

CONSISTENT INITIALIZATION AND PERTURBATION ANALYSIS FOR ABSTRACT DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. In this paper we consider linear and time-invariant differential-algebraic equations (DAEs) $E\dot{x}(t) = Ax(t) + f(t)$, $x(0) = x_0$, where $x(\cdot)$ and $f(\cdot)$ are functions with values in separable Hilbert spaces X and Z . $E : X \rightarrow Z$ is assumed to be a bounded operator, whereas A is closed and defined on some dense subspace $D(A)$ which is in general a proper subset of X . Based on a decoupling of the algebraic and the differential part, the set of initial values being consistent with the given inhomogeneity will be parameterized. As a consequence of these results, we will derive estimates for the trajectory $x(\cdot)$ in dependence of the initial state x_0 and the inhomogeneity $f(\cdot)$. In the theory of differential-algebraic equations, this is commonly known as perturbation analysis.

Key words. abstract differential-algebraic equations, partial differential-algebraic equations, infinite dimensional linear systems theory, consistent initialization, perturbation analysis

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1. Introduction. Coupled systems of differential-algebraic and partial differential equations, or - often called - partial differential-algebraic equations (PDAEs), are the outcome when modelling jointly interacting - for instance technical - configurations. A spatial discretization of the involved partial differential equations (PDEs) usually leads to a high dimensional differential-algebraic equation (DAE). However, for a better understanding of the underlying problem, an analytical investigation of the original equations is important. A systematic and general study of these problems leads to systems $F(\dot{x}(t), x(t), t) = 0$, $x(0) = x_0$ in an abstract setting, the so-called abstract DAEs (ADAEs). The unknown function $x(\cdot)$ now takes its values in an infinite dimensional space, and the Fréchet derivative $\frac{d}{dx}F(\dot{x}, x, t)$ has a non-trivial nullspace, in general. Typical examples where this kind of equations occurs is in electrical engineering, when circuits with distributed components like transmission lines [11, 14] or semiconductor devices [2, 17] are considered. Coupled systems also appear in other disciplines like mechanical or chemical engineering [9, 10].

In this work, we focus on linear and time-invariant ADAEs

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + f(t), \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $E : X \rightarrow Z$ is a bounded linear operator from a separable Hilbert space X to another space Z . In many practical cases, X and Z are product spaces of Lebesgue spaces and \mathbb{R}^n . The operator A often contains differential operators with respect to time and thus it is natural to assume that it may be unbounded and that it is defined on some dense subspace $D(A)$. The first concrete question we address in this work is the classification of the initial states $x_0 \in X$ for which (1.1) possesses a solution. This kind of problem already arises in the theory of finite dimensional DAEs and is there known as *consistent initialization*. The second task we treat is the *perturbation analysis* of ADAEs. The aim this is the development of estimates of the form

$$\|x(t_f)\| \leq c \cdot \left(\|x_0\|_k + \sum_{j=0}^{\nu_{pert}} \|f^{(j)}\|_{L_p([0, t_f], Z)} \right), \tag{1.2}$$

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where $f^{(j)}(\cdot)$ denotes the j -th derivative of $f(\cdot)$ with respect to time, $L_p([0, t_f], Z)$ is a Lebesgue space of Z -valued functions, $\|\cdot\|_k$ is some norm on a subspace of X and some constant $c > 0$ being independent of x_0 and $f(\cdot)$ but dependent of t_f .

In the case where $E, A \in \mathbb{R}^{n \times n}$, the Kronecker normal form [3, 7] is useful for the determination of the initial states $x_0 \in \mathbb{R}^n$ being consistent with a given (sufficiently smooth) inhomogeneity $f(\cdot)$. To be more precise, if $\det(sE - A)$ is not the zero polynomial, then there exist invertible matrices $W, T \in \mathbb{R}^{n \times n}$ such that $WET = \text{diag}(N, I)$, $WAT = \text{diag}(I, \bar{A})$, where $N \in \mathbb{R}^{r \times r}$ is a nilpotent matrix and \bar{A} is some matrix of dimensions $n - r \times n - r$. By a coordinate transformation

$$\begin{bmatrix} x_1(\cdot) \\ x_2(\cdot) \end{bmatrix} := T^{-1}x(\cdot), \quad \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix} := Wf(\cdot),$$

we get that (1.1) is equivalent to the following two decoupled equations

$$N\dot{x}_1(t) = x_1(t) + f_1(t), \quad (1.3a)$$

$$\dot{x}_2(t) = \bar{A}x_2(t) + f_2(t). \quad (1.3b)$$

Here, (1.3a) contains the algebraic relations and some further - hidden algebraic - constraints becoming purely algebraic when (1.3a) is formally differentiated with respect to time. If ν is the *Kronecker index* of the DAE, i.e. the nilpotency index of N , the solution is given by $x_1(t) = -\sum_{k=0}^{\nu-1} N^k f^{(k)}(t)$. Equation (1.3b) is an ordinary differential equation (ODE) that is solved by the variation of constants formula $x_2(t) = e^{\bar{A}t}x_2(0) + \int_0^t e^{\bar{A}(t-\tau)}f_2(t-\tau)d\tau$. Altogether, for the existence of a solution, the inhomogeneity has to be $\nu-1$ -times differentiable in some sense and the initial value $x_1(0)$ has to satisfy (1.3a) for $t = 0$ whereas $x_2(0)$ can be chosen arbitrarily. Furthermore, (1.3) implies that for $t_f > 0$ there exists a constant $c > 0$ such that for all sufficiently smooth inhomogeneities $f(\cdot)$ and consistent initial values $x_0 \in \mathbb{R}^n$ the norm inequality

$$\|x(t_f)\| \leq c \cdot \left(\|x_0\| + \sum_{k=0}^{\nu} \|f^{(k)}\|_{L_p([0, t_f], \mathbb{R}^n)} \right) \quad (1.4)$$

holds. Inspired by [13], a projector-based approach for the generalization of the Kronecker normal form to ADAEs was presented in [14, 15]. Indeed, this result is the basis for our investigations in the fields of consistent initialization and perturbation analysis. We now briefly present some known results about decoupling of operator pairs. We remark that $L_b(X, Z)$ denotes the space of bounded linear operators mapping from X to Z and we abbreviate $L_b(X) := L_b(X, X)$. The identity operator is usually denoted by $I \in L_b(X)$ and the domain $D(A)$ of a closed linear operator $A : D(A) \subset X \rightarrow Y$ is equipped with the graph norm $\|\cdot\|_{D(A)}$ defined by $\|x\|_{D(A)} := \sqrt{\|x\|^2 + \|Ax\|^2}$.

THEOREM 1.1 ([15], Theorem 4.1). *Let X, Z be Hilbert spaces and let (E, A) be a regular operator pair such that $E \in L_b(X, Z)$ has closed range and $A : D(A) \subset X \rightarrow Z$ is closed and densely defined. Moreover, let an operator chain with*

$$\begin{aligned} E_0 &:= E, \quad A_0 := A, \\ Q_i &\in L_b(X) \cap L_b(D(A)), \quad Q_i^2 = Q_i, \quad \text{im } Q_i = \ker E_i, \quad \sum_{j=0}^{i-1} \ker E_j \subset \ker Q_i, \\ P_i &:= I - Q_i, \\ E_{i+1} &= E_i - A_i Q_i, \quad D(E_{i+1}) = D(E_i) \cap (D(A_i) + \ker Q_i), \\ A_{i+1} &= A_i P_i, \quad D(A_{i+1}) = D(A_i) + \text{im } Q_i \end{aligned}$$

exist that is stationary, i.e. there exists a $\nu \in \mathbb{N}$ with $\ker E_\nu = \{0\}$. Furthermore, let

$$\operatorname{im} E + A \left(\sum_{k=0}^{\nu-1} \ker E_k \right) \quad (1.5)$$

be closed. Then, there exist Hilbert spaces X_1, X_2, X_3 as well as bounded operators $W \in L_b(Z, X_1 \times X_2 \times X_3)$, $T \in L_b(X_1 \times X_2, X)$, where T is bijective and W is injective and has dense range, i.e. $\overline{\operatorname{im} W} = X_1 \times X_2 \times X_3$, such that

$$WET = \begin{bmatrix} N & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad WAT = \begin{bmatrix} I & K \\ 0 & \mathfrak{U} \\ 0 & \mathfrak{P} \end{bmatrix}. \quad (1.6)$$

$N \in L_b(X_1)$ is a nilpotent operator with nilpotency order ν and $\mathfrak{P}, \mathfrak{U}, K$ are linear operators being defined on $[0 \ I] T^{-1} D(A)$. The operator $WET : X_1 \times X_2 \rightarrow X_1 \times X_2 \times X_3$ is bounded and WAT extends to a closed operator with domain

$$D(WAT) = X_1 \times [0 \ I] T^{-1} D(A). \quad (1.7)$$

The proof in [14, 15] is constructive and denoting $P_{\Sigma\nu} := P_0 \cdots P_{\nu-1}$, the spaces X_1, X_2 and X_3 are given by $X_1 := \operatorname{im} Q_0 \times \cdots \times \operatorname{im} Q_{\nu-1}$, $X_2 = \operatorname{im} P_{\Sigma\nu}$ and $X_3 = \operatorname{im}(I - E_\nu E_\nu^-)$, where E_ν^- is a left inverse of E_ν . The transformations W and T are defined by

$$T \cdot [x_0, x_1, \dots, x_{\nu-1}, x_\nu] = x_\nu + \sum_{i=0}^{\nu-1} P_0 \cdots P_{i-1} Q_i x_i \quad (1.8)$$

and

$$Wz = (-Q_0 P_1 \cdots P_{\nu-1} E_\nu^- z, -Q_1 P_2 \cdots P_{\nu-1} E_\nu^- z, \dots, -Q_{\nu-1} E_\nu^- z, P_{\Sigma\nu} E_\nu^- z, (I - E_\nu E_\nu^-) z). \quad (1.9)$$

The operators in (1.6) then can be expressed as

$$N = \begin{bmatrix} 0 & Q_0 Q_1 & Q_0 P_1 Q_2 & Q_0 P_1 P_2 Q_3 & \cdots & Q_0 P_1 \cdots P_{\nu-2} Q_{\nu-1} \\ & \ddots & Q_1 Q_2 & Q_1 P_2 Q_3 & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & Q_{\nu-3} P_{\nu-2} Q_{\nu-1} \\ & & & & \ddots & Q_{\nu-2} Q_{\nu-1} \\ & & & & & 0 \end{bmatrix}, \quad (1.10a)$$

$$K = - \begin{bmatrix} Q_0 P_1 \cdots P_{\nu-1} E_\nu^- A_\nu \\ Q_1 P_2 \cdots P_{\nu-1} E_\nu^- A_\nu \\ \vdots \\ Q_{\nu-1} E_\nu^- A_\nu \end{bmatrix}, \quad (1.10b)$$

$$\mathfrak{U} = P_{\Sigma\nu} E_\nu^- A_\nu, \quad (1.10c)$$

$$\mathfrak{P} = (I - E_\nu E_\nu^-) A_\nu \quad (1.10d)$$

and the domain of WAT is given by $D(WAT) = D(A_\nu) = \ker P_{\Sigma\nu} \oplus (\operatorname{im} P_{\Sigma\nu} \cap D(A))$. In [13, 14, 15], the natural number ν is called *ADAE index*. It takes the role of the

Kronecker index in the abstract setting. An ADAE in decoupling form is then given by

$$\begin{aligned} \begin{bmatrix} N & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} I & K \\ 0 & \mathfrak{U} \\ 0 & \mathfrak{P} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}, \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}. \end{aligned} \quad (1.11)$$

System (1.11) comprehends the boundary control system

$$\begin{aligned} \dot{x}_2(t) &= \mathfrak{U}x_2(t) + f_2(t), \\ 0 &= \mathfrak{P}x_2(t) + f_3(t), \\ y(t) &= Kx_2(t), \\ x_2(0) &= x_{2,0}. \end{aligned} \quad (1.12)$$

This type of systems is intensively studied in [4, 6]. By backsubstitution, $x_1(\cdot)$ is then the solution of

$$\begin{aligned} N\dot{x}_1(t) &= x_1(t) + y(t) + f_1(t), \\ x_1(0) &= x_{1,0}. \end{aligned} \quad (1.13)$$

Comparing this with the corresponding result in the finite dimensional case, there are essentially two differences: There appears a "third row" containing the operator \mathfrak{P} which can be interpreted as a *boundary control term*. By the representation (1.10d) we see that this is caused by the fact that, in general, the injectivity of E_ν does not imply its surjectivity in contrast to the case where E_ν is a square matrix. The *coupling term* K is also an effect of the infinite dimensionality of the problem. In [14, 15], an example of an ADAE is given for which it was shown that no decoupling form exists with $K = 0$. The following result gives a more detailed characterization of this phenomenon.

THEOREM 1.2. *Let (E, A) be an operator pair satisfying the assumptions of Theorem 1.1 and let $k \in \mathbb{N}$. Then, there exist transformations \bar{W}, \bar{T} , such that*

$$(\bar{W}E\bar{T}, \bar{W}A\bar{T}) = \left(\begin{bmatrix} \bar{N} & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & \bar{K} \\ 0 & \bar{\mathfrak{U}} \\ 0 & \bar{\mathfrak{P}} \end{bmatrix} \right) \quad (1.14)$$

with $\bar{N}^k \bar{K} = 0$, if there are transformations W, T such that (WET, WAT) has the decoupling form (1.6), and there exists an operator $H \in L_b(X_3, X_1)$ such that

$$N^j (K + H\mathfrak{P})\mathfrak{U}^j \in L_b(X_2, X_1) \text{ for } j = 0, \dots, \nu - k - 1. \quad (1.15)$$

Proof. Without loss of generality, we may assume that (E, A) already is in decoupling form (1.6). We now define the operator $L := \sum_{j=0}^{\nu-k-1} N^j (K + H\mathfrak{P})\mathfrak{U}^j$ which is bounded due to the assumption (1.15) and apply the transformations

$$\bar{W} = \begin{bmatrix} I & NL & H \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \bar{T} = \begin{bmatrix} I & -L \\ 0 & I \end{bmatrix}. \quad (1.16)$$

We then get a form (1.14) with $\bar{N} = N$, $\bar{\mathfrak{U}} = \mathfrak{U}$, $\bar{\mathfrak{P}} = \mathfrak{P}$ and $\bar{K} = N^{\nu-k}(K + H\mathfrak{P})\mathfrak{U}^{\nu-k}$ and we calculate $\bar{N}^k \bar{K} = N^k N^{\nu-k} K \mathfrak{U}^{\nu-k} = 0$. \square

In [14, 15], also a projector-based approach to the decoupling with $\bar{N}^k \bar{K} = 0$ is considered. Then the kernels of Q_i have to be chosen in a particular way leading to the notion of *canonical projectors*. In the case where the preliminaries of Theorem 1.2 are violated, the canonical projectors may be unbounded and therefore unapt for the decoupling. A conclusion of Theorem 1.2 is that a *complete decoupling*, i.e. a form (1.14) with $\bar{K} = 0$ exists if (1.15) is fulfilled for $k = 0$. For trivial reasons, this holds true in finite dimensions.

The remaining part of this work is organized as follows: In the following section, solution concepts for ADAEs are clarified. We define the notions of weak and classical solutions and consistent initial values. Some solvability criteria are presented. Section 3 gives the main result about consistent initialization of ADAEs. It is shown that initial values not only have to fulfill the (hidden) algebraic relations but there are some further constraints which are due to the appearance of the operators K and \mathfrak{P} in the decoupling form (1.6). Thereafter, Section 4 treats the perturbation analysis of ADAEs. We will show that the solution continuously depends upon the Sobolev norms of the inhomogeneity and some particular norm of the initial value. Before this work is concluded, we consider an example from electrical engineering, apply the results of this work and give practical interpretations in Section 5.

2. Solvability of Abstract Differential-Algebraic Equations. In this section we introduce solution concepts for ADAEs. First, some necessary function spaces are introduced.

DEFINITION 2.1. *Let X be a separable Hilbert space and let $\mathcal{I} \subset \mathbb{R}$ be an interval. Then for $k \in \mathbb{N} \cup \{\infty\}$ we define $C^k(\mathcal{I}, X)$ to be the space of k -times continuously differentiable functions $f(\cdot) : \mathcal{I} \rightarrow X$ and we identify $C(\mathcal{I}, X) := C^0(\mathcal{I}, X)$. For $p \in [1, \infty]$, the Lebesgue spaces of X -valued functions are abbreviated by $L_p(\mathcal{I}, X)$. The Sobolev space of k -times weakly differentiable functions is given by $H^{k,p}(\mathcal{I}, X) := \{f(\cdot) \in L_p(\mathcal{I}, X), f^{(l)}(\cdot) \in L_p(\mathcal{I}, X) \text{ for } 0 \leq l \leq k\}$. For negative exponents k , $H^{k,p}(\mathcal{I}, X)$ is defined to be the dual space of $H^{-k,p}(\mathcal{I}, X)$. We identify the Hilbert space $H^k(\mathcal{I}, X)$ with $H^{k,2}(\mathcal{I}, X)$ and in the case where the argument X in the above spaces is missing, we mean $X = \mathbb{R}$.*

The separability of X is needed for the availability of common assertions like the denseness of $C^k(\mathcal{I}, X) \cap L_p(\mathcal{I}, X)$ and $H^k(\mathcal{I}, X)$ in $L_p(\mathcal{I}, X)$ for $k \geq 0$. However, the separability assumption is nonrestrictive, since practically motivated spaces usually have this property. For more details in the field of spaces of functions with values in abstract spaces, we refer to [8] and the bibliography therein.

In the following we clarify the solution types. As in the case of abstract ODEs, i.e. $X = Z$ and E is the identity, certain trajectories $x(\cdot)$ make sense even if they are not continuously differentiable and $x(t) \notin D(A)$ for some t . This leads to concept of weak solutions. Our definition is motivated by the corresponding concept for abstract ODEs in [1].

DEFINITION 2.2. *Let an ADAE (1.1) with $E \in L_b(X, Z)$ and a densely defined linear $A : D(A) \subset X \rightarrow Z$ be given.*

For $f(\cdot) \in C([0, t_f], Z)$, we call $x(\cdot)$ a classical solution, if $x(\cdot) \in C^1([0, t_f], X)$, $x(t) \in D(A)$ and (1.1) is fulfilled pointwise for all $t \in [0, t_f]$.

For $f(\cdot) \in L_1([0, t_f], Z)$, the function $x(\cdot)$ is called a weak solution if it is continuous

and for all $z^* \in D(A^*)$ holds that $\langle x(\cdot), E^* z^* \rangle \in H^{1,1}([0, t_f])$ with

$$\frac{d}{dt} \langle x(\cdot), E^* z^* \rangle = \langle x(\cdot), A^* z^* \rangle + \langle f(\cdot), z^* \rangle, \quad (2.1)$$

where $\frac{d}{dt}$ means the weak derivative. A vector x_0 is called a consistent initial value if of (1.1) possesses a weak solution.

In the above definition of weak solutions, the inhomogeneity was assumed to be in the space $L_1([0, t_f], Z)$. One could extend the definition in a manner that $L_p([0, t_f], Z)$ is considered instead. However, since these spaces are contained in $L_1([0, t_f], Z)$, Definition 2.2 comprehends these cases. In the remaining part we are mainly interested in weak solutions of ADAEs. Consequently, weak solutions are meant if just spoken about solutions. In [15], a uniqueness result for solutions of ADAEs is given. We further state a criterion for the existence of solutions in the case where the ADAE is trivially initialized, i.e. $x(0) = 0$ whose proof is based on Laplace transformation. The more general approach to initialization is considered in Section 3.

THEOREM 2.3 ([15], Theorem 3.1). *Let an ADAE (1.1) be given and assume that there exists $k \in \mathbb{N}$, $M, \omega \in \mathbb{R}$ such that $(sE - A)^{-1}$ exists and is bounded for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega$ with particularly $\|(sE - A)^{-1}\|_{L_b(Z, X)} \leq M(1 + |s|)^k$. Furthermore, let $x_0 = 0$ and $f(\cdot)$ with $f(\cdot) \in H^{k+1}([0, t_f], Z)$ and $f(0) = \dots = f^{(k)}(0) = 0$. Then (1.1) has a weak solution.*

3. Parameterization of the Consistent Initial Values. Taking a look at an ADAE in decoupling form (1.11) and recalling the results from finite dimensional DAEs, one could naively deduce that for the initialization, $x_2(0) \in X_2$ could be chosen arbitrarily and $x_1(0)$ is then fixed by

$$x_1(0) = - \sum_{j=0}^{\nu-1} N^j f_1^{(j)}(0) - \sum_{j=0}^{\nu-1} N^j \frac{d^j}{dt^j} K x_2(t)|_{t=0}. \quad (3.1)$$

The problem is that due to possible unboundedness of K , the second sum in the right hand side of (3.1) does not always exist and even if it makes sense, the initial value consisting of $x_1(0)$ and $x_2(0)$ may be inconsistent as the following example shows. For $\xi \in [0, t_f]$, we denote by $C_\xi \in L_b(H^1([0, t_f]), \mathbb{R})$ the evaluation at ξ , i.e. $C_\xi f = f(\xi)$. Consider the ADAE in decoupling form

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -C_1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{\partial}{\partial \xi} \\ 0 & 0 & C_0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}, \quad (3.2)$$

where - in the notation of (1.1) - the spaces X , Z and $D(A)$ are given by $X = \mathbb{R}^2 \times L_2([0, 1])$, $D(A) = \mathbb{R}^2 \times H^1([0, 1])$, $Z = \mathbb{R}^2 \times L_2([0, 1]) \times \mathbb{R}$. We assume a trivial inhomogeneity, i.e. $f_i(\cdot) \equiv 0$ for $i = 1, \dots, 4$, and take the initial value with $x_1(0) = x_3(0) = 0$ and $x_2(0) \in L_2([0, 1], X)$ defined by

$$(x_2(0))(\xi) = \begin{cases} 0: & \xi \in [0, \frac{1}{3}], \\ 1: & \xi \in (\frac{1}{3}, \frac{2}{3}), \\ 0: & \xi \in [\frac{2}{3}, 1]. \end{cases} \quad (3.3)$$

The third row of (3.2) is given by $\frac{d}{dt}x_3(t, \xi) = -\frac{\partial}{\partial \xi}x_3(t, \xi)$ and together with boundary condition $x_3(t, 0) = 0$ and initial value (3.3), it has the solution

$$x_3(t, \xi) = \begin{cases} x_3(0, \xi - t) & : \xi - t \in [0, 1], \\ 0 & : \text{else.} \end{cases}$$

Hence, we get

$$x_1(t) = \begin{cases} 0 & : t \in [0, \frac{1}{3}], \\ 1 & : t \in (\frac{1}{3}, \frac{2}{3}), \\ 0 & : t \in \mathbb{R}^{\geq \frac{2}{3}}. \end{cases} \quad (3.4)$$

Thus, for $t_f \geq \frac{1}{3}$ we have $x_1(\cdot) \in L_2([0, t_f]) \setminus H^1([0, t_f])$ and, since $x_2(\cdot)$ is the derivative of $x_1(\cdot)$, the second component of the state fulfills $x_2(\cdot) = \dot{x}_1(\cdot) \notin C([0, t_f])$. Altogether, we can conclude that (3.2) with the initial values $x_1(0) = x_2(0) = 0$ and $x_3(0)$ as in (3.3) has no weak solution in the sense of Definition 2.2.

The reason that an ADAE in decoupling form cannot be initialized in the above proposed way is that the boundary control system (1.12) may produce an output that is not smooth enough. This additional problem does not appear for ordinary finite dimensional systems where any initial value results into an infinitely often differentiable output of the free system (i.e. $f(\cdot) \equiv 0$). Indeed, if $x_3(\cdot)$ in (3.2) is initialized by a smoother function, it possesses a weak solution. An interpretation of this is that the boundary control term delivers some further relations, which can be considered as (*hidden*) *boundary conditions* in analogy to the corresponding notion for the algebraic constraints. Nevertheless, the above example has a weak solution if we study (3.2) on a time interval $[0, t_f] \subset [0, \frac{1}{3}]$. We are not separately considering this effect but consistent initialization such that a weak solution for an arbitrary time horizon is contemplated. Another conclusion that can be drawn from this example is that it is not possible to infer an inequality

$$\begin{aligned} & \left\| \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \end{bmatrix} \right\|_{\mathbb{R}^2 \times L^2([0,1])} \\ & \leq c \left(\left\| \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} \right\|_{\mathbb{R}^2 \times L^2([0,1])} + \left\| \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \\ f_3(\cdot) \\ f_4(\cdot) \end{bmatrix} \right\|_{H^2([0, t_f], \mathbb{R}^2 \times L^2([0,1]) \times \mathbb{R})} \right) \end{aligned} \quad (3.5)$$

analogous to the results in finite dimensions. One has to provide the initial value with some stronger norm, such that a uniform estimate similar to (3.5) is possible.

Before we present the result about consistent initialization, a definition from [16] is briefly reviewed. In that reference, boundary control systems (1.12) with - in our notation - $f_2(\cdot) \equiv 0$ are considered.

DEFINITION 3.1. *A boundary control system (1.12) is said to be well-posed if the following holds:*

- (i) *The operator $\mathfrak{P} : D(\mathfrak{U}) \rightarrow X_3$ is onto, $\ker \mathfrak{P}$ is dense in X_2 , $\mathfrak{P} \in L_b(D(\mathfrak{U}), X_2)$ and there exists a $\mu \in \mathbb{R}$ such that $\ker(\mu I - \mathfrak{U}) \cap \ker \mathfrak{P} = \{0\}$ and $\mu I - \mathfrak{U}$ is onto.*
- (ii) *For every $x_{2,0} \in D(\mathfrak{U}) \cap \ker \mathfrak{P}$, (1.12) with additionally $f_2(\cdot) \equiv 0$, $f_3(\cdot) \equiv 0$ possesses a unique solution $x_2(\cdot) \in C([0, t_f], D(\mathfrak{U})) \cap C^1([0, t_f], X)$ depending*

continuously on $x_{2,0} \in D(\mathfrak{U})$, i.e. there exists a constant $c > 0$ such that

$$\max_{t \in [0, t_f]} \{ \|x(t)\|_{D(\mathfrak{U})}, \|\dot{x}(t)\| \} \leq c \|x_{2,0}\|_{D(\mathfrak{U})}.$$

(iii) For all $f_2(\cdot) \in H^1([0, t_f], X_2)$ with $f_2(0) = 0$, $f_3(\cdot) \in H^1([0, t_f], X_3)$ and $x_{2,0} \in D(\mathfrak{U})$ with $\mathfrak{P}x_{2,0} = -f_3(0)$, (1.12) possesses a unique solution $x_2(\cdot) \in C([0, t_f], D(\mathfrak{U})) \cap C^1([0, t_f], X)$ depending continuously on $x_{2,0} \in D(\mathfrak{U})$ and $\dot{f}_2(\cdot) \in L_2([0, t_f], X_2)$, $\dot{f}_3(\cdot) \in L_2([0, t_f], X_3)$, i.e. there exists a constant $c > 0$ such that

$$\begin{aligned} & \max_{t \in [0, t_f]} \{ \|x(t)\|_{D(\mathfrak{U})}, \|\dot{x}(t)\| \} \\ & \leq c \left(\|x_{2,0}\|_{D(\mathfrak{U})} + \|\dot{f}_2(\cdot)\|_{L_2([0, t_f], X_2)} + \|\dot{f}_3(\cdot)\|_{L_2([0, t_f], X_3)} \right). \end{aligned}$$

(iv) The operator K satisfies $K \in L_b(D(\mathfrak{U}), Y)$ and there exists a constant $c > 0$ such that for all $x_{2,0} \in D(\mathfrak{U}) \cap \ker \mathfrak{P}$, the solution $x_2(\cdot)$ of (1.12) with the specific inhomogeneity $f_2(\cdot) \equiv 0$, $f_3(\cdot) \equiv 0$ satisfies

$$\|x_2(\cdot)\|_{L_2([0, t_f], X_2)} \leq c \|x_{2,0}\|.$$

(v) There exists a constant $c > 0$ such that for all $f_2(\cdot) \in H^2([0, t_f], X_2)$ with $f_2(0) = 0$ and $f_3(\cdot) \in H^2([0, t_f], X_3)$ with $f_3(0) = 0$, the output $y(\cdot)$ of (1.12) with $x_{2,0} = 0$ satisfies

$$\|y(\cdot)\|_{L_2([0, t_f], X_1)} \leq c \left(\|f_2(\cdot)\|_{L_2([0, t_f], X_2)} + \|f_3(\cdot)\|_{L_2([0, t_f], X_3)} \right).$$

The well-posedness of the boundary control system constructed from an ADAE in decoupling form will be an essential prerequisite in the remaining part of this work. In [14], this property was shown for ADAEs arising from the equations of electrical circuits with transmission lines.

We now present the main result on the consistent initialization of ADAEs in decoupling form.

THEOREM 3.2. *Let an ADAE in decoupling form (1.11) with index ν be given and let $k \in \{0, \dots, \nu\}$ with $N^k K = 0$. Further assume that the boundary control system (1.12) is well-posed according to Definition 3.1. Then for $f_1(\cdot)$ with $N^j f_1(\cdot) \in H^{\nu-j}([0, t_f], X_1)$ for $j = 0, \dots, \nu - 1$, $f_2(\cdot) \in H^k([0, t_f], X_2)$, $f_3(\cdot) \in H^k([0, t_f], X_3)$ and $x_{1,0} \in X_1$, $x_{2,0} \in X_2$ satisfying*

$$\begin{aligned} f_3(0) &= -\mathfrak{P}x_{2,0}, & x_{2,1} &:= f_2(0) + \mathfrak{U}x_{2,0} \in D(\mathfrak{U}), \\ \dot{f}_3(0) &= -\mathfrak{P}\dot{x}_{2,1}, & x_{2,2} &:= \dot{f}_2(0) + \mathfrak{U}\dot{x}_{2,1} \in D(\mathfrak{U}), \\ &\vdots & &\vdots \\ f_3^{(k-2)}(0) &= -\mathfrak{P}x_{2,k-2}, & x_{2,k-1} &:= f_2^{(k-2)}(0) + \mathfrak{U}x_{2,k-2} \in D(\mathfrak{U}), \\ f_3^{(k-1)}(0) &= -\mathfrak{P}\dot{x}_{2,k-1}, & x_{2,k} &:= f_2^{(k-1)}(0) + \mathfrak{U}\dot{x}_{2,k-1} \in X, \end{aligned} \tag{3.6a}$$

$$x_{1,0} = - \sum_{j=0}^{\nu-1} N^j f_1^{(j)}(0) - \sum_{j=0}^{k-1} N^j K x_{2,j}, \tag{3.6b}$$

there exists a weak solution with $x_1(0) = x_{1,0}$ and $x_2(0) = x_{2,0}$.

For the proof of Theorem 3.2, the subsequent lemma is needed. It gives conditions

for the inhomogeneity and the initial state of boundary control systems such that a smooth output is delivered.

LEMMA 3.3. *Let a well-posed boundary control system (1.12) be given with $f_2(\cdot) \in H^k([0, t_f], X_2)$, $f_3(\cdot) \in H^k([0, t_f], X_3)$ and the initial value $x_{2,0}$ fulfill (3.6a). Then the output satisfies $y(\cdot) \in H^k([0, t_f], X_1)$ and there exists a constant $c \in \mathbb{R}^+$ such that the following uniform estimate is valid for all $x_{2,k}$*

$$\|y(\cdot)\|_{H^k([0, t_f], X_1)} \leq c \left(\|f_2(\cdot)\|_{H^k([0, t_f], X_2)} + \|f_3(\cdot)\|_{H^k([0, t_f], X_3)} + \|x_{2,k}\| \right). \quad (3.7)$$

Moreover, the initial values of the first $k-1$ derivatives of the output are given by

$$y^{(j)}(0) = Kx_{2,j}. \quad (3.8)$$

Proof. We show the above statement while additionally assuming that $f_2(\cdot) \in H^{k+2}([0, t_f], X_2)$ with $f_2^{(k)}(0) = 0$, $f_3(\cdot) \in H^{k+2}([0, t_f], X_3)$ and $x_{2,k} \in D(\mathfrak{U})$ with $\mathfrak{P}x_{2,k} = -f_2^{(k)}(0)$. By continuity arguments, the more general assumptions of Lemma 3.3 follow from the denseness of $H^{k+2}([0, t_f], X_i)$ in $H^k([0, t_f], X_i)$ and $\ker \mathfrak{P}$ in X_2 . Consider the boundary control system

$$\dot{\bar{x}}_{2,k}(t) = \mathfrak{U}\bar{x}_{2,k}(t) + f_2^{(k)}(t), \quad (3.9a)$$

$$0 = \mathfrak{P}\bar{x}_{2,k}(t) + f_3^{(k)}(t), \quad (3.9b)$$

$$\bar{y}_k(t) = K\bar{x}_{2,k}, \quad (3.9c)$$

$$\bar{x}_{2,k}(0) = x_{2,k}. \quad (3.9d)$$

Integrating (3.9a), we obtain for $t \in [0, t_f]$

$$\bar{x}_{2,k}(t) - \bar{x}_{2,k}(0) = \int_0^t \mathfrak{U}\bar{x}_{2,k}(\tau) d\tau + f_2^{(k-1)}(t) - f_2^{(k-1)}(0).$$

Since $\mathfrak{U} \in L_b(D(\mathfrak{U}), X_1)$ and $x_{2,k} = f_2^{(k-1)}(0) + \mathfrak{U}x_{2,k-1}$ by definition, this equation simplifies to

$$\bar{x}_{2,k}(t) = \mathfrak{U} \left(\int_0^t \bar{x}_{2,k}(\tau) d\tau + x_{2,k-1} \right) + f_2^{(k-1)}(t).$$

Hence, $\bar{x}_{2,k-1}(t) := x_{2,k-1} + \int_0^t \bar{x}_{2,k}(\tau) d\tau$ satisfies $\dot{\bar{x}}_{2,k-1}(t) = \mathfrak{U}\bar{x}_{2,k-1}(t) + f_2^{(k)}(t)$ pointwise. Moreover, due to $\mathfrak{P} \in L_b(D(\mathfrak{U}), X_3)$, we can calculate

$$\begin{aligned} \mathfrak{P}\bar{x}_{2,k-1}(t) + f_3^{(k-1)}(t) &= \int_0^t \mathfrak{P}\bar{x}_{2,k}(\tau) d\tau + \mathfrak{P}x_{2,k-1} + f_3^{(k-1)}(t) \\ &= \int_0^t f_3^{(k)}(\tau) d\tau - f_3^{(k-1)}(0) + f_3^{(k-1)}(t) = 0. \end{aligned}$$

Defining $\bar{y}_{k-1}(t) := \int_0^t \bar{y}_k(\tau) d\tau + Kx_{2,k}$, we obtain $\dot{\bar{y}}_{k-1}(t) = \bar{y}_k(t)$. Then $K \in L_b(D(\mathfrak{U}), X_1)$ leads to $\bar{y}_{k-1}(t) = K\bar{x}_{2,k-1}(t)$. The above defined $\bar{x}_{k-1}(\cdot)$ and $\bar{y}_{k-1}(\cdot)$ therefore solve the boundary control system

$$\dot{\bar{x}}_{2,k-1}(t) = \mathfrak{U}\bar{x}_{2,k-1} + f_2^{(k-1)}(t), \quad (3.10a)$$

$$0 = \mathfrak{P}\bar{x}_{2,k-1}(t) + f_3^{(k-1)}(t), \quad (3.10b)$$

$$\bar{y}_{k-1}(t) = K\bar{x}_{2,k-1}, \quad (3.10c)$$

$$\bar{x}_{2,k-1}(0) = x_{2,k-1}. \quad (3.10d)$$

By the definition of well-posedness and the linearity of the boundary control system (3.9), there exists a constant $c > 0$ such that for all $f_2(\cdot) \in H^{k+2}([0, t_f], X_2)$, $f_3(\cdot) \in H^{k+2}([0, t_f], X_3)$, $x_{2,k} \in D(\mathfrak{A})$ with $\mathfrak{B}x_{2,k} = -f_3^{(k)}(0)$, the output $\bar{y}_k(\cdot)$ satisfies

$$\|\bar{y}_k(\cdot)\|_{L_2([0, t_f], X_1)} \leq c \left(\|x_{2,k}\| + \|f_2(\cdot)\|_{H^k([0, t_f], X_2)} + \|f_3(\cdot)\|_{H^k([0, t_f], X_3)} \right).$$

Since $\frac{d}{dt}\bar{y}_{k-1}(\cdot) = \bar{y}_k(\cdot)$, then there exists $c > 0$ such that the output of (3.10) fulfills

$$\|\bar{y}_{k-1}(\cdot)\|_{H^1([0, t_f], X_1)} \leq c \left(\|x_{2,k}\| + \|f_2(\cdot)\|_{H^k([0, t_f], X_2)} + \|f_3(\cdot)\|_{H^k([0, t_f], X_3)} \right).$$

Iterating this argumentation $k-1$ -times, we are led to the system (1.12) with $y(\cdot) = \bar{y}_0 \in H^k([0, t_f], X_1)$, $y^{(j)}(0) = \bar{y}_j(0) = Kx_{2,j}$ and an estimate

$$\|y(\cdot)\|_{H^k([0, t_f], X_1)} \leq c \left(\|x_{2,k}\| + \|f_2(\cdot)\|_{H^k([0, t_f], X_2)} + \|f_3(\cdot)\|_{H^k([0, t_f], X_3)} \right). \quad \square$$

The conditions (3.6a) also follow by formally building the first $k-1$ derivatives of (1.12) yielding

$$\begin{aligned} \dot{x}_2^{(j+1)}(t) &= \mathfrak{A}x_2^{(j)}(t) + f_2^{(j)}(t) \\ 0 &= \mathfrak{B}x_2^{(j)}(t) + f_3^{(j)}(t) \\ y^{(j)}(t) &= Kx_2^{(j)}(t) \\ x(0) &= x_{2,j} \end{aligned} \tag{3.11}$$

for $i = 0, \dots, k-1$. By a successive insertion of these relations, we obtain (3.6a). This further justifies the nomenclature of (*hidden*) *boundary constraints* for these conditions. With Lemma 3.3, we are now able to prove Theorem 3.2.

Proof. The unique candidate for the weak solution $(x_1(\cdot), x_2(\cdot))$ of (1.11), consists of $x_2(\cdot)$, the solution of (1.12) and

$$x_1(t) = - \sum_{j=0}^{k-1} N^j y^{(j)}(t) - \sum_{j=0}^{\nu-1} N^j f_1^{(j)}(t). \tag{3.12}$$

It remains to show that both $x_1(\cdot)$ and $x_2(\cdot)$ are continuous and satisfy the prescribed initial conditions. Lemma 3.3 implies that $y(\cdot) \in H^k([0, t_f], X_1)$ with $y^{(j)}(0) = Kx_{2,j}$. Additionally, $N^j f_1(\cdot) \in H^{\nu-j}([0, t_f], X_1)$ for $j = 0, \dots, \nu-1$ and (3.12) imply $x_1(\cdot) \in H^1([0, t_f], X_1)$, especially that $x_1(\cdot) \in C([0, t_f], X_1)$. It can be seen that $x_2(0)$ reads as in (3.6b), and this completes the proof. \square

Remark 3.4. By the argumentation in the above proof, we could also assume that $N^j f_1(\cdot) \in C^j([0, t_f], X_1)$ for the existence of a weak solution. In terms of better overview, this is not further specified. It was further assumed in Theorem 3.2 that $f_2(\cdot) \in H^{k,p}([0, t_f], X_2)$, $f_3(\cdot) \in H^{k,p}([0, t_f], X_3)$ for especially $p = 2$. The corresponding result for general $p \in [1, \infty]$ can be yielded by a suitable extension of the notion of well-posedness of boundary control systems. Similar problems are considered in [18]. This is also not deepened in this work. It should be further remarked that Theorem 3.2 only gives sufficient conditions for the consistency of initial values. In the example (3.2), the initial value is consistent for $t_f \leq \frac{1}{3}$ although (3.6a) is violated. However, the given conditions in Theorem 3.2 are independent of t_f , i.e. they are also sufficient for the existence of solutions on an infinite time horizon.

Briefly revisiting the example (3.2), we have - in the notation of Theorem 3.2 - that

$\mathfrak{U} = -\frac{\partial}{\partial \xi}$, $D(\mathfrak{U}) = H^1([0, 1])$, $\mathfrak{B} = C_0$ and $\nu = k = 2$. The (hidden) boundary constraints therefore read $(x_3(0))(0) = -f_4(0)$, $(f_3(0) - \frac{\partial}{\partial \xi} x_3(0))(\cdot) \in H^1([0, 1])$, $(\frac{\partial}{\partial \xi} x_3(0) - f_3(0))(0) = \dot{f}_4(0)$. In the case where the inhomogeneity would be absent, these constraints simplify to $(x_3(0))(\cdot) \in H^2([0, 1])$ with $(x_3(0))(0) = (\frac{\partial}{\partial \xi} x_3(0))(0) = 0$.

We now present the consequences of consistent initialization and perturbation analysis for abstract differential-algebraic equations (1.1) decoupled by the projector approach in the proof of Theorem 1.1. Let $x(0) = x_0$ be the initial value. According to (1.10c) and (1.10d), the condition (3.6a) on the (hidden) boundary constraints read

$$\begin{aligned}
 (I - E_\nu E_\nu^-)f(0) &= -(I - E_\nu E_\nu^-)A_\nu x_0, \\
 (I - E_\nu E_\nu^-)\dot{f}(0) &= -(I - E_\nu E_\nu^-)A_\nu x_1, \\
 &\vdots \\
 (I - E_\nu E_\nu^-)f^{(k-2)}(0) &= -(I - E_\nu E_\nu^-)A_\nu x_{k-2}, \\
 (I - E_\nu E_\nu^-)f^{(k-1)}(0) &= -(I - E_\nu E_\nu^-)A_\nu x_{k-1},
 \end{aligned} \tag{3.13a}$$

where the x_i are recursively defined by

$$\begin{aligned}
 x_1 &:= P_{\Sigma\nu} E_\nu^- f(0) + P_{\Sigma\nu} E_\nu^- A_\nu x_0 \in D(A), \\
 x_2 &:= P_{\Sigma\nu} E_\nu^- \dot{f}(0) + P_{\Sigma\nu} E_\nu^- A_\nu x_1 \in D(A), \\
 &\vdots \\
 x_{k-1} &:= P_{\Sigma\nu} E_\nu^- f^{(k-2)}(0) + P_{\Sigma\nu} E_\nu^- A_\nu x_{k-2} \in D(A), \\
 x_k &:= P_{\Sigma\nu} E_\nu^- f^{(k-1)}(0) + P_{\Sigma\nu} E_\nu^- A_\nu x_{k-1} \in X.
 \end{aligned} \tag{3.13b}$$

We now determine $Q_j x(0)$ for $j = 0, \dots, \nu - 1$, i.e. we translate the condition (3.6b) into the projector-based approach to decoupling. The assumption $N^k K = 0$ means that $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$ for $j = k, \dots, \nu - 1$. For the time being, we assume that $k = \nu$, meaning that K in (1.10b) is full. According to the transformations (1.8) and (1.9), we then have

$$\begin{aligned}
 Q_j x_0 &= Q_j Q_{j+1} \dot{x}(0) + Q_j P_{j+1} Q_{j+2} \dot{x}(0) + \dots + Q_j P_{j+1} \cdots P_{\nu-2} Q_{\nu-1} \dot{x}(0) \\
 &\quad + Q_j P_{j+1} \cdots P_{\nu-2} P_{\nu-1} E_\nu^- A_\nu x_0 + Q_j P_{j+1} \cdots P_{\nu-2} P_{\nu-1} E_\nu^- f(0).
 \end{aligned} \tag{3.14a}$$

Inserting the relations

$$\begin{aligned}
 Q_{j+k} x^{(l)}(0) &= Q_{j+k} Q_{j+i+1} x^{(l+1)}(0) + Q_{j+i} P_{j+i+1} Q_{j+i+2} x^{(l+1)}(0) + \dots \\
 &\quad + Q_{j+i} P_{j+i+1} \cdots P_{\nu-2} Q_{\nu-1} x^{(l+1)}(0) \\
 &\quad + Q_{j+i} P_{j+i+1} \cdots P_{\nu-2} P_{\nu-1} E_\nu^- A_\nu x_l \\
 &\quad + Q_{j+i} P_{j+i+1} \cdots P_{\nu-2} P_{\nu-1} E_\nu^- f^{(l)}(0)
 \end{aligned} \tag{3.14b}$$

for $l = 1, \dots, i$ leads to conditions for the consistent initial values.

Remark 3.5. We could also derive explicit representations for $Q_j x(0)$ with combinatorial methods, but the resulting expressions are very complicated and therefore hard to use for general $\nu \in \mathbb{N}$. In many examples of practical relevance like e.g. electrical circuits with transmission lines [14], the ADAE index does not exceed 2. Then the above representation simplifies to

$$\begin{aligned}
 Q_1 x_0 &= Q_1 E_2^- A_2 x_0 + Q_1 E_2^- f(0), \\
 Q_0 x_0 &= Q_0 P_1 E_2^- A_2 x(0) + Q_0 Q_1 E_2^- A_2 x_1 \\
 &\quad + Q_0 P_1 E_2^- f(0) + Q_0 Q_1 E_2^- \dot{f}(0).
 \end{aligned} \tag{3.15}$$

If we assume that $N^k K = 0$ and $N^{k-1} K \neq 0$ for an arbitrary $k \in \{0, \dots, \nu\}$, we can use (3.13) as well. The fact that x_l is not defined for $l = k+1, \dots, \nu-1$ is compensated by $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$ for $j = k, \dots, \nu-1$. Hence, the expression (3.14b) then even makes sense.

Summarizing the above considerations, we can formulate the following result.

COROLLARY 3.6. *Let an ADAE (1.1) be given for which the assumptions of Theorem 1.1 are valid and let the boundary control system (1.12) with \mathfrak{A} , \mathfrak{B} and K as in (1.10) be well-posed. Further, let $k \in \{0, \dots, \nu\}$ be such that $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$ for $j = k, \dots, \nu-1$ and $(I - E_\nu E_\nu^-) f(\cdot) \in H^k([0, t_f], Z)$, $P_{\Sigma_\nu} E_\nu^- f(\cdot) \in H^k([0, t_f], X)$, $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- f(\cdot) \in H^{\nu-j}([0, t_f], X)$. Then, for an initial value x_0 satisfying (3.13) and (3.14), the ADAE (1.1) possesses a unique solution.*

Since both the (hidden) algebraic constraints and the (hidden) boundary constraints can be obtained by a formal differentiation, this technique could be also applied to the entire system (1.1) without performing a transformation into decoupling form. Then we get a so-called *derivative array*

$$\begin{aligned} E\dot{x}(0) &= Ax(0) + f(0), \\ &\vdots \\ Ex^{(\nu+1)}(0) &= Ax^{(\nu)}(0) + f^{(\nu)}(0) \end{aligned} \tag{3.16}$$

and solving for $x(0)$. Derivative arrays are well-known in the theory of finite dimensional DAEs (see e.g. [3]). However, this procedure in general delivers more constraints than the needed ones, since the possible fact that in the decoupling form holds $N^k K = 0$ for some integer $k < \nu$ is not taken into account.

4. Perturbation Analysis. We now consider an adequate generalization of the perturbation estimates for the finite dimensional case in (1.4). In the example (3.2) we constituted that an estimate like (3.5) is not possible if $x_3(0)(\cdot)$ is provided with the Lebesgue norm. It will turn out that a similar estimate is possible if we take a Sobolev norm of $x_3(0)$ instead. In that example, we can argue by the transport equation $\frac{\partial}{\partial t} x_3(t, \xi) = -\frac{\partial}{\partial \xi} x_3(t, \xi)$ that the derivative with respect to time equals the spatial one. We first present a result about the perturbation analysis of ADAEs in decoupling form. Later on, we state the consequences for general DAEs which can be decoupled by the projector approach presented after Theorem 1.1. An essential prerequisite in the following result will be that the system is initialized with a vector fulfilling (3.6).

THEOREM 4.1. *Let an ADAE system in decoupling form (1.11) with ADAE index ν be given and assume that the boundary control system (1.12) is well-posed. Moreover, assume that $N^k K = 0$ for some $k \in \{0, \dots, \nu\}$. Then there exists a $c \in \mathbb{R}^+$ such that for all $f_1(\cdot)$ with $N^j f_1(\cdot) \in H^{\nu-j}([0, t_f], X_1)$ for $j = 0, \dots, \nu-1$, $f_2(\cdot) \in H^k([0, t_f], X_2)$, $f_3(\cdot) \in H^k([0, t_f], X_3)$ and $x_{1,0} \in X_1$, $x_{2,0} \in X_2$ satisfying (3.6), we have*

$$\begin{aligned} \left\| \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} \right\| &\leq c \left(\|x_{2,k}\| + \sum_{j=0}^{\nu-1} \|N^j f_1(\cdot)\|_{H^{j+1}([0, t_f], X_1)} \right. \\ &\quad \left. + \|f_2(\cdot)\|_{H^k([0, t_f], X_2)} + \|f_3(\cdot)\|_{H^k([0, t_f], X_3)} \right). \end{aligned} \tag{4.1}$$

Proof. As a conclusion from (3.7) in Lemma 3.3 (take the identity as output operator), we verify the estimate

$$\|x_2(t_f)\| \leq c_2 \left(\|x_{2,k}\| + \|f_2(\cdot)\|_{H^k([0,t_f],X_2)} + \|f_3(\cdot)\|_{H^k([0,t_f],X_3)} \right) \quad (4.2)$$

for some $c_2 \in \mathbb{R}^+$. Furthermore, for $y(\cdot) = Kx_2(\cdot)$, a combination of Lemma 3.3 with the boundedness of the evaluation operators, that is the existence of $\tilde{c} > 0$ with $\|y^{(j)}(t_f)\| \leq \tilde{c}\|y(\cdot)\|_{H^k([0,t_f],X_1)}$ for all $j < k$, $y(\cdot) \in H^k([0,t_f],X_1)$, we get that there exist constants \bar{c}_j with

$$\|y^{(j)}(t_f)\|_{X_1} \leq \bar{c}_j \left(\|x_{2,k}\| + \|f_2(\cdot)\|_{H^k([0,t_f],X_2)} + \|f_3(\cdot)\|_{H^k([0,t_f],X_3)} \right). \quad (4.3)$$

Using the expression (3.12) for $x_1(t)$, the triangular inequality yields

$$\|x_1(t_f)\| \leq \sum_{i=0}^{\nu-1} \left\| N^i f_1^{(i)}(t_f) \right\| + \sum_{i=0}^{k-1} \|y^{(i)}(t_f)\|.$$

A combination of this relation with (4.3) and making use of the boundedness of the evaluation operator yields the existence of $c_1 \in \mathbb{R}^+$ with

$$\|x_1(t_f)\| \leq c_1 \left(\|x_{2,k}\| + \|f_2(\cdot)\|_{H^k([0,t_f],X_2)} + \|f_3(\cdot)\|_{H^k([0,t_f],X_3)} \right). \quad (4.4)$$

Altogether, from (4.2) and (4.4), the desired result is obtained for $c = c_1 + c_2$. \square
We briefly discuss Theorem 4.1 by means of the example (3.2). An application of (4.1) leads to

$$\begin{aligned} & \left\| \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \end{bmatrix} \right\|_{\mathbb{R}^2 \times L^2([0,1])} \\ & \leq c \left(\left\| \left(\dot{f}_3(0) - \frac{\partial}{\partial \xi} f_3(0) + \frac{\partial^2}{\partial \xi^2} x_3(0) \right) (\cdot) \right\|_{L^2([0,1])} + \|f_1(\cdot)\|_{H^2([0,t_f])} \right. \\ & \quad \left. + \|f_2(\cdot)\|_{H^1([0,t_f])} + \|f_3(\cdot)\|_{H^2([0,t_f],L^2([0,1]))} + \|f_4(\cdot)\|_{H^2([0,t_f])} \right). \end{aligned} \quad (4.5)$$

In the case where the inhomogeneity vanishes, the following simpler estimate is valid

$$\left\| \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \end{bmatrix} \right\|_{\mathbb{R}^2 \times L^2([0,1])} \leq c \|x_3(0)(\cdot)\|_{H^2([0,1])}. \quad (4.6)$$

From this example, it can be drawn that $x_{2,k}$ in (4.1) may correspond to spatial derivatives. Aspects of spatial perturbation are e.g. considered in [12] for a special class of partial differential-algebraic equations. The results of this work give a more general approach in an abstract framework.

Subsequently, we present the corresponding perturbation result for general ADAEs decoupled by the projector-based transformations (1.8) and (1.9). The proof consists of an application of Theorem 4.1 to these special transformastions and is therefore omitted.

COROLLARY 4.2. *Let an ADAE (1.1) be given for which the assumptions of Theorem 1.1 are valid and let the boundary control system (1.12) with \mathfrak{U} , \mathfrak{F} and K as in*

(1.10) be well-posed. Further, let $k \in \{0, \dots, \nu\}$ be such that $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$ for $j = k, \dots, \nu - 1$. Then there exists a $c > 0$ such that for all $f(\cdot)$ with $(I - E_\nu E_\nu^-)f(\cdot) \in H^k([0, t_f], Z)$, $P_{\Sigma\nu} E_\nu^- f(\cdot) \in H^k([0, t_f], X)$, $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- f(\cdot) \in H^{j+1}([0, t_f], X)$ and an initial value x_0 satisfying (3.13) and (3.14) holds that

$$\|x(t_f)\| \leq c \left(\|x_k\| + \sum_{j=0}^{\nu-1} \|Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- f(\cdot)\|_{H^{j+1}([0, t_f], X)} + \|P_{\Sigma\nu} E_\nu^- f(\cdot)\|_{H^k([0, t_f], X)} + \|(I - E_\nu E_\nu^-)f(\cdot)\|_{H^k([0, t_f], Z)} \right). \quad (4.7)$$

In the case where the entire inhomogeneity is ν -times weakly differentiable, the following simpler estimate can be formulated

$$\|x(t_f)\| \leq c (\|x_k\| + \|f(\cdot)\|_{H^\nu([0, t_f], Z)}). \quad (4.8)$$

If additionally the ADAE possesses a complete decoupling, i.e. there exists a decoupling form with $K = 0$, then for some $c > 0$, the inequality

$$\|x(t_f)\| \leq c (\|x_0\| + \|f(\cdot)\|_{H^\nu([0, t_f], Z)}) \quad (4.9)$$

holds true.

5. Example: An Electrical Circuit with a Transmission Line . Consider the electrical circuit of Fig. 5.1 containing a transmission line of length ℓ . The voltage

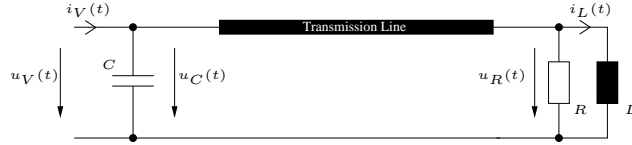


FIG. 5.1. Electrical circuit with transmission line

and current courses $V(\xi, t)$, $I(\xi, t)$ along the transmission line satisfy the telegraph equations

$$\begin{aligned} C_T \frac{\partial}{\partial t} V(\xi, t) &= -G_T I(\xi, t) - \frac{\partial}{\partial \xi} V(\xi, t), \\ L_T \frac{\partial}{\partial t} I(\xi, t) &= -\frac{\partial}{\partial \xi} I(\xi, t) - R_T V(\xi, t) \end{aligned}$$

for some constants $G_T, R_T \geq 0$, $C_T, L_T > 0$. Further, due to element relations and the Kirchhoff laws [5], we get

$$\begin{aligned} C \dot{u}_C(t) &= i_V(t) - I(0, t), & 0 &= u_C(t) - u_V(t), \\ L \dot{i}_L(t) &= u_R(t), & 0 &= R^{-1} u_R(t) + i_L(t) - I(\ell, t), \\ 0 &= u_C(t) - V(0, t), & 0 &= u_R(t) - V(\ell, t). \end{aligned}$$

This system can be rewritten as an abstract differential algebraic system (1.1) as follows. The state $x(t)$ and the inhomogeneity $f(t)$ are chosen to be

$$x(t) = \begin{bmatrix} u_C(t) \\ i_V(t) \\ i_L(t) \\ u_R(t) \\ V(t) \\ I(t) \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ -u_V(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $V(t)$ and $I(t)$ are functions describing of the voltage and current along the transmission line, i.e. $(V(t))(\xi) := V(\xi, t)$ and $(I(t))(\xi) := I(\xi, t)$. As spaces X and Z , we take $X := \mathbb{R}^4 \times (L_2([0, \ell]))^2$ $Z := \mathbb{R}^6 \times (L^2([0, \ell]))^2$ and we use the operators

$$E = \begin{bmatrix} C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_T & 0 \\ 0 & 0 & 0 & 0 & 0 & L_T \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -C_0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & R^{-1} & 0 & -C_\ell \\ 1 & 0 & 0 & 0 & -C_0 & 0 \\ 0 & 0 & 0 & 1 & -C_\ell & 0 \\ 0 & 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial \xi} \\ 0 & 0 & 0 & 0 & -\frac{\partial}{\partial \xi} & -R_T \end{bmatrix},$$

where C_p denotes the evaluation of a function at the point $p \in [0, \ell]$. The domain of A reads $D(A) = \mathbb{R}^5 \times (H([0, \ell]))^2$. It turns out that this system has ADAE index 2 and decoupling projectors are given by

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

A left inverse of $E_2 = E - AQ_0 - AP_0Q_1$ can be chosen as

$$E_2^- = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L^{-1} & -RL^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_T^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & L_T^{-1} \end{bmatrix}$$

Now using the expressions (1.10), we obtain that the inherent dynamics live in the space $\{0\}^2 \times \mathbb{R} \times \{0\} \times (L^2([0, 1]))^2$. More precisely, the inherent abstract ODE is given by

$$\frac{d}{dt} \begin{bmatrix} i_L(t) \\ V(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} -L^{-1}R & 0 & L^{-1}RC_1 \\ 0 & -\frac{G_T}{C_T} & -\frac{1}{C_T} \frac{\partial}{\partial \xi} \\ 0 & -\frac{1}{L_T} \frac{\partial}{\partial \xi} & -\frac{R_T}{L_T} \end{bmatrix} \begin{bmatrix} i_L(t) \\ V(t) \\ I(t) \end{bmatrix} \quad (5.1a)$$

with boundary conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -C_0 & 0 \\ -R & -C_\ell & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ V(t) \\ I(t) \end{bmatrix} + \begin{bmatrix} u_V(t) \\ 0 \end{bmatrix}. \quad (5.1b)$$

Moreover, we obtain the identity $Q_1 E_2^- A_2 = 0$ implying that we have a decoupling form with $NK = 0$. However, K is unbounded due to the identity

$$Q_0 P_1 E_2^- A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R & 0 & 0 & -RC_\ell \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, the preliminaries of Theorem 1.2 are violated and thus, we cannot obtain a decoupling form with $K = 0$. We will not go into the detail into the proof that the boundary control system extracted from the circuit equations is well-posed. In [14], this is shown for general electrical circuits containing transmission lines. By only sketching the proof, the transmission line and the other (finite dimensional dynamics) are first considered separately and afterwards representing the given boundary control system as the separate systems with additional input-feedback. Then an application of the results of [19] leads to the well-posedness.

We now collect the (hidden) algebraic relations. $0 = Q_1 x(t) - Q_1 E_2^- f(t)$ is equivalent to

$$0 = u_C(t) - u_V(t), \quad (5.2a)$$

whereas $Q_0 Q_1 \dot{x}(t) = Q_0 x + Q_0 P_1 E_2^- A_2 x(t) - Q_0 P_1 E_2^- f(t)$ yields

$$C \dot{u}_C(t) = i_V(t) - C_0 I(t), \quad (5.2b)$$

$$0 = u_R(t) + R i_L(t) - RC_\ell I(t). \quad (5.2c)$$

We now make use of Theorem 3.2 by giving conditions on an initial value with $u_C(0) = u_{C0}$, $i_V(0) = i_{V0}$, $i_L(0) = i_{L0}$, $u_R(0) = u_{R0}$, $V_T(0) = V_{T0}$, $I_T(0) = I_{T0}$ being consistent for an inhomogeneity containing $u_V(\cdot) \in H^2([0, t])$. For a general ADAE in decoupling form (1.11) with ADAE index 2, $NK = 0$ and an inhomogeneity satisfying $f_2(\cdot) \equiv 0$, the boundary conditions (3.6a) simplify to $x_{2,0} \in D(\mathfrak{U})$, $f_3(0) = -\mathfrak{P}x_{2,0}$. Taking a closer look at (5.1), we see that the domain of \mathfrak{U} is given by $\mathbb{R} \times H^1([0, \ell]) \times H^1([0, \ell])$ and the graph norm is equivalent to this Sobolev norm. This means that i_{L0} , V_0 and I_0 have to fulfill

$$V_0, I_0 \in H^1([0, \ell]), \quad V_0(0) = u_V(0), \quad V_0(\ell) = -R i_{L0}. \quad (5.3a)$$

The algebraic relations are then given by

$$u_{V0} = u_V(0), \quad i_{V0} = C \dot{u}_V(0) + I_0(0), \quad u_{R0} = R I_0(\ell) - R i_{L0}. \quad (5.3b)$$

Furthermore, for all $u_V(\cdot) \in H^2([0, t])$ and initial values fulfilling (5.2) and (5.3), Theorem 4.1 yields the existence of a constant $c > 0$ such that

$$\left\| \begin{bmatrix} u_C(t) \\ i_V(t) \\ i_L(t) \\ u_R(t) \\ V(t) \\ I(t) \end{bmatrix} \right\|_{\mathbb{R}^4 \times (L_2([0, \ell]))^2} < c (\|V_0(\cdot)\|_{H^1([0, \ell])} + \|I_0(\cdot)\|_{H^1([0, \ell])} + |i_{L0}| + \|u_V(\cdot)\|_{H^2([0, \ell])}). \quad (5.4)$$

6. Conclusion. In this paper we have considered the consistent initialization and perturbation analysis of differential-algebraic systems on infinite dimensional Hilbert spaces, the so-called abstract DAEs. Based on a decoupling form, we have formulated sufficient criteria for an initial value being consistent with a given inhomogeneity. The initial value has to fulfill not only algebraic relations being hidden in the system but there are some further conditions, the (hidden) boundary constraints. For systems which are consistently initialized, perturbation results are formulated. In these estimates, the initial value has been provided with some stronger norm, which has been interpreted as spatial perturbation in the given examples.

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