A note on scenario reduction for two-stage stochastic programs

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We extend earlier work on scenario reduction by relying directly on Fortet-Mourier metrics instead of using upper bounds given in terms of mass transportation problems. The importance of Fortet-Mourier metrics for quantitative stability of two-stage models is reviewed and some numerical results are also provided.

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1. Introduction

In the papers [2,5] a stability-based methodology is developed for reducing the set of scenarios in convex stochastic programming models. Such a reduction may be desirable in some situations when the underlying optimization models already happen to be large scale and the incorporation of a large number of scenarios might lead to huge programs and, hence, to high computation times. The idea of the scenario reduction framework in [2,5] is to compute the (nearly) best approximation of the underlying discrete probability distribution by a measure with smaller support in terms of a probability metric which is associated to the stochastic program in a natural way. Such "natural" (or canonical) metrics for probability measures are known for (linear) two-stage stochastic programs: the r-th order Fourier-Mourier metrics, where the choice of $r \geq 1$ depends on the specific structure of the programs (see Section 3 and [10,11]).

However, the strategies for scenario reduction developed in [2,5] are not based on Fourier-Mourier metrics, but on their upper bounds in form of certain mass transportation problems which enjoy specific properties and representations. In the present note we remove this drawback and develop scenario reduction algorithms that are rigorously based on Fortet-Mourier metrics. The key step in this direction is that we do no longer use the (generalized) distances c for scenarios as in [2,5], but so-called reduced distances (or costs) $\hat{c}$ which, indeed, are distances in the finite-dimensional scenario space and represent infima of certain optimization problems.

Our paper is organized as follows. In Section 2 we discuss distances of (multivariate) probability measures that are based on mass transportation problems. We review some of their topological properties, duality results and representations that are needed in the

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sequel. Section 3 reviews stability properties of multiperiod two-stage stochastic programs with respect to the distances introduced in the previous section. In Section 4 we extend our earlier theory and heuristic algorithms for optimal scenario reduction to the relevant metrics. Finally, we present some numerical experience for the new forward algorithm of scenario reduction. It is tested on realistic data from electricity portfolio management.

2. Distances of probability distributions

Let $\mathcal{P}(\Xi)$ denote the set of all Borel probability measures on a closed subset Ξ of $\mathbb{R}^s$. Let $\hat{\mu}_c$ and $\mathring{\mu}_c$ denote the Monge-Kantorovich and Kantorovich-Rubinstein functionals, respectively ([6,8]), on $\mathcal{P}(\Xi) \times \mathcal{P}(\Xi)$. They have the form

$$\hat{\mu}_c(P,Q) := \inf\left\{\int_{\Xi\times\Xi} c(\xi,\tilde{\xi})\eta(d\xi,d\tilde{\xi}) : \eta \in \mathcal{P}(\Xi\times\Xi),\ \pi_1\eta = P,\ \pi_2\eta = Q\right\}, \tag{1}$$

$$\mathring{\mu}_c\,(P,Q) := \inf\left\{\int_{\Xi\times\Xi} c(\xi,\tilde{\xi})\eta(d\xi,d\tilde{\xi}) : \eta \in \mathcal{M}(\Xi\times\Xi),\ \pi_1\eta - \pi_2\eta = P - Q\right\}, \tag{2}$$

where $\mathcal{M}(\Xi\times\Xi)$ denotes the set of all finite measures on $\Xi\times\Xi$ and π_1 and π_2 denote the projections onto the first and second components, respectively.
We assume that the cost function $c : \Xi\times\Xi \to \mathbb{R}$ is nonnegative, symmetric and continuous and satisfies the properties:
(i) $c(\xi,\tilde{\xi}) = 0$ if and only if $\xi = \tilde{\xi}$,
(ii) $\sup\{c(\xi,\tilde{\xi}) : \xi,\tilde{\xi} \in B, \|\xi - \tilde{\xi}\| \le \delta\}$ tends to 0 as $\delta \to 0$ for each bounded $B \subseteq \Xi$,
(iii) c allows the estimate $c(\xi,\tilde{\xi}) \le \lambda(\xi) + \lambda(\tilde{\xi})$ for all $\xi,\tilde{\xi} \in \Xi$ with a measurable function $\lambda_c : \Xi \to \mathbb{R}_+$ that is bounded on bounded subsets of Ξ.
Both functionals are finite if we assume that the probability measures P, Q belong to

$$\mathcal{P}_c(\Xi) := \left\{Q \in \mathcal{P}(\Xi) : \int_\Xi \lambda_c(\xi)Q(d\xi) < \infty\right\}. \tag{3}$$

An important example for cost functions is

$$c_r(\xi,\tilde{\xi}) := \max\{1, \|\xi - \xi_0\|^{r-1}, \|\tilde{\xi} - \xi_0\|^{r-1}\}\|\xi - \tilde{\xi}\| \quad (\xi,\tilde{\xi} \in \Xi), \tag{4}$$

for some $r \ge 1$ and $\xi_0 \in \Xi$. In this case, we set $\lambda_{c_r}(\xi) := 2\|\xi - \xi_0\| \max\{1, \|\xi - \xi_0\|^{r-1}\}$ and know that $\mathcal{P}_{c_r}(\Xi)$ coincides with the set of all probability measures having absolute moments of order r. We also use the notation $\mathcal{P}_r(\Xi)$ for $\mathcal{P}_{c_r}(\Xi)$, $\mathring{\mu}_r$ for $\mathring{\mu}_{c_r}$ etc.
The following dual representation and characterizations of Monge-Kantorovich and Kantorovich-Rubinstein functionals are of special interest here. The results are derived in [9, Section 2], [8, Section 4.3] and [6, Theorems 6.2.5 and 6.3.1].

Proposition 2.1 *For all probability measures $P, Q \in \mathcal{P}_c(\Xi)$ the Kantorovich-Rubinstein functional $\mathring{\mu}_c$ admits the dual representation*

$$\mathring{\mu}_c\,(P,Q) = \sup_{f\in\mathcal{F}_c}\left|\int_\Xi f(\xi)P(d\xi) - \int_\Xi f(\xi)Q(d\xi)\right|, \tag{5}$$

where $\mathcal{F}_c$ is the class of all functions $f : \Xi \to \mathbb{R}$ satisfying $f(\xi) - f(\tilde{\xi}) \le c(\xi,\tilde{\xi})$ for all $\xi,\tilde{\xi} \in \Xi$.

Proposition 2.2 *Let Ξ be compact. Then every Kantorovich-Rubinstein functional with cost function c coincides with a Monge-Kantorovich functional with reduced cost $\hat{c}$. More precisely, it holds*

$$\mathring{\mu}_c\,(P,Q) = \mathring{\mu}_{\hat{c}}\,(P,Q) = \hat{\mu}_{\hat{c}}(P,Q) \le \hat{\mu}_c(P,Q), \tag{6}$$

where the real-valued function $\hat{c}$ on $\Xi \times \Xi$ is given by

$$\hat{c}(\xi,\tilde{\xi}) := \inf\Big\{ \sum_{i=1}^{n-1} c(\xi_i,\xi_{i+1}) : n \in \mathbb{N}, \xi_i \in \Xi, \xi_1 = \xi, \xi_n = \tilde{\xi}\Big\}. \tag{7}$$

The function $\hat{c}$ is a metric on Ξ with $\hat{c} \le c$ and coincides with c if c satisfies the triangle inequality.

Proposition 2.3 *The Kantorovich-Rubinstein functional $\mathring{\mu}_r$ is a metric on $\mathcal{P}_r(\Xi)$ and the estimate*

$$\left| \int_\Xi \|\xi\|^r P(d\xi) - \int_\Xi \|\xi\|^r Q(d\xi) \right| \le r\ \mathring{\mu}_r\,(P,Q) \tag{8}$$

holds for all $P, Q \in \mathcal{P}_r(\Xi)$.
Convergence of a sequence (P_n) of probability measures in this metric space to some limit P is equivalent to any of the following conditions:

(a) $(\hat{\mu}_r(P_n,P))$ tends to 0 as $n \to \infty$,

(b) (P_n) converges weakly to P and the r-th order absolute moments of P_n converge to those of P.

The distance $\mathring{\mu}_r$ is also called *Fortet-Mourier metric* of order r. Such metrics were first studied in [3] and introduced by its dual representation (5). The compactness assumption in Proposition 2.2 is not restrictive here since it will be used for probability measures with finite support. Its importance in the present context is due to the fact that Kantorovich-Rubinstein functionals are appropriate for stability issues (see Section 3), but Monge-Kantorovich functionals, i.e., mass transportation problems, allow special representations.

Now, let us consider the case of two discrete probability measures P and Q having finitely many scenarios ξ_i (with probabilities p_i), $i = 1,\dots,N$, and $\xi_{N+j} = \tilde{\xi}_j$ (with probabilities q_j), $j = 1,\dots,M$, respectively. Then we have

$$\hat{\mu}_c(P,Q) = \inf\left\{ \sum_{i=1}^{N}\sum_{j=1}^{M} \eta_{ij} c(\xi_i,\tilde{\xi}_j) \,\middle|\, \eta_{ij} \ge 0, \sum_{j=1}^{M} \eta_{ij} = p_i, \sum_{i=1}^{N} \eta_{ij} = q_j \right\},$$

i.e. $\hat{\mu}_c(P,Q)$ is the optimal value of a *linear transportation problem*, and

$$\mathring{\mu}_c\,(P,Q) = \inf\left\{ \sum_{i,j=1}^{N+M} c(\xi_i,\xi_j)\eta_{ij} \,\middle|\, \eta_{ij} \ge 0, \sum_{j=1}^{N+M} \eta_{ij} - \sum_{j=1}^{N+M} \eta_{ji} = P(\{\xi_i\}) - Q(\{\xi_i\}) \right\},$$

i.e. $\mathring{\mu}_c\,(P,Q)$ is the optimal value of a *minimum cost flow problem* and can be reformulated as a linear transportation problem with reduced cost $\hat{c}$ according to Proposition 2.2. Hence, for discrete measures with finite support the functionals $\hat{\mu}_c$ and $\mathring{\mu}_c$ are computationally accessible.

3. A review of stability for two-stage models

If the second stage of a linear stochastic program with recourse models a (stochastic) dynamical decision process, as is the case in a variety of applications, the two-stage problem takes on the form

$$\min\left\{\int_{\Xi} f_0(\xi,x)P(d\xi) : x\in X\right\}, \tag{9}$$

where X is a polyhedral subset of $\mathbb{R}^m$, Ξ a closed subset of $\mathbb{R}^s$, P is a Borel probability measure on Ξ and the integrand f_0 is of the form

$$f_0(\xi,x) = \langle c,x\rangle + \inf\left\{\sum_{j=1}^{\ell}\langle q_j(\xi),y_j\rangle : W_jy_j = h_j(\xi) - T_j(\xi)y_{j-1}, y_j\in Y_j, j=1,\dots,\ell\right\}$$

with $c\in\mathbb{R}^m$, polyhedral subsets Y_j of $\mathbb{R}^{\overline{m}_j}$, recourse costs $q_j(\xi)\in\mathbb{R}^{\overline{m}_j}$, right-hand sides $h_j(\xi)\in\mathbb{R}^{r_j}$, technology matrices $T_j\in\mathbb{R}^{r_j\times\overline{m}_{j-1}}$ and recourse matrices $W_j\in\mathbb{R}^{r_j\times\overline{m}_j}$ for $j=1,\dots,\ell$ and some $\ell\in\mathbb{N}$; the vectors $q_j(\cdot)$, $h_j(\cdot)$ and the matrices $T_j(\cdot)$ are (potentially) stochastic and affine functions of ξ. Then the second stage program has separable block structure and the recourse variable y has the form $y=(y_1,\dots,y_\ell)$. When rewriting the model as a two-stage stochastic programming model with recourse decision $y=(y_1,\dots,y_\ell)$, the recourse matrix has separable block structure with $W_1,\dots,W_\ell$ and the matrices $T_1(\xi),\dots,T_\ell(\xi)$ appearing as its main and lower diagonal blocks.

The following stability result for optimal values $v(P)$ and ε-approximate solution sets $S_\varepsilon(P)$ of (9) is derived in the recent paper [11].

Proposition 3.1 *Let $P\in\mathcal{P}_{\ell+1}(\Xi)$ and the solution set $S(P)$ of (9) be nonempty and bounded. Assume that $h_j(\xi)-T_j(\xi)x\in W_j(Y_j)$ holds for each $j=1,\dots,\ell$ and all pairs $(\xi,x)\in\Xi\times X$ (*relatively complete recourse*). Moreover, assume* $\ker(W_j)\cap Y_j^\infty=\{0\}$ *for $j=1,\dots,\ell-1$, where Y_j^∞ denotes the (polyhedral) horizon cone to Y_j.*
Then there exist constants $L>0$ and $\hat\varepsilon>0$ such that for any $\varepsilon\in(0,\hat\varepsilon)$ the estimates

$$\begin{aligned} |v(P)-v(Q)| &\le L\ \mathring{\mu}_{\ell+1}(P,Q)\\ d\!l_\infty(S_\varepsilon(P),S_\varepsilon(Q)) &\le \frac{L}{\varepsilon}\,\mathring{\mu}_{\ell+1}(P,Q)\end{aligned}$$

hold whenever $Q\in\mathcal{P}_{\ell+1}(\Xi)$ and $\mathring{\mu}_{\ell+1}(P,Q)<\varepsilon$. Here, $d\!l_\infty$ denotes the Pompeiu-Hausdorff distance on closed and bounded subsets of $\mathbb{R}^m$.

The case $\ell=1$ corresponds to the situation of linear two-stage models with fixed recourse (see [10, Theorem 24]). Hence, together with the results in [7,10], we have that the number $r\ge 1$ should be selected as

$$r=\begin{cases} 1 & \text{, if either costs or right-hand sides in (9) are random}\\ 2 & \text{, if only costs and right-hand sides are random in (9)}\\ \ell+1 & \text{, if, in addition, all technology matrices are random.}\end{cases}$$

4. Optimal scenario reduction

Let P be a discrete probability distribution with scenarios ξ_i and probabilities p_i, $i = 1, \ldots, n$. If the number n of scenarios is large, one might wish to delete scenarios of P in a best possible way, i.e., such that the original problem or, more precisely, its optimal value admits minimal changes. To make this requirement precise, we denote by Q_J a discrete distributon whose support consists of a subset of scenarios ξ_j, $j \in \{1, \ldots, n\} \setminus J$, of P having probabilities q_j, $j \notin J$. Hence, it is of interest to determine a subset J of $\{1, \ldots, n\}$ and probabilities q_j, $j \notin J$, such that the distance $|v(P) - v(Q_J)|$ of optimal values is minimal with respect to all subsets of given cardinality. But, in general, this distance is difficult to handle. According to Proposition 3.1 we know, however, that, for two-stage models, $|v(P) - v(Q_J)|$ can be estimated by a multiple of some metric or functional μ of P and Q_J. Hence, one might consider $\mu(P, Q_J)$ instead and arrives at the principle of *optimal scenario reduction*: Fix $k \in \mathbb{N}$, $k < n$, and determine a solution of the minimization problem

$$\min \left\{ \mu(P, Q_J) | J \subset \{1, \ldots, n\}, \# J = n - k, q_j \geq 0, \sum_{j \notin J} q_j = 1 \right\}. \tag{10}$$

In a first step, it is of interest to fix J and to determine the optimal weights q_j, $j \notin J$, such that Q_J is a probability measure, i.e., to solve the best approximation problem.

$$\min \left\{ \mu(P, Q_J) \mid q_j \geq 0, \sum_{j \notin J} q_j = 1 \right\}. \tag{11}$$

The next result asserts that the latter problem (11) is solvable and provides an explicit representation of the infimum in case $\mu = \overset{\circ}{\mu}_c$.

Theorem 4.1 *For given nonempty subset J of $\{1, \ldots, n\}$ the problem (11) has a solution $Q_J^* = \sum_{j \notin J} q_j^* \delta_{\xi_j}$ and it holds*

$$\begin{aligned} D_J &:= \overset{\circ}{\mu}_c (P, Q_J^*) = \min \left\{ \overset{\circ}{\mu}_c (P, Q_J) : q_j \geq 0, \sum_{j \notin J} q_j = 1 \right\} = \sum_{i \in J} p_i \min_{j \notin J} \hat{c}(\xi_i, \xi_j) \qquad (12) \\ &= \sum_{i \in J} p_i \min \left\{ \sum_{\ell=1}^{m-1} c(\xi_{l_\ell}, \xi_{l_{\ell+1}}) : m \in \mathbb{N}, l_\ell \in \{1, \ldots, n\}, l_1 = i, l_m = j \notin J \right\}, \end{aligned}$$

where $q_j^* = p_j + \sum_{i \in J_j} p_i$, $\forall j \notin J$, with $J_j := \{i \in J | j = j(i)\}$ and the index $j(i)$ belonging to $\arg\min_{j \notin J} \hat{c}(\xi_i, \xi_j)$, $\forall i \in J$, *i.e., the* optimal redistribution *consists in adding each deleted scenario weight to that of some of those scenarios being closest w.r.t. $\hat{c}$.*

Proof: Due to Proposition 2.2 we have the identity

$$\overset{\circ}{\mu}_c (P, Q_J) = \hat{\mu}_{\hat{c}}(P, Q_J),$$

where the reduced cost function $\hat{c}$ is a metric on the support Ξ of P. Since [2, Theorem 2] is established for the Monge-Kantorovich functional, it implies the desired representation

$$\min\left\{\hat{\mu}_{\hat{c}}(P,Q_J)\mid q_j\geq 0,\sum_{j\notin J}q_j=1\right\}=\sum_{i\in J}p_i\min_{j\notin J}\hat{c}(\xi_i,\xi_j).$$

together with the asserted redistribution rule. □

The preceding result coincides with [2, Theorem 2] if c is a metric. Using the explicit formula (12), the problem (10) of optimal scenario reduction is of the form

$$\min\left\{D_J=\sum_{i\in J}p_i\min_{j\notin J}\hat{c}(\xi_i,\xi_j)\,|J\subset\{1,...,n\},\#J=n-k\right\},\tag{13}$$

i.e., it represents a metric k-median problem in the metric space $(\Xi,\hat{c})$. The problem is known to be NP-hard, hence, (polynomial-time) approximation algorithms and heuristics become important. The approximation algorithms for the metric k-median problem in [1] and [12, Chapter 25] achieve guarantees of $6\frac{2}{3}$ and 6 times the optimal.

Simple heuristics may be derived by extending the two extremal cases $k=n-1$ and $k=1$ of problem (13). These problems correspond to solving

$$\min_{l\in\{1,...,n\}}p_l\min_{j\neq l}\hat{c}(\xi_l,\xi_j)\quad(k=n-1)\qquad\text{and}\qquad\min_{u\in\{1,...,n\}}\sum_{\substack{i=1\\ i\neq u}}^{n}p_i\hat{c}(\xi_u,\xi_i)\quad(k=1).$$

Their solutions are the index sets $J=\{l_1\}$ and $J=\{1,\ldots,n\}\setminus\{u_1\}$, respectively. The two sets arise from different algorithmic ideas: backward reduction and forward selection. Both ideas can be extended and lead to backward and forward heuristics for finding approximate solutions of (13). For example, the forward selection procedure determines an index set $J^{[k]}$ of deleted scenarios having cardinality $n-k$.

Algorithm 4.2 *(Forward selection)*

Step [0]: $J^{[0]}:=\{1,\ldots,n\}$.

Step [i]: $u_i\in\arg\min_{u\in J^{[i-1]}}\sum_{k\in J^{[i-1]}\setminus\{u\}}p_k\min_{j\notin J^{[i-1]}\setminus\{u\}}\hat{c}(\xi_k,\xi_j)$,

$J^{[i]}:=J^{[i-1]}\setminus\{u_i\}$.

Step [k+1]: *Optimal redistribution.*

This algorithm was first studied in [5] for the case $\hat{c}=c$. There it is shown that the algorithm requires $O(k\,n^2)$ operations. Although the algorithm does not lead to optimality in general, the performance evaluation of its implementation in [5] is very encouraging.

5. Numerical experience

We consider again the scenario tree in [2,5] representing the increasing uncertainty of electrical load in a stochastic electrical power production model for a weekly time horizon

(see [4] for additional information). The scenario tree was obtained by calibrating a time series model for the electrical load, by simulating a large number of load realizations, and by constructing an initial ternary load scenario tree based on sample means and standard deviations of the simulated realizations. The initial load scenario tree represents a discrete probability distribution P that consists of $3^6 = 729$ uniformly distributed scenarios and enters a 7-period two-stage stochastic programming model. Table 1 presents our com-

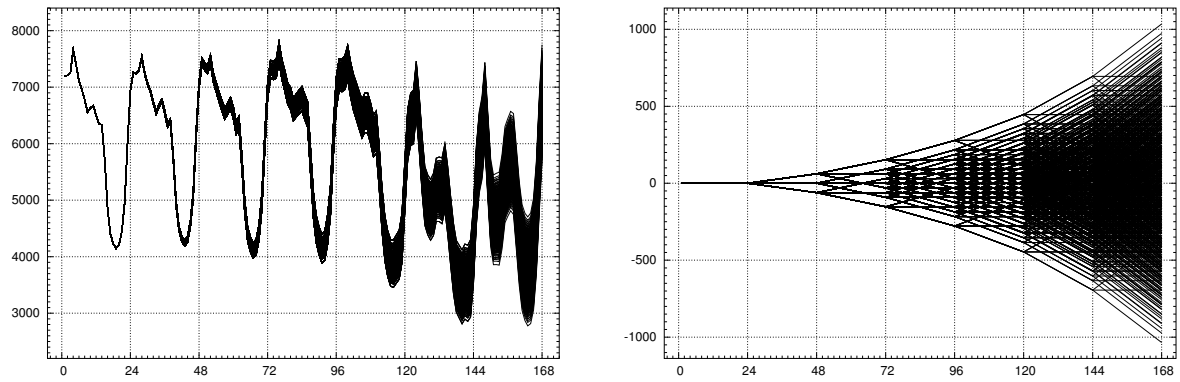

Figure 1. Load scenarios for one week and mean shifted initial load scenario tree

putational results for optimal scenario reduction of the initial load scenario tree by using Algorithm 4.2. A comparison with the results in Table 2 shows the improvement of using $\overset{\circ}{\mu}_r$ instead of $\hat{\mu}_r$. Both tables display the relative distances between the original load tree and some of the reduced ones, and the effects of varying the order r of the Kantorovich-Rubinstein distances $\overset{\circ}{\mu}_r$ and Monge-Kantorovich functionals $\hat{\mu}_r$, respectively. The relative distances are computed by dividing all distances by the Kantorovich-Rubinstein distance between the initial load distribution P and the Dirac measure at the scenario obtained in the first forward selection step, i.e., by $\overset{\circ}{\mu}_r\ (P, \delta_{\xi_{u_1}})$. To compute a reduced tree for $r = 1$, the running time on a PC equipped with a 3 GHz processor is less than 10 seconds including about 4 seconds to compute the scenario distances $c_r(\cdot, \cdot)$. For $r > 1$ about 9 seconds are needed in addition to compute the reduced cost $\hat{c}_r(\cdot, \cdot)$. Figures 2 illustrates the structure of the reduced scenario trees consisting of 20 scenarios for varying order r. Recall that approximations of probability distributions with respect to $\overset{\circ}{\mu}_r$ have the property that r-th order absolute moments are approximately recovered (see Proposition 2.3). This leads to the selection of different scenarios for different r with a tendency to outer scenarios for growing r. Figure 3 illustrates the behaviour of the relative $\overset{\circ}{\mu}_r$-distances with respect to the number of reduced scenarios for increasing order r.

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Table 1
Numerical results for optimal scenario reduction based on $\mathring{\mu}_r$

Number of scenarios	**Relative $\mathring{\mu}_r$-distances**						
	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5	0.522	0.646	0.684	0.696	0.687	0.682	0.668
10	0.419	0.536	0.589	0.577	0.582	0.556	0.535
20	0.323	0.420	0.469	0.472	0.466	0.431	0.395
50	0.230	0.305	0.335	0.337	0.301	0.256	0.210
100	0.169	0.220	0.242	0.222	0.180	0.133	0.094
150	0.137	0.178	0.185	0.156	0.114	0.077	0.049
200	0.117	0.148	0.143	0.112	0.076	0.045	0.025
300	0.094	0.102	0.085	0.057	0.032	0.016	0.008
400	0.072	0.067	0.049	0.028	0.013	0.006	0.002
500	0.050	0.039	0.024	0.017	0.005	0.002	0.001
600	0.028	0.018	0.009	0.004	0.001	0.000	0.000

Table 2
Numerical results for optimal scenario reduction based on $\hat{\mu}_r$

Number of scenarios	**Relative $\hat{\mu}_r$-distances**						
	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
1	1.000	1.609	2.354	3.146	3.910	4.627	5.302
5	0.522	0.738	0.940	1.079	1.209	1.217	1.257
10	0.419	0.574	0.713	0.787	0.820	0.803	0.794
20	0.323	0.448	0.538	0.600	0.617	0.601	0.565
50	0.230	0.308	0.359	0.378	0.369	0.331	0.286
100	0.169	0.221	0.253	0.248	0.211	0.168	0.130
150	0.137	0.179	0.192	0.171	0.134	0.097	0.066
200	0.117	0.149	0.147	0.121	0.088	0.058	0.035
300	0.094	0.102	0.088	0.062	0.037	0.021	0.011
400	0.072	0.067	0.050	0.03	0.015	0.007	0.003
500	0.050	0.039	0.025	0.012	0.005	0.002	0.001
600	0.028	0.018	0.009	0.004	0.001	0.000	0.000

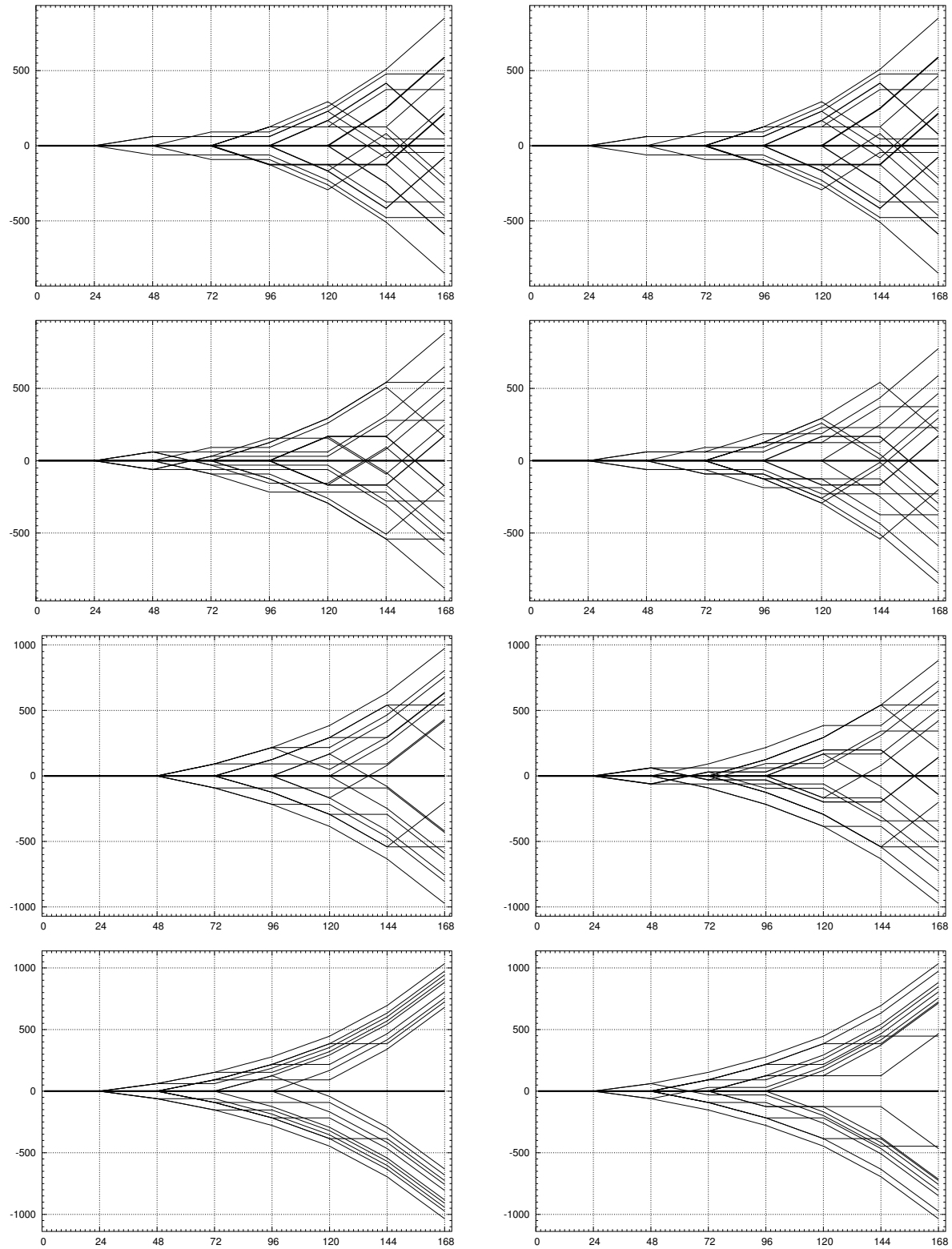

Figure 2. Reduced trees containing $k = 20$ scenarios obtained by using $\mathring{\mu}_r$ (left column) and $\hat{\mu}_r$ (right column) for $r = 1, 2, 4, 7$

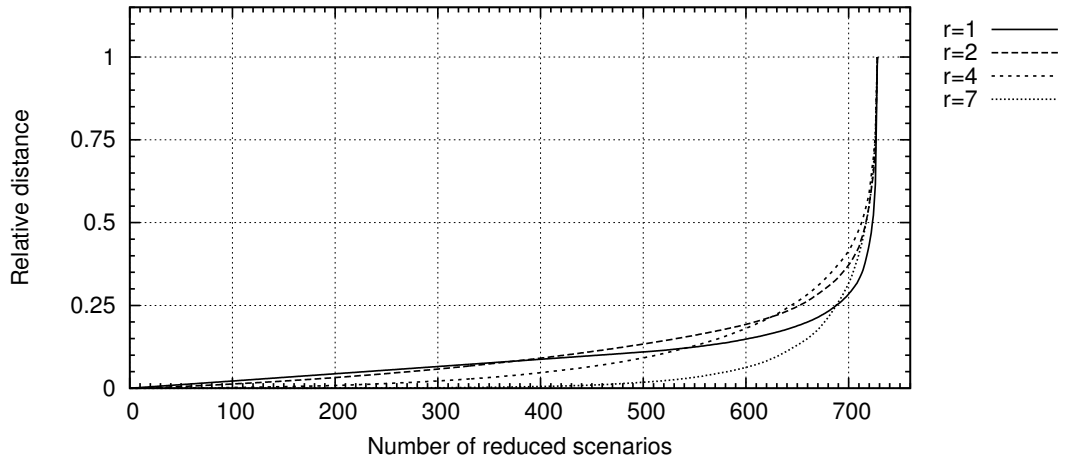

Figure 3. Behaviour of the $\mathring{\mu}_r$-distance for $r = 1, 2, 7$

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