

NUMERICAL VERIFICATION OF OPTIMALITY CONDITIONS

ARND RÖSCH¹, DANIEL WACHSMUTH²

Abstract. A class of optimal control problem for a semilinear elliptic partial differential equation with control constraints is considered. It is well known that sufficient second-order conditions ensure the stability of optimal solutions, the convergence of numerical methods. Otherwise, such conditions are very difficult to verify (analytically or numerically). We will propose a new approach: Starting with a numerical solution for a fixed mesh we will show the existence of a local minimizer of the continuous problem. Moreover, we will prove that this minimizer satisfies the sufficient second-order conditions.

1. Introduction. In this paper, we consider the optimal control problem (P) of minimizing $J(y, u)$ given by

$$J(y, u) := \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|u\|_{L^2(\Omega)}^2 \quad (1.1)$$

subject to the semilinear boundary value problem

$$\begin{aligned} (Ay)(x) + f(y(x)) &= u(x) && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma, \end{aligned} \quad (1.2)$$

and to the control constraints

$$a \leq u(x) \leq b \text{ a.e. in } \Omega. \quad (1.3)$$

In this setting, A is a uniformly bounded elliptic differential operator and Ω is a bounded domain of \mathbb{R}^N , $N = 2, 3$, with boundary Γ . Moreover ν is a fixed positive number. Precise assumptions on and definitions of the quantities introduced above are formulated at the end of this section.

Sufficient second-order optimality conditions are well established mathematical tools. The development of such conditions for control constrained problems governed by partial differential equations started with the papers of Goldberg and Tröltzsch [7, 8]. Two new trends lowered the gap between necessary and sufficient optimality conditions a few years later. Dontchev, Hager, Poore and Yang [6] introduced strongly active sets in the theory of sufficient second-order conditions. The idea is that no coercivity is needed on subspaces where the first-order conditions are sufficient. The equivalence of the coercivity condition with a positivity condition for the second derivative of the Lagrangian for a class of semilinear elliptic problems was proven by Bonnans [4].

Lipschitz stability results for optimal control problems governed by PDEs can be proved using second-order optimality conditions, see for instance the papers of Malanowski and Tröltzsch [11, 12] and Tröltzsch [14]. Moreover, it is possible to show locally quadratic convergence of the SQP-method, see Arada, Raymond and Tröltzsch [3]. These conditions are necessary to derive error estimates for discretized optimal control problems, see Arada, Casas, and Tröltzsch [2], and Casas, Mateos and Tröltzsch [5].

¹ Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria.

²Institut für Mathematik, Technische Universität Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany.

Although sufficient second-order optimality conditions have great importance, there is no method for verification (analytically or numerically) up to now. The numerical verification of such conditions is the main concern of this paper. Our starting point is a very realistic situation: A numerical solution of a discretized optimal control problem is given with information on the mesh size and the discretization error. Under certain conditions, we will show that a local minimum of the continuous problem exists in a neighborhood of the numerical solution.

The paper is organized as follows: Section 2 is devoted to optimality conditions of the continuous problem. In Section 3 we sketch our strategy. State equation, discretization, and objective functional are estimated in the following sections 4–6. The main theorem on verification of optimality conditions is stated and proven in Section 7. We will discuss all assumptions and the results for a numerical example in Section 8.

Assumptions. At first, we want to specify the assumptions on the various ingredients of the considered optimal control problem.

(A1) $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$ is a bounded domain that is either convex and polygonal or of the class $C^{1,1}$.

We denote by $\|\cdot\|_{H^1(\Omega)}$ the full H^1 norm: $\|y\|_{H^1(\Omega)}^2 = \|y\|_{L^2(\Omega)}^2 + \|\nabla y\|_{L^2(\Omega)}^2$. We will denote the imbedding constants from $H_0^1(\Omega)$ to $L^p(\Omega)$ by I_p , e.g.

$$\|y\|_{L^p(\Omega)} \leq I_p \|y\|_{H^1(\Omega)} \quad \forall y \in H_0^1(\Omega).$$

A is a uniformly elliptic differential operator defined by

$$(Ay)(x) = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j} y(x)) + c_0(x)y(x)$$

with functions a_{ij} that belong to $C^{0,1}(\bar{\Omega})$, satisfying the condition $a_{ij}(x) = a_{ji}(x)$ and

$$\delta_0 \|y\|_{H^1(\Omega)}^2 \leq (Ay, y), \quad (Ay_1, y_2) \leq \delta_1 \|y_1\|_{H^1(\Omega)} \|y_2\|_{H^1(\Omega)} \quad \forall y \in H_0^1(\Omega).$$

Let us denote by $a(\cdot, \cdot)$ the bilinear form induced by A

$$a(u, v) = (Au, v).$$

(A2) The function $f = f(y) : \Omega \rightarrow \mathbb{R}$ is of class C^2 with $f(0) = 0$. It satisfies the following conditions on boundedness and Lipschitz-continuity: For all $\tilde{M} > 0$ there are constants $c_f, c_{f'}, c_{f''} > 0$ such that

$$\begin{aligned} |f(y_1) - f(y_2)| &\leq c_f |y_1 - y_2| \\ |f'(y_1) - f'(y_2)| &\leq c_{f'} |y_1 - y_2| \\ |f''(y_1) - f''(y_2)| &\leq c_{f''} |y_1 - y_2| \end{aligned}$$

hold for all $|y_i| \leq \tilde{M}$, $i = 1, 2$. Moreover we require $f'(y) \geq 0$ in \mathbb{R} .

2. Optimality conditions for the continuous problem. A function y is called weak solution of the semilinear elliptic equation (4), if it satisfies

$$a(y, v) + (f(y), v) = (u, v) \quad \forall v \in V = H^1(\Omega) \cap C(\bar{\Omega}). \quad (2.1)$$

Solvability as well as regularity results for the weak formulation of the state equation are stated in Section 4.

We define the Lagrange functional by

$$L(y, u, p) = J(y, u) - a(y, p) - (f(y), p) + (u, p). \quad (2.2)$$

Then the necessary first-order optimality conditions are given by

$$L_y(\bar{y}, \bar{u}, \bar{p})(v) = 0 \quad \text{for } v \in V, \quad (2.3)$$

$$L_u(\bar{y}, \bar{u}, \bar{p})(u - \bar{u}) \geq 0 \quad \text{for } u \in U_{ad}. \quad (2.4)$$

Equation of (2.3) is equivalent to the adjoint equation defined by

$$a(v, p) + (f'(\bar{y})p, v) = (y - y_d, v) \quad \forall v \in V = H^1(\Omega) \cap C(\bar{\Omega}). \quad (2.5)$$

Existence and regularity results for the adjoint equation can be found in Section 4.

A sufficient second-order condition, henceforth called (SSC), is given by

$$L''_{(y,u)}(\bar{y}, \bar{u}, \bar{p})(y, u) \geq \delta \|u\|_{L^2(\Omega)}^2 \quad (2.6)$$

for all (y, u) satisfying the linearized equation

$$a(y, v) + (f'(\bar{y})y, v) = (u, v) \quad \forall v \in V. \quad (2.7)$$

Here, the second derivative of the Lagrangian is given by

$$L''_{(y,u)}(\bar{y}, \bar{u}, \bar{p})(y, u) = \|y\|_{L^2(\Omega)}^2 + \nu \|u\|_{L^2(\Omega)}^2 + (f''(\bar{y})y^2, \bar{p}). \quad (2.8)$$

However, the solution $(\bar{y}, \bar{u}, \bar{p})$ is unknown. Hence, it is impossible to show that (SSC) holds in advance.

3. The discretized problem. Typically, control and state are discretized (for instance by finite elements) to obtain numerical approximations $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$ of solutions of the continuous problem. Here, the following questions arise:

- Is the numerical solution close to a stationary point of the undiscretized problem?
- Is the numerical solution close to a local minimum of the undiscretized problem?

In the article, we will derive conditions for the solution of the discretized problem $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$ that ensure the existence of a local minimizer of (P) in a specified neighborhood of the numerical solution $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$. Under additional assumptions, we can even show that this unknown minimizer fulfills (SSC).

We will formulate general assumptions for the discretization of the optimal control problem. The assumption fits to finite element discretizations of the problem: The state equation (2.1) is replaced by

$$a(y_h, v_h) + (f(y_h), v_h) = (u, v_h) \quad \forall v_h \in V_h, \quad (3.1)$$

where $V_h \subset V$ is a finite dimensional space. The set of admissible controls is given by

$$U_{ad}^h := \{u_h \in U_h : a \leq u_h(x) \leq b\}, \quad (3.2)$$

where U_h is the space of functions that are piecewise constant over the elements of a triangulation of Ω . We refer to Section 5 for the precise formulation of the assumptions on the discretization.

The corresponding discretized adjoint equation is given by

$$a(v_h, p_h) + (f'(\bar{y}_h)p_h, v_h) = (y - y_d, v_h) \quad \forall v_h \in V_h. \quad (3.3)$$

The discretized optimal control problem is given by: minimize $J(y_h, u_h)$ subject to the discretized equation (3.1) and to the control constraint $u \in U_{ad}^h$.

In the sequel, we assume that $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$ solves that discretized problem without numerical errors. However, it would be sufficient for all results that the numerical error is smaller than the discretization error.

Now, let us sketch our strategy: We *assume* a condition similar to (2.6): namely,

$$L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y, u) \geq \delta \|u\|_{L^2(\Omega)}^2 \quad (3.4)$$

holds for all $(y, u) \in V \times U_{ad}$ fulfilling the linearized equation³

$$a(y, v) + (f'(\bar{y}_h)y, v) = (u, v) \quad \forall v \in V. \quad (3.5)$$

So we assume the coercivity for the discretized solution instead of the continuous one. Note that this condition is required for all $u \in U_{ad}$ (and not only in U_{ad}^h). The inequality (3.4) means

$$\|y\|_{L^2(\Omega)}^2 + \nu \|u\|_{L^2(\Omega)}^2 + (f''(\bar{y}_h)y^2, \bar{p}_h) \geq \delta \|u\|_{L^2(\Omega)}^2.$$

Since \bar{y}_h, \bar{p}_h are known, this can be verified easily. It would suffice for instance that $f''(\bar{y}_h(x))\bar{p}_h(x) > 0$ holds on Ω . Moreover, if $(\bar{y}, \bar{u}, \bar{p})$ satisfy (SSC), then such an assumption will be true for sufficiently fine discretizations.

The pair (\bar{u}_h, \bar{y}_h) is not admissible for the undiscretized problem. Therefore, we introduce the auxiliary state y^h as the solution of the elliptic equation with right-hand side \bar{u}_h

$$a(y^h, v) + (f(y^h), v) = (\bar{u}_h, v) \quad \forall v \in V. \quad (3.6)$$

Under the assumption that δ in (3.4) is large enough, we will show the existence of a radius $r > 0$ such that

$$J(y, u) - J(y^h, \bar{u}_h) > 0 \quad \text{if } \|u - \bar{u}_h\|_{L^2(\Omega)} = r.$$

This result will be the key for our argumentation: Since (y^h, \bar{u}_h) is an admissible pair for the undiscretized problem, we obtain the existence of a local minimizer \hat{u} of (P) in the neighborhood $\{u \in U_{ad} : \|u - \bar{u}_h\|_{L^2(\Omega)} < r\}$ of \bar{u}_h , see Section 6. Under additional assumptions, this local minimizer is unique and fulfills the sufficient condition (SSC). That is, we get the fulfillment of (SSC) as an *a-posteriori* result and not as an *a-priori* assumption.

In order to get a computable bound for the radius r , all estimations has to carried out carefully, and all constants have to be known. A representative collection of that constants includes

³Let us remark, that we could also deal with a linearized equation in V_h instead of V .

- embedding constants $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$,
- interpolation constants for the finite elements,
- global bounds of the state,
- norm of the solution operators of the PDEs.

In the following sections, we will estimate the several ingredients of the optimal control problem and its discretization. The solution operator of the semilinear elliptic equation is studied in the next Section 4. The finite element discretization and associated error estimates for the state equations can be found in Section 5. The difference of the objective functionals $J(y, u) - J(y^h, \bar{u}_h)$ is investigated and estimated in Section 6. The existence of a local minimizer \hat{u} of (P) in the neighborhood of \bar{u}_h is proven in Section 7. Also the proof that \hat{u} satisfies (SSC) is contained in that section.

4. State equation. Now, let us study briefly the state equation.

THEOREM 4.1. *The semilinear state equation admits a unique solution $y \in V \cap H^2(\Omega)$ for all $u \in L^2$ and satisfies the estimates*

$$\|y\|_{H^2(\Omega)} \leq c_S \|u\|_{L^2(\Omega)}, \quad (4.1)$$

$$\|y\|_{L^\infty(\Omega)} \leq c_{SL^\infty} \|u\|_{L^2(\Omega)}. \quad (4.2)$$

Since U_{ad} is bounded, we have in addition

$$\|y\|_{L^\infty(\Omega)} < M \quad (4.3)$$

for all solutions y associated to admissible controls u .

For the proof we refer to Grisvard [9].

Thanks to Theorem 4.1, we can interpret all assumptions of **(A2)**, which were defined on bounded sets, as global estimates:

(A2') The function $f = f(y) : \Omega \rightarrow \mathbb{R}$ is of class C^2 with $f(0) = 0$. It satisfies the following conditions on boundedness and Lipschitz-continuity: There are constants $c_f, c_{f'}, c_{f''} > 0$ such that

$$\begin{aligned} |f(y_1) - f(y_2)| &\leq c_f |y_1 - y_2| \\ |f'(y_1) - f'(y_2)| &\leq c_{f'} |y_1 - y_2| \\ |f''(y_1) - f''(y_2)| &\leq c_{f''} |y_1 - y_2| \end{aligned}$$

hold for all $y_1, y_2 \in \mathbb{R}$.

This assumption implies the boundedness of the derivatives of f : $|f'(y)| \leq c_f$ and $|f''(y)| \leq c_{f'}$ for all $y \in \mathbb{R}$.

Now, let us prove Lipschitz continuity of the control-to-state mapping.

LEMMA 4.2. *Let y_1 and y_2 be the solutions of (2.1) for controls u_1 and u_2 , respectively. Then the estimate*

$$\|y_1 - y_2\|_{L^2(\Omega)} \leq \|y_1 - y_2\|_{H^1(\Omega)} \leq c_L \|u_1 - u_2\|_{L^2(\Omega)} \quad (4.4)$$

is valid.

Proof. Subtracting the weak formulations for y_1 and y_2 and testing with the difference $V = y_1 - y_2$, we find

$$a(y_1 - y_2, y_1 - y_2) + (f(y_1) - f(y_2), y_1 - y_2) = (u_1 - u_2, y_1 - y_2).$$

Using the coercivity of the bilinear form and the monotonicity of f , we get

$$a(y_1 - y_2, y_1 - y_2) + (f(y_1) - f(y_2), y_1 - y_2) \geq \delta_0 \|y_1 - y_2\|_{H^1(\Omega)}^2.$$

This implies

$$\delta_0 \|y_1 - y_2\|_{H^1(\Omega)}^2 \leq \|u_1 - u_2\|_{L^2(\Omega)} \|y_1 - y_2\|_{L^2(\Omega)}.$$

Consequently,

$$\|y_1 - y_2\|_{H^1(\Omega)} \leq c_L \|u_1 - u_2\|_{L^2(\Omega)}$$

is valid with $c_L = \frac{I_2}{\delta_0}$. \square

A similar estimate is available for solutions of the linearized system

$$a(y, v) + (f'(\bar{y})y, v) = (u, v) \quad \forall v \in V = H^1(\Omega) \cap C(\bar{\Omega}). \quad (4.5)$$

Here, \bar{y} is a given function of V .

COROLLARY 4.3. *The linearized state equation (4.5) admits a unique solution y for each $u \in U$, and it holds*

$$\|y_1 - y_2\|_{L^2(\Omega)} \leq \|y_1 - y_2\|_{H^1(\Omega)} \leq c_L \|u_1 - u_2\|_{L^2(\Omega)}$$

with the same constant c_L as in (4.4) above.

We conclude this section with the state of an existence result for the adjoint equation (2.5).

LEMMA 4.4. *The adjoint equation (2.5) admits a unique solution. Moreover the estimate*

$$\|p\|_{H^2(\Omega)} \leq c_p \|y - y_d\|_{L^2(\Omega)} \quad (4.6)$$

is valid. For the proof we refer to Grisvard [9].

5. Finite element discretization. The state V is approximated by a finite-dimensional subspace $V_h \subset V$. Here, we impose the following requirement.

(A3) The mesh parameter h is supposed to be smaller than 1. We assume the existence of an operator $i_h : V \rightarrow V_h$ with the following properties

$$\|y - i_h y\|_{L^2(\Omega)} \leq c_1 h^2 \|y\|_{H^2(\Omega)} \quad (5.1)$$

$$\|y - i_h y\|_{H^1(\Omega)} \leq c_2 h \|y\|_{H^2(\Omega)}. \quad (5.2)$$

This assumption is a standard property of conform finite element discretizations.

The control is discretized piecewise constant on a mesh \mathcal{T}_h containing open sets T (finite elements):

$$U_h := \{q_h \in L^2(\Omega) : q_h|_T = \text{const for all } T \in \mathcal{T}_h\}.$$

We require for the mesh \mathcal{T}_h :

(A4) The diameter of the largest element of \mathcal{T}_h is bounded by h . Moreover,

$$\bigcup_{T \in \mathcal{T}_h} \bar{T} = \bar{\Omega}, \quad T_i \cap T_j = \emptyset \text{ for all } T_i, T_j \in \mathcal{T}_h, i \neq j.$$

Now, we give an error estimate for the semilinear equation.

LEMMA 5.1. *Let y and y_h be the unique solutions of the semilinear equations (2.1) and (3.1), respectively. Then, the following error estimates are valid*

$$\|y - y_h\|_{L^2(\Omega)} \leq c_{L^2} h^2 \|y\|_{H^2(\Omega)}, \quad (5.3)$$

$$\|y - y_h\|_{H^1(\Omega)} \leq c_{H^1} h \|y\|_{H^2(\Omega)}. \quad (5.4)$$

Proof. Using the ellipticity assumption **(A1)** and the monotonicity of f in **(A2')**, we find

$$\delta_0 \|y - y_h\|_{H^1(\Omega)} \leq a(y - y_h, y - y_h) + (f(y) - f(y_h), y - y_h). \quad (5.5)$$

The weak formulations for y and y_h imply

$$a(y - y_h, v_h) + (f(y) - f(y_h), v_h) = 0 \quad \forall v_h \in V_h. \quad (5.6)$$

We use (5.6) with $v_h = y_h - i_h y$. Inserting this equation in (5.5), we obtain

$$\delta_0 \|y - y_h\|_{H^1(\Omega)}^2 \leq a(y - y_h, y - i_h y) + (f(y) - f(y_h), y - i_h y). \quad (5.7)$$

Moreover, we can estimate

$$\begin{aligned} a(y - y_h, y - i_h y) + (f(y) - f(y_h), y - i_h y) &\leq \delta_1 \|y - y_h\|_{H^1(\Omega)} \|y - i_h y\|_{H^1(\Omega)} \\ &\quad + c_f \|y - y_h\|_{L^2(\Omega)} \|y - i_h y\|_{L^2(\Omega)}. \end{aligned}$$

Combining this inequality with (5.7), we find

$$\delta_0 \|y - y_h\|_{H^1(\Omega)}^2 \leq \delta_1 \|y - y_h\|_{H^1(\Omega)} \|y - i_h y\|_{H^1(\Omega)} + c_f \|y - y_h\|_{L^2(\Omega)} \|y - i_h y\|_{L^2(\Omega)}. \quad (5.8)$$

From this, we get easily

$$\delta_0 \|y - y_h\|_{H^1(\Omega)} \leq \delta_1 \|y - i_h y\|_{H^1(\Omega)} + c_f \|y - i_h y\|_{L^2(\Omega)}$$

or using assumption **(A3)**

$$\delta_0 \|y - y_h\|_{H^1(\Omega)} \leq \delta_1 c_2 h \|y\|_{H^2(\Omega)} + c_f c_1 h^2 \|y\|_{H^2(\Omega)}.$$

Hence, for $h < 1$ the constant c_{H^1} can be chosen as

$$c_{H^1} = (\delta_1 c_2 + c_f c_1) / \delta_0.$$

Next, we introduce a solution g by

$$a(g, v) + (f_y^{y_h} g, v) = (e, v)$$

with

$$e = \frac{y - y_h}{\|y - y_h\|_{L^2(\Omega)}}.$$

The expression $f_y^{y_h} g$ is defined by

$$f_y^{y_h} g = \int_0^1 f'(y_h + s(y - y_h)) g \, ds.$$

Note, that $\|e\|_{L^2(\Omega)} = 1$. Since f' is globally bounded, we have

$$\|g\|_{H^2(\Omega)} \leq c_M$$

with some constant $c_M > 0$. Using these definitions, we find

$$\begin{aligned} \|y - y_h\|_{L^2(\Omega)} &= (e, y - y_h) \\ &= a(g, y - y_h) + (f_y^{y_h} g, y - y_h) \\ &= a(g, y - y_h) + (g, f_y^{y_h}(y - y_h)) \\ &= a(g, y - y_h) + (g, f(y) - f(y_h)). \end{aligned}$$

From (5.6) we obtain with $v_h = i_h g$

$$\begin{aligned} \|y - y_h\|_{L^2(\Omega)} &= a(g - i_h g, y - y_h) + (g - i_h g, f(y) - f(y_h)) \\ &\leq \delta_1 \|g - i_h g\|_{H^1(\Omega)} \|y - y_h\|_{H^1(\Omega)} + c_f \|g - i_h g\|_{L^2(\Omega)} \|y - y_h\|_{L^2(\Omega)} \\ &\leq \delta_1 c_2 c_{H^1} h^2 \|g\|_{H^2(\Omega)} \|y\|_{H^2(\Omega)} + c_f c_1 h^2 \|g\|_{H^2(\Omega)} \|y - y_h\|_{L^2(\Omega)} \\ &\leq (\delta_1 c_2 c_{H^1} c_M h^2 + c_f c_1 c_{H^1} c_M h^3) \|y\|_{H^2(\Omega)}. \end{aligned}$$

Now, for $h < 1$ we have

$$c_{L^2} = (\delta_1 c_2 + c_f c_1) c_{H^1} c_M.$$

□

Applying these results, we can give error estimates for the auxiliary function y^h introduced in (3.6).

COROLLARY 5.2. *It holds for the difference $y^h - \bar{y}_h$*

$$\|y^h - \bar{y}_h\|_{L^2(\Omega)} \leq c_{L^2} c_S h^2 \|\bar{u}_h\|_{L^2(\Omega)}, \quad (5.9)$$

$$\|y^h - \bar{y}_h\|_{H^1(\Omega)} \leq c_{H^1} c_S h \|\bar{u}_h\|_{L^2(\Omega)}. \quad (5.10)$$

Proof. The claim follows directly from Lemma 5.1 and Theorem 4.1. □

6. Estimation of the objective functional, existence of a local minimizer. Throughout the following sections, we assume the fulfillment of **(A1)**, **(A3)**, and **(A4)**, which are assumptions concerning the state equation and the discretization.

Furthermore, we impose the following conditions on the discrete solution $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$:

(A5) The solution $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$ satisfies the discretized state equation (3.1), the discretized adjoint equation (3.3), and the variational inequality

$$(\nu \bar{u}_h + \bar{p}_h, u_h - \bar{u}_h) \geq 0 \quad \forall u_h \in U_{ad}^h.$$

We assume additionally that the coercivity condition (3.4)-(3.5) holds at $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$ with some $\delta > 0$. We require the regularities $\bar{y}_h \in L^\infty(\Omega)$ and $\bar{p}_h \in W^{1,\infty}(\Omega)$.

The latter assumption allows us to use the modified assumption **(A2')** on the bounded set $\{y \in L^\infty(\Omega) : \|y\|_\infty \leq \max(M, \|\bar{y}_h\|_\infty)\}$. Here, M is the global bound provided by Theorem 4.1.

Now, we will investigate the behavior of the objective functional in the neighborhood of \bar{u}_h . The following lemma expresses $J(y, u) - J(y^h, \bar{u}_h)$ as the sum of several addends, which will be estimated in the sequel.

LEMMA 6.1. *Let u be an admissible control and y the associated state (solution of (2.1)). Then we can represent the difference of the objective values as*

$$J(y, u) - J(y^h, \bar{u}_h) = L(\bar{y}_h, \bar{u}_h, \bar{p}_h) - L(y^h, \bar{u}_h, \bar{p}_h) \quad (6.1)$$

$$+ L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h) \quad (6.2)$$

$$+ L_u(\bar{y}_h, \bar{u}_h, \bar{p}_h)(u - \bar{u}_h) \quad (6.3)$$

$$+ \frac{1}{2} L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h, u - \bar{u}_h)^2 \quad (6.4)$$

$$+ r_2 \quad (6.5)$$

where r_2 denotes a second order remainder term.

Proof. Since (y, u) and (y^h, \bar{u}_h) fulfill (2.1), we find

$$\begin{aligned} J(y, u) - J(y^h, \bar{u}_h) &= L(y, u, \bar{p}_h) - L(y^h, \bar{u}_h, \bar{p}_h) \\ &= L(y, u, \bar{p}_h) - L(\bar{y}_h, \bar{u}_h, \bar{p}_h) \\ &\quad + L(\bar{y}_h, \bar{u}_h, \bar{p}_h) - L(y^h, \bar{u}_h, \bar{p}_h) \end{aligned} \quad (6.6)$$

The assertion is now obtained by a Taylor expansion

$$\begin{aligned} L(y, u, \bar{p}_h) - L(\bar{y}_h, \bar{u}_h, \bar{p}_h) &= L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h) + L_u(\bar{y}_h, \bar{u}_h, \bar{p}_h)(u - \bar{u}_h) \\ &\quad + \frac{1}{2} L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h, u - \bar{u}_h)^2 + r_2. \end{aligned} \quad (6.7)$$

□

Now, we will estimate the terms (6.1)-(6.5).

LEMMA 6.2. *The discretization error in the Lagrange function can be estimated by*

$$|L(y^h, \bar{u}_h, \bar{p}_h) - L(\bar{y}_h, \bar{u}_h, \bar{p}_h)| < c_j h^2 \|\bar{u}_h\|_{L^2(\Omega)}. \quad (6.8)$$

Proof. Here, we find for the objective

$$\begin{aligned} |J(y^h, \bar{u}_h) - J(\bar{y}_h, \bar{u}_h)| &= \left| \frac{1}{2} \|y^h - y_d\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\bar{y}_h - y_d\|_{L^2(\Omega)}^2 \right| \\ &= \left| \frac{1}{2} \|y^h - \bar{y}_h\|_{L^2(\Omega)}^2 + (\bar{y}_h - y_d, y^h - \bar{y}_h) \right| \\ &\leq c_{L^2} c_S h^2 \|\bar{u}_h\|_{L^2(\Omega)} (\|\bar{y}_h - y_d\|_{L^2(\Omega)} + \frac{1}{2} c_{L^2} c_S h^2 \|\bar{u}_h\|_{L^2(\Omega)}) \end{aligned}$$

It remains to estimate the term associated to the semilinear equation in the difference $L(y^h, \bar{u}_h, \bar{p}_h) - L(\bar{y}_h, \bar{u}_h, \bar{p}_h)$. The semilinear equation is satisfied for the pair (y^h, \bar{u}_h) . Consequently, this term vanishes. The pair (\bar{y}_h, \bar{u}_h) fulfills only the equation in V_h . However, the test function \bar{p}_h belongs to V_h . Hence, this term vanishes, too. Consequently, the assertion is true with

$$c_j = c_{L^2} c_S (\|\bar{y}_h - y_d\|_{L^2(\Omega)} + \frac{1}{2} c_{L^2} c_S h^2 \|\bar{u}_h\|_{L^2(\Omega)}).$$

□

LEMMA 6.3. *The error in the adjoint equation can be estimated by*

$$|L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h)| \leq c_y h^2 \|\bar{y}_h - y_d\|_{L^2(\Omega)} \|y\|_{H^2(\Omega)} \quad \forall y \in V. \quad (6.9)$$

Proof. We start with

$$L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h) = (\bar{y}_h - y_d, y - \bar{y}_h) - a(\bar{p}_h, y - \bar{y}_h) - (f'(\bar{y}_h)\bar{p}_h, y - \bar{y}_h).$$

Moreover, we define a function p^h as solution of

$$a(p^h, v) + (f'(\bar{y}_h)p^h, v) = (\bar{y}_h - y_d, v) \quad \forall v \in V. \quad (6.10)$$

We obtain for $v = y - \bar{y}_h$

$$(\bar{y}_h - y_d, y - \bar{y}_h) = a(p^h, y - \bar{y}_h) + (f'(\bar{y}_h)p^h, y - \bar{y}_h).$$

Therefore, we can continue using Galerkin orthogonality

$$\begin{aligned} L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h) &= a(p^h - \bar{p}_h, y - \bar{y}_h) + (f'(\bar{y}_h)(p^h - \bar{p}_h), y - \bar{y}_h) \\ &= a(p^h - \bar{p}_h, y - i_h y) + (f'(\bar{y}_h)(p^h - \bar{p}_h), y - i_h y) \\ &\quad + a(p^h - \bar{p}_h, i_h y - \bar{y}_h) + (f'(\bar{y}_h)(p^h - \bar{p}_h), i_h y - \bar{y}_h) \\ &= a(p^h - \bar{p}_h, y - i_h y) + (f'(\bar{y}_h)(p^h - \bar{p}_h), y - i_h y). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} |L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h)| &\leq \delta_1 \|p^h - \bar{p}_h\|_{H^1(\Omega)} \|y - i_h y\|_{H^1(\Omega)} \\ &\quad + c_f \|p^h - \bar{p}_h\|_{L^2(\Omega)} \|y - i_h y\|_{L^2(\Omega)}. \end{aligned}$$

The proving technique of Lemma 5.1 delivers

$$\|p^h - \bar{p}_h\|_{H^1(\Omega)} \leq c_{H^1} h \|p^h\|_{H^2(\Omega)} \quad \text{and} \quad \|p^h - \bar{p}_h\|_{L^2(\Omega)} \leq c_{L^2} h^2 \|p^h\|_{H^2(\Omega)}. \quad (6.11)$$

Moreover, we can apply the inequalities (5.1) and (5.2) for the interpolation error of y . Finally, we get

$$|L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h)| \leq (c_{H^1} \delta_1 + c_{L^2} c_f h^2) h^2 \|p^h\|_{H^2(\Omega)} \|y\|_{H^2(\Omega)}$$

Of course, the norm $\|p^h\|_{H^2(\Omega)}$ can be estimated by means of Lemma 4.4

$$\|p^h\|_{H^2(\Omega)} \leq c_p \|\bar{y}_h - y_d\|_{L^2(\Omega)}, \quad (6.12)$$

and (6.9) is obtained with $c_y = c_p(c_{H^1} \delta_1 + c_{L^2} c_f)$ if $h \leq 1$. □

LEMMA 6.4. *The error in the optimality condition can be estimated by*

$$L_u(\bar{y}_h, \bar{u}_h, \bar{p}_h)(u - \bar{u}_h) \geq -c_u h \|u - \bar{u}_h\|_{L^2(\Omega)} \|\bar{p}_h\|_{W^{1,\infty}(\Omega)} \quad \forall u \in U_{ad}. \quad (6.13)$$

Proof. The optimality condition for \bar{u}_h is given by

$$(\bar{p}_h + \nu \bar{u}_h, u_h - \bar{u}_h) \geq 0 \quad \forall u_h \in U_{ad}^h. \quad (6.14)$$

This implies $u_h = a$ on all elements T_h where $\bar{p}_h + \nu\bar{u}_h$ is a.e. positive. Analogously $u_h = b$ holds on all elements T_h where $\bar{p}_h + \nu\bar{u}_h$ is a.e. negative. For the set T_h^a of all such elements we find for arbitrary $u \in U_{ad}$

$$(\bar{p}_h + \nu\bar{u}_h, u - \bar{u}_h)_{L^2(T_h^a)} \geq 0 \quad \forall u \in U_{ad}. \quad (6.15)$$

It remains to estimate the error on the set T_h^i of elements where the expression $\bar{p}_h + \nu\bar{u}_h$ changes the sign. Since \bar{u}_h is constant on each element, we find

$$|\bar{p}_h + \nu\bar{u}_h| \leq h\|\bar{p}_h\|_{W^{1,\infty}(\Omega)} \quad \text{on } T_h^i.$$

From this, we conclude

$$(\bar{p}_h + \nu\bar{u}_h, u - \bar{u}_h)_{L^2(T_h^i)} \geq -|T_h^i|^{1/2}h\|u - \bar{u}_h\|_{L^2(T_h^i)}\|\bar{p}_h\|_{W^{1,\infty}(\Omega)} \quad \forall u \in U_{ad}. \quad (6.16)$$

Combining (6.15) and (6.16), (6.13) is obtained with $c_u = |T_h^i|^{1/2}$. \square

LEMMA 6.5. *Let $u \in U_{ad}$ be given together with the associated solution y of the semilinear state equation (2.1). Then it holds with $r = \|u - \bar{u}_h\|_{L^2(\Omega)}$*

$$L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h, u - \bar{u}_h)^2 \geq \delta r^2 - d_1 h r^2 - d_2 h^2 r - d_3 r^3 \quad (6.17)$$

with constants d_i specified below.

Proof. At first, let us define d as the solution of the linearized equation

$$a(d, v) + (f'(\bar{y}_h)d, v) = (u - \bar{u}_h, v) \quad \forall v \in V.$$

Observe that d can be used as test function in the coercivity condition (3.4) whereas $y - \bar{y}_h$ would not be suitable there. Furthermore, we have by Corollary 4.3 the estimate

$$\|d\|_{L^2(\Omega)} \leq c_L \|u - \bar{u}_h\|_{L^2(\Omega)}. \quad (6.18)$$

Now, we rewrite the left-hand side in (6.17) as

$$\begin{aligned} & L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h, u - \bar{u}_h)^2 = \\ & L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(d, u - \bar{u}_h)^2 + L_{yy}(\bar{y}_h, \bar{u}_h, \bar{p}_h)[y - \bar{y}_h - d, y - \bar{y}_h + d]. \end{aligned} \quad (6.19)$$

The first addend gives the coercivity $\geq \delta r^2$ using the sufficient condition (3.4). The second one can be estimated using (2.8) by

$$\begin{aligned} & |L_{yy}(\bar{y}_h, \bar{u}_h, \bar{p}_h)[y - \bar{y}_h - d, y - \bar{y}_h + d]| \\ & \leq (1 + c_{f'}\|\bar{p}_h\|_{L^\infty(\Omega)}) \|y - \bar{y}_h - d\|_{L^2(\Omega)} \|y - \bar{y}_h + d\|_{L^2(\Omega)}. \end{aligned} \quad (6.20)$$

Using (4.4), (5.9), and (6.18), we find

$$\begin{aligned} \|y - \bar{y}_h + d\|_{L^2(\Omega)} & \leq \|y - y^h\|_{L^2(\Omega)} + \|y^h - \bar{y}_h\|_{L^2(\Omega)} + \|d\|_{L^2(\Omega)} \\ & \leq c_L \|u - \bar{u}_h\|_{L^2(\Omega)} + c_{L^2} c_S h^2 \|\bar{u}_h\|_{L^2(\Omega)} + c_L \|u - \bar{u}_h\|_{L^2(\Omega)} \\ & = 2c_L r + c_{L^2} c_S \|\bar{u}_h\|_{L^2(\Omega)} h^2. \end{aligned} \quad (6.21)$$

We could treat $y - \bar{y}_h - d$ in the same way. However, we need a sharper result. To this end, we use the splitting $y - \bar{y}_h - d = (y - y^h - d) + (y^h - \bar{y}_h)$.

The first function $y - y^h - d =: y_1$ is the weak solution of

$$a(y_1, v) + (f'(\bar{y}_h)y_1, v) = - (f(y) - f(y^h) - f'(\bar{y}_h)(y - y^h), v) \quad \forall v \in V.$$

The right-hand side we transform to

$$\begin{aligned} f(y) - f(y^h) - f'(\bar{y}_h)(y - y^h) &= (f'(y^h) - f'(\bar{y}_h))(y - y^h) \\ &\quad + \int_0^1 [f'(y^h + s(y - y^h)) - f'(y^h)](y - y^h) ds. \end{aligned}$$

Its L^2 -norm is estimated by

$$\begin{aligned} \|f(y) - f(y^h) - f'(\bar{y}_h)(y - y^h)\|_{L^2(\Omega)} \\ \leq c_{f'} \left(\|y^h - \bar{y}_h\|_{L^4(\Omega)} \|y - y^h\|_{L^4(\Omega)} + \frac{1}{2} \|y - y^h\|_{L^4(\Omega)}^2 \right). \end{aligned}$$

Hence, we can estimate

$$\begin{aligned} \|y_1\|_{L^2(\Omega)} &= \|y - d - y^h\|_{L^2(\Omega)} \leq c_L \|f(y) - f(y^h) - f'(\bar{y}_h)(y - y^h)\|_{L^2(\Omega)} \\ &\leq c_L c_{f'} \left(\|y^h - \bar{y}_h\|_{L^4(\Omega)} \|y - y^h\|_{L^4(\Omega)} + \frac{1}{2} \|y - y^h\|_{L^4(\Omega)}^2 \right). \end{aligned}$$

Analogously to (5.9) we get a L^4 -error estimate for $y - y^h$ by Lemma 5.1

$$\|y^h - \bar{y}_h\|_{L^4(\Omega)} \leq I_4 c_{H^1} h c_S \|\bar{u}_h\|_{L^2(\Omega)}.$$

Applying Corollary 4.3, we find

$$\|y - y^h\|_{L^4(\Omega)} \leq I_4 \|y - y^h\|_{H^1(\Omega)} \leq I_4 c_L r.$$

Altogether, we derived the estimate

$$\|y - d - y^h\|_{L^2(\Omega)} \leq c_L^2 c_{f'} I_4^2 \left(c_{H^1} c_S \|\bar{u}_h\|_{L^2(\Omega)} h r + \frac{1}{2} c_L r^2 \right).$$

Now we can proceed with the estimation of L_{yy} already started in (6.20)

$$\begin{aligned} |L_{yy}(\bar{y}_h, \bar{u}_h, \bar{p}_h)[y - \bar{y}_h - d, y - \bar{y}_h + d]| \\ \leq (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) \|y - \bar{y}_h - d\|_{L^2(\Omega)} \|y - \bar{y}_h + d\|_{L^2(\Omega)} \\ \leq (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) c_L^2 c_{f'} I_4^2 \left(c_{H^1} c_S \|\bar{u}_h\|_{L^2(\Omega)} h r + \frac{1}{2} c_L r^2 \right) \\ \cdot (2c_L r + c_{L^2} c_S \|\bar{u}_h\|_{L^2(\Omega)} h^2) \\ = d_1 h r^2 + d_2 h^2 r + d_3 r^3 \end{aligned}$$

with constants defined by

$$\begin{aligned} d_0 &= (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) c_L^2 c_{f'} I_4^2, \\ d_1 &= d_0 \cdot \left(c_{H^1} c_S \|\bar{u}_h\|_{L^2(\Omega)} \cdot 2c_L + \frac{1}{2} c_L c_{L^2} c_S \|\bar{u}_h\|_{L^2(\Omega)} \right), \\ d_2 &= d_0 \cdot c_{H^1} c_S \|\bar{u}_h\|_{L^2(\Omega)} \cdot c_{L^2} c_S \|\bar{u}_h\|_{L^2(\Omega)}, \\ d_3 &= d_0 \cdot c_L^2. \end{aligned}$$

Here, we used again $h < 1$. Finally, we obtain for the second derivative

$$L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h, u - \bar{u}_h)^2 \geq \delta r^2 - d_1 h r^2 - d_2 h^2 r - d_3 r^3,$$

and the claim is proven. \square

LEMMA 6.6. *Assume that (2.6) holds. Then, the estimate*

$$J(y, u) - J(y^h, \bar{u}_h) \geq \delta r^2 - a_1 h^2 - a_2 h r - a_3 h^2 r - a_4 r^3 - a_5 h r^2 \quad (6.22)$$

is valid for all (y, u) satisfying (2.1) and $\|u - \bar{u}_h\|_{L^2(\Omega)} = r$.

Proof. Since (y, u) and (y^h, \bar{u}_h) fulfill (2.1), we find

$$\begin{aligned} J(y, u) - J(y^h, \bar{u}_h) &= L(y, u, \bar{p}_h) - L(y^h, \bar{u}_h, \bar{p}_h) \\ &= L(y, u, \bar{p}_h) - L(\bar{y}_h, \bar{u}_h, \bar{p}_h) \\ &\quad + L(\bar{y}_h, \bar{u}_h, \bar{p}_h) - L(y^h, \bar{u}_h, \bar{p}_h) \end{aligned} \quad (6.23)$$

The second difference was already estimated in Lemma 6.2. We will now focus on the first difference

$$\begin{aligned} L(y, u, \bar{p}_h) - L(\bar{y}_h, \bar{u}_h, \bar{p}_h) &= L_y(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h) + L_u(\bar{y}_h, \bar{u}_h, \bar{p}_h)(u - \bar{u}_h) \\ &\quad + \frac{1}{2} L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y - \bar{y}_h, u - \bar{u}_h)^2 + r_2. \end{aligned} \quad (6.24)$$

Note that the quadratical part of the objective is approximated exactly. Only the nonlinear function f causes the remainder part r_2 . We estimated all addends in the Lemmas 6.3, 6.4 and 6.5. It remains to investigate the remainder term. Here, we find

$$\begin{aligned} |r_2| &= \left| \int_{\Omega} \int_0^1 \int_0^s (f''(\bar{y}_h + t(y - \bar{y}_h)) - f''(\bar{y}_h)) (y - \bar{y}_h)^2 dt ds \bar{p}_h dx \right| \\ &\leq \frac{1}{6} c_{f''} \|y - \bar{y}_h\|_{L^3(\Omega)}^3 \|\bar{p}_h\|_{L^\infty(\Omega)}. \end{aligned} \quad (6.25)$$

We estimate the right-hand side of (6.25) using Lemma 4.2 and Corollary 5.2 as

$$\begin{aligned} \|y - \bar{y}_h\|_{L^3(\Omega)} &\leq \|y - y^h\|_{L^3(\Omega)} + \|y^h - \bar{y}_h\|_{L^3(\Omega)} \\ &\leq c_L I_3 \|u - \bar{u}_h\|_{L^2(\Omega)} + c_{H^1} c_S I_3 h \|\bar{u}_h\|_{L^2(\Omega)}. \end{aligned}$$

Using $\|u - \bar{u}_h\|_{L^2(\Omega)} = r$ and $(a + b)^3 \leq 4(a^3 + b^3)$ for positive a, b , we obtain

$$|r_2| \leq \frac{2}{3} c_{f''} \|\bar{p}_h\|_{L^\infty(\Omega)} (c_L^3 I_3^3 r^3 + (c_{H^1} c_S I_3 \|\bar{u}_h\|_{L^2(\Omega)})^3 h^3) =: c_{r1} r^3 + c_{r2} h^3.$$

By means of Lemma 6.2, 6.3, 6.4 and 6.5, and (6.25) we get

$$\begin{aligned} J(y, u) - J(y^h, \bar{u}_h) &\geq -c_j \|\bar{u}_h\|_{L^2(\Omega)} h^2 - c_y \|\bar{y}_h - y_d\|_{L^2(\Omega)} \|y\|_{H^2(\Omega)} h^2 \\ &\quad - c_u \|\bar{p}_h\|_{W^{1,\infty}(\Omega)} h r + \delta r^2 - d_1 h r^2 - d_2 h^2 r - d_3 r^3 - c_{r1} r^3 - c_{r2} h^3. \end{aligned}$$

Here, it remains to estimate $\|y\|_{H^2(\Omega)}$:

$$\|y\|_{H^2(\Omega)} \leq c_S \|u\|_{L^2(\Omega)} \leq c_S (\|\bar{u}_h\|_{L^2(\Omega)} + r).$$

Consequently, we obtain

$$\begin{aligned} J(y, u) - J(y^h, \bar{u}_h) &\geq \delta r^2 - (c_y \|\bar{y}_h - y_d\|_{L^2(\Omega)} c_S \|\bar{u}_h\|_{L^2(\Omega)} + c_{r2}) h^2 \\ &\quad - (c_u \|\bar{p}_h\|_{W^{1,\infty}(\Omega)}) hr - (c_y \|\bar{y}_h - y_d\|_{L^2(\Omega)} c_S + d_2) h^2 r - (d_3 + c_{r1}) r^3 - d_1 h r^2. \end{aligned}$$

Setting

$$\begin{aligned} a_1 &:= c_j \|\bar{u}_h\|_{L^2(\Omega)} + c_y \|\bar{y}_h - y_d\|_{L^2(\Omega)} c_S \|\bar{u}_h\|_{L^2(\Omega)} + c_{r2}, \\ a_2 &:= c_u \|\bar{p}_h\|_{W^{1,\infty}(\Omega)}, \\ a_3 &:= c_y \|\bar{y}_h - y_d\|_{L^2(\Omega)} c_S + d_2 \\ a_4 &:= d_3 + c_{r1} \\ a_5 &:= d_1, \end{aligned}$$

the assertion is obtained. \square

Now, let us fix a mesh-size h with associated solution \bar{u}_h . Suppose that the polynomial given by (6.22) is positive for some $r > 0$. Then we have that the value $J(y, u)$ is greater than the $J(y^h, \bar{u}_h)$ for all u having L^2 -distance r to \bar{u}_h . Hence, there exists a local minimum of the optimal control problem (1.1)–(1.3) inside the neighborhood $\{u : \|u - \bar{u}_h\|_{L^2(\Omega)} < r\}$ of \bar{u}_h .

COROLLARY 6.7. *Assume that there exists a positive value r such that*

$$-a_4 r^3 + (\delta - a_5 h) r^2 - (a_2 h + a_3 h^2) r - a_1 h^2 > 0.$$

Then

$$J(y, u) - J(y^h, \bar{u}_h) > 0 \tag{6.26}$$

holds for all (y, u) satisfying (2.1) and $\|u - \bar{u}_h\|_{L^2(\Omega)} = r$

A sufficient condition that the assumption of the previous Corollary 6.7 is fulfilled is given as the last result in this section.

COROLLARY 6.8. *Let us assume that $\sigma := \delta - a_5 h > 0$ holds. Let us suppose further that there exists $r_+ > 0$ that fulfills*

$$r_+ > \max \left\{ \frac{3(a_2 h + a_3 h^2)}{\sigma}, \sqrt{\frac{3a_1 h^2}{\sigma}} \right\} \tag{6.27}$$

and if $a_4 > 0$ additionally

$$r_+ < \frac{\sigma}{3a_4}. \tag{6.28}$$

Then the pre-requisite of Corollary 6.7 is satisfied.

Proof. The first condition (6.27) gives

$$\frac{2}{3}(\delta - a_5 h) r_+^2 = \frac{2}{3} \sigma r_+^2 > (a_2 h + a_3 h^2) r_+ + a_1 h^2 \geq 0.$$

If a_4 is not zero then the second one implies $\frac{1}{3} \sigma r_+^2 > a_4 r_+^3 > 0$. Now, the polynomial $-a_4 r^3 + (\delta - a_5 h) r^2 - (a_2 h + a_3 h^2) r - a_1 h^2$ admits a positive value for r_+ . Thus, it satisfies the assumptions of the previous Corollary 6.7. \square

REMARK 6.9. *The assumptions of Corollary 6.7 directly link the discretization parameter h to the coercivity factor δ : the discretization has to be fine enough. If the sufficient condition (2.6) holds for the solution of the original, undiscretized problem then for sufficiently small h the assumptions of Corollary 6.7 will be satisfied.*

7. Verification of optimality conditions. In the following, r denotes a fixed radius fulfilling the assumptions of Corollary 6.7. We are now going to prove that in a r -neighborhood of \bar{u}_h exists a solution of the optimality system connected with the continuous problem (1.1)–(1.3).

THEOREM 7.1. *There exist at least one control \hat{u} with associated state \hat{y} and adjoint state \hat{p} in a r -neighborhood of \bar{u}_h fulfilling the first-order necessary optimality conditions (2.3).*

Proof. We investigate the optimal control problem for

$$U_{ad}^r := U_{ad} \cap \{u \in U : \|u - \bar{u}_h\|_{L^2(\Omega)} \leq r\}.$$

This set is weakly compact. Hence, there exists at least one solution \hat{u} of the modified problem. Since \bar{u}_h is feasible and

$$J(y, u) - J(y^h, \bar{u}_h) > 0$$

is satisfied for all controls u with $\|u - \bar{u}_h\|_{L^2(\Omega)} = r$, we have $\|\hat{u} - \bar{u}_h\|_{L^2(\Omega)} < r$. Consequently, the local minimizer \hat{u} of the modified problem is also a local minimizer for the original problem. In particular, \hat{u} has to fulfill the first-order necessary optimality conditions. \square

As a consequence, we can give an error estimate for the associated state \hat{y} and adjoint state \hat{p} in terms of r and h .

LEMMA 7.2. *Let $(\hat{y}, \hat{u}, \hat{p})$ fulfill the first-order necessary optimality conditions. Then one can estimate the distance to the discrete solution by*

$$\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \leq c_{y1} r + c_{y2} h^2 \quad (7.1)$$

and

$$\|\hat{p} - \bar{p}_h\|_{L^2(\Omega)} \leq c_{p1} r + c_{p2} h^2, \quad (7.2)$$

with constants c_{yi} and c_{pi} independent of h , r , and $(\hat{y}, \hat{u}, \hat{p})$.

Proof. The difference $\hat{y} - \bar{y}_h$ can be treated using the auxiliary function y^h defined in (3.6)

$$\begin{aligned} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq \|\hat{y} - y^h\|_{L^2(\Omega)} + \|y^h - \bar{y}_h\|_{L^2(\Omega)} \\ &\leq c_L r + c_{L^2} c_S h^2 \|\bar{u}_h\|_{L^2(\Omega)}. \end{aligned}$$

Here, we applied the estimate (5.9). The claim (7.1) follows with $c_{y1} = c_L$ and $c_{y2} = c_{L^2} c_S \|\bar{u}_h\|_{L^2(\Omega)}$.

For the estimation of the adjoint states recall the definition of p^h in (6.10) and the corresponding estimate (6.11)

$$\|p^h - \bar{p}_h\|_{L^2(\Omega)} \leq c_L h^2 \|p^h\|_{H^2(\Omega)} \leq c_p c_{L^2} h^2 \|\bar{y}_h - y_d\|_{L^2(\Omega)}.$$

Now, we introduce the splitting

$$\|\hat{p} - \bar{p}_h\|_{L^2(\Omega)} \leq \|\hat{p} - p^h\|_{L^2(\Omega)} + \|p^h - \bar{p}_h\|_{L^2(\Omega)},$$

and it remains to investigate $d := \hat{p} - p^h$. It is a solution of

$$a(d, v) + (f'(\hat{y})d, v) = (\hat{y} - \bar{y}_h, v) - ((f'(\hat{y}) - f'(\bar{y}_h))\bar{p}_h, v) \quad \forall v \in V.$$

Hence, we obtain

$$\begin{aligned} \|\hat{p} - p^h\|_{L^2(\Omega)} &\leq c_L (\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} + \|(f'(\hat{y}) - f'(\bar{y}_h))\bar{p}_h\|_{L^2(\Omega)}) \\ &\leq c_L (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)}. \end{aligned}$$

And the estimate (7.2) is satisfied with $c_{p1} = c_L (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) c_{y1}$ and $c_{p2} = c_L (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) c_{y2} + c_p c_{L^2} \|\bar{y}_h - y_d\|_{L^2(\Omega)}$. \square

With the same technique, we get a H^1 -error estimate for state and adjoint state.

COROLLARY 7.3. *Under the assumptions of the previous Lemma we have*

$$\|\hat{y} - \bar{y}_h\|_{H^1(\Omega)} \leq c_L r + c_{H^1} c_S \|\bar{u}_h\|_{L^2(\Omega)} h$$

and

$$\|\hat{p} - \bar{p}_h\|_{H^1(\Omega)} \leq c_L (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) \|\hat{y} - \bar{y}_h\|_{H^1(\Omega)} + c_p c_{H^1} \|\bar{y}_h - y_d\|_{L^2(\Omega)} h.$$

Next, we will introduce additional assumptions on r that guarantees that the control \hat{u} fulfills a second-order sufficient optimality condition.

THEOREM 7.4. *Under the assumption*

$$r < \frac{\delta - c_h h^2}{c_r}$$

every control \hat{u} in a r -neighborhood of \bar{u}_h fulfills together with its associated state \hat{y} and adjoint \hat{p} the coercivity

$$L''_{(y,u)}(\hat{y}, \hat{u}, \hat{p})(y, u)^2 \geq \delta' \|u\|_{L^2(\Omega)}^2 \quad (7.3)$$

for all y given as solution of

$$a(y, v) + (f'(\hat{y})y, v) = (u, v) \quad \forall v \in V. \quad (7.4)$$

Moreover, there exist only one local minimum in this neighborhood.

The constants c_r and c_h will be determined in the course of the proof.

Proof. Let \hat{u} be a stationary point with associated state \hat{y} and adjoint state \hat{p} . Further, let $u \in U$ be an arbitrary control with associated solutions y and y^h of the linearized equations (7.4) and (3.5) respectively. Then the pair (y, u) is suitable in (7.3), whereas (y^h, u) can be utilized as test functions in (3.4).

At first, we have to estimate the difference $d := y - y^h$ as well as the sum $y + y^h$ in terms of u and h . The difference fulfills the equation

$$0 = a(d, v) + (f'(\hat{y})y, v) - (f'(\bar{y}_h)y^h, v) = a(d, v) + (f'(\hat{y})d, v) + ((f'(\hat{y}) - f'(\bar{y}_h))y^h, v)$$

for all $v \in V$. Testing with d itself, we obtain

$$\|d\|_{H^1(\Omega)} \leq \frac{I_4}{\delta_0} \|(f'(\hat{y}) - f'(\bar{y}_h))y^h\|_{L^{4/3}(\Omega)}.$$

Hence, we can estimate d as solution of a linearized equation with right-hand side $-(f'(\hat{y}) - f'(\bar{y}_h))y^h$, which is estimated by

$$\begin{aligned} \|(f'(\hat{y}) - f'(\bar{y}_h))y^h\|_{L^{4/3}(\Omega)} &\leq c_{f'} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \|y^h\|_{L^4(\Omega)} \\ &\leq c_{f'} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} c_L I_4 \|u\|_{L^2(\Omega)}. \end{aligned}$$

Collecting all these inequalities, we find for $d = y - y^h$

$$\begin{aligned} \|y - y^h\|_{L^2(\Omega)} &\leq c_L I_4 \|(f'(\hat{y}) - f'(\bar{y}_h))y^h\|_{L^2(\Omega)} \\ &\leq c_L \delta_0^{-1} I_4^2 c_{f'} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}. \end{aligned}$$

The sum $y + y^h$ is estimated by

$$\|y + y^h\|_{L^2(\Omega)} \leq 2c_L \|u\|_{L^2(\Omega)}.$$

Now, we can investigate the second derivative of the Lagrangian. We start with the decomposition

$$\begin{aligned} L''_{(y,u)}(\hat{y}, \hat{u}, \hat{p})(y, u)^2 &= L''_{(y,u)}(\hat{y}, \hat{u}, \hat{p})(y, u) - L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y, u)^2 \\ &\quad + L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y, u)^2 - L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y^h, u)^2 \\ &\quad + L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y^h, u)^2. \end{aligned}$$

The last addend gives us the desired coercivity by (3.4). So we have to estimate the two differences in this equation. The first one yields

$$\begin{aligned} &\left| L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y, u)^2 - L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y^h, u)^2 \right| \\ &= \left| (y - y^h, y + y^h) + \int_{\Omega} f''(\bar{y}_h) \bar{p}_h (y - y^h)(y + y^h) \right| \\ &\leq (1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) \|y - y^h\|_{L^2(\Omega)} \|y + y^h\|_{L^2(\Omega)} \\ &\leq \underbrace{(1 + c_{f'} \|\bar{p}_h\|_{L^\infty(\Omega)}) 2c_L^2 \delta_0^{-1} I_4^2 c_{f'}}_{C_1} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}^2 \\ &\leq C_1 \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

Let us proceed with the second difference

$$\begin{aligned} &\left| L''_{(y,u)}(\hat{y}, \hat{u}, \hat{p})(y, u) - L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y, u)^2 \right| \\ &= \left| \int_{\Omega} (f''(\hat{y}) \hat{p} - f''(\bar{y}_h) \bar{p}_h) y^2 dx \right| \leq \|f''(\hat{y}) \hat{p} - f''(\bar{y}_h) \bar{p}_h\|_{L^2(\Omega)} \|y\|_{L^4(\Omega)}^2 \\ &\leq c_L^2 I_4^2 \|f''(\hat{y}) \hat{p} - f''(\bar{y}_h) \bar{p}_h\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

Using Lipschitz estimates, we obtain for the right-hand side

$$\begin{aligned} \|f''(\hat{y}) \hat{p} - f''(\bar{y}_h) \bar{p}_h\|_{L^2(\Omega)} &\leq \|f''(\hat{y}) \hat{p} - f''(\hat{y}) \bar{p}_h\|_{L^2(\Omega)} + \|f''(\hat{y}) \bar{p}_h - f''(\bar{y}_h) \bar{p}_h\|_{L^2(\Omega)} \\ &\leq c_{f'} \|\hat{p} - \bar{p}_h\|_{L^2(\Omega)} + c_{f''} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \|\bar{p}_h\|_{L^\infty(\Omega)}. \end{aligned}$$

Hence, we arrive at

$$\begin{aligned} &\left| L''_{(y,u)}(\hat{y}, \hat{u}, \hat{p})(y, u) - L''_{(y,u)}(\bar{y}_h, \bar{u}_h, \bar{p}_h)(y, u)^2 \right| \\ &\leq c_L^2 I_4^2 (c_{f'} \|\hat{p} - \bar{p}_h\|_{L^2(\Omega)} + c_{f''} \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \|\bar{p}_h\|_{L^\infty(\Omega)}) \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

So, we find

$$\begin{aligned} L''_{(y,u)}(\hat{y}, \hat{u}, \hat{p})(y, u)^2 &\geq \left\{ \delta - (c_L^2 I_4^2 c_{f'}) \|\hat{p} - \bar{p}_h\|_{L^2(\Omega)} \right. \\ &\quad \left. - (C_1 + c_L^2 I_4^2 c_{f''} \|\bar{p}_h\|_{L^\infty(\Omega)}) \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)} \right\} \|u\|_{L^2(\Omega)}^2 \end{aligned}$$

Here, we can apply the estimates of $\|\hat{p} - \bar{p}_h\|_{L^2(\Omega)}$ and $\|\hat{y} - \bar{y}_h\|_{L^2(\Omega)}$ of Lemma 7.2. We get consequently

$$L''_{(y,u)}(\hat{y}, \hat{u}, \hat{p})(y, u)^2 \geq \{\delta - c_r r - c_h h^2\} \|u\|_{L^2(\Omega)}^2$$

with

$$c_r = C_1 c_{y_1} + c_L^2 I_4^2 (c_{f'} c_{p_1} + c_{f''} \|\bar{p}_h\|_{L^\infty(\Omega)} c_{y_1})$$

and

$$c_h = C_1 c_{y_2} + c_L^2 I_4^2 (c_{f'} c_{p_2} + c_{f''} \|\bar{p}_h\|_{L^\infty(\Omega)} c_{y_2}),$$

and the claim is proven. \square

Finally, we will give an estimate of the neighborhood of \hat{u} , where \hat{u} is locally optimal.

THEOREM 7.5. *Let the assumptions of the previous Theorem 7.4 be satisfied. Then it holds with some $\hat{\delta} > 0$*

$$J(y, u) - J(\hat{y}, \hat{u}) \geq \hat{\delta} \|u - \hat{u}\|_{L^2(\Omega)}^2$$

for all admissible $u \in U_{ad}$ with $\|u - \hat{u}\|_{L^2(\Omega)} < r'$ for some sufficiently small r' .

Proof. By assumption $(\hat{y}, \hat{u}, \hat{p})$ fulfills the necessary optimality conditions together with the coercivity relation (7.3). Let (y, u) be another admissible pair. We have

$$J(\hat{y}, \hat{u}) = L(\hat{y}, \hat{u}, \hat{p}) \quad \text{and} \quad J(y, u) = L(y, u, \hat{p}),$$

since (\hat{y}, \hat{u}) and (y, u) are admissible. Taylor-expansion of the Lagrange-function yields

$$\begin{aligned} J(y, u) - J(\hat{y}, \hat{u}) &= L(y, u, \hat{p}) - L(\hat{y}, \hat{u}, \hat{p}) \\ &= L_y(\hat{y}, \hat{u}, \hat{p})(y - \hat{y}) + L_u(\hat{y}, \hat{u}, \hat{p})(u - \hat{u}) \\ &\quad + \frac{1}{2} L''(\hat{y}, \hat{u}, \hat{p})[(y - \hat{y}, u - \hat{u})]^2 + r_2. \end{aligned} \tag{7.5}$$

Since the necessary conditions (2.3) are satisfied at \hat{y}, \hat{u} with adjoint state \hat{p} , the first term vanishes. The second addend is nonnegative due to the variational inequality. The remainder term r_2 satisfies, confer (6.25),

$$|r_2| \leq \frac{1}{6} c_{f''} \|y - \hat{y}\|_{L^4(\Omega)}^3 \|\hat{p}\|_{L^4(\Omega)} \leq \frac{1}{6} c_{f''} c_L^3 I_4^4 \|u - \hat{u}\|_{L^2(\Omega)}^3 \|\hat{p}\|_{H^1(\Omega)}. \tag{7.6}$$

The pair $(y - \hat{y}, u - \hat{u})$ is not suitable as test function in (7.3), since $y - \hat{y}$ is not the solution of a linearized equation. Let us introduce an auxiliary state d as the weak solution of

$$a(d, v) + (f'(\hat{y})d, v) = (u - \hat{u}, v) \quad \forall v \in V.$$

When we use d instead of $y - \hat{y}$, we make the small error $r_1 := (y - \hat{y}) - d$, which is itself the weak solution of

$$a(r_1, v) + (f_y^{\hat{y}} r_1, v) = ((f_y^{\hat{y}} - f'(\hat{y}))d, v) \quad \forall v \in V.$$

with $f_y^{\hat{y}} = \int_0^1 f'(\hat{y} + s(y - \hat{y})) ds$. Since f' and $f_y^{\hat{y}}$ are non-negative, we get the estimate

$$\begin{aligned} \|r_1\|_{H^1(\Omega)} &\leq c_L \|(f_y^{\hat{y}} - f'(\hat{y}))d\|_{L^2(\Omega)} \leq \frac{1}{2} c_L c_{f'} \|y - \hat{y}\|_{L^4(\Omega)} \|d\|_{L^4(\Omega)} \\ &\leq \frac{1}{2} c_L^3 c_{f'} I_4^2 \|u - \hat{u}\|_{L^2(\Omega)}. \end{aligned} \quad (7.7)$$

Substituting $y - \hat{y} = d + r_1$, we obtain

$$\begin{aligned} \frac{1}{2} L_{yy}(\hat{y}, \hat{u}, \hat{p}) [y - \hat{y}]^2 &= \frac{1}{2} L_{yy}(\hat{y}, \hat{u}, \hat{p}) [d]^2 + L_{yy}(\hat{y}, \hat{u}, \hat{p}) [d, r_1] + \frac{1}{2} L_{yy}(\hat{y}, \hat{u}, \hat{p}) [r_1]^2 \\ &= \frac{1}{2} L_{yy}(\hat{y}, \hat{u}, \hat{p}) [d]^2 + \frac{1}{2} \tilde{r}_2. \end{aligned}$$

The remainder term is given by

$$\tilde{r}_2 = (r_1, 2d + r_1) + \int_{\Omega} f''(\hat{y}) \hat{p} r_1 (2d + r_1) dx$$

and can be estimated by

$$\begin{aligned} |\tilde{r}_2| &\leq (1 + c_{f'} I_4^3 \|\hat{p}\|_{H^1(\Omega)}) \|r_1\|_{H^1(\Omega)} (\|d\|_{H^1(\Omega)} + \|r_1\|_{H^1(\Omega)}) \\ &\leq \underbrace{\frac{1}{2} (1 + c_{f'} I_4^3 \|\hat{p}\|_{H^1(\Omega)}) (2c_L \|u - \hat{u}\|_{L^2(\Omega)} + \|r_1\|_{H^1(\Omega)}) c_L^3 c_{f'} I_4^2}_{r_0} \|u - \hat{u}\|_{L^2(\Omega)}. \end{aligned} \quad (7.8)$$

with $r_0 \rightarrow 0$ as $\|u - \hat{u}\|_{L^2(\Omega)} \rightarrow 0$.

So far, we achieved the following estimate for the difference of the objective values

$$J(y, u) - J(\hat{y}, \hat{u}) \geq \frac{1}{2} L''(\hat{y}, \hat{u}, \hat{p}) [d, u]^2 - |r_2| - r_0 \|u - \hat{u}\|_{L^2(\Omega)}^2.$$

In the next step, we apply the coercivity (7.3) given by Theorem 7.4. Furthermore, we utilize the estimates of r_0 , r_1 , and r_2 in (7.6), (7.7) and (7.8), respectively. To shorten the estimates, let us assume $\|u - \hat{u}\|_{L^2(\Omega)} \leq R$. And we obtain

$$\begin{aligned} J(y, u) - J(\hat{y}, \hat{u}) &\geq \left\{ \frac{\delta'}{2} - \frac{1}{2} c_L^3 c_{f'} I_4^2 (1 + c_{f'} I_4^3 \|\hat{p}\|_{H^1(\Omega)}) (2c_L R + \frac{1}{2} c_L^3 c_{f'} I_4^2 R^2) \right. \\ &\quad \left. - \frac{1}{6} c_{f''} I_4^4 R^3 \|\hat{p}\|_{H^1(\Omega)} \right\} \|u - \hat{u}\|_{L^2(\Omega)}^2 \end{aligned} \quad (7.9)$$

It remains to estimate $\|\hat{p}\|_{H^1(\Omega)}$. Using the splitting

$$\|\hat{p}\|_{H^1(\Omega)} \leq \|\hat{p} - \bar{p}_h\|_{H^1(\Omega)} + \|\bar{p}_h\|_{H^1(\Omega)}$$

Corollary 7.3 provides us with a computable estimate of that norm.

For $R < R_0$ small enough, the factor in braces in (7.9) is greater than zero. This implies quadratic growth of the objective functional in the neighborhood $\{u \in U_{ad} : \|u - \hat{u}\|_{L^2(\Omega)} \leq R_0\}$. \square

8. Example. In this section, we will apply our results to the consider the optimal control problem of minimizing $J(y, u)$ given by

$$J(y, u) := \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|u\|_{L^2(\Omega)}^2 \quad (8.1)$$

subject to the semilinear boundary value problem

$$\begin{aligned} -(\Delta y)(x) + y^3(x) &= u(x) && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma, \end{aligned} \quad (8.2)$$

and to the control constraints

$$-0.1 \leq u(x) \leq 0.5 \text{ a.e. in } \Omega. \quad (8.3)$$

The domain Ω is the unit square $\Omega = (0, 1)^2$, its boundary denoted by Γ . The parameter ν was chosen as $\nu = 0.1$. The desired state y_d is given by

$$y_d = 8 \sin \pi x_1 \sin \pi x_2 - 4. \quad (8.4)$$

We find that the set of admissible controls is bounded in $L^2(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq 0.5 =: M_U \quad \forall u \in U_{ad}.$$

8.1. The solution. At first, let us show the computed solutions for a fixed discretization. The state and adjoint state were discretized using piecewise linear and continuous functions on a regular triangulation of the domain. The control was discretized according to **(A3)** by piecewise constant functions. See also Section 8.2.4 below for the discretization details. The grid consists of 20,000 triangles with 10,201 nodes. The discretization parameter was $h = \sqrt{2}/100 = 0.01414\dots$. Here and in the sequel '...' means truncation of floating point numbers after 4 leading digits.

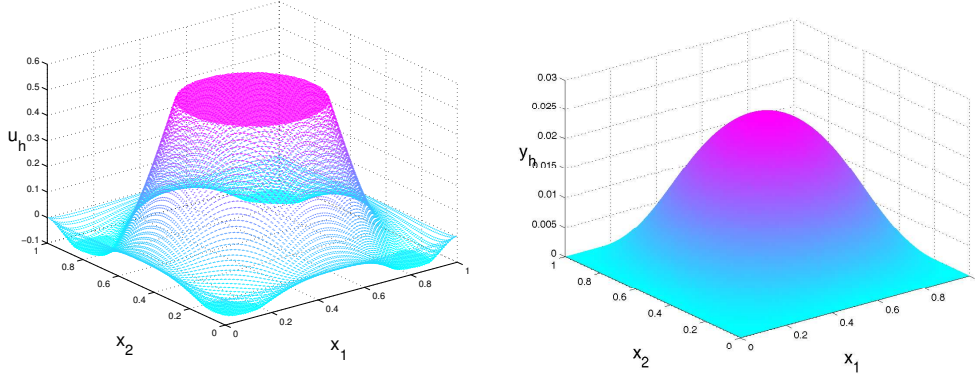


FIG. 8.1. Discrete solution \bar{u}_h and \bar{y}_h .

Now let us report about the norms of the solution, which are needed for the following computations:

$$\|y_h\|_{L^\infty(\Omega)} = 0.02580\dots, \quad \|y_h\|_{H^1(\Omega)} = 0.05638\dots, \quad \|y_h - y_d\|_{L^2(\Omega)} = 2.371\dots,$$

$$\|u_h\|_{L^2(\Omega)} = 0.2808\dots,$$

$$\|p_h\|_{L^\infty(\Omega)} = 0.1053\dots, \quad \|p_h\|_{H^1(\Omega)} = 0.2334\dots, \quad \|p_h\|_{W^{1,\infty}(\Omega)} = 0.3441\dots$$

The set of elements T_h^i , where the control constraint is inactive, see Lemma 6.4, has measure $|T_h^i| = 0.7267\dots$

8.2. Computation of the constants. In the following small sections, we will explain how all those constants involved in the proofs are computed.

8.2.1. Imbedding constants. At first, we will compute the imbedding constants I_p of the imbeddings $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. We obtain with the use of eigenfunction of $-\Delta$ the constant of the imbedding in $L^2(\Omega)$ as

$$I_2 = \frac{1}{\pi + 1} = 0.2415\dots$$

For the imbedding constants I_4 and I_6 we have the following. Because of the inequalities $\|y\|_{L^4(\Omega)}^4 \leq \frac{1}{2}\|y\|_{L^2(\Omega)}^2\|\nabla y\|_{L^2(\Omega)}^2$ and $\|y\|_{L^6(\Omega)}^6 \leq \frac{9}{8}\|y\|_{L^4(\Omega)}^4\|\nabla y\|_{L^2(\Omega)}^2$, see [13], the imbedding constants I_4 and I_6 can be computed as

$$I_4 = \left(\frac{1}{2}I_2^2\right)^{1/4} = 2^{-1/4}I_2^{1/2} = 0.4132\dots, \quad I_6 = \left(\frac{9}{8}I_4^4\right)^{1/6} = 0.5658\dots$$

Now, it remains to compute I_3 for the imbedding in $L^3(\Omega)$. Using the interpolation inequality $\|y\|_{L^3(\Omega)} \leq \|y\|_{L^2(\Omega)}^{1/3}\|y\|_{L^4(\Omega)}^{2/3}$, we find

$$I_3 = I_2^{1/3}I_4^{2/3} = 0.3454\dots$$

The interpolation between $L^2(\Omega)$ and $L^6(\Omega)$ would give a larger imbedding constant $\tilde{I}_3 = I_2^{1/2}I_6^{1/2} = 0.3696\dots$ in our case.

8.2.2. Solution mapping $u \mapsto y$. At first, we investigate the bilinear form a . We have $|a(y_1, y_2)| \leq \|y_1\|_{H^1(\Omega)}\|y_2\|_{H^1(\Omega)}$, which gives $\delta_1 = 1$. Furthermore it holds

$$a(y, y) = \|\nabla y\|_{L^2(\Omega)}^2 = \|y\|_{H^1(\Omega)}^2 - \|y\|_{L^2(\Omega)}^2 \geq (1 - I_2^2)\|y\|_{H^1(\Omega)}^2,$$

and we obtain $\delta_0 = 1 - I_2^2 = 0.9417\dots$

The Lipschitz constant of the solution mapping now is given by Lemma 4.2 as

$$c_L = \frac{I_2}{\delta_0} = 0.2564\dots$$

In the following, we will apply the identity $|y|_{H^2(\Omega)} = \|-\Delta y\|_{L^2(\Omega)}$, which can easily be proved by Fourier expansion. Now, let us estimate the constant c_S of Theorem 4.1. Since $f(0) = 0$ holds, we find for the nonlinear equation

$$\|y\|_{H^1(\Omega)} \leq c_L\|u\|_{L^2(\Omega)}.$$

Hence, we can derive

$$\begin{aligned} \|\Delta y\|_{L^2(\Omega)} &\leq \|f(y)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \\ &\leq \|y\|_{L^6(\Omega)}^3 + \|u\|_{L^2(\Omega)} \leq (I_6^3 c_L^3 \|u\|_{L^2(\Omega)}^2 + 1)\|u\|_{L^2(\Omega)} \\ &\leq (I_6^3 c_L^3 M_U^2 + 1)\|u\|_{L^2(\Omega)}, \end{aligned}$$

which gives finally

$$\|y\|_{H^2(\Omega)} \leq \sqrt{c_L^2 + (I_6^3 c_L^3 M_U^2 + 1)^2}\|u\|_{L^2(\Omega)}$$

and the value of the constant $c_S = \sqrt{c_L^2 + (I_6^3 c_L^3 M_U^2 + 1)^2} = 1.033\dots$

8.2.3. Global estimates of the nonlinearity. Furthermore, we need a L^∞ -bound of the solutions to the state equation. Here, we will Stampacchia's result, see [10, 15].

COROLLARY 8.1. *Let y be the solution of the nonlinear state equation (2.1) for a given right-hand side $u \in L^2(\Omega)$. Then it holds $y \in L^\infty(\Omega)$ with*

$$\|y\|_{L^\infty(\Omega)} \leq 8 \frac{I_4 I_6}{\sqrt{\delta_0}} |\Omega|^{1/12} \|u\|_{L^2(\Omega)}.$$

Proof. Let k be a real number. Define

$$v(x) = \begin{cases} y(x) - k & \text{if } y(x) \geq k \\ 0 & \text{if } |y(x)| < k \\ y(x) + k & \text{if } y(x) \leq -k \end{cases}, \quad \Omega(k) = \{x : |y(x)| \geq k\}.$$

Now, we test (2.1) by v ,

$$a(y, v) + (f(y), v) = (u, v).$$

Because of $f(0) = 0$ and the monotonicity of f , it holds $(f(y), v) \geq 0$. The right-hand side we estimate by

$$\begin{aligned} |(u, v)| &\leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} \|v\|_{L^6(\Omega)} |\Omega(k)|^{1/3} \\ &\leq \|u\|_{L^2(\Omega)} |\Omega(k)|^{1/3} I_6 \|v\|_{H^1(\Omega)} \leq \frac{\delta_0}{2} \|v\|_{H^1(\Omega)}^2 + \frac{|\Omega(k)|^{2/3} I_6^2}{2} \|u\|_{L^2(\Omega)}^2. \end{aligned} \quad (8.5)$$

Further, it holds

$$|a(y, v)| \geq a(v, v) \geq \delta_0 \|v\|_{H^1(\Omega)}^2.$$

Now, we proceed with

$$\|v\|_{H^1(\Omega)}^2 \geq \frac{1}{I_6^2} \|v\|_{L^6(\Omega)}^2 = \left(\int_{\Omega(k)} (|y| - k)^6 dx \right)^{1/3}. \quad (8.6)$$

For $h > k$ we find since $\Omega(h) \subset \Omega(k)$

$$\begin{aligned} \left(\int_{\Omega(k)} (|y| - k)^6 dx \right)^{1/3} &\geq \left(\int_{\Omega(h)} (|y| - k)^6 dx \right)^{1/3} \geq \left(\int_{\Omega(h)} (h - k)^6 dx \right)^{1/3} \\ &= (h - k)^2 |\Omega(h)|^{1/3}. \end{aligned}$$

Altogether, we obtain

$$\frac{\delta_0}{I_6^2} (h - k)^2 |\Omega(h)|^{1/3} \leq |\Omega(k)|^{2/3} I_6^2 \|u\|_{L^2(\Omega)}^2,$$

or, equivalently,

$$|\Omega(h)| \leq \left(\frac{I_6^4 \|u\|_{L^2(\Omega)}^2}{\delta_0} \right)^3 \frac{1}{(h - k)^6} |\Omega(k)|^2.$$

Then by a result of Stampacchia it holds $|\Omega(d)| = 0$ for

$$d = 4 \frac{I_6^2 \|u\|_{L^2(\Omega)}}{\sqrt{\delta_0}} |\Omega|^{1/6}.$$

This implies

$$\|y\|_{L^\infty(\Omega)} \leq d = 4 \frac{I_6^2}{\sqrt{\delta_0}} |\Omega|^{1/6} \|u\|_{L^2(\Omega)},$$

and the claim is proven. Due to the assumptions on f this estimate is valid for every suitable choice of the semilinearity. \square

In our case, the constant $c_{SL^\infty} = 4 \frac{I_6^2 \|u\|_{L^2(\Omega)}}{\sqrt{\delta_0}} |\Omega|^{1/6}$, see Theorem 4.1, has the value $c_{SL^\infty} = 1.3194\dots$. Other choices of L^p -spaces than L^6 in the estimates (8.5) and (8.6) yield other values of c_{SL^∞} . Using the imbedding constants computed in this section, $p = 6$ gave the smallest constant over all possible combinations from $\{3, 4, 6\}$.

Now, we can compute the global bound of the L^∞ -norms of all possible states by

$$M = c_{SL^\infty} M_U = 0.7916\dots$$

Then the Lipschitz estimates of f holds with the following constants:

$$\begin{aligned} c_f &= 3M^2 = 1.880\dots, \\ c_{f'} &= 6M = 4.750\dots, \\ c_{f''} &= 6. \end{aligned}$$

8.2.4. Finite elements constants. The discrete space V_h is chosen to be the set of continuous functions that are linear on the triangles of the triangulation T , e.g.

$$V_h = \{v \in C(\bar{\Omega}) : v|_{T_h} \in P_1(T_h) \quad \forall T_h \in T, v|_\Gamma = 0\}.$$

The unit square was triangulated uniformly by orthogonal and congruent triangles.

It is known that the interpolation operator fulfills the requirements (5.1) and (5.2) of **(A3)**. However, we could not find any estimate of the associated constant c_1 in the literature. So we decided to define the operator i_h as follows.

For $y \in H_2(\Omega) \cap H_0^1(\Omega)$, we set $y_h := i_h y$ to be the solution of

$$a(y_h, v) = (-\Delta y, v) \quad \forall v \in V_h.$$

Then, upper bounds for the values of c_1 and c_2 can be found in [1, Theorem 5]:

$$c_1 = 0.2381\dots, \quad c_2 = 0.4888\dots$$

Before we can derive the constants of Lemma 5.1, we have to compute the constant c_M that appears in the proof of that Lemma. The norm of the auxiliary function g defined there can be estimated by $\|g\|_{H^1(\Omega)} \leq c_L$. The L^2 -norm of its Laplacian is

$$\|\Delta g\|_{L^2(\Omega)} \leq \|e\|_{L^2(\Omega)} + \|f_y^{y_h} g\|_{L^2(\Omega)} \leq 1 + \|g\|_{L^2(\Omega)} c_f \leq 1 + c_f c_L.$$

Hence, we obtain $c_M = \sqrt{c_L^2 + (1 + c_f c_L)^2} = 1.504\dots$

Now, we can compute the desired constants as

$$\begin{aligned} c_{L^2} &= (\delta_1 c_2 + c_f c_1) c_{H^1} c_M = 1.398\dots, \\ c_{H^1} &= (\delta_1 c_2 + c_f c_1) / \delta_0 = 0.9936\dots \end{aligned}$$

8.2.5. Auxiliary constants. Now, let us investigate the auxiliary adjoint state p^h defined in (6.10). Here we want to compute the constant c_p as used in (6.12). By Corollary 4.3, we find

$$\|p^h\|_{H^1(\Omega)} \leq c_L \|\bar{y}_h - y_d\|_{L^2(\Omega)}.$$

And with

$$\begin{aligned} \|\Delta p^h\|_{L^2(\Omega)} &= \|f'(\bar{y}_h)p^h\|_{L^2(\Omega)} + \|\bar{y}_h - y_d\|_{L^2(\Omega)} \\ &\leq 3I_2 \|\bar{y}_h\|_{L^\infty(\Omega)}^2 \|p^h\|_{H^1(\Omega)} + \|\bar{y}_h - y_d\|_{L^2(\Omega)} \\ &\leq (3I_2 c_L \|\bar{y}_h\|_{L^\infty(\Omega)}^2 + 1) \|\bar{y}_h - y_d\|_{L^2(\Omega)} \end{aligned}$$

we find $c_p = \sqrt{c_L^2 + (3I_2 c_L \|\bar{y}_h\|_{L^\infty(\Omega)}^2 + 1)^2} = 1.032\dots$

8.3. Verification results. Finally, we report about the estimates of the objective functional. The coefficients in the polynomial (6.22) were computed to

$$a_1 = 3.578\dots, a_2 = 0.2933\dots, a_3 = 9.173\dots, a_4 = 0.003854\dots, a_5 = 0.00805\dots,$$

which leads to the estimate

$$J(y, u) - J(y^h, \bar{u}_h) \geq -0.003854r^3 + 0.09989r^2 - 0.005983r - 0.0007157.$$

This polynomial admits positive values for $r \in [0.1202, 25.85]$. That implies the existence of a local minimizer \hat{u} of J within the set

$$B_h = \{u \in U_{ad} : \|u - \bar{u}_h\|_{L^2(\Omega)} < 0.1202\}.$$

Furthermore, it follows that the global minimizer of J belongs to that neighborhood, since the upper bound 25.85 of the positivity interval is much larger than the diameter of the set of admissible controls.

Regarding the check of the sufficient condition — coercivity of the second-derivative of the Lagrangian — we achieved the following. For every $r < 5.188$, Theorem 7.4 gives coercivity of $L''_{(y,u)}$ with a positive δ' . Finally, Theorem 7.5 yields quadratic growth of the cost functional in a L^2 -neighborhood of \hat{u} with radius $r = 0.8214$. Hence, it follows that the local minimizer \hat{u} of J , whose existence is proven above, is unique in the specified neighborhood B_h . Moreover, we know already that the global minimizer belongs to that neighborhood. Consequently, the function \hat{u} is the *unique globally optimal control* of our original problem (P). That is, the optimal control problem (8.1)–(8.4) is uniquely solvable.

Let us emphasize, that we proved a-posteriori that the sufficient optimality condition holds at the still unknown control \hat{u} . And we computed a approximation \bar{u}_h of that control with a computable error bound $\|\hat{u} - \bar{u}_h\|_{L^2(\Omega)} < 0.1202$.

8.4. Dependence on h . We computed the solution of the discrete problem for a sequence of discretizations. Then we performed the calculation of the different radii as above.

In the table, r_+ denotes the smallest positive number, such that the polynomial in the estimate in the estimation of the cost functional admits a positive value. That is, r_+ is an upper bound for the approximation error in the controls. Our computable error bound decreases linearly with h , which is the expected convergence order for piecewise constant trial functions for the control, [2].

h	r_+
0.01414	0.1202
0.00707	0.05725
0.00354	0.02793
0.00177	0.01379

REFERENCES

- [1] T. Apel and M. Dobrowolski. Anisotropic interpolation with applications to the finite element method. *Computing*, 47:277–293, 1992.
- [2] N. Arada, E. Casas, and F. Tröltzsch. Error estimates for a semilinear elliptic control problem. *Computational Optimization and Application*, 23:201–229, 2002.
- [3] N. Arada, J.-P. Raymond, and F. Tröltzsch. On an augmented Lagrangian SQP method for a class of optimal control problems in Banach spaces. *Computational Optimization and Applications*, 22:369–398, 2002.
- [4] J. F. Bonnans. Second-order analysis for control constrained optimal control problems of semilinear elliptic equations. *Appl. Math. Optim.*, 38:303–325, 1998.
- [5] E. Casas, M. Mateos, and F. Tröltzsch. Error estimates for the numerical approximation of boundary semilinear elliptic control problems. *Comput. Optim. Appl.*, 31:193–219, 2005.
- [6] A. L. Dontchev, W. W. Hager, A. B. Poore, and B. Yang. Optimality, stability, and convergence in optimal control. *Appl. Math. Optim.*, 31:297–326, 1995.
- [7] H. Goldberg and F. Tröltzsch. Second order optimality conditions for a class of control problems governed by nonlinear integral equations with application to parabolic boundary control. *Optimization*, 20(5):687–698, 1989.
- [8] H. Goldberg and F. Tröltzsch. Second-order sufficient optimality conditions for a class of nonlinear parabolic boundary control problems. *SIAM J. Control Optim.*, 31(4):1007–1025, 1993.
- [9] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman, London, 1985.
- [10] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*. Academic Press, New York, 1980.
- [11] K. Malanowski and F. Tröltzsch. Lipschitz stability of solutions to parametric optimal control for parabolic equations. *ZAA*, 18:469–489, 1999.
- [12] K. Malanowski and F. Tröltzsch. Lipschitz stability of solutions to parametric optimal control for elliptic equations. *Control and Cybernetics*, 29:237–256, 2000.
- [13] M. Plum and C. Wiens. New solutions of the Gelfand problem. *J. Math. Anal. Appl.*, 269(2):588–606, 2002.
- [14] F. Tröltzsch. Lipschitz stability of solutions of linear-quadratic parabolic control problems with respect to perturbations. *Dyn. Contin. Discrete Impulsive Syst.*, 7:289–306, 2000.
- [15] F. Tröltzsch. *Optimale Steuerung partieller Differentialgleichungen*. Vieweg, Wiesbaden, 2005.