A SURVEY ON MODEL REDUCTION OF COUPLED SYSTEMS

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Abstract. In this paper we give an overview of model order reduction techniques for coupled systems. We consider linear time-invariant control systems that are coupled through input-output relations and discuss model reduction of such systems using moment matching and balanced truncation. Structure-preserving approaches to model order reduction of coupled systems are also presented. Numerical examples are given.

Key words. Coupled systems, model reduction, balanced truncation, moment matching

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1. Introduction. Modelling and simulation of complex physical and technical processes yield coupled systems that consist of ordinary differential equations, differential-algebraic equations and partial differential equations. Such systems arise in many practical applications including very large system integrated (VLSI) chip design and micro-electro-mechanical systems (MEMS), e.g., [10, 14, 21, 52, 58]. As the number and density of components on a single chip increase and feature sizes decrease, different physical effects such as thermal interaction, electromagnetic radiation, substrate noise and crosstalk cannot be ignored anymore. Furthermore, the design of micro- and nano-structures requires the development of new multi-physical models describing their complex internal behavior. Another application area of coupled systems is in subdomain decomposition. Partial differential equations on complicated spatial geometries may be represented as a system of partial differential equations on simpler domains coupled, for example, through boundary conditions.

As the mathematical models get more detailed and different coupling effects have to be included, the development of efficient simulation and optimization tools for large-scale coupled systems is a challenging task. Such systems consist of several subsystems whose inputs and outputs are coupled via additional algebraic relations. The subsystems usually have a high number of internal variables that leads to large memory requirements and computational complexity. To handle such large systems in simulation, control and optimization, their *model order reduction* (or *reduced-order modelling*) is indispensable. A general idea of model order reduction is to approximate a large-scale system by a reduced model of lower state space dimension that has the same behavior as the original system.

In the last years, many different model reduction methods have been developed in computational fluid dynamics, control design and electrical and mechanical engineering, see [3, 11, 47] for books on this topic. In this paper we review recent progress in dimension reduction of coupled systems. In structural dynamics, model reduction methods based on subsystem structuring have been of interest already for a long time [16, 35, 43]. Here, we will not consider these method, but will rather focus on general concepts of model reduction of coupled systems developed in [42, 53, 60].

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This paper is organized as follows. In Section 2 we introduce linear time-invariant coupled systems and give their closed-loop formulation. Section 3 deals with model order reduction of coupled systems. To make the paper self-contained, we briefly review model reduction techniques of balanced truncation and moment matching approximation. Furthermore, we report two approaches for reduced-order modelling of coupled systems based on the reduction of closed-loop systems (Section 3.1) and on structure-preserving model reduction (Section 3.2). The discussion of the advantages and disadvantages of these approaches is presented in Section 3.3. Finally, in Section 4 we consider some numerical examples.

2. Coupled systems. Consider a system of k coupled linear time-invariant generalized state space subsystems in the first-order form

$$\begin{aligned} E_j\dot{x}_j(t) &= A_jx_j(t) + B_ju_j(t), \\ y_j(t) &= C_jx_j(t), \end{aligned} \tag{2.1}$$

or in the second-order form

$$\begin{aligned} M_j\ddot{x}_j(t) + D_j\,\dot{x}_j(t) + S_j\,x_j(t) &= B_ju_j(t), \\ C_{j2}\dot{x}_j(t) + C_{j1}x_j(t) &= y_j(t), \end{aligned} \tag{2.2}$$

that are coupled through the relations

$$u_j(t) = K_{j1}y_1(t) + \ldots + K_{jk}y_k(t) + H_ju(t), \quad j = 1,\ldots,k, \tag{2.3}$$

$$y(t) = R_1\,y_1(t) + \ldots + R_k\,y_k(t). \tag{2.4}$$

Here $E_j, A_j, M_j, D_j, S_j \in \mathbb{R}^{n_j,n_j}$, $B_j \in \mathbb{R}^{n_j,m_j}$, $C_j, C_{j1}, C_{j2} \in \mathbb{R}^{p_j,n_j}$, $x_j(t) \in \mathbb{R}^{n_j}$ are internal state vectors, $u_j(t) \in \mathbb{R}^{m_j}$ are internal inputs and $y_j(t) \in \mathbb{R}^{p_j}$ are internal outputs. Furthermore, $K_{jl} \in \mathbb{R}^{m_j,p_l}$, $H_j \in \mathbb{R}^{m_j,m}$, $R_j \in \mathbb{R}^{p,p_j}$, $u(t) \in \mathbb{R}^m$ is an external input and $y(t) \in \mathbb{R}^p$ is an external output. Coupled systems of the form (2.1)–(2.4) are also known as *interconnected* or *composite systems*. The first-order systems of the form (2.1) arise in simulation of linear RCL circuits that consist of resistors, capacitors, inductors, voltage and current sources only [2, 34, 62]. In this case the components of the state vector $x_j(t)$ are the nodal voltages, the inductor currents and the currents through the voltage sources, $u_j(t)$ contains the currents and voltages of the current and voltage sources, respectively, and $y_j(t)$ consists of the voltages across the current sources and the currents through the voltage sources. The linear RCL circuits are often used to model the interconnections of VLSI networks. They can also be described by the second-order systems (2.2), where $x_j(t)$ consists of the nodal voltages only. Systems of the form (2.2) appear also in mechanical and structural dynamics. In this case, $x_j(t)$ is the displacement vector and $u_j(t)$ is the acting force. Furthermore, systems (2.1) and (2.2) arise from spatial discretization of instationary linear partial differential equations that describe, for example, heat transfer, vibrations, electromagnetic radiation or fluid flow.

Since the second-order system (2.2) can be rewritten as an equivalent first-order system of the form (2.1), in the following we will consider the coupled system (2.1), (2.3), (2.4) only adding comments on (2.2) if necessary. The matrices E_j in (2.1) may be singular, but we will assume that the pencils $\lambda E_j - A_j$ are *regular*, i.e., $\det(\lambda E_j - A_j) \not\equiv 0$ for $j = 1,\ldots,k$. In this case we can consider the transfer functions of (2.1) given by $\boldsymbol{G}_j(s) = C_j(sE_j - A_j)^{-1}B_j$. If $E_jx_j(0) = 0$, then applying the Laplace transform to (2.1), we find that $\boldsymbol{y}_j(s) = \boldsymbol{G}_j(s)\boldsymbol{u}_j(s)$, where $\boldsymbol{y}_j(s)$ and $\boldsymbol{u}_j(s)$

are the Laplace transforms of $y_j(t)$ and $u_j(t)$. Thus, $\boldsymbol{G}_j(s)$ describes the input-output relation of system (2.1) in the frequency domain.

The transfer function $\boldsymbol{G}_j(s)$ is called *proper* if $\lim_{s\to\infty} \boldsymbol{G}_j(s) < \infty$, and *improper*, otherwise. System (2.1) is *asymptotically stable* if the pencil $\lambda E_j - A_j$ is *stable*, i.e., all its finite eigenvalues have negative real part. The transfer function $\boldsymbol{G}_j(s)$ of (2.1) is called *stable* if it has no poles in the closed right half-plane. Clearly, the asymptotically stable system (2.1) has the stable transfer function $\boldsymbol{G}_j(s)$. Note that the stability of $\boldsymbol{G}_j(s)$ does not, in general, imply that $\lambda E_j - A_j$ is stable. However, for any stable transfer function $\boldsymbol{G}_j(s)$ one can find a generalized state space representation (2.1) such that $\boldsymbol{G}_j(s) = C_j(sE_j - A_j)^{-1}B_j$ and $\lambda E_j - A_j$ is stable, see [36]. Let $\mathbb{H}_\infty$ be the space of all proper and stable rational transfer functions. We provide this space with the $\mathbb{H}_\infty$-*norm* defined for $\boldsymbol{G} \in \mathbb{H}_\infty$ by

$$\|\boldsymbol{G}\|_{\mathbb{H}_\infty} := \sup_{\Re e(s)>0} \|\boldsymbol{G}(s)\|_2 = \sup_{\omega\in\mathbb{R}} \|\boldsymbol{G}(i\omega)\|_2,$$

where $\|\cdot\|_2$ denotes the matrix spectral norm.

Let $n = n_1 + \ldots + n_k$, $p_0 = p_1 + \ldots + p_k$ and $m_0 = m_1 + \ldots + m_k$. Consider the coupling block matrices

$$R = [\, R_1,\, \ldots,\, R_k \,] \in \mathbb{R}^{p,p_0}, \qquad H = [\, H_1^T,\, \ldots,\, H_k^T \,]^T \in \mathbb{R}^{m_0,m},$$

and $K = [K_{j,l}]_{j,l=1}^k \in \mathbb{R}^{m_0,p_0}$ together with the block diagonal matrices

$$\begin{aligned} E &= \operatorname{diag}(E_1, \ldots, E_k) \in \mathbb{R}^{n,n}, & A &= \operatorname{diag}(A_1, \ldots, A_k) \in \mathbb{R}^{n,n}, \\ B &= \operatorname{diag}(B_1, \ldots, B_k) \in \mathbb{R}^{n,m_0}, & C &= \operatorname{diag}(C_1, \ldots, C_k) \in \mathbb{R}^{p_0,n}. \end{aligned} \tag{2.5}$$

Let $\boldsymbol{G}(s) = C(sE - A)^{-1}B = \operatorname{diag}\big(\boldsymbol{G}_1(s), \ldots, \boldsymbol{G}_k(s)\big)$. If $I - \boldsymbol{G}(s)K$ is invertible, then the input-output relation of the coupled system (2.1), (2.3), (2.4) can be written as $\boldsymbol{y}(s) = \boldsymbol{\mathcal{G}}(s)\boldsymbol{u}(s)$, where $\boldsymbol{y}(s)$ and $\boldsymbol{u}(s)$ are the Laplace transforms of the external output $y(t)$ and the external input $u(t)$, respectively, and the closed-loop transfer function $\boldsymbol{\mathcal{G}}(s)$ has the form

$$\boldsymbol{\mathcal{G}}(s) = R\big(I - \boldsymbol{G}(s)K\big)^{-1}\boldsymbol{G}(s)H = R\,\boldsymbol{G}(s)\big(I - K\boldsymbol{G}(s)\big)^{-1}H. \tag{2.6}$$

A generalized state space realization of $\boldsymbol{\mathcal{G}}(s)$ is given by

$$\begin{aligned} \mathcal{E}\,\dot{x}(t) &= \mathcal{A}\,x(t) + \mathcal{B}\,u(t), \\ y(t) &= \mathcal{C}\,x(t), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \mathcal{E} &= E \in \mathbb{R}^{n,n}, & \mathcal{A} &= A + BKC \in \mathbb{R}^{n,n}, \\ \mathcal{B} &= BH \in \mathbb{R}^{n,m}, & \mathcal{C} &= RC \in \mathbb{R}^{p,n}. \end{aligned} \tag{2.8}$$

Note that $I - \boldsymbol{G}(s)K$ is invertible if and only if the pencil $\lambda E - A - BKC$ is regular. Moreover, if $\boldsymbol{G}(s)$ and $(I - \boldsymbol{G}(s)K)^{-1}$ are proper, then the coupled system (2.1), (2.3), (2.4) is well-posed in the sense that the closed-loop transfer function $\boldsymbol{\mathcal{G}}(s)$ exists and it is proper. In a schematic way, an example of a coupled system is shown in Figure 2.1.

The model reduction problem for the coupled system (2.1), (2.3), (2.4) consists in an approximation of the global mapping from the external input $u(t)$ to the external output $y(t)$. In other words, we want to find a reduced-order model

$$\begin{aligned} \widetilde{\mathcal{E}}\,\dot{\widetilde{x}}(t) &= \widetilde{\mathcal{A}}\,\widetilde{x}(t) + \widetilde{\mathcal{B}}\,u(t), \\ y(t) &= \widetilde{\mathcal{C}}\,\widetilde{x}(t), \end{aligned} \tag{2.9}$$

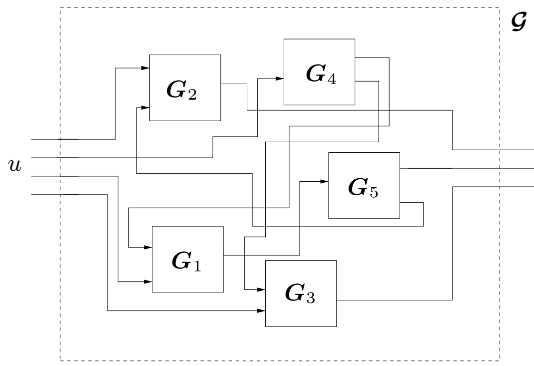


FIG. 2.1. *Coupled system.*

with $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{A}} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{\mathcal{B}} \in \mathbb{R}^{\ell,m}$, $\widetilde{\mathcal{C}} \in \mathbb{R}^{\ell,n}$ and $\ell \ll n$ that approximates the closed-loop system (2.7). In the frequency domain, the model reduction problem can be reformulated as follows: for given $\boldsymbol{\mathcal{G}}(s) = \mathcal{C}(s\,\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$, find an approximation $\widetilde{\boldsymbol{\mathcal{G}}}(s) = \widetilde{\mathcal{C}}(s\,\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}})^{-1}\widetilde{\mathcal{B}}$ such that $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{A}} \in \mathbb{R}^{\ell,\ell}$ and $\|\,\widetilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{\mathcal{G}}\,\|$ is small in some system norm. For instance, the approximation error can be estimated in the $\mathbb{H}_\infty$-norm. Apart from having a small state space dimension ℓ, it is also required that the reduced-order system (2.9) preserves essential properties of (2.7) like stability and passivity. Note that passivity, in general, means that the system does not produce energy and it is important system property, especially in circuit design [2].

3. Model reduction approaches for coupled systems. There exist two main possibilities for model order reduction of coupled systems. The first approach is to consider all subsystems together in the closed-loop form (2.7) and to compute the reduced-order system (2.9) by applying any model reduction method to (2.7). The second approach consists in replacing subsystems (2.1) by reduced-order models that are coupled then through the same interconnection relations. In this section we discuss these two approaches in more detail and mention their advantages and disadvantages.

3.1. Model reduction of the closed-loop system. Most of the model reduction methods for linear time-invariant dynamical systems are based on the projection of the system onto lower dimensional subspaces. Using these methods for the closed-loop system (2.7), we can compute the reduced-order model (2.9) by projection

$$\widetilde{\mathcal{E}} = \mathcal{W}^T\mathcal{E}\,\mathcal{T}, \qquad \widetilde{\mathcal{A}} = \mathcal{W}^T\mathcal{A}\,\mathcal{T}, \qquad \widetilde{\mathcal{B}} = \mathcal{W}^T\mathcal{B}, \qquad \widetilde{\mathcal{C}} = \mathcal{C}\,\mathcal{T}, \tag{3.1}$$

where the projection matrices $\mathcal{W}$, $\mathcal{T} \in \mathbb{R}^{n,\ell}$ determine the subspaces of interest. For example, in modal model reduction the columns of $\mathcal{W}$ and $\mathcal{T}$ span, respectively, the left and right deflating subspaces of the pencil $\lambda\,\mathcal{E} - \mathcal{A}$ corresponding to the dominant eigenvalues [18, 44]. Balanced truncation model reduction is based on the projection of system (2.7) onto the subspaces corresponding to the dominant Hankel singular values of (2.7), see [46, 54]. In the moment matching approximation, one choose the projection matrices $\mathcal{W}$ and $\mathcal{T}$ whose columns form the bases of certain Krylov subspaces associated with (2.7), e.g., [5, 22]. In the next subsections we briefly describe balanced truncation and moment matching methods.

3.1.1. Balanced truncation. One of the most studied model reduction techniques is balanced truncation, an approach firstly proposed for standard state space systems in [19, 27, 46, 54] and then extended to generalized state space systems in [45, 50, 56]. An important property of balanced truncation model reduction methods is that stability is preserved in the reduced-order system. Moreover, the existence of computable error bounds allows an adaptive choice of the state space dimension ℓ of the approximate model. A disadvantage of these methods is that (generalized) Lyapunov equations have to be solved. However, recent results on low rank approximations to the solutions of matrix equations [9, 12, 29, 33, 40, 49] make the balanced truncation model reduction approach attractive for large-scale problems.

Consider the closed-loop system (2.7) with the stable pencil $\lambda\mathcal{E} - \mathcal{A}$. For simplicity, we will assume that the matrix $\mathcal{E}$ is nonsingular. However, all results of this subsection can also be extended for systems with singular $\mathcal{E}$, see [45, 56] for details. The balanced truncation model reduction method is closely related to the *controllability Gramians* $\mathcal{P}$ and the *observability Gramian* $\mathcal{Q}$ that are unique symmetric, positive semidefinite solutions of the generalized Lyapunov equations

$$\mathcal{E}\,\mathcal{P}\mathcal{A}^T + \mathcal{A}\,\mathcal{P}\,\mathcal{E}^T = -\mathcal{B}\,\mathcal{B}^T, \tag{3.2}$$

$$\mathcal{E}^T\mathcal{Q}\,\mathcal{A} + \mathcal{A}^T\mathcal{Q}\,\mathcal{E} = -\mathcal{C}^T\mathcal{C}\,. \tag{3.3}$$

The matrix $\mathcal{P}\,\mathcal{E}^T\mathcal{Q}\,\mathcal{E}$ has nonnegative eigenvalues, and the square roots of these eigenvalues $\sigma_j = \sqrt{\lambda_j(\mathcal{P}\,\mathcal{E}^T\mathcal{Q}\,\mathcal{E})}$ define the *Hankel singular values* of system (2.7). We will assume that σ_j are ordered decreasingly. System (2.7) is called *balanced* if $\mathcal{P} = \mathcal{Q} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. The Hankel singular values characterize the 'importance' of state variables in (2.7). States of the balanced system corresponding to the small Hankel singular values are difficult to reach and to observe at the same time. Such states are less involved in the energy transfer from inputs to outputs, and, therefore, they can be truncated without changing the system properties significantly [46]. Thus, a general idea of balanced truncation is to transform system (2.7) into a balanced form and to truncate the states that correspond to the small Hankel singular values. In practice, balancing and truncation can be combined by projecting system (2.7) onto the dominant subspaces of the matrix $\mathcal{P}\,\mathcal{E}^T\mathcal{Q}\,\mathcal{E}$. This can be done in a numerically efficient way using the following algorithm that is an obvious generalization of the square root method [39, 59].

ALGORITHM 3.1. *Generalized square root balanced truncation method.*
Given system (2.7) *with the transfer function* $\mathcal{G}(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$, *compute the reduced-order system* (2.9).

1. *Compute the Cholesky factors* $L_\mathcal{P}$ *and* $L_\mathcal{Q}$ *of the Gramians* $\mathcal{P} = L_\mathcal{P}L_\mathcal{P}^T$ *and* $\mathcal{Q} = L_\mathcal{Q}L_\mathcal{Q}^T$, *that satisfy the generalized Lyapunov equations* (3.2) *and* (3.3).
2. *Compute the singular value decomposition*
$$L_\mathcal{P}^T\,\mathcal{E}^T L_\mathcal{Q} = [\,U_1,\,U_2\,]\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}[\,V_1,\,V_2\,]^T\,,$$
where $[\,U_1, U_2\,]$ *and* $[\,V_1, V_2\,]$ *have orthonormal columns,* $\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_\ell)$ *and* $\Sigma_2 = \mathrm{diag}(\sigma_{\ell+1}, \ldots, \sigma_r)$ *with* $r = \mathrm{rank}(L_\mathcal{P}^T\,\mathcal{E}^T L_\mathcal{Q})$.
3. *Compute the reduced system* (2.9) *with* $\widetilde{\mathcal{E}} = \mathcal{W}^T\mathcal{E}\,\mathcal{T}$, $\widetilde{\mathcal{A}} = \mathcal{W}^T\mathcal{A}\,\mathcal{T}$, $\widetilde{\mathcal{B}} = \mathcal{W}^T\mathcal{B}$ *and* $\widetilde{\mathcal{C}} = \mathcal{C}\,\mathcal{T}$, *where* $\mathcal{W} = L_\mathcal{Q}V_1\Sigma_1^{-1/2}$ *and* $\mathcal{T} = L_\mathcal{P}U_1\Sigma_1^{-1/2}$.

One can show that the reduced-order system $\widetilde{\mathcal{G}}(s) = \widetilde{\mathcal{C}}(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}})^{-1}\widetilde{\mathcal{B}}$ computed by this algorithm is stable and the $\mathbb{H}_\infty$-norm error bound

$$\|\,\widetilde{\mathcal{G}} - \mathcal{G}\,\|_{\mathbb{H}_\infty} \leq 2(\sigma_{\ell+1} + \ldots + \sigma_r)$$

holds, where $\sigma_{\ell+1}, \ldots, \sigma_r$ are the truncated Hankel singular values of system (2.7), see [19, 27, 51]. To solve the large-scale generalized Lyapunov equations (3.2) and (3.3) for the Cholesky factors without forming the Gramians $\mathcal{P}$ and $\mathcal{Q}$ explicitly, we can use the ADI method [40, 49], the cyclic Smith method [33, 49] or the sign function method [9, 12].

Apart from the balanced truncation method considered here, other balancing-based model reduction techniques have been developed, see [32, 47]. These are LQG balancing, stochastic balancing, positive real balancing and bounded real balancing. All these techniques are related to algebraic Riccati equations and aim to capture specific system properties like closed-loop performance, minimum phase property, passivity and $\mathbb{H}_\infty$-gain.

3.1.2. Moment matching approximation. An alternative model reduction approach for linear time-invariant systems is a moment matching approximation based on Krylov subspace methods, see [3, 5, 22] for recent surveys on these methods. Suppose that $s_0 \in \mathbb{C}$ is not an eigenvalue of the pencil $\lambda\mathcal{E} - \mathcal{A}$. Then the matrix $\mathcal{A} - s_0\mathcal{E}$ is nonsingular, and the transfer function $\boldsymbol{\mathcal{G}}(s) = \mathcal{C}(s\,\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ of the closed-loop system (2.7) can be expanded into a Taylor series at s_0 as

$$\begin{aligned}\boldsymbol{\mathcal{G}}(s) &= -\mathcal{C}\big(I - (s-s_0)(\mathcal{A} - s_0\mathcal{E})^{-1}\mathcal{E}\big)^{-1}(\mathcal{A} - s_0\mathcal{E})^{-1}\mathcal{B}\\ &= \mathcal{M}_0 + \mathcal{M}_1(s-s_0) + \mathcal{M}_2(s-s_0)^2 + \ldots\,,\end{aligned}$$

where the matrices

$$\mathcal{M}_j = -\mathcal{C}\left((\mathcal{A} - s_0\mathcal{E})^{-1}\mathcal{E}\right)^j(\mathcal{A} - s_0\mathcal{E})^{-1}\mathcal{B}$$

are called the *moments* of system (2.7) at the expansion point s_0. The moment matching approximation problem consists in determining a reduced-order system (2.9) whose the transfer function $\widetilde{\boldsymbol{\mathcal{G}}}(s) = \widetilde{\mathcal{C}}(s\,\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}})^{-1}\widetilde{\mathcal{B}}$ has the Taylor series expansion at s_0 of the form

$$\widetilde{\boldsymbol{\mathcal{G}}}(s) = \widetilde{\mathcal{M}}_0 + \widetilde{\mathcal{M}}_1(s-s_0) + \widetilde{\mathcal{M}}_2(s-s_0)^2 + \ldots\,, \tag{3.4}$$

where the moments $\widetilde{\mathcal{M}}_j$ satisfy the moment matching conditions

$$\mathcal{M}_j = \widetilde{\mathcal{M}}_j, \qquad j = 0, 1, \ldots, q. \tag{3.5}$$

For $s_0 = 0$ and q as large as possible, the approximation (3.4), (3.5) coincides with the matrix *Padé approximation* of $\boldsymbol{\mathcal{G}}(s)$, e.g., [8]. For an arbitrary complex number $s_0 \neq 0$, the moment matching approximation is the problem of *rational interpolation* [1]. Besides a single interpolation point, it is also possible to construct a reduced-order system with the transfer function $\widetilde{\boldsymbol{\mathcal{G}}}(s)$ that matches $\boldsymbol{\mathcal{G}}(s)$ at multiple points $\{s_0, s_1, \ldots, s_l\}$. Such an approximation is called a *multi-point rational interpolant* and has been studied in [25, 30]. Furthermore, one can consider the Laurent expansion of $\boldsymbol{\mathcal{G}}(s)$ at $s_0 = \infty$ given by

$$\boldsymbol{\mathcal{G}}(s) = \mathcal{M}_{-d+1}s^{d-1} + \ldots + \mathcal{M}_{-1}s + \mathcal{M}_0 + \mathcal{M}_1 s^{-1} + \mathcal{M}_2 s^{-2} + \ldots\,,$$

where the coefficients $\mathcal{M}_j$ are known as *Markov parameters* of system (2.7). In this case, computing the approximation

$$\widetilde{\boldsymbol{\mathcal{G}}}(s) = \widetilde{\mathcal{M}}_{-d+1}s^{d-1} + \ldots + \widetilde{\mathcal{M}}_{-1}s + \widetilde{\mathcal{M}}_0 + \widetilde{\mathcal{M}}_1 s^{-1} + \widetilde{\mathcal{M}}_2 s^{-2} + \ldots$$

with $\mathcal{M}_j = \widetilde{\mathcal{M}}_j$ for $j = -(d-1), \ldots, -1, 0, 1, \ldots, q$ reduces to the *partial realization* problem [13, 28].

In order to determine the reduced-order system (2.9) satisfying the moment matching condition (3.5), the explicit computation of the moments can be avoided by using the following connection between the Padé (or Padé-type) approximation and the right and left block Krylov subspaces

$$\mathcal{K}_{q_r}(\,\mathcal{A}_0^{-1}\mathcal{E},\ \mathcal{A}_0^{-1}\mathcal{B}\,) = \mathrm{Im}\,[\,\mathcal{A}_0^{-1}\mathcal{B},\ \ \mathcal{A}_0^{-1}\mathcal{E}\mathcal{A}_0^{-1}\mathcal{B},\ \ \ldots\ ,\ (\mathcal{A}_0^{-1}\mathcal{E})^{q_r-1}\mathcal{A}_0^{-1}\mathcal{B}\ ],$$
$$\mathcal{K}_{q_l}(\mathcal{A}_0^{-T}\mathcal{E}^T, \mathcal{A}_0^{-T}\mathcal{C}^T) = \mathrm{Im}\,[\,\mathcal{A}_0^{-T}\mathcal{C}^T,\ \mathcal{A}_0^{-T}\mathcal{E}^T\mathcal{A}_0^{-T}\mathcal{C}^T,\ \ldots\ ,\ (\mathcal{A}_0^{-T}\mathcal{E}^T)^{q_l-1}\mathcal{A}_0^{-T}\mathcal{C}^T],$$

with $\mathcal{A}_0 = \mathcal{A} - s_0\mathcal{E}$.

THEOREM 3.1. [26, 30] *Consider the closed-loop system* (2.7) *and the reduced-order system* (2.9), (3.1) *with some projection matrices $\mathcal{W}$, $\mathcal{T} \in \mathbb{R}^{n,\ell}$. Let $s_0 \in \mathbb{C}$ be not an eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$ and $\lambda\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$, and let $\mathcal{M}_j$ and $\widetilde{\mathcal{M}}_j$ be the moments of systems* (2.7) *and* (2.9), (3.1), *respectively.*

1. *If $\mathcal{K}_{q_r}(\mathcal{A}_0^{-1}\mathcal{E}, \mathcal{A}_0^{-1}\mathcal{B}) \subseteq \mathrm{Im}\,\mathcal{T}$ and $\mathcal{W} = \mathcal{T}$, then $\mathcal{M}_j = \widetilde{\mathcal{M}}_j$ for $j = 0, \ldots, q_r-1$.*
2. *If $\mathcal{K}_{q_r}(\mathcal{A}_0^{-1}\mathcal{E}, \mathcal{A}_0^{-1}\mathcal{B}) \subseteq \mathrm{Im}\,\mathcal{T}$ and $\mathcal{K}_{q_l}(\mathcal{A}_0^{-T}\mathcal{E}^T, \mathcal{A}_0^{-T}\mathcal{C}^T) \subseteq \mathrm{Im}\,\mathcal{W}$, then $\mathcal{M}_j = \widetilde{\mathcal{M}}_j$ for $j = 0, \ldots, q_r + q_l - 1$.*

This theorem proposes to take the projection matrices $\mathcal{T}$ and $\mathcal{W}$ as the bases of the Krylov subspaces $\mathcal{K}_{q_r}(\mathcal{A}_0^{-1}\mathcal{E}, \mathcal{A}_0^{-1}\mathcal{B})$ and $\mathcal{K}_{q_l}(\mathcal{A}_0^{-T}\mathcal{E}^T, \mathcal{A}_0^{-T}\mathcal{C}^T)$, respectively. Such bases can be efficiently computed by Lanczos or Arnoldi process [5, 20, 25] in the single-input single-output case and Lanczos- or Arnoldi-type methods [22, 24, 30, 48] in the multi-input multi-output case.

While the Krylov-based moment matching methods are efficient for very large sparse problems, the reduced-order systems computed by these methods have only local good approximation properties. So far, there is no global error bound, see [5, 6, 31] for recent contributions on this topic. The location of the interpolation points strongly influences the approximation quality. The optimal choice of these points remains an open problem. Another drawback of the moment matching methods is that stability and passivity are not necessarily preserved in the resulting reduced-order model, so that usually a post-processing is needed to guarantee these properties. Recently, passivity-preserving model reduction methods based on Krylov subspaces have been developed for standard state space systems [4, 55] and also for structured generalized state space systems arising in circuit simulation [23, 24, 38, 48].

3.2. Structure-preserving model reduction. Model order reduction of the closed-loop system (2.7) does not preserve the interconnection structure in the approximate system (2.9). Although many different model reduction methods have been developed for linear dynamical systems, *structure-preserving reduced-order modelling* of coupled systems only recently has received an attention [42, 43, 53, 60]. Instead of reduction of the entire system (2.7), one can replace each subsystem (2.1), or some of them, by a reduced-order model

$$\begin{aligned} \widetilde{E}_j\dot{\widetilde{x}}_j(t) &= \widetilde{A}_j\widetilde{x}_j(t) + \widetilde{B}_j\widetilde{u}_j(t), \\ \widetilde{y}_j(t) &= \widetilde{C}_j\widetilde{x}_j(t), \end{aligned} \tag{3.6}$$

where $\widetilde{E}_j,\ \widetilde{A}_j \in \mathbb{R}^{\ell_j,\ell_j}$, $\widetilde{B}_j \in \mathbb{R}^{\ell_j,m_j}$, $\widetilde{C}_j \in \mathbb{R}^{p_j,\ell_j}$ with $\ell_j \ll n_j$, and then to couple these subsystems through the same interconnection relations

$$\widetilde{u}_j(t) = K_{j1}\widetilde{y}_1(t) + \ldots + K_{jk}\widetilde{y}_k(t) + H_ju(t), \quad j = 1, \ldots, k, \tag{3.7}$$
$$\widetilde{y}(t) = R_1\,\widetilde{y}_1(t) + \ldots + R_k\,\widetilde{y}_k(t). \tag{3.8}$$

Note that since the internal outputs $y_j(t)$ are replaced by the approximate outputs $\widetilde{y}_j(t)$, due to (3.7), the internal inputs $u_j(t)$ in (3.6) should also be replaced by the approximate inputs $\widetilde{u}_j(t)$. Let

$$\widetilde{E} = \mathrm{diag}(\widetilde{E}_1, \ldots, \widetilde{E}_k), \qquad \widetilde{A} = \mathrm{diag}(\widetilde{A}_1, \ldots, \widetilde{A}_k),$$
$$\widetilde{B} = \mathrm{diag}(\widetilde{B}_1, \ldots, \widetilde{B}_k), \qquad \widetilde{C} = \mathrm{diag}(\widetilde{C}_1, \ldots, \widetilde{C}_k).$$

If the reduced-order pencils $\lambda\widetilde{E}-\widetilde{A}$ and $\lambda\widetilde{E}-\widetilde{A}-\widetilde{B}K\widetilde{C}$ are regular, then the reduced-order closed-loop system has the form (2.9) with

$$\widetilde{\mathcal{E}} = \widetilde{E}, \qquad \widetilde{\mathcal{A}} = \widetilde{A} + \widetilde{B}K\widetilde{C}, \qquad \widetilde{\mathcal{B}} = \widetilde{B}H, \qquad \widetilde{\mathcal{C}} = R\widetilde{C}. \tag{3.9}$$

The transfer function of this system is given by

$$\widetilde{\boldsymbol{\mathcal{G}}}(s) = R\big(I - \widetilde{\boldsymbol{G}}(s)K\big)^{-1}\widetilde{\boldsymbol{G}}(s)H = R\,\widetilde{\boldsymbol{G}}(s)\big(I - K\widetilde{\boldsymbol{G}}(s)\big)^{-1}H,$$

where $\widetilde{\boldsymbol{G}}(s) = \mathrm{diag}(\widetilde{\boldsymbol{G}}_1(s), \ldots, \widetilde{\boldsymbol{G}}_k(s))$ with $\widetilde{\boldsymbol{G}}_j(s) = \widetilde{C}_j(s\,\widetilde{E}_j - \widetilde{A}_j)^{-1}\widetilde{B}_j$, $j = 1, \ldots, k$.

The reduced-order subsystems (3.6) can be computed by projection

$$\widetilde{E}_j = W_j^T E_j T_j, \qquad \widetilde{A}_j = W_j^T A_j T_j, \qquad \widetilde{B}_j = W_j^T B_j, \qquad \widetilde{C}_j = C_j T_j, \tag{3.10}$$

where the projection matrices $W_j,\ T_j \in \mathbb{R}^{\ell_j, \ell_j}$ are determined for every subsystem either independently or using interconnection structure as it was proposed in [42, 60]. Note that in this case the matrix coefficients of the reduced-order system (2.9) have the form (3.1) with the block diagonal projection matrices

$$\mathcal{W} = \mathrm{diag}(W_1, \ldots, W_k), \qquad \mathcal{T} = \mathrm{diag}(T_1, \ldots, T_k). \tag{3.11}$$

The following theorem gives a bound on the $\mathbb{H}_\infty$-norm of the error $\widetilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{\mathcal{G}}$. For the time being, we assume that all the subsystems are asymptotically stable.

THEOREM 3.2. *Consider the coupled system* (2.1)–(2.4) *with asymptotically stable subsystems and consider the reduced-order coupled system* (3.6)–(3.8). *Let*

$$\mathit{\Pi}_l = \mathrm{diag}(\xi_1 I_{p_1}, \ldots, \xi_k I_{p_k}), \qquad \mathit{\Pi}_r = \mathrm{diag}(\xi_1 I_{m_1}, \ldots, \xi_k I_{m_k}),$$

where $\xi_j = 1$ *if* $\widetilde{\boldsymbol{G}}_j \neq \boldsymbol{G}_j$ *and* $\xi_j = 0$*, otherwise. Let*

$$g_1 = \|\mathit{\Pi}_r K(I - \boldsymbol{G}K)^{-1}\|_{\mathbb{H}_\infty}, \qquad g_2 = \|R(I - \boldsymbol{G}K)^{-1}\mathit{\Pi}_l\|_{\mathbb{H}_\infty},$$
$$g_3 = \|(I - K\boldsymbol{G})^{-1}K\mathit{\Pi}_l\|_{\mathbb{H}_\infty}, \qquad g_4 = \|\mathit{\Pi}_r(I - K\boldsymbol{G})^{-1}H\|_{\mathbb{H}_\infty}.$$

If

$$2\max\{g_1, g_3\}\max_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty} < 1, \tag{3.12}$$

then the absolute error $\widetilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{\mathcal{G}}$ *is bounded as*

$$\|\,\widetilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{\mathcal{G}}\,\|_{\mathbb{H}_\infty} \le \min\{c_1, c_2\}\max_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty}, \tag{3.13}$$

where $c_1 = 2g_2(\|H\|_2 + g_1\|\boldsymbol{G}H\|_{\mathbb{H}_\infty})$ *and* $c_2 = 2g_4(\|R\|_2 + g_3\|R\,\boldsymbol{G}\|_{\mathbb{H}_\infty})$.

Proof. The result immediately follows from [53, Theorem 3.1]. □

Note that Theorem 3.2 provides not only the approximation error bounds but also gives sufficient criteria for the stability of the reduced-order system. Indeed, if $\boldsymbol{\mathcal{G}}$ is stable, $\|\boldsymbol{G}H\|_{\mathbb{H}_\infty}$ or $\|R\,\boldsymbol{G}\|_{\mathbb{H}_\infty}$ is bounded and condition (3.12) holds, then Theorem 3.2 implies that $\widetilde{\boldsymbol{\mathcal{G}}}$ is also stable. Further aspects of stability of coupled systems can be found in [37, 53].

3.2.1. Subsystem model reduction by balanced truncation. Now we apply the $\mathbb{H}_\infty$-norm estimates provided by balanced truncation to the coupled system (2.1)–(2.4), where all subsystems are asymptotically stable. As a consequence of Theorem 3.2 we obtain the following error bounds for the closed-loop system (2.9) computed by the balanced truncation model reduction method applied to the subsystems.

COROLLARY 3.3. *Consider the coupled system* (2.1)–(2.4) *with asymptotically stable subsystems and consider the reduced-order coupled system* (3.6)–(3.8)*, where subsystems* (3.6) *are computed by Algorithm* 3.1 *applied to* (2.1)*. Let*

$$\gamma = 2 \max_{1 \le j \le k} (\sigma^{(j)}_{\ell_j+1} + \ldots + \sigma^{(j)}_{n_j}),$$

where $\sigma^{(j)}_{\ell_j+1}, \ldots, \sigma^{(j)}_{n_j}$ *denote the truncated Hankel singular values of the* j*th subsystem* (2.1)*. Further, let* g_1*,* g_3*,* c_1 *and* c_2 *be as in Theorem* 3.2*. If* $2\gamma \max\{g_1, g_3\} < 1$*, then the* $\mathbb{H}_\infty$*-norm of the error* $\widetilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{\mathcal{G}}$ *can be bounded as*

$$\| \widetilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{\mathcal{G}} \|_{\mathbb{H}_\infty} \le \gamma \min\{c_1, c_2\}. \tag{3.14}$$

Note that the computation of the a priori error bounds (3.13) and (3.14) for large-scale systems is expensive, since we need to calculate the $\mathbb{H}_\infty$-norm of the transfer functions of the state space dimension $n_1 + \ldots + n_k$. Similar to Theorem 3.2 and Corollary 3.3, we can also obtain the a posteriori error bounds like (3.13) and (3.14), with $\boldsymbol{G}$ replaced by $\widetilde{\boldsymbol{G}}$.

An essential assumption in Theorem 3.2 and Corollary 3.3 was the asymptotic stability of the subsystems (2.1). However, the asymptotic stability of the involved subsystems is neither necessary nor sufficient for the asymptotic stability of the closed-loop system (2.7). Nevertheless, unstable subsystems can be artificially represented as a coupling of stable subsystems, and we are then in the situation of Theorem 3.2 and Corollary 3.3. A possibility for the representation of an unstable subsystem (2.1) as a coupling of stable ones is based on the *coprime factorization*.

Consider the transfer function $\boldsymbol{G}_j(s) = C_j(sE_j - A_j)^{-1}B_j$ which is not necessary in $\mathbb{H}_\infty$. Such a transfer function admits a representation $\boldsymbol{G}_j(s) = \boldsymbol{N}_j(s)\boldsymbol{D}_j(s)^{-1}$, where $\boldsymbol{D}_j \in \mathbb{H}_\infty$ is square and $\boldsymbol{N}_j \in \mathbb{H}_\infty$ has the same matrix dimensions as $\boldsymbol{G}_j$. If, additionally, there exist $\boldsymbol{X}_j, \boldsymbol{Y}_j \in \mathbb{H}_\infty$ such that $\boldsymbol{X}_j(s)\boldsymbol{D}_j(s)+\boldsymbol{Y}_j(s)\boldsymbol{N}_j(s) = I$, then $\boldsymbol{D}_j$ and $\boldsymbol{N}_j$ are called *right coprime factors* of $\boldsymbol{G}_j$. For system (2.1) with no unstable and coevally uncontrollable modes, the coprime factors can be determined via a state feedback matrix $F_j \in \mathbb{R}^{m_j, n_j}$ with the property that the pencil $sE_j - A_j - B_jF_j$ is stable and of index at most one [15, 61]. In this case, $\boldsymbol{N}_j$ and $\boldsymbol{D}_j$ can be chosen as

$$\begin{aligned} \boldsymbol{N}_j(s) &= C_j(sE_j - A_j - B_jF_j)^{-1}B_j, \\ \boldsymbol{D}_j(s) &= F_j(sE_j - A_j - B_jF_j)^{-1}B_j + I. \end{aligned}$$

Then the extended transfer function

$$\boldsymbol{G}_{ext,j}(s) = \begin{bmatrix} \boldsymbol{N}_j(s) \\ \boldsymbol{D}_j(s) - I \end{bmatrix} \tag{3.15}$$

is stable and has the generalized state space representation

$$\begin{aligned} E_j\dot{x}_j(t) &= (A_j + B_jF_j)x(t) + B_jv_j(t), \\ \begin{bmatrix} y_{1j}(t) \\ y_{2j}(t) \end{bmatrix} &= \begin{bmatrix} C_j \\ F_j \end{bmatrix} x_j(t). \end{aligned} \tag{3.16}$$

Coupling this system with itself by the relations

$$\begin{aligned} v_j(t) &= -y_{2j}(t) + u_j(t) = [\,0,\ \ -I\,] \begin{bmatrix} y_{1j}(t) \\ y_{2j}(t) \end{bmatrix} + u_j(t), \\ y_j(t) &= y_{1j}(t) = [\,I,\ \ 0\,] \begin{bmatrix} y_{1j}(t) \\ y_{2j}(t) \end{bmatrix}, \end{aligned} \tag{3.17}$$

we obtain the coupled system which has the same transfer function $\boldsymbol{G}_j(s)$ as (2.1). Such a coupled system is shown in Figure 3.1. Note that the state space dimension of (3.16) coincides with that of (2.1).

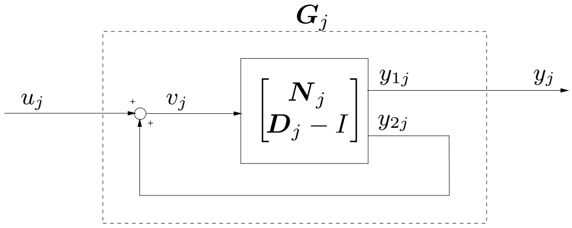


FIG. 3.1. *Coprime factorization as a coupled system.*

In the following, we discuss the benefit of the coprime factorization to the structure-preserving model reduction of the coupled system (2.1), (2.3), (2.4) with possibly unstable subsystems. Without loss of generality, we may assume that the first q subsystems are unstable and the corresponding right coprime factorizations are given by $\boldsymbol{G}_j(s) = \boldsymbol{N}_j(s)\boldsymbol{D}_j(s)^{-1}$ for $j = 1, \ldots, q$. The unstable subsystems can now be replaced by the asymptotically stable models (3.16) with the internal inputs and outputs satisfying (3.17). In this case, the coupling relations (2.3) and (2.4) take the form

$$\begin{aligned} v_j(t) &= -y_{2j}(t) + u_j(t) & 1 \le j \le q, \\ &= K_{j1}y_{11}(t) + \ldots + K_{jq}y_{1q}(t) - y_{2j}(t) \\ &\quad + K_{j,q+1}y_{q+1}(t) + \ldots + K_{jk}y_k(t) + H_j u(t), \\ u_j(t) &= K_{j1}y_{11}(t) + \ldots + K_{jq}y_{1q}(t) & q < j \le k, \\ &\quad + K_{j,q+1}y_{q+1}(t) + \ldots + K_{jk}y_k(t) + H_j u(t), \\ y(t) &= R_1 y_{11}(t) + \ldots + R_q y_{1q}(t) + R_{q+1}y_{q+1}(t) + \ldots + R_k y_k(t). \end{aligned}$$

The closed-loop transfer function of the new extended coupled system is given by $\boldsymbol{\mathcal{G}}_{ext}(s) = R_{ext}(I - \boldsymbol{G}_{ext}(s)K_{ext})^{-1}\boldsymbol{G}_{ext}(s)H$, where

$$\begin{aligned} K_{ext} &= K\,\mathrm{diag}\big([I_{p_1},\ 0], \ldots, [I_{p_q},\ 0],\ I\big) - \mathrm{diag}\big([0,\ I_{m_1}], \ldots, [0,\ I_{m_q}],\ 0\big), \\ R_{ext} &= \big[\, R_1\ 0\ \cdots\ R_q\ 0\ R_{q+1}\ R_{q+2}\ \cdots\ R_k \,\big] \end{aligned}$$

and

$$\boldsymbol{G}_{ext}(s) = \mathrm{diag}\big(\boldsymbol{G}_{ext,1}(s), \ldots, \boldsymbol{G}_{ext,q}(s), \boldsymbol{G}_{q+1}(s), \ldots, \boldsymbol{G}_k(s)\big)$$

with $\boldsymbol{G}_{ext,j}(s)$ as in (3.15). It has been shown in [53] that $\boldsymbol{\mathcal{G}}_{ext}(s)$ coincides with the transfer function $\boldsymbol{\mathcal{G}}(s)$ of the closed-loop system (2.7). This allows us to apply Theorem 3.2 and Corollary 3.3 to the extended coupled system with all subsystems being asymptotically stable in order to obtain the error bounds for the reduced-order system, see [53] for details.

Another structure-preserving balancing-based model reduction method for coupled systems has been considered in [60]. It has been proposed there to project the subsystems (2.1) with $E_j = I$ onto the dominant eigenspaces of the matrices $P_{jj}Q_{jj}$, where P_{jj} and Q_{jj} are the diagonal blocks of the controllability and observability Gramians $\mathcal{P} = [P_{jl}]_{j,l=1}^k$ and $\mathcal{Q} = [Q_{jl}]_{j,l=1}^k$ of the closed-loop system (2.7). Clearly, in the generalized state space case we should consider the matrices $P_{jj}E_j^TQ_{jj}E_j$. A drawback of this approach is that the stability is not necessarily preserved in the reduced-order subsystems (3.6). Furthermore, we cannot make use of the error bound (3.13) since there are no global error estimates on $\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j$.

3.2.2. Krylov subspace structure-preserving techniques. In this subsection we review structure-preserving model reduction methods based on Krylov subspaces. These methods have been previously proposed for second-order systems from structural dynamics, MEMS simulation and electronic circuit design [7, 23, 57] and then extended to coupled systems in [60]. A general framework for Krylov-based structure-preserving model reduction methods for partitioned systems can be found in [41, 42].

As it was mentioned above, for general projection matrices $\mathcal{W}$ and $\mathcal{T}$, the reduced-order system (2.9), (3.1) does not preserve the interconnection structure. This can be avoided if we take the block diagonal projection matrices $\mathcal{W}$ and $\mathcal{T}$ as in (3.11). However, in order to guarantee the moment matching conditions (3.5), the diagonal blocks in $\mathcal{W}$ and $\mathcal{T}$ have to satisfy certain subspace conditions given in the following theorem.

THEOREM 3.4. *Let $\hat{\mathcal{W}} = [\,\hat{W}_1^T, \ldots, \hat{W}_k^T\,]^T$ and $\hat{\mathcal{T}} = [\,\hat{T}_1^T, \ldots, \hat{T}_k^T\,]^T$ with $\hat{W}_j$, $\hat{T}_j \in \mathbb{R}^{n_j,\ell}$. Assume that the reduced-order systems* (3.6) *are computed by projection* (3.10)*, where W_j, $T_j \in \mathbb{R}^{n_j,\ell_j}$ have full column rank and satisfy*

$$\operatorname{Im}\hat{W}_j \subseteq \operatorname{Im} W_j, \qquad \operatorname{Im}\hat{T}_j \subseteq \operatorname{Im} T_j.$$

Let $\mathcal{M}_j$ and $\widetilde{\mathcal{M}}_j$ be the moments of the closed-loop systems (2.7), (2.8) *and* (2.9), (3.9)*, respectively.*

1. *If $\mathcal{K}_{q_r}(\mathcal{A}_0^{-1}\mathcal{E}, \mathcal{A}_0^{-1}\mathcal{B}) \subseteq \operatorname{Im}\hat{\mathcal{T}}$ and $W_i = T_i$ for $i = 1, \ldots, k$, then $\mathcal{M}_j = \widetilde{\mathcal{M}}_j$ for $j = 0, \ldots, q_r - 1$.*
2. *If $\mathcal{K}_{q_r}(\mathcal{A}_0^{-1}\mathcal{E}, \mathcal{A}_0^{-1}\mathcal{B}) \subseteq \operatorname{Im}\hat{\mathcal{T}}$ and $\mathcal{K}_{q_l}(\mathcal{A}_0^{-T}\mathcal{E}^T, \mathcal{A}_0^{-T}\mathcal{C}^T) \subseteq \operatorname{Im}\hat{\mathcal{W}}$, then $\mathcal{M}_j = \widetilde{\mathcal{M}}_j$ for $j = 0, \ldots, q_r + q_l - 1$.*

Proof. See [42, Theorem 4.1] and [60, Lemma 7]. ◻

A natural way to determine the projection matrices T_j and W_j is to compute the QR decomposition or the singular value decomposition of the matrices $\hat{T}_j$ and $\hat{W}_j$ such that the columns of $\hat{\mathcal{T}} = [\,\hat{T}_1^T, \ldots, \hat{T}_k^T\,]^T$ and $\hat{\mathcal{W}} = [\,\hat{W}_1^T, \ldots, \hat{W}_k^T\,]^T$ span the Krylov subspaces $\mathcal{K}_{q_r}(\mathcal{A}_0^{-1}\mathcal{E}, \mathcal{A}_0^{-1}\mathcal{B})$ and $\mathcal{K}_{q_l}(\mathcal{A}_0^{-T}\mathcal{E}^T, \mathcal{A}_0^{-T}\mathcal{C}^T)$, respectively. The matrices $\hat{T}_j$ and $\hat{W}_j$, in turn, can be computed simultaneously by applying Lanczos- or Arnoldi-type method to the closed-loop system (2.7). The following theorem shows that $\hat{T}_j$ and $\hat{W}_j$ can also be generated by Krylov subspace methods applied to the subsystems (2.1) separately.

THEOREM 3.5. *Consider the closed-loop system* (2.7), (2.8). *Let* $s_0 \in \mathbb{C}$ *be neither an eigenvalue of the pencil* $\lambda E - A$ *nor an eigenvalue of the pencil* $\lambda\,\mathcal{E} - \mathcal{A}$. *Then*

$$\mathcal{K}_{q_r}(\,\mathcal{A}_0^{-1}\mathcal{E},\ \mathcal{A}_0^{-1}\mathcal{B}\,) \subseteq \mathcal{K}_{q_r}((s_0E-A)^{-1}E,\ (s_0E-A)^{-1}B\,),$$
$$\mathcal{K}_{q_l}(\mathcal{A}_0^{-T}\mathcal{E}^T, \mathcal{A}_0^{-T}\mathcal{C}^T\,) \subseteq \mathcal{K}_{q_l}((s_0E-A)^{-T}E^T, (s_0E-A)^{-T}C^T).$$

Proof. These inclusions can be proved similarly to the case $E = I$, see [60, Lemma 6]. ∎

3.3. Comparison of two approaches for model reduction of coupled systems. The computation of the reduced-order model (2.9) by applying a model reduction method to the closed-loop system (2.7) has a couple of disadvantages. First of all note that the behavior of coupled systems is determined by different interconnected subsystems that are usually governed by entirely different physical laws and they often act in different spaces and time scales. There is no general model reduction technique, which can be considered as optimal, since the reliability, computation time and approximation quality of reduced-order models strongly depend on system properties. In model reduction of the closed-loop system (2.7), we ignore the special properties of the subsystems and destroy the coupling structure. Also in structure-preserving model reduction, where the projection matrices W_j and T_j are determined from the closed-loop system (2.7), we do not make use of subsystem s properties. If we slightly change the coupled system, for example, by adding new subsystems, by replacing some of them by new ones or by changing the coupling configuration, we have to re-compute the reduced-order model again.

Subsystem model reduction, where the projection matrices W_j and T_j are computed separately from the subsystems (2.1), is free of these difficulties. In this approach, every subsystem can be reduced by a most suitable model reduction method that takes into consideration the structure and properties of the subsystem. If error estimates for subsystems are available, then using bound (3.13) we can evaluate how well the subsystems should be approximated to attain a prescribed accuracy in the reduced-order closed-loop system (2.9). Finally, subsystem model reduction is attractive for parallelization, since all k subsystems may be reduced simultaneously using k processors.

On the other hand, separate reduction of the subsystems usually yields the approximate model (2.9) of larger state space dimension than the system computed by projection of the closed-loop system (2.7). Furthermore, subsystem model reduction is often restricted to coupled systems whose subsystems have a small number of internal inputs and outputs.

4. Numerical examples. In this section we present two numerical examples to demonstrate the properties of the discussed model reduction approaches for coupled systems. The computations were performed using MATLAB 7.

Example 4.1. Consider a heated beam whose temperature is steered by a PI-controller as shown in Figure 4.1. The transfer function of the PI-controller is given by $\boldsymbol{G}_1(s) = k_P + k_I s^{-1}$ and it is realized by the descriptor system

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}_1(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} k_I \\ -k_P \end{bmatrix} u_1(t), \\ y_1(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_1(t). \end{aligned} \tag{4.1}$$

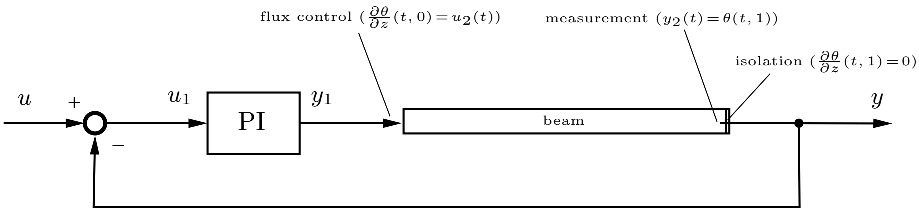


FIG. 4.1. *A heated beam with a PI-controller.*

The heat transfer along the 1D beam of length 1 is described by

$$\frac{\partial\theta}{\partial t}(t,z) = \kappa\frac{\partial^2\theta}{\partial z^2}(t,z),$$

where $t > 0$ is the time, $z \in [0,1]$ is the position, $\theta(t,z)$ is the temperature distribution and κ is the heat conductivity of the material. On the left-hand side of the beam, the temperature flux is controlled by an input $u_2(t)$, whereas the beam is assumed to be perfectly isolated on the right-hand side. From this, we get the boundary conditions

$$\frac{\partial\theta}{\partial z}(t,0) = u_2(t), \qquad \frac{\partial\theta}{\partial z}(t,1) = 0.$$

The temperature is measured at $z = 1$ and it forms the output of the system, i.e., $y_2(t) = \theta(t,1)$ and $y(t) = y_2(t)$. By a spatial discretization of the heat equation with $n_2 + 1$ equidistant grid points, we obtain the system

$$\begin{aligned} E_2\dot{x}_2(t) &= A_2x_2(t) + B_2u_2(t), \\ y_2(t) &= C_2x_2(t), \end{aligned} \tag{4.2}$$

where $E_2 = I_{n_2}$ and

$$A_2 = \kappa(n_2+1)^2\begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \kappa(n_2+1) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T.$$

The interconnection of the PI-controller and the beam is expressed by the relations

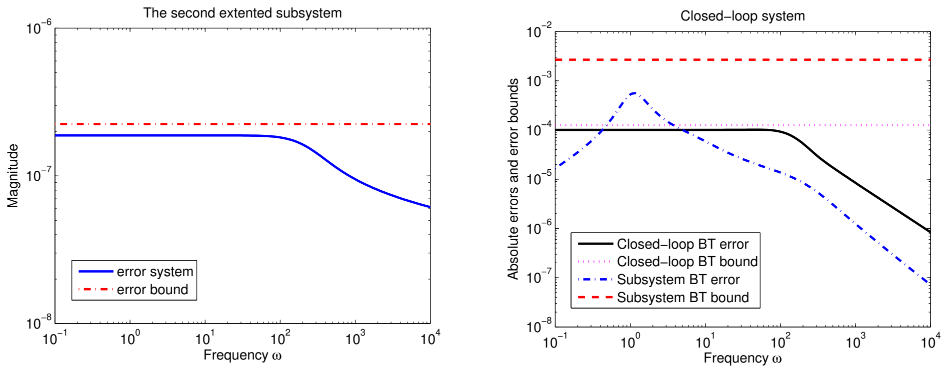
$$u_1(t) = u(t) - y_2(t), \quad u_2(t) = y_1(t).$$

Note that both the subsystems (4.1) and (4.2) are not asymptotically stable, since their transfer functions $\boldsymbol{G}_1(s) = C_1(sE_1 - A_1)^{-1}B_1$ and $\boldsymbol{G}_2(s) = C_2(sE_2 - A_2)^{-1}B_2$ have a pole at the origin. The stabilizing state feedback matrices can be chosen as

$$F_1 = [\,0,\ -1\,], \quad F_2 = [\,-n_2 - 1,\ 0,\ \cdots,\ 0\,].$$

In this case, we obtain an extended coupled system with the stable subsystems $\boldsymbol{G}_{ext,1}(s)$ and $\boldsymbol{G}_{ext,2}(s)$ as in (3.15) and the interconnection matrices

$$K_{ext} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad R_{ext} = [0,\ 0,\ 1,\ 0].$$



(a) (b)

FIG. 4.2. *Example 4.1: (a) the absolute error* $\|\widetilde{\boldsymbol{G}}_{ext,2}(i\omega)-\boldsymbol{G}_{ext,2}(i\omega)\|_2$ *and the error bound* γ*; (b) the absolute errors* $\|\widetilde{\boldsymbol{\mathcal{G}}}(i\omega)-\boldsymbol{\mathcal{G}}(i\omega)\|_2$ *and the error bounds* γ_{ext} *and* γ_{cl}*.*

In our experiments, we took the numerical values $k_P = k_I = \kappa = 1$, $n_2 = 1000$. The second subsystem $\boldsymbol{G}_{ext,2}$ has been approximated by a reduced model $\widetilde{\boldsymbol{G}}_{ext,2}$ of order $\ell_2 = 21$ computed by balanced truncation. Figure 4.2 (a) shows the absolute error $\|\widetilde{\boldsymbol{G}}_{ext,2}(i\omega) - \boldsymbol{G}_{ext,2}(i\omega)\|_2$ for the frequency range $\omega \in [\,10^{-1},\,10^4\,]$ and the error bound γ that is twice the sum of the truncated Hankel singular values of $\boldsymbol{G}_{ext,2}$. We chose ℓ_2 such that $\gamma < 10^{-6}$. The resulting approximate closed-loop system with the transfer function $\widetilde{\boldsymbol{\mathcal{G}}}(s)$ has order $\ell = 23$. Figure 4.2 (b) shows the absolute error $\|\widetilde{\boldsymbol{\mathcal{G}}}(i\omega)-\boldsymbol{\mathcal{G}}(i\omega)\|_2$ and the a posteriori error bound $\gamma_{ext} = \gamma \min\{c_1, c_2\}$, where c_1 and c_2 are as in Theorem 3.2 with $\boldsymbol{G}$, K and R replaced by $\widetilde{\boldsymbol{G}}_{ext} = \operatorname{diag}(\boldsymbol{G}_{ext,1}, \widetilde{\boldsymbol{G}}_{ext,2})$, K_{ext} and R_{ext}, respectively. Comparing the approximation errors, we see that due the coupling the error in the closed-loop system is larger than the error in the subsystem.

Furthermore, we applied the balanced truncation method to the closed-loop system and selected the order of the reduced model as a minimal integer ℓ such that the error bound $\gamma_{cl} = 2(\sigma_{\ell+1} + \ldots + \sigma_n)$ is smaller than γ_{ext}. We obtained the reduced model of order $\ell = 5$ with the approximation error comparable with the error in subsystem model reduction, see Figure 4.2 (b). Note, however, if we change the parameters k_P, k_I and κ, then the closed-loop system is also changed, and we need to re-compute the reduced model. On the other hand, the reduced closed-loop system computed by subsystem model reduction can easily be modified by changing the first subsystem and re-scaling the matrix coefficients in the reduced-order second subsystem.

Example 4.2. Consider the delay-differential system

$$\begin{aligned} \dot{x}(t) &= -x(t-1) + u(t), \\ y(t) &= x(t). \end{aligned} \tag{4.3}$$

This system can be represented as an interconnection of

$$\begin{aligned} \dot{x}_1(t) &= 0 \cdot x_1(t) + \begin{bmatrix} 1, & -1 \end{bmatrix} u_1(t), \\ y_1(t) &= \phantom{0 \cdot{}} x_1(t) \end{aligned} \tag{4.4}$$

with the system $\hat{\boldsymbol{G}}_2$ representing the pure unit delay $\hat{y}_2(t) = u_2(t-1)$. The coupling relations are

$$u_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{y}_2(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$
$$u_2(t) = y_1(t), \qquad y(t) = y_1(t).$$

The second subsystem has the transfer function $\hat{\boldsymbol{G}}_2(s) = e^{-s}$ and, due to its irrationality, its system realizations have an infinite dimensional state space [17, 52]. The delay can be achieved by the following partial differential equation with boundary control and observation

$$\begin{aligned} \frac{\partial f}{\partial t}(t,z) &= -\frac{\partial f}{\partial z}(t,z), \\ f(t,0) &= u_2(t), \\ f(t,1) &= \hat{y}_2(t). \end{aligned} \tag{4.5}$$

A spatial discretization of this equation with n_2 equidistant grid points leads to the subsystem as in (4.2) with $E_2 = I_{n_2}$ and

$$A_2 = n_2 \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ n_2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T.$$

The transfer function of this subsystem is given by

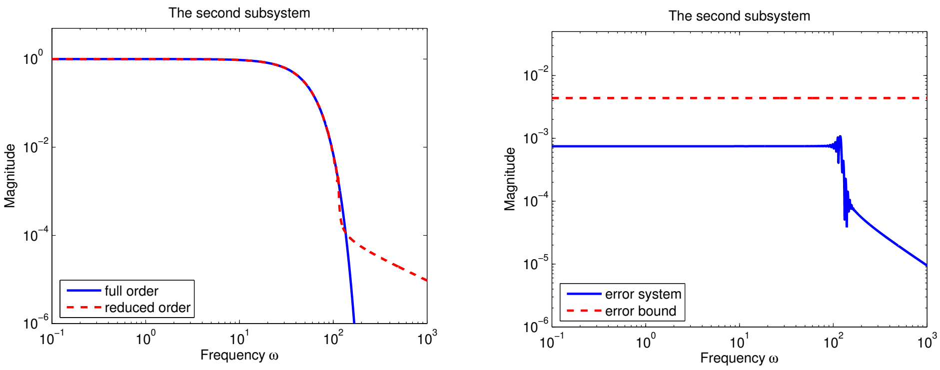
$$\boldsymbol{G}_2(s) = \frac{n_2}{(n_2+s)^{n_2}}.$$

The closed-loop system is a finite dimensional approximant of the originally infinite dimensional delay-differential system (4.3). The estimation of the discretization error in the $\mathbb{H}_\infty$-norm is treated in [52].

The first subsystem (4.4) is not asymptotically stable since $\boldsymbol{G}_1(s)$ has a pole at the origin, whereas the second subsystem satisfies $\boldsymbol{G}_2 \in \mathbb{H}_\infty$. A stabilizing state feedback matrix can be taken as $F_1 = [\,0,\ 2\,]^T$. The interconnection matrices are

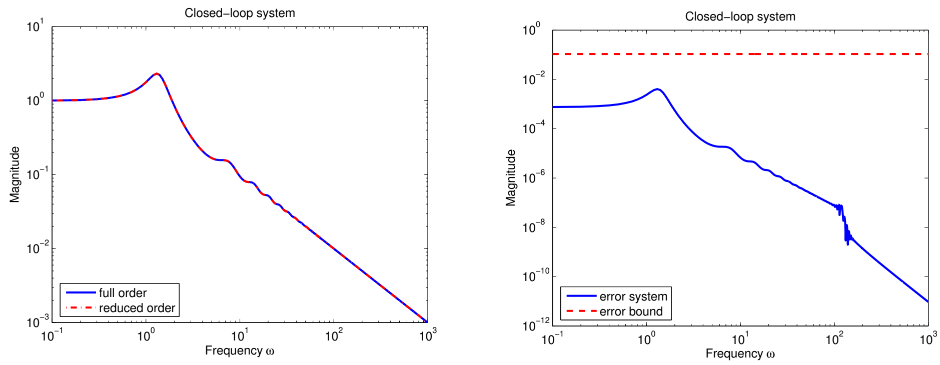
$$K_{ext} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad R_{ext} = [\,1,\ 0,\ 0,\ 0\,].$$

For the second subsystem, we chose $n_2 = 1000$. This subsystem has been approximated by a reduced model of order $\ell_2 = 41$ computed by the balanced truncation method. The absolute values of the frequency responses $\widetilde{\boldsymbol{G}}_2(i\omega)$ and $\boldsymbol{G}_2(i\omega)$ of the original and reduced-order subsystems are given in Figure 4.3 (a), whereas the absolute error $\|\widetilde{\boldsymbol{G}}_2(i\omega) - \boldsymbol{G}_2(i\omega)\|_2$ and the error bound $\gamma = 2(\sigma^{(2)}_{\ell_2+1} + \ldots + \sigma^{(2)}_{n_2})$ are presented in Figure 4.3 (b). In Figure 4.4 (a) we plotted the absolute values of the frequency responses $\boldsymbol{\mathcal{G}}(i\omega)$ and $\widetilde{\boldsymbol{\mathcal{G}}}(i\omega)$ of the original and the reduced-order closed-loop systems. Figure 4.4 (b) shows the error $\|\widetilde{\boldsymbol{\mathcal{G}}}(i\omega) - \boldsymbol{\mathcal{G}}(i\omega)\|_2$ and the a posteriori error bound $\gamma_{ext} = \gamma \min\{c_1, c_2\}$, where c_1 and c_2 are as in Theorem 3.2 with $\boldsymbol{G}$, K and R replaced by $\widetilde{\boldsymbol{G}}_{ext} = \mathrm{diag}(\boldsymbol{G}_{ext,1}, \widetilde{\boldsymbol{G}}_2)$, K_{ext} and R_{ext}, respectively.



(a) (b)

FIG. 4.3. *Example 4.2: (a) the absolute values of frequency responses* $\boldsymbol{G}_2(i\omega)$ *and* $\widetilde{\boldsymbol{G}}_2(i\omega)$*; (b) the absolute error* $\|\widetilde{\boldsymbol{G}}_2(i\omega) - \boldsymbol{G}_2(i\omega)\|_2$ *and the error bound* γ*.*



(a) (b)

FIG. 4.4. *Example 4.2: (a) the absolute values of frequency responses* $\boldsymbol{\mathcal{G}}(i\omega)$ *and* $\widetilde{\boldsymbol{\mathcal{G}}}(i\omega)$*; (b) the absolute error* $\|\widetilde{\boldsymbol{\mathcal{G}}}(i\omega) - \boldsymbol{\mathcal{G}}(i\omega)\|_2$ *and the error bound* γ_{ext}*.*

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