +\infty. \tag{5.25}$$

Moreover, by virtue of (5.4) and (3.1) we have

$$\|u_n\|_{L^\infty(0,T;H)} + \|Bu_n\|_{L^\infty(0,T;V)} \leq C \tag{5.26}$$

and consequently, thanks to (5.5) and by comparison in (5.20), we get

$$\|v_n\|_{L^2(0,T;V)} + \|\xi_n\|_{L^2(0,T;V)} \leq C(T) \tag{5.27}$$

for every $n \in \mathbb{N}$. Hence, by standard compactness arguments we deduce the existence of functions $\bar{u}, \bar{w}, \bar{v}, \bar{\xi} : (0,T) \to H$ such that, possibly taking a subsequence of n as $n \nearrow +\infty$, the convergences

$$u_n \to \bar{u} \quad \text{strongly in } H^1(0,T;V') \text{ and weakly star in } L^\infty(0,T;H) \tag{5.28}$$

$$Bu_n \to \bar{w} \quad \text{weakly star in } L^\infty(0,T;V) \tag{5.29}$$

$$v_n \to \bar{v} \quad \text{and} \quad \xi_n \to \bar{\xi} \quad \text{weakly in } L^2(0,T;V) \tag{5.30}$$

hold as $n \nearrow +\infty$. Concerning (5.28), we point out that the boundedness properties in (5.25) and (5.26) allow us to see (as for (4.21)) that $u_n \to \bar{u}$ weakly star in

$H^1(0,T;V') \cap L^\infty(0,T;H)$ and strongly in $C^0([0,T];V')$, but the fact that $u_n' \to 0$ strongly in $L^2(0,T;V')$ (cf. again (5.25)) yield (5.28) and, in addition, $\bar{u}' = 0$. Then, the function $\bar{u}$ does not depend on t; besides, (5.22) and (5.18) imply that $\bar{u} = u_\infty$. Moreover, as $P_0 u(t) = P_0 u_0$ for all $t \geq 0$ (cf. Remark 3.5), (5.16) follows easily.

As a next step, we check that

$$\bar{\xi}(t) \in \partial\Phi(u_\infty) \quad \text{for a.e. } t \in (0,T)\,. \tag{5.31}$$

Indeed, as $u_n \to u_\infty$ strongly in $L^2(0,T;V')$ and $\xi_n \to \bar{\xi}$ weakly in $L^2(0,T;V)$ (cf. (5.28) and (5.30)), we may repeat the argument from Subsection 4.3 and infer, as a counterpart to (4.31), that

$$\langle \bar{\xi}(t) - \eta, u_\infty - z\rangle_H \geq 0 \quad \text{a.e. in } (0,T)\,, \quad \forall z \in \mathrm{Dom}(\partial\Phi), \quad \forall \eta \in \partial\Phi(z). \tag{5.32}$$

Since $\partial\Phi$ is maximal monotone, we obtain the assertion.

It remains to check that $\bar{w} = Bu_\infty$. To this aim, we have to distinguish between the two cases in which either (3.6) or (3.7) holds.

Case $\partial\Phi$ strongly monotone. Take two different integers n and n' and test the difference of equations (5.19) written for n, n' by $A^{-1}(u_n - u_{n'})$. Note that this is possible since $(u_n - u_{n'})(t) = u(t_n + t) - u(t_{n'} + t) \in V'_*$ for all $t \geq 0$. Then, integrating the resulting equation over $(0,T)$, with the help of (5.20), (3.6), and (5.22) we obtain

$$\begin{aligned} \frac{1}{2}\|(u_n - u_{n'})(T)\|_{V'}^2 + C''_\Phi \|u_n - u_{n'}\|^2_{L^2(0,T;H)} &\leq \frac{1}{2}\|(u(t_n) - u(t_{n'})\|^2_{V'} \\ &+ \int_0^T \langle u_n - u_{n'}, -(Bu_n - Bu_{n'}) + A^{-1}(g_n - g_{n'})\rangle dt\,. \end{aligned} \tag{5.33}$$

Owing to (5.18), (5.28)–(5.29), and (5.23), the right hand side of (5.33) tends to 0 as $n \nearrow +\infty$: in particular, note the strong convergence of $\{u_n - u_{n'}\}$ to 0 in $L^\infty(0,T;V')$ against the boundedness of $\{-(Bu_n - Bu_{n'}) + A^{-1}(g_n - g_{n'})\}$ in $L^1(0,T;V)$. Hence, we infer that

$$u_n \to u_\infty \quad \text{strongly in } L^2(0,T;H) \tag{5.34}$$

as $n \nearrow +\infty$. Moreover, arguing as in (4.34)–(4.35), we also derive

$$Bu_n \to Bu_\infty \quad \text{strongly in } L^2(0,T;V), \tag{5.35}$$

whence (cf. (5.29)) $Bu_\infty = \bar{w}$.

Case B linear. If (3.7) holds, then the equality $Bu_\infty = \bar{w}$ is a straightforward consequence of the weak star convergence $u_n \to u_\infty$ in $L^\infty(0,T;H)$ in (5.28) and the linearity of B.

Therefore, thanks to the established convergences, passing to the limit as $n \nearrow \infty$ in (5.19)–(5.20), we find out that

$$A\bar{v}(t) = g_\infty \quad \text{and} \quad \bar{v}(t) = \bar{\xi}(t) + Bu_\infty \quad \text{for a.e. } t \in (0,T). \tag{5.36}$$

It suffices now to select any $t \in (0,T)$ such that $\bar{\xi}(t) \in \partial\Phi(u_\infty)$ in H by virtue of (5.31), and set $\xi_\infty := \bar{\xi}(t)$. Then, (5.17) results from (5.36) and Theorem 5.5 is completely proved. ∎

Remark 5.6. Let us note that for the proof of Theorem 5.5 we did not use the bound for $|\Phi(u(t))|$ contained in (5.4), but it is always interesting to have it, because for some potential Φ such bound may give further information on the long-time behaviour of the solution u. For instance, if the domain of Φ is as in Proposition 2.11 and the set Z used there is bounded in $\mathbb{R}^N$, then weak star convergence in L^∞ can be inferred for $\{u(t_n)\}$ and $\{u_n\}$ in the respective space and space-time domains.

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