HYDRODYNAMIC LIMIT FOR THE $A+B \to \emptyset$ MODEL

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Abstract. We study a two-species interacting particle model on a subset of $\mathbb{Z}$ with open boundaries. The two species are injected with time dependent rate on the left, resp. right boundary. Particles of different species annihilate when they try to occupy the same site. This model has been proposed as a simple model for the dynamics of an "order book" on a stock market. We consider the hydrodynamic scaling limit for the empirical process and prove a large deviation principle that implies convergence to the solution of a non-linear parabolic equation.

1. Introduction and results

1.1. **Introduction.** In this article we study the hydrodynamic behaviour of a one-dimensional stochastic particle model that is a variant of what is known as the $A+B \to \emptyset$ model for a reaction front or phase boundary. In this model, two types A, B, of particles live on a finite subset $\Lambda_N \equiv \{-N, -N+1, \dots, N-1, N\}$ of the integer lattice. B-type particle enter Λ_N at the left boundary of the interval with some (time dependent) rate $\lambda^-(t)$, and A-particles enter at the right boundary with rate $\lambda^+(t)$. Within Λ_N particles perform random walk with drift (depending on the type); the main feature of the model is that if two particles of different type meet, they annihilate instantly[1]. As a result of this dynamics, A and B particles occupy two disjoint domains: there will be a right-most site, $p_b(t)$, that is occupied by a B-particle, and a left-most site, $p_a(t) > p_b(t)$, that is occupied by an A-particle. The empty space between $p_a(t)$ and $p_b(t)$ can be seen as a phase boundary, and its dynamic is of primary interest

Our main motivation to study this model comes from mathematical finance, where the model can be seen as a very simple model for the dynamics of an *order book*. Here the values $i \in \Lambda_N$ represent the logarithm of a price (in some units), and the particles of type A and B represent the orders placed at a (stock) market to sell, respectively buy, a unit of stock at a given price. In this context, the numbers $p_a(t), p_b(t)$ represent what are called the "ask" and the "bid" prices of the stock at time t, while the quoted price may be the mid-point between these values. The objective of the model is to understand how the dynamics of a price process arises through trading on a stock market. A basic version of this model was proposed originally by Bak et al. [1] and further investigated by by Eliezer et al. [5, 4], and Tang and Tian [9]. A different version of the order book dynanamics was considered in [6]. In [2], the issue of order book dynamics was developed further, and a more complex model was developed. In this larger context, the simple $A+B \to \emptyset$ model can be viewed as an effective model for a small part of the order book, namely those orders that are near the current price, the "active zone". The arrival (and disappearance) of orders at the boundary of the system represents then the effective interaction between the "active zone" and the rest of the system. The main interest is to see how a time-dependent

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[1]In the literature a slightly different model is often studies where particles of different type annihilate with rate one; this leads to the existence of a "coexistence region" which in our model is excluded.

external injection rate — representing e.g. changes in the macro-economic environment — effects the price process through trading.

In this paper we present some rigorous results in this direction by considering a scaling limit in which both the size, N, of the interval Λ_N, and the number of particles per "price" tends to infinity. We will see that under appropriate scaling the "empirical distribution" of the order book converges to a continuum profile whose dynamic is governed by a parabolic, non-linear drift-diffusion equation. This had been predicted heuristically in [9] in the time-independent case. Here we will derive this result from a large deviation principle for the empirical distribution in the spirit of the usual proofs of hydrodynamic limits in particle systems (see [8, 7]). While in main respects our proof follows the conventional lines, some of the particularities of our model require special attention and are apparently not covered by existing proofs. Our presentation will concentrate on these aspects.

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1.2. **Definitions.** Let us now formally introduce our model. For a given positive integer, N, the configurations of this model are elements of the set $X_N = \mathbb{Z}^{\Lambda_N}$, equipped with the discrete topology. We use η, $\eta = \{\eta(x) \in \mathbb{Z} : x \in \Lambda_N\}$, to denote these configurations. Here we used a very convenient trick to represent the number of A and B particles at a site x by the single $\mathbb{Z}$-valued variable $\eta(x)$: If $\eta(x) \geq 0$, then there are $\eta(x)$ A-particles at site x, and if $\eta(x) \leq 0$, then there are $-\eta(x)$ B-particles at site x. Due to the fact that the two types of particles cannot co-exist at the same site, this notation is unambiguous.

We further set $\Lambda = [-1, 1]$. Points in Λ are denoted by u, v.

Given a (not necessary positive) continuous function, $\gamma : \Lambda \to \mathbb{R}$, such that $\|\gamma\|_\infty < +\infty$, we denote by ν_γ^N the product measure on X_N with Poisson marginals corresponding to the profile γ. This means that, under ν_γ^N, the random variables $|\eta(x)|$ are independent, have Poisson distribution with mean $N|\gamma(x/N)|$, and $\operatorname{sign}\eta(x) = \operatorname{sign}\gamma(x/N)$ ν_γ^N-a.s.

The (accelerated) $A + B \to \emptyset$ model is a time-inhomogeneous Markov process, $\eta_t^N = \eta_t$, with state space X_N whose generator, $L_{N,t}$, acts on functions, f, as

$$\begin{aligned} L_{N,t} f(\eta) \equiv N^2 &\sum_{x\in\Lambda_N} \eta^+(x)\big[p_N\big(f(\eta^{x,x-1}) - f(\eta)\big) + q_N\big(f(\eta^{x,x+1}) - f(\eta)\big)\big] \\ &+ N^2 \sum_{x\in\Lambda_N} \eta^-(x)\big[q_N\big(f(\eta^{x,x-1}) - f(\eta)\big) + p_N\big(f(\eta^{x,x+1}) - f(\eta)\big)\big] \\ &+ N^3\lambda^+(t)\big[f(\eta^{N,+}) - f(\eta)\big] \\ &+ N^3\lambda^-(t)\big[f(\eta^{-N,-}) - f(\eta)\big]. \end{aligned} \tag{1}$$

Here $\eta^+(x) = \eta(x) \vee 0$, $\eta^-(x) = |\eta(x) \wedge 0|$, and $p_N = e^{d/N}/2 = 1/q_n$, $d \in \mathbb{R}$. The configurations, $\eta^{x,\cdot}$, with one particle moved from position x, are defined by

$$\eta^{x,x\pm1}(y) \equiv \begin{cases} \eta(y) & \text{for } y \notin \{x, x\pm 1\}, \\ \eta(x) - \operatorname{sign}\eta(x) & \text{for } y = x, \\ \eta(y) + \operatorname{sign}\eta(x) & \text{for } y = x \pm 1, \end{cases} \tag{2}$$

$$\eta^{x,\pm}(y) \equiv \begin{cases} \eta(y) & \text{for } y \neq x, \\ \eta(x) \pm 1 & \text{for } y = x. \end{cases} \tag{3}$$

Finally, $\lambda^\pm(t)$ are uniformly bounded, continuous functions on $[0, \infty)$ (i.e., there exists $c > 0$ such that $1/c \leq \lambda^\pm(t) \leq c$ for all $t \in [0, \infty)$). Observe that the time-inhomogeneity of the process is only due to non-constant injection rates $\lambda^\pm$.

Remark. Note that $e^{d/N}-1\sim d/N$ is a drift; it acts in opposite direction in the two populations and thus represents a drift towards the "price" (if $d>0$). Note also that the strength of the drift scales with $1/N$; this choice ensures, as we will see, convergence to a parabolic equation. A stronger drift towards the price would lead formally to an equation of first order.

Let $T>0$ be a fixed time whose value does not change in the paper. We use $P^{d,\lambda}_{N,\gamma}$ to denote the probability measure induced by η^N_t on the path space, $D([0,T],X_N)$, (the Skorokhod space of càdlàg functions from $[0,T]$ to X_N), the initial measure being ν^N_γ. If no confusion can arise, we will occasionally skip some of the indices of $P^{d,\lambda}_{N,\gamma}$.

To simplify the reasoning (and in view of the financial application) we consider only $\gamma\in\tilde C(\Lambda)$, where

$$\tilde C(\Lambda)=\big\{\gamma\in C(\Lambda):\sup\{u\in\Lambda:\gamma(u)<0\}\le\inf\{u\in\Lambda:\gamma(u)>0\}\big\}. \tag{4}$$

If this is the case, the process η_t starts in and never leaves its invariant set, $\Sigma_N\subset X_N$,

$$\Sigma_N\equiv\big\{\eta:\max\{x:\eta(x)<0\}<\min\{x:\eta(x)>0\}\big\}. \tag{5}$$

We also consider an auxiliary weakly asymmetric model. Let $H(u,t)$ be a continuous function on the set $\Lambda\times[0,T]$. We sometimes write $H_t(\cdot)$ for $H(\cdot,t)$ to make the notation more compact. The weakly asymmetric model is generated by

$$\begin{aligned}
L^H_{N,t}f(\eta)\equiv N^2\sum_{x\in\Lambda_N}\eta^+(x)\Big[&p_Ne^{H_t((x-1)/N)-H_t(x/N)}\big(f(\eta^{x,x-1})-f(\eta)\big)\\
&+q_Ne^{H_t((x+1)/N)-H_t(x/N)}\big(f(\eta^{x,x+1})-f(\eta)\big)\Big]\\
+N^2\sum_{x\in\Lambda_N}\eta^-(x)\Big[&q_Ne^{-H_t((x-1)/N)+H_t(x/N)}\big(f(\eta^{x,x-1})-f(\eta)\big)\\
&+p_Ne^{-H_t((x+1)/N)+H_t(x/N)}\big(f(\eta^{x,x+1})-f(\eta)\big)\Big]\\
+N^3\lambda^+(t)[f(\eta^{N,+})-f(\eta)]&+N^3\lambda^-(t)[f(\eta^{-N,-})-f(\eta)].
\end{aligned} \tag{6}$$

The corresponding measure on $D([0,T],X_N)$ is denoted by $P^{H,d,\lambda}_{N,\gamma}$, with ν^N_γ as initial distribution.

For a Borel measures, ν, on Λ, and measurable functions, f, we set $\langle\nu,f\rangle\equiv\int_\Lambda f(u)\,\nu(\mathrm{d}u)$. We write $\mathcal{M}$ for the space of pairs, $\mu=(\mu^+,\mu^-)$, where μ^+ and μ^- are non-negative Borel measures on Λ that satisfy

$$u\in\operatorname{supp}\mu^+\text{ and }v\in\operatorname{supp}\mu^-\implies v\le u. \tag{7}$$

The space $\mathcal{M}$ is endowed with the topology that makes a sequence, μ_n, converge to μ, iff both $\langle\mu^+_n,f\rangle\to\langle\mu^+,f\rangle$ and $\langle\mu^-_n,f\rangle\to\langle\mu^-,f\rangle$ hold for all $f\in C(\Lambda)$. The space $\mathcal{M}$ is the completion of the space of signed Borel measures on Λ that satisfy (7) with respect to the considered topology (in this case μ^+ and μ^- should be interpreted as the canonical decomposition of a signed measure to its positive and negative part). The space $\mathcal{M}$ is larger than this space, since μ^+ and μ^- can both have an atom at the (at most one) point in the intersection of their supports. We define $\langle\mu,f\rangle\equiv\langle\mu^+,f\rangle-\langle\mu^-,f\rangle$ and $\langle|\mu|,f\rangle\equiv\langle\mu^+,f\rangle+\langle\mu^-,f\rangle$. We use $\mathcal{M}_0\subset\mathcal{M}$ to denote the set of μ's such that both μ^+ and μ^- are absolutely continuous with respect to the Lebesgue measure. The elements of $\mathcal{M}_0$ are signed measures. Note that $\mathcal{M}_0$ is not a closed subset of $\mathcal{M}$.

Finally, let us define a map, $\pi^N : \Sigma_N \to \mathcal{M}$, through

$$\pi^N(\eta; \mathrm{d}u) \equiv N^{-2} \sum_{x\in\Lambda_N} \eta(x)\delta_{x/N}(\mathrm{d}u). \tag{8}$$

We commonly abbreviate $\pi^N(\eta^N_s)$ by π^N_s.

1.3. **The rate functional.** We start to introduce notations that are required to define the rate functional of the large deviation principle we are going to present at the end of this section. As usually, there are two distinct types of large deviations, a "static part", arising from the large deviations of the initial product measure, and a "dynamic one".

The static part is rather easy to understand, but a small complication arises from the fact that we work with "signed" Poisson distributions. Fix $\gamma \in \tilde{C}(\Lambda)$ and define $\mathcal{M}(\gamma) \subset \mathcal{M}$ by

$$\mathcal{M}(\gamma) \equiv \big\{\mu \in \mathcal{M} : \operatorname{supp}\mu^{\pm} \subset \overline{\{u : \gamma(u) \gtrless 0\}}\big\}. \tag{9}$$

The set $\mathcal{M}(\gamma)$ is closed subset of $\mathcal{M}$. For each $\gamma \in \tilde{C}(\Lambda)$ we define $h_\gamma : \mathcal{M} \to \mathbb{R}$ by

$$h_\gamma(\mu) \equiv \begin{cases} \Big\langle \mathrm{d}u, |g(u)| \log\Big|\frac{g(u)}{\gamma(u)}\Big|\Big\rangle + \langle \mathrm{d}u, |\gamma| - |g|\rangle, & \text{if } \mu \in \mathcal{M}_0 \cap \mathcal{M}(\gamma), \mu(\mathrm{d}u) = g(u)\mathrm{d}u, \\ +\infty, & \text{otherwise.} \end{cases} \tag{10}$$

It is known fact that $h_\gamma(\mu)$ can be also defined using the following variational formula. Let $\gamma' \in C(\Lambda)$. We define $h_{\gamma'|\gamma} : \mathcal{M} \to \mathbb{R}$ by

$$h_{\gamma'|\gamma}(\mu) \equiv \Big\langle |\mu|, \log\Big|\frac{\gamma'}{\gamma}\Big|\Big\rangle + \langle \mathrm{d}u, |\gamma| - |\gamma'|\rangle. \tag{11}$$

Then

$$h_\gamma(\mu) = \sup_{\gamma' \in C(\Lambda)} h_{\gamma'|\gamma}(\mu), \qquad \text{if } \mu \in \mathcal{M}(\gamma), \tag{12}$$

and $h_\gamma(\mu) = +\infty$ otherwise. Since $h_{\gamma'|\gamma}$ is continuous in the considered topology and $\mathcal{M}(\gamma)$ is closed, the functional h_γ is lower semi-continuous on $\mathcal{M}$. It is, however, not convex.

Let us now turn to the dynamic part of the rate functional. Let $H \in C_0^{2,1}(\Lambda\times[0,T])$, the space of functions that have continuous derivative of the second order in the space variable and of the first order in the time variable and that satisfy $H(\pm 1, t) = 0$ for all $t \in [0,T]$. We define the functionals ℓ_H, J_H, $I_0 : D([0,T],\mathcal{M}) \to \mathbb{R}$ by

$$\begin{aligned} \ell_H(\pi) &= \langle \pi_T, H_t\rangle - \langle \pi_0, H_0\rangle - \int_0^T \langle \pi_t, \tfrac12 \Delta H_t + \partial_t H_t\rangle \,\mathrm{d}t \\ &\qquad + \sum_{\alpha=\pm1} \int_0^T \lambda^\alpha(t)(\partial_u H_t)(\alpha)\,\mathrm{d}t + d\int_0^T \langle |\pi_t|, \partial_u H_t\rangle\,\mathrm{d}t, \\ J_H(\pi) &= \ell_H(\pi) - \int_0^T \langle |\pi_t|, \tfrac12 (\partial_u H_t)^2\rangle\,\mathrm{d}t, \\ I_0(\pi) &= \sup\big\{J_H(\pi) : H \in C_0^{2,1}(\Lambda\times[0,T])\big\}. \end{aligned} \tag{13}$$

Here, λ^α, $\alpha = \pm 1$, stands for λ^+ or λ^-, and H_t for $H(\cdot, t)$, as above. The topology that we use makes the functional J_H continuous and I_0 lower semi-continuous on $D([0,T],\mathcal{M})$. The functional I_0 is also not convex.

Putting both pieces together, we now define, for all $\gamma \in \tilde{C}(\Lambda)$ the rate function, $I_\gamma : D([0,T],\mathcal{M}) \to \mathbb{R}$, by

$$I_\gamma(\pi) = I_0(\pi) + h_\gamma(\pi_0). \tag{14}$$

Finally, let $\mathcal{A}\subset D([0,T],\mathcal{M})$ be the set of all $\pi_t=\rho(u,t)\mathrm{d}u$ such that their density $\rho(u,t)$ is for some $H\in C_0^{2,1}(\Lambda\times[0,T])$ and $\gamma\in\tilde{C}(\Lambda)$ a unique weak solution of the system

$$\begin{aligned}\partial_t\rho(u,t)&=\frac{1}{2}\Delta\rho(u,t)+\partial_u\big(|\rho(u,t)|(d-\partial_uH(u,t))\big),\\ \rho(u,0)&=\gamma(u),\\ \rho(\pm1,t)&=\pm2\lambda^{\pm}(t).\end{aligned}\tag{15}$$

The uniqueness of the solution of system (15) can be established easily from Gronwall's lemma. It suffices to prove

Lemma 1.1. *Let ρ_1,ρ_2 be two solutions of (15). Then there exists $C<\infty$ depending only on H, such that*

$$\|\rho_1(\cdot,t)-\rho_2(\cdot,t)\|_2^2\le 2(d^2+C)\int_0^t\|\rho_1(\cdot,s)-\rho_2(\cdot,s)\|_2^2\,\mathrm{d}s.\tag{16}$$

Proof. If ρ_1,ρ_2 are solutions, then $\rho_1-\rho_2$ satisfies zero-boundary conditions. Hence

$$\begin{aligned}&\int_0^t\int_{-1}^1(\rho_1-\rho_2)\partial_s(\rho_1-\rho_2)\,\mathrm{d}u\mathrm{d}s\\ &=-\frac{1}{2}\int_0^t\int_{-1}^1\left(\partial_u(\rho_1-\rho_2)\right)^2\mathrm{d}s\mathrm{d}u-\int_0^t\int_{-1}^1(|\rho_1|-|\rho_2|)(d-\partial_uH)\partial_u(\rho_1-\rho_2)\,\mathrm{d}s\mathrm{d}u.\end{aligned}\tag{17}$$

Using partial integration, the left-hand side equals

$$\frac{1}{2}\int_0^t\int_{-1}^1\partial_s\left(\rho_1(s,u)-\rho_2(s,u)\right)^2\mathrm{d}s\mathrm{d}u=\frac{1}{2}\|\rho_1(\cdot,t)-\rho_2(\cdot,t)\|_2^2.\tag{18}$$

To bound the right-hand side, completing the square we get the integrand is not larger than

$$\frac{1}{2}\left(d-\partial_uH(s,u)\right)^2\left(|\rho_1(s,u)|-|\rho_2(s,u)|\right)^2\le\frac{1}{2}(d^2+C)\left(\rho_1(s,u)-\rho_2(s,u)\right)^2,\tag{19}$$

where $C=\|\partial_uH\|_\infty^2$. From here the claimed estimate is obvious. □

We can now state our main theorem:

Theorem 1.2. *Fix $T>0$ and initial profile $\gamma\in\tilde{C}(\Lambda)$. The sequence of measures $Q_{N,\gamma}^{d,\lambda}\equiv P_{N,\gamma}^{d,\lambda}\circ(\pi^N)^{-1}$ on $D([0,T],\mathcal{M})$ satisfies a LDP with rate functional I_γ and speed N^2. Namely, for each closed set $C\subset D([0,T],\mathcal{M})$ and for each open set $O\subset D([0,T],\mathcal{M})$,*

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N^2}\log Q_{N,\gamma}^{d,\lambda}[C]&\le-\inf_{\pi\in C}I_\gamma(\pi),\\ \limsup_{N\to\infty}\frac{1}{N^2}\log Q_{N,\gamma}^{d,\lambda}[O]&\ge-\inf_{\pi\in O\cap\mathcal{A}}I_\gamma(\pi).\end{aligned}\tag{20}$$

Remark. 1. We believe that the restriction to the set $\mathcal{A}\cap O$ in the lower bound is not necessary, however since the rate functional I_γ is not convex we do not know how to leave it out.

The large deviation estimate implies the following law of large numbers.

Theorem 1.3. *Let γ be an initial density profile, $\gamma\in\tilde{C}(\Lambda)$, and let $\rho(u,t)$ be the unique weak solution of the system* (15) *with $H\equiv0$. Then for any $t\in[0,T]$ the sequence of random measures $\pi_t^N\in\mathcal{M}$ converges in probability to the measure $\pi_t(\mathrm{d}u)=\rho(u,t)\mathrm{d}u$ in the topology of the set $\mathcal{M}$.*

2. Coupling of one-type-of-particle processes

To prove both theorems it is convenient to introduce two additional processes where only one type of particles is present. The original $A+B\to\emptyset$ model is then constructed by coupling of those. We use $\eta_t^{(+)}$ and $\eta_t^{(-)}$ to denote these processes.

The first one, $\eta_t^{(+)}$, is generated by $L_{N,t}$ (resp. by $L^H_{N,t}$ if a weakly asymmetric variant is considered) with $\lambda^-(t)\equiv 0$. To define $\eta_t^{(-)}$ we first set $\lambda^+(t)\equiv 0$ in the generator $L_{N,t}$. The generated process, say $\bar\eta_t$, stays always non-positive if the initial configuration is also non-positive. We, therefore, define $\eta_t^{(-)}\equiv -\bar\eta_t$. We use $P^{(\pm)}_{N,\gamma}=P^{d,\lambda,(\pm)}_{N,\gamma}$ to denote the distribution of these processes on $D([0,T],X_N)$, the initial distribution being $\nu^N_{|\gamma \mathbb{1}\{\gamma\gtrless 0\}|}$.

Let us now construct the coupling. Since the formal generator of it would not be transparent, we give only a verbal, but rigorous, description. Consider two processes $\eta_t^{(+)}$, $\eta_t^{(-)}$ as defined above on a common probability space. Additionally, let every particle of both processes carry a mark when it enters Λ_N (i.e. every particle in the initial configuration and every newly injected particle are marked). During the dynamics, when a marked particle from $\eta_t^{(+)}$ meets a marked particle from $\eta_t^{(-)}$, both their marks are erased instantaneously but the particles stay in Λ_N, they do not annihilate each other. We use $\eta_t^{(\pm\bullet)}(x)$, resp. $\eta_t^{(\pm\circ)}(x)$, to denote the numbers of marked, resp. non-marked, particles of type $+$, or $-$ at x, t. Using the introduced notation it is easy to see that the process

$$\tilde\eta_t(x)\equiv \eta_t^{(+\bullet)}(x)-\eta_t^{(-\bullet)}(x)=\eta_t^{(+)}(x)-\eta_t^{(+\circ)}(x)-\big(\eta_t^{(-)}(x)-\eta_t^{(-\circ)}(x)\big) \tag{21}$$

has the same law as $\eta_t(x)$.

We use this coupling frequently. As its first application we prove one (technical) lemma.

Lemma 2.1. *For all $T>0$, d, γ and λ*

$$\lim_{a\to\infty}\frac1a \limsup_{N\to\infty}\frac{1}{N^2}\log Q^{d,\lambda}_{N,\gamma}\big[\sup_{t\le T}\langle|\pi^N_t|,1\rangle\ge a\big]=-\infty. \tag{22}$$

Proof. By definition of π^N_t and (21)

$$\langle|\pi^N_t|,1\rangle=N^{-2}\sum_{x\in\Lambda_N}|\eta_t(x)|\le N^{-2}\sum_{x\in\Lambda_N}\{\eta_t^{(+)}(x)+\eta_t^{(-)}(x)\}. \tag{23}$$

It is therefore sufficient to prove (22) for e.g. $N^{-2}\sum\eta_t^{(+)}(x)$. Let $A_N=[0,T]\cap N^{-1}\mathbb{Z}$ and let $\mathcal{I}_N(t)$ denotes the number of injected particles in the system $\eta^{(+)}$ during the time interval $[t,t+N^{-1}]$. Then

$$\begin{aligned}P^\lambda_N\Big[\sup_{t\le T}N^{-2}\sum_{x\in\Lambda_N}\eta_t^{(+)}(x)\ge a\Big]&\\ \le P_N^{\|\lambda\|_\infty}\Big[\sup_{t\in A_N}\sum_{x\in\Lambda_N}\eta_t^{(+)}(x)\ge\frac{aN^2}{2}\Big]&+P_N^{\|\lambda\|_\infty}\Big[\sup_{t\in A_N}\mathcal{I}_N(t)\ge\frac{aN^2}{2}\Big].\end{aligned} \tag{24}$$

We first control the second term. Since the injection rate was made constant, $\mathcal{I}_N(t)$, $t\in A_N$, are i.i.d. Poisson random variables with mean $N^2\|\lambda\|_\infty$. Therefore

$$P_N^{\|\lambda\|_\infty}\Big[\sup_{t\in A_N}\mathcal{I}_N(t)\ge\frac{aN^2}{2}\Big]\le NT\mathbb{P}\Big[\Pi(\|\lambda\|_\infty N^2)\ge\frac{aN^2}{2}\Big]. \tag{25}$$

(Here and below we use $\Pi(m)$ to denote a Poisson random variable with mean m.) It is then easy to show using the properties of the Poisson random variable (mainly the fact that the rate function of the Poisson distribution increases faster than linearly) that

$$\lim_{a\to\infty}\frac{1}{a}\limsup_{N\to\infty}\frac{1}{N^2}\log P_N^{\|\lambda\|_\infty}\Big[\sup_{t\in A_N}\mathcal{I}_N(t)\geq\frac{aN^2}{2}\Big]=-\infty. \tag{26}$$

To control the first term in (24) we divide every $\eta_t^{(+)}(x)$ into two parts, $\eta_t^{(+)}(x)=\eta_t^i(x)+\eta_t^0(x)$, where $\eta_t^0(x)$ is the number of particles from the initial configuration η_0 that are located at t, x, and $\eta_t^i(x)$ is the number of particles *injected* later at the same place. The contribution of initial particles, $\sum_x\eta_t^0(x)$, decreases with t. Moreover, $\sum_x\eta_0^0(x)$ has the Poisson distribution with mean bounded by $C(\gamma)N^2$. Therefore,

$$\lim_{a\to\infty}\frac{1}{a}\limsup_{N\to\infty}\frac{1}{N^2}\log P_N^{\|\lambda\|_\infty}\Big[\sup_{t\in A_N}N^{-2}\sum_{x\in\Lambda_N}\eta_t^0(x)\geq\frac{a}{4}\Big]=-\infty. \tag{27}$$

For the injected particles we have

$$P_N^{\|\lambda\|_\infty}\Big[\sup_{t\in A_N}N^{-2}\sum_{x\in\Lambda_N}\eta_t^i(x)\geq\frac{a}{4}\Big]\leq NP_N^{\|\lambda\|_\infty}\Big[N^{-2}\sum_{x\in\Lambda_N}\eta_T^i(x)\geq\frac{a}{4}\Big]. \tag{28}$$

It is therefore sufficient to show

$$\lim_{a\to\infty}\frac{1}{a}\limsup_{N\to\infty}\frac{1}{N^2}\log P_N^{\|\lambda\|_\infty}\Big[N^{-2}\sum_{x\in\Lambda_N}\eta_T^i(x)\geq\frac{a}{4}\Big]=-\infty. \tag{29}$$

The probability in the last display equals to

$$\mathbb{P}\Big[\sum_{i=1}^{\Pi(\|\lambda\|_\infty TN^3)}Y_i\geq\frac{aN^2}{4}\Big], \tag{30}$$

where $Y_i=Y_i^N$ are i.i.d. Bernoulli random variables with $\mathbb{P}[Y_i=1]$ being equal to the probability that a particle exposed to the drift d and injected into Λ_N at a time uniformly distributed in $[0,T]$ does not exit Λ_N before time T. Consider now one particle started at site N at time 0 in the system with drift ($\mathbb{P}^d$), resp. without drift ($\mathbb{P}^0$). Let $\tau=\tau_N$ be its exit time from Λ_N. Then, the random variables Y_i satisfy

$$\mathbb{E}[\exp(\omega Y_i)]=T^{-1}\int_0^T 1+(e^\omega-1)\mathbb{P}^d(\tau>t)\,\mathrm{d}t\leq 1+T^{-1}c(d)(e^\omega-1)\int_0^T\mathbb{P}^0(\tau>t)\,\mathrm{d}t. \tag{31}$$

In this inequality we used the fact that in Λ_N the Radon-Nikodym derivative between distributions of trajectory of one particle with and without the drift is easily computed (using, e.g., Proposition 2.6 on page 320 of [7]), and is bounded by a constant. Indeed,

$$\frac{\mathrm{d}\mathbb{P}^d}{\mathrm{d}\mathbb{P}^0}(T)=\exp\Big\{TN^2\Big(\cosh\frac{d}{N}-1\Big)+\frac{d}{N}J\Big\}, \tag{32}$$

where J is the difference between the number of the jumps to the right and to the left, which is always smaller than $2N+1$ (otherwise the particle exits Λ_N).

The probability $\mathbb{P}^0(\tau>t)$ is smaller than the probability that a continuous-time simple random walk with jumping rate 1 started at 0 does not hit the negative half-line before time N^2t. This probability behaves, as is well known, as $(N^2t)^{-1/2}$. Therefore, (31) is bounded by $1+CN^{-1}(e^\omega-1)$. A straightforward application of Chebyshev's exponential inequality proves (29). The computation is standard. For the further reference we formulate one of its steps as a lemma whose easy proof is omitted.

Lemma 2.2. *Let $\Pi(m)$ be a Poisson random variable with mean m and Y_i a sequence of i.i.d. random variables independent of $\Pi(m)$ satisfying $\mathbb{E}[e^{\omega Y_i}] = f(\omega)$ then*

$$\mathbb{E}\Big[\exp\Big(\omega\sum_{i=1}^{\Pi(m)} Y_i\Big)\Big] = \exp\big(m(f(\omega)-1)\big). \tag{33}$$

Lemma 2.1 is then consequence of (24), (26), (27) and (29). □

From Lemma 2.1 follows an easy corollary that we need later.

Corollary 2.3. *Let $A_\delta = \{(t,s)\in[0,T]^2 : 0\le t-s\le\delta\}$. Then for all $a>0$.*

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log Q^{d,\lambda}_{N,\gamma}\Big[\sup_{(t,s)\in A_\delta}\int_s^t \langle|\pi^N_\tau|,1\rangle\,\mathrm{d}\tau \ge a\Big] = -\infty. \tag{34}$$

3. Super-exponential estimate

One of the main steps in proving the large deviations from the hydrodynamic limit is a so-called super-exponential estimate, which allows the replacement of local functions by functions of the empirical density π^N_t and external parameters of the process. Since in our model the mutual interaction between the particles is relatively weak, the only non-trivial part comes from the boundary effects. As we will see later, we need to control deviations of $\eta_t(\pm N)/2N$ from $\pm\lambda^\pm(t)$. To this end we prove

Proposition 3.1. *Let $G\in C([0,T])$ and define*

$$W^\pm_G = \int_0^T G(t)\Big(\frac{\eta_t(\pm N)}{2N}\mp\lambda^\pm(t)\Big)\,\mathrm{d}t. \tag{35}$$

Then for all $\delta>0$ and G fixed

$$\limsup_{N\to\infty}\frac{1}{N^2}\log P^d_N[|W^\pm_G|>\delta] = -\infty. \tag{36}$$

Remark. To our knowledge, large deviations from the hydrodynamic limit for a system with time dependent boundary condition were never studied rigorously. It was not evident for us how to generalise the usual spectral methods used to prove the super-exponential estimate for constant boundary condition to the time-dependent case. The main complication stems in the fact that the process has no natural (equilibrium) measure that can serve as a basis for constructing all required functional vector spaces. That is why we use different methods to prove the proposition. They however only apply in our case and use substantially the fact that the particles are almost non-interacting.

Proof. To simplify the notation we prove claim (36) only for W^+_G. We use the coupling between the two systems with only one type of particles that is introduced in Section 2.

Define $\lambda^{(+)}(t)=\lambda^+(t)$, $\lambda^{(-)}=\lambda^{(+\circ)}=\lambda^{(-\circ)}\equiv 0$, and for $\iota\in\{+,-,+\circ,-\circ\}$

$$W^{(\iota)}_G = \int_0^T G(t)\Big(\frac{\eta^{(\iota)}_t(N)}{2N}-\lambda^{(\iota)}(t)\Big)\,\mathrm{d}t. \tag{37}$$

From (21), it can be seen easily that the claim of the proposition is implied by

$$\limsup_{N\to\infty}\frac{1}{N^2}\log P^d_N[|W^{(\iota)}_G|\ge\delta/4] = -\infty, \qquad \text{for all } \iota\in\{+,-,+\circ,-\circ\}. \tag{38}$$

In the system with drift $d\neq 0$ it is relatively straightforward to control $\eta^{(+)}_t(N)$ and $\eta^{(-)}_t(N)$. Further, since $\eta^{(-\circ)}_t(x)\le\eta^{(-)}_t(x)$ for all x and t, the estimate for $\iota=-\circ$ can be deduced from the one for $\iota=-$. However, it is not clear how to control $\eta^{(+\circ)}_t(N)$ in the presence of the drift. This becomes trivial when the system without drift ($d=0$) is considered, since in this case the coupling can be strengthened: When two marked particles of the opposite types meet, they become non-marked and coalesce together. This is possible only for $d=0$ because the individual dynamics of both types of particles is the same. Using this stronger coupling we have always $\eta_t^{(+\circ)}=\eta_t^{(-\circ)}$ and therefore for $d=0$ the estimate for $\iota=+\circ$ also follows from the one for $\iota=-$. To prove claim (38) we, therefore, need to show that (a) the replacing of the system with drift by the system without drift is permitted, (b) the system without drift satisfies (38).

To prove (a) we compute the Radon-Nikodym derivative of both distributions. It is easy to find that

$$\begin{aligned}\frac{\mathrm{d}P_N^{d,(+)}}{\mathrm{d}P_N^{0,(+)}}(\eta)&=\exp\Big\{N^4\Big(\cosh\Big(\frac{d}{N}\Big)-1\Big)\int_0^T\langle|\pi_s^N|,1\rangle\mathrm{d}s+J_l^+(\eta_t)d/N-J_r^+(\eta_t)d/N\Big\},\\ \frac{\mathrm{d}P_N^{d,(-)}}{\mathrm{d}P_N^{0,(-)}}(\eta)&=\exp\Big\{N^4\Big(\cosh\Big(\frac{d}{N}\Big)-1\Big)\int_0^T\langle|\pi_s^N|,1\rangle\mathrm{d}s+J_r^-(\eta_t)d/N-J_l^-(\eta_t)d/N\Big\},\end{aligned}\tag{39}$$

where $J_r^+(\eta_t)$, $(J_l^+(\eta_t))$ is the number of jumps of $+$ particles to the right (left) in η_t up to time T. Similarly we define $J_r^-(\eta_t)$, $J_l^-(\eta_t)$. Setting $R_T^+=J_l^+(\eta_t)-J_r^+(\eta_t)$ and $R_T^-=J_r^-(\eta_t)-J_l^-(\eta_t)$ we get for $\iota\in\{+,-\}$

$$\frac{\mathrm{d}P_N^{d,(\iota)}}{\mathrm{d}P_N^{0,(\iota)}}(\eta)=\exp\{R_T^\iota dN^{-1}\}\exp\Big\{c(d)N^2\int_0^T\langle|\pi_s^N|,1\rangle\mathrm{d}s\Big\}.\tag{40}$$

The following lemma, whose proof can be found later in this section, controls R_T^+, R_T^-.

Lemma 3.2. *For all functions γ, λ^+, and every $d\in\mathbb{R}$ there is a function $f=f_{\gamma,d,\lambda}$ satisfying $\lim_{K\to\infty}f(K)=\infty$, such that*

$$\limsup_{N\to\infty}\frac{1}{N^2}\log P_N^d[R_T^\iota>KN^3]=-f(K),\qquad\iota\in\{+,-\}.\tag{41}$$

In the system without drift the estimates for $\eta^{(\pm\circ)}$ follow from the estimate for $\eta^{(-)}$ as we have already remarked. Therefore, to prove claim (b) it is sufficient to show

Lemma 3.3. *For $\iota\in\{+,-\}$, $G\in C([0,T])$ and $\delta>0$*

$$\limsup_{N\to\infty}\frac{1}{N^2}\log P_N^0[|W_G^{(\iota)}|\geq\delta/4]=-\infty.\tag{42}$$

We first use the last two lemmas to finish the proof of Proposition 3.1. We define R_T by $R_T\equiv\max R_T^+,R_T^-$. Then for any $K>0$ and all N large enough

$$\begin{aligned}P_N^d\big[|W_G^{(\iota)}|\geq\delta/4\big]&\leq P_N^d\Big[|W_G^{(\iota)}|\geq\delta/4,R_T\leq KN^3,\int_0^T\langle|\pi_s^N|,1\rangle\mathrm{d}s\leq K\Big]\\&\qquad+P_N^d[R_T>KN^3]+P_N^d\Big[\int_0^T\langle|\pi_s^N|,1\rangle\mathrm{d}s>K\Big].\end{aligned}\tag{43}$$

This is by Lemmas 2.1 and 3.2 bounded by

$$\begin{aligned}&\leq E_N^0\Big[\frac{\mathrm{d}P_N^d}{\mathrm{d}P_N^0}\mathbb{1}\Big\{|W_G^{(\iota)}|\geq\delta/4,R_T\leq KN^3,\int_0^T\langle|\pi_s^N|,1\rangle\mathrm{d}s\leq K\Big\}\Big]+e^{-f(K)N^2/2}\\&\leq e^{c(d)KN^2}P_{N,\gamma}^0\big[|W_G^{(\iota)}|\geq\delta/4\big]+e^{-f(K)N^2/2}.\end{aligned}\tag{44}$$

The claim (38) and therefore Proposition 3.1 then follows easily taking first $N\to\infty$, using Lemma 3.3, and finally taking K arbitrarily large. $\square$

Proof of Lemma 3.2. Without loss of generality we consider only R_T^+. We decompose it into two parts, $R_T^+ = R_T^0 + R_T^i$, where R_T^0 is the part of R_T^+ that is due to particles present in system at time $t = 0$, and R_T^i is due to particles injected later.

The number of particles existing at $t = 0$ has the Poisson distribution with mean $C_\gamma N^2(1 + o(1))$. In Λ_N the difference between the numbers of steps of one particle to the right and the left is at most $2N + 1$. Therefore, it is easy to get using large deviation upper bound for the Poisson random variable that

$$P^d_{N,\gamma}\big[R_T^0 > KN^3/2\big] \le e^{-f_\gamma(K)N^2}, \tag{45}$$

with $f_\gamma(K)$ diverging as $K \to \infty$.

Any particle injected to Λ_N at site N contributes to R_T either by 0 if it exists Λ_N at N before time T, or by at most $2N + 1$ if it exits at $-N$ or if it stays in Λ_N up to time T. Therefore

$$P^d_N[R^i_T > KN^3/2] \le \mathbb{P}\Bigg[\sum_{i=1}^{\Pi(C_{\lambda^+}N^3)} Y_i > KN^2/8\Bigg] + P^d_N\Bigg[\sum_{x\in\Lambda_N} \eta^i_T(x) \ge KN^2/8\Bigg], \tag{46}$$

where $\eta^i_T(x)$ is the number of the injected particles at x, T as in the proof of Lemma 2.1, and $Y_i = Y_i^N$ are i.i.d. Bernoulli random variables such that $\mathbb{P}[Y_i = 1]$ is equal to the probability that a particle injected at N exits at $-N$ (in the system with drift). This probability can be bounded from above by C_d/N. An easy proof of this claim is left to the reader. It is then possible to use Chebyshev's inequality and Lemma 2.2 to bound the first term in the previous display by

$$\mathbb{P}\Bigg[\sum_{i=1}^{\Pi(C_{\lambda^+}N^3)} Y_i \ge KN^2/8\Bigg] \le e^{-f_{\lambda_+,d}(K)N^2}, \tag{47}$$

with diverging $f_{\lambda_+,d}$. The bound for the second term follows from (29). This finishes the proof of Lemma 3.2. □

Proof of Lemma 3.3. We want to show that for all initial profiles γ, all injection intensities λ and every $\delta > 0$

$$\begin{aligned} &P^0_{N,\gamma}\Bigg[\bigg|\int_0^T G(t)\Big\{\frac{\eta_t^+(N)}{2N} - \lambda^+(t)\Big\}\,\mathrm{d}t \ge \delta\bigg|\Bigg] \le e^{-KN^2},\\ &P^0_{N,\gamma}\Bigg[\int_0^T G(t)\frac{\eta_t^-(N)}{2N}\,\mathrm{d}t \ge \delta\Bigg] \le e^{-KN^2} \end{aligned} \tag{48}$$

for all $K > 0$ and N large enough. For sake of brevity we prove only the first inequality, the second one can be proved using very similar methods. Without loss of generality we choose G to be an indicator function, $G(t) = \mathbb{1}[t_1, t_2](t)$ for $0 < t_1 < t_2 \le T$.

We again treat separately the initial and the injected particles. Let $x_1, \dots, x_k$ be the positions of the particles being in Λ_N at time $t = 0$. Here k has Poisson distribution with mean $N\sum_{x\in\Lambda_N}\gamma(x/N)$. Let further $s_1, \dots, s_l$ be the times when the other particles are injected, l has again Poisson distribution with mean $N^3\int_0^T \lambda(t)\,\mathrm{d}t$. We use $\ell_i(x,t)$, $i \le k + l$, to denote the time spent by initial particle i or the injected particle $i - k$ at x before the time t. Then for G chosen as above we can write

$$\frac{1}{2N}\int_0^T G(t)\eta_t^+(N)\,\mathrm{d}t = \frac{1}{2N}\sum_{i=1}^{k+l} \ell_i(N,t_2) - \ell_i(N,t_1). \tag{49}$$

We use $\ell_N(x,t;x_0,t_0)$ to denote the random variables with same law as the local time at x, t of the continuous-time non-accelerated simple random walk that is started at x_0,

t_0 and killed on exit from Λ_N. For two random variables X, Y we write $X \preceq Y$ if $P[X \geq u] \leq P[Y \geq u]$ for all $u \in \mathbb{R}$.

We first estimate the contribution of the initial particles to the right-hand side of (49). Observe that for these particles

$$\ell_i(N, t_2) - \ell_i(N, t_1) \leq \ell_i(N, T) \preceq N^{-2}\ell_N(N, TN^2; N, 0) \preceq N^{-2}\ell_N(N, \infty; N, 0). \tag{50}$$

The second inequality follows from the decomposition on the first visit of N. It is easy to see that $\ell_N(N, \infty; N, 0)$ is exponentially distributed with mean $2(1 - 1/N)$. Therefore, the contribution of the initial particles satisfies (denoting by e_i i.i.d. mean-one exponentials)

$$\frac{1}{2N}\sum_{i=0}^{k} \ell_i(N, t_2) - \ell_i(N, t_1) \preceq N^{-3} \sum_{i=0}^{\Pi[C(\gamma)N^2]} e_i. \tag{51}$$

Using Chebyshev's inequality and Lemma 2.2 we get

$$P^0_{N,\gamma}\Big[\frac{1}{2N}\sum_{i=1}^{k} \ell_i(N, t_2) - \ell_i(N, t_1) \geq \delta\Big] \leq \exp(-\delta\omega N^3 + cC(\gamma)N^2(1-\omega)^{-1}) \ll e^{-KN^2}. \tag{52}$$

We further treat the injected particles. We first prove that the contribution of those with injection times $s_k < t_1 - (N)^{-1/2}$ and also of those with $s_k \geq t_1 - (N)^{-1/2}$ after $s_k + (N)^{-1/2}$ is negligible. Let

$$E_N \equiv \frac{1}{2N}\sum_{i=1}^{l} \ell_i(N, t_2) - \ell_i\big(N, t_1 \vee (s_i + (N)^{-1/2})\big). \tag{53}$$

It is easy to see that

$$\begin{aligned} E_N &\leq \frac{1}{2N}\sum_{i=1}^{l} \ell_i(N, \infty) - \ell_i\big(N, t_1 \vee (s_i + (N)^{-1/2})\big) \\ &\preceq \frac{1}{2N^3}\sum_{i=1}^{l} \ell_N(N, \infty; N, 0) - \ell_N(N, N^{3/2}; N, 0) \equiv \frac{1}{2N^3}\sum_{i=1}^{l} Y_i. \end{aligned} \tag{54}$$

Decomposing on the first hit of N after $N^{3/2}$, it can be seen that Y_i's have the same law as $Z_i\ell_N(N, \infty; N, 0)$, where Z_i are i.i.d., Bernoulli random variables with success probability r_N equal to the probability that the non-accelerated random walk in Λ_N started at N survives up to time $N^{3/2}$ and then returns to N. The probability r_N tends to 0 as N increases. Indeed, e.g.,

$$\begin{aligned} r_N \leq\ & \mathbb{P}[\text{SRW reaches } N - N^{3/8} \text{ before dying}] \\ & + \mathbb{P}[\text{SRW survives time } N^{3/2} \text{ in box of size } N^{3/8}], \end{aligned} \tag{55}$$

and the both terms on the right hand side converge to 0 as can be proved easily. A standard application of Chebyshev's inequality and Lemma 2.2 gives

$$P_{N,\gamma}[E_n \geq \delta] \leq \exp\{-2\omega N^3 + C(\lambda) r_N N^3((1 - 2\omega)^{-1} - 1)\} \ll e^{-KN^2}. \tag{56}$$

The only non-negligible contribution comes from the particles injected in the time interval $[t_1, t_2]$. As can be seen from the result of the previous paragraph, all these contribution should originate from first $(N)^{-1/2}$ time units of their life. We estimate only the contribution of those with $s_i \in [t_1, t_2 - (N)^{-1/2}] \equiv I_N$. The proof of the fact that particles with

$s_i \in [t_1 - (N)^{-1/2}, t_1]$ and $s_i \in [t_2 - (N)^{-1/2}, t_2]$ can be also neglected (because there is too little of them) is left to the reader. We want to estimate

$$\begin{aligned} F_N &\equiv \frac{1}{2N} \sum_{i=1}^{l} \mathbb{1}\{s_i \in [t_1, t_2 - (N)^{-1/2}]\}\big(\ell_i(N, s_i + (N)^{-1/2}) - \ell_i(N, s_i)\big) \\ &\overset{law}{=} \frac{1}{2N^3} \sum_{i=1}^{\Pi\left(N^3 \int_{I_N} \lambda(t)\,\mathrm{d}t\right)} Y_i^N, \end{aligned} \tag{57}$$

where Y_i^N are this time i.i.d. with the same law as $\ell_N(N, N^{3/2}; N, 0)$. Clearly, $Y_i^N \preceq Y_i^{N+1}$ and Y_i^N is asymptotically exponentially distributed with mean 2. In particular, the Laplace transform of Y_i^N converges to $(1-2\omega)^{-1}$ uniformly on any compact interval. A trivial large deviation argument then gives

$$\mathbb{P}_{N,\gamma}\Big[\Big|F_N - \int_{t_1}^{t_2} \lambda(t)\,\mathrm{d}t\Big| \geq \delta\Big] \leq e^{-KN^2} \tag{58}$$

for all $K > 0$ and N large enough. The proof of the lemma then follows from the last inequality together with (52) and (56). □

4. The upper bound

4.1. **Bound for compact sets.** The proof of the LDP upper bound is essentially the same as in [8]. For $H \in C_0^{2,1}(\Lambda \times [0, T])$ we consider the exponential martingale $M_N^H(t)$ defined by

$$\begin{aligned} M_N^H(t) = \frac{\mathrm{d}Q_{N,\gamma}}{\mathrm{d}Q_{N,\gamma}^H} = \exp\Big\{ &- N^2\Big[\langle \pi_t^N, H_t\rangle - \langle \pi_0^N, H_0\rangle \\ &- N^{-2}\int_0^t e^{-N^2\langle \pi_s^N, H_s\rangle}(\partial_s + L_{N,s})e^{N^2\langle \pi_s^N, H_s\rangle}\,\mathrm{d}s\Big]\Big\}. \end{aligned} \tag{59}$$

A standard but rather lengthy calculation shows that

$$\begin{aligned} M_N^H(T) = \exp\Big\{ - N^2\Big[&\langle \pi_T^N, H_T\rangle - \langle \pi_0^N, H_0\rangle - \int_0^T \langle \pi_t^N, (\partial_t + \tfrac{1}{2}\Delta_N)H_t\rangle\,\mathrm{d}t \\ &- \int_0^T \langle |\pi_t^N|, \tfrac{1}{2}(\nabla_N H_t)^2\rangle\,\mathrm{d}t + d\int_0^T \langle |\pi_t^N|, \nabla_N H_t\rangle\,\mathrm{d}t \\ &+ \sum_{\alpha=\pm 1} \alpha \int_0^T \frac{\eta_t(\alpha N)}{2N}\nabla_N H_t(\alpha)\,\mathrm{d}t + O_H(N^{-1})C(\pi^N)\Big]\Big\} \\ = \exp\big\{ - N^2\big[&J_H(\pi^N) + V_N^H(\eta^N) + C(\pi^N)O_H(N^{-1})\big]\big\}, \end{aligned} \tag{60}$$

where we use ∇_N and Δ_N to denote the discrete derivative, resp. Laplace operator with mesh size N^{-1},

$$V_N^H(\eta) \equiv \sum_{\alpha=\pm 1} \int_0^T \Big(\alpha\frac{\eta_t(\alpha N)}{2N} - \lambda^\alpha(t)\Big)\partial_u H_t(\alpha)\,\mathrm{d}t, \tag{61}$$

$C(\pi) = \int_0^T \langle |\pi_t|, 1\rangle\,\mathrm{d}t$ and $|O_H(N^{-1})| \leq C(H)N^{-1}$. For all $\delta > 0$ we define the set

$$B_{N,\delta}^H \equiv \{\eta \in D([0, T], X_N) : |V_N^H(\eta)| \leq \delta\}. \tag{62}$$

By Proposition 3.1

$$\limsup_{N\to\infty} \frac{1}{N^2} \log P_{N,\gamma}[(B_{N,\delta}^H)^c] = -\infty. \tag{63}$$

Let $O \subset D([0,T], \mathcal{M})$ be a open set such that $C(O) \equiv \sup_{\pi \in O} C(\pi) < \infty$. Using $O \subset (O \cap B^H_{N,\delta}) \cup (B^H_{N,\delta})^c$ and (63) we get

$$\limsup_{n\to\infty} \frac{1}{N^2} \log Q_{N,\gamma}[O] = \limsup_{n\to\infty} \frac{1}{N^2} \log Q_{N,\gamma}[O \cap B_{H,\delta}]. \tag{64}$$

The probability on the right hand side can be written as

$$\begin{aligned} Q_{N,\gamma}[O \cap B_{H,\delta}] &= E^H_{N,\gamma'}\Big[M^H_N(T) \frac{\mathrm{d}\nu^N_\gamma}{\mathrm{d}\nu^N_{\gamma'}} \mathbb{1}\{O \cap B_{H,\delta}\}\Big] \\ &\le \sup_{\pi \in O} \exp\{N^2(-J_H(\pi) - h_{\gamma'|\gamma}(\pi_0) + O(\delta) + C(\pi)C(\gamma,\gamma',H)N^{-1})\}, \end{aligned} \tag{65}$$

where the γ'-dependence of the error term comes from the approximation of the $\mathrm{d}\nu^N_\gamma / \mathrm{d}\nu^N_{\gamma'}$ by $\exp(-N^2 h_{\gamma'|\gamma})$. Letting $N \to \infty$ and optimising over H, δ and γ' we get

$$\limsup_{N\to\infty} \frac{1}{N^2} \log Q_{N,\gamma}[O] \le \inf_{H,\gamma',\delta} \sup_{\pi\in O} \big\{ - J_H(\pi) - h_{\gamma'|\gamma}(\pi_0) + O(\delta) + C(O)C(\gamma,\gamma',H)N^{-1}\big\}. \tag{66}$$

Since J_H and $h_{\gamma'|\gamma}$ are lower semi-continuous we can use e.g. Lemma A2.3.3. from [7]. Therefore for each compact set $K \subset D([0,T], \mathcal{M})$

$$\limsup_{N\to\infty} \frac{1}{N^2} \log Q_{N,\gamma}[K] \le \sup_{\pi\in K} \inf_{H,\gamma',\delta} \big\{ - J_H(\pi) - h_{\gamma'|\gamma}(\pi_0) + O(\delta) + C(K)C(\gamma,\gamma',H)N^{-1}\big\}. \tag{67}$$

The lemma should be applied with a little bit of care of the error term $C(O)$. It is however easy to see that for any compact set K, the constant $C(K) = \sup_{\pi \in K} C(\pi) < \infty$ and therefore one can cover this set with open sets O with $C(O) < (1+\delta)C(K)$.

4.2. **Exponential tightness.** To pass from compact to closed sets one should prove the exponential tightness of the sequence Q^N. The proof on pages 271–273 of [7] can be easily adapted to our case. We only sketch the differences.

First, we need to show that for every continuous function $H : \Lambda \to \mathbb{R}$ and every $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N^2} \log Q_N \Big[\sup_{|t-s|\le\delta} \big|\langle \pi^N_t, H\rangle - \langle \pi^N_s, H\rangle\big| > \varepsilon \Big] = -\infty. \tag{68}$$

We have

$$Q_N \Big[\sup_{|t-s|\le\delta} \big|\langle \pi^N_t, H\rangle - \langle \pi^N_s, H\rangle\big| > \varepsilon \Big] \le \sum_{i=0}^{\delta^{-1}T} Q_N \Big[\sup_{\tau\le 2\delta} \big|\langle \pi^N_{\delta i}, H\rangle - \langle \pi^N_{\delta i+\tau}, H\rangle\big| > \varepsilon/2 \Big]. \tag{69}$$

Therefore, it is sufficient to prove a bound (uniform in i) of type (68) for all summands in (69). For all $s > 0$ and $a > 0$ we define martingales (similarly as in (59))

$$\begin{aligned} M^{aH}_N(t;s) = \exp\Big\{ N^2 \Big[& a\langle \pi^N_t, H\rangle - a\langle \pi^N_0, H\rangle \\ & - N^{-2} \int_s^t e^{-N^2 \langle \pi^N_\tau, aH\rangle} (\partial_\tau + L_{N,\tau}) e^{N^2\langle \pi^N_\tau, aH\rangle} \,\mathrm{d}\tau \Big] \Big\}. \end{aligned} \tag{70}$$

We use $A_{t,s}(a)$ to denote the integral inside of the exponential. In order to prove the required bound it is enough to prove a statement of type (68) (with $a\varepsilon$ instead of ε) for $N^{-2} \log M^{aH}_N(t;s)$ and for $N^{-2} A_{t,s}(a)$ uniformly for $s \in [0,T] \cap \delta\mathbb{Z}$. Since $M^{aH}_N(\cdot;s)$ are mean one positive martingales, they can be treated using Chebyshev's exponential inequality as in [7]. Using the same computation as in (60), we observe that $N^{-2}A_{t,s}(a)$ is bounded by

$$C(a,H)\Big\{\int_s^t \langle|\pi_\tau^N|,1\rangle + \frac{\eta_\tau(N)+\eta_\tau(-N)}{2N}\,\mathrm{d}\tau\Big\} \tag{71}$$

Therefore the estimate for $N^{-2}A_{t,s}(a)$ follows from Corollary 2.3 and Proposition 3.1. This proves (68). The proof of the exponential tightness can be then continued exactly as in [7]; only change is that Lemma 2.1 should replace the bound $\eta_t(x)\le 1$ which is valid for the exclusion processes treated there.

4.3. **Proof of the law of large numbers.** Let $\rho(u,t)$ be the unique solution of the system (15) with $H\equiv 0$. (Uniqueness is guaranteed by Lemma 1.1.) To prove the law of large numbers it is sufficient to show that $\rho(u,t)\,\mathrm{d}x$ is the unique element of $D([0,T],\mathcal{M})$ such that I_γ is equal to 0.

To this end we define for a given absolutely continuous path π in $D([0,T],\mathcal{M})$ the inner product $\langle\cdot,\cdot\rangle_\pi$ on $C^{2,1}(\Lambda\times[0,T])$ by

$$\langle G,H\rangle_\pi = \int_0^T \langle|\pi_t|,\partial_u G\,\partial_u H\rangle\,\mathrm{d}t. \tag{72}$$

If $\mathcal{N}(\pi)$ denotes the kernel of this inner product, we define $\mathcal{H}_1(\pi)$ as the completion of $C^{2,1}(\Lambda\times[0,T])|_{\mathcal{N}(\pi)}$ with respect to the corresponding norm. It can be then proved as in [7], pp. 274, that there exists $F_\pi\in\mathcal{H}_1$ such that for all $G\in\mathcal{H}_1$

$$\ell_G(\pi)=\langle G,F_\pi\rangle_\pi, \qquad J_G(\pi)=\langle G,F_\pi\rangle_\pi-\frac12\langle G,G\rangle_\pi, \tag{73}$$

and

$$I_\gamma(\pi)=h_\gamma(\pi)+\frac12\langle F_\pi,F_\pi\rangle_\pi. \tag{74}$$

So, I_γ can be 0 only if $\pi_0(\mathrm{d}u)=\gamma(u)\mathrm{d}u$ and $F_\pi\equiv 0$, which in turns implies that $\ell_G(\pi)=0$ for every G and therefore π has a density that is the solution of (15) with $H\equiv 0$.

5. The lower bound

5.1. **Laws of large numbers for weakly asymmetric systems.** To prove the lower bound we should prove a family of laws of large numbers for weakly asymmetric systems with generator (6). First, we extend the validity of the super-exponential estimate (Proposition 3.1) to these systems.

Lemma 5.1. *For all $H\in C_0^{2,1}(\Lambda\times[0,T])$, $G\in C([0,T])$ and $\delta>0$*

$$\limsup_{N\to\infty}\frac{1}{N^2}\log P_N^{d,H}[|W_G^\pm|>\delta]=-\infty. \tag{75}$$

Proof. The probability in question can be written as

$$P_N^{d,H}[|W_G^\pm|>\delta]=E_N^d\Big[\frac{\mathrm{d}P_N^{d,H}}{\mathrm{d}P_N^d}\mathbb{1}\{|W_G^\pm|>\delta\}\Big]. \tag{76}$$

By Hölder inequality this is bounded by

$$E_N^d\Big[\Big(\frac{\mathrm{d}P_N^{d,H}}{\mathrm{d}P_N^d}\Big)^2\Big]^{1/2}\big(P_N^d[|W_G^\pm|>\delta]\big)^{1/2}. \tag{77}$$

If we show that the first factor is bounded by $e^{C_H N^2}$ for some C_H not depending on N, then the proof is easily finished using Proposition 3.1.

To achieve this we use the explicit expression for the Radon-Nikodym derivative (60),

$$E_N^d\Big[\Big(\frac{\mathrm{d}P_N^{d,H}}{\mathrm{d}P_N^d}\Big)^2\Big] \le E_N^d\Big[\exp\Big\{2N^2C(H)C(\pi^N)+2N^2C(H)\int_0^T\sum_{\alpha=\pm1}\frac{|\eta_t(\alpha N)|}{2N}\,\mathrm{d}t\Big\}\Big]$$
$$\le\Big(E_N^d\big[\exp\{4N^2C(H)C(\pi^N)\}\big]E_N^d\Big[\exp\Big\{4N^2C(H)\int_0^T\sum_{\alpha=\pm1}\frac{|\eta_t(\alpha N)|}{2N}\,\mathrm{d}t\Big\}\Big]\Big)^{1/2} \tag{78}$$

again by Hölder inequality. The both factors can be bounded by e^{cN^2}. For first one it is enough to use Lemma 2.1 (namely the fact that $Q_N[\sup_t\langle|\pi_t^N|,1\rangle\ge a]\le e^{-f(a)N^2}$ for $f(a)$ such that $f(a)/a\to\infty$) together with Laplace-Varadhan type of argument. The same argument applies for the second term, Proposition 3.1 gives enough control of of probability that $|\eta_t(\alpha N)|$ becomes large. □

Being equipped with the super-exponential estimate we can proceed as in the proof of the upper bound. Namely, (65) becomes

$$Q_{N,\gamma}^H[O\cap B_{G,\delta}]=E_{N,\gamma'}^G\Big[M_N^G(T)M_N^H(T)^{-1}\frac{\mathrm{d}\nu_\gamma^N}{\mathrm{d}\nu_{\gamma'}^N}\mathbb{1}\{O\cap B_{G,\delta}\}\Big]. \tag{79}$$

Further steps that give a large deviation upper bound: For any closed set C

$$\limsup_{N\to\infty}\frac{1}{N^2}\log Q_{N,\gamma}^H[C]\le-\inf_{\pi\in C}\sup_{G\in C_0^{2,1}(\Lambda\times[0,T])}\big\{J_G(\pi)-J_H(\pi)+h_\gamma(\pi_0)\big\}. \tag{80}$$

Using the formula (73) it is not difficult to see that the right-hand side of the last display equals

$$-\inf_{\pi\in C}\big\{h_\gamma(\pi_0)+\langle F_\pi-H,F_\pi-H\rangle_\pi\big\}. \tag{81}$$

The minimiser of this functional thus satisfy $\ell_G(\pi)=\langle G,H\rangle_\pi$ for all $G\in C_0^{2,1}(\Lambda\times[0,T])$ and therefore it is weak solution of system (15) for the given H. This prove the law of large numbers for the weakly asymmetric systems.

5.2. **The lower bound.** The proof of the lower bound is then standard, see e.g. [7], pp. 275. Again Lemma 2.1 should be used to keep $\langle|\pi_t^N|,1\rangle$ bounded.

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