

# A dual approach to regularity in thin film micromagnetics

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## Abstract

We prove an optimal regularity result in the two dimensional theory of soft ferromagnetic films. The associate Euler-Lagrange equation is given by a microlocally degenerate variational inequality involving fractional derivatives. A difference quotient type argument based on a dual formulation in terms of magnetostatic potentials yields a Hölder estimate for the uniquely determined gradient projection of the magnetization field.

## 1 Introduction

We prove an optimal regularity result in the two dimensional theory of soft ferromagnetic films. Soft means that the crystalline anisotropy is negligible in the micromagnetic energy. Recently, a thin film model was derived by A. DeSimone, R.V. Kohn, S. Müller, and F. Otto, see [7]. The variational principle emerged as the  $\Gamma$ -limit on the scale of the square of the film thickness. It consists in the competition of the limiting magnetostatic energy, the alignment with the external field, and a relaxed saturation constraint:

$$\mathcal{E}(m) = \frac{1}{4} \|\mathcal{H}(m)\|_{\dot{H}^{1/2}}^2 - \int_{\Omega} h \cdot m \rightarrow \min \quad (1.1)$$
$$m : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ with } |m| \leq 1 \text{ and } \mathcal{H}(m) \in H^{1/2}(\mathbb{R}^2; \mathbb{R}^2),$$

where  $\mathcal{H}(m)$  denotes the two-dimensional Helmholtz projection on gradients (here  $m = m\chi_{\Omega} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is understood to be trivially extended) and  $\|f\|_{\dot{H}^{1/2}}^2 = \int |\xi| |\hat{f}(\xi)|^2 d\xi$  is given by the homogeneous semi-norm on the fractional Sobolev space  $H^{1/2}(\mathbb{R}^2)$ . Note that the magnetostatic interaction term can equivalently be written as  $\|\mathcal{H}(m)\|_{\dot{H}^{1/2}}^2 = \|\nabla' \cdot m\|_{\dot{H}^{-1/2}}^2$ . Thus, the main feature of this variational principle is that for a gradient field  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , the energy only depends on the magnetization  $m$  via the distributional charge density  $\rho = \nabla' \cdot m : \mathbb{R}^2 \rightarrow \mathbb{R}$ , and it

is strictly convex in  $\rho$ . Moreover the saturation constraint is relaxed to the convex constraint  $|m| \leq 1$ . The set of planar magnetization fields  $\{|m| \leq 1 \text{ in } \Omega\}$  with prescribed magnetic charge density is large. We can add a gauge, i.e. a rotated gradient  $\nabla^\perp \psi$  with  $\psi|_{\partial\Omega} = 0$ , which leaves the charge density unchanged, i.e.  $\mathcal{E}(m + \nabla^\perp \psi) = \mathcal{E}(m)$ . Indeed, given a minimizer  $m_0$  of (1.1), one is mainly interested in restoring saturation, which leads to the following Hamilton-Jacobi equation

$$|\nabla' \psi + m_0^\perp| = 1 \quad \text{in } \Omega \quad \text{and} \quad \psi = 0 \quad \text{on } \partial\Omega. \quad (1.2)$$

Clearly the solution of this (fully nonlinear) first order equation will give rise to the singular domain structure, that one would expect, cf, [10]. The solvability of the Dirichlet problem (1.2) highly depends on the regularity of the data, i.e. the regularity of the variational solution  $m_0$ . Indeed finding a continuous representative  $m_0$ , we can solve (1.2) in the sense of viscosity solutions, see e.g. [3].

From the variational principle (1.1), however, we only expect improved regularity for the uniquely determined gradient portion  $\mathcal{H}(m_0)$  of  $m_0$ . Accordingly, we show

**Theorem.** *The uniquely determined gradient part  $\mathcal{H}(m_0)$  of a minimizer  $m_0$  of the reduced thin film variational principle (1.1) corresponding to a smooth gradient field  $h = -\nabla' H$  is locally Hölder continuous with Hölder exponent  $\alpha = 1/2$ .*

A similar regularity result was achieved in [7] section 5 and covers crystalline anisotropy as well. The argument, however, is a global one and restricted to simply connected domains. Our argument relies on a dual formulation of (1.1) in terms of a magnetostatic potential  $u : \mathbb{R}^3 \rightarrow \mathbb{R}$  correlated via a minimax principle

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\Omega} |\nabla' u - h| dx' \rightarrow \min \quad \text{for } u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \quad (1.3)$$

The trace of the magnetostatic potential  $u$  is in turn correlated with  $\mathcal{H}(m)$  by the vectorial Riesz transform, i.e.  $u(\cdot, 0) = \mathcal{R} \cdot m = \mathcal{R} \cdot \mathcal{H}(m)$ . Thus we can infer continuity for  $\mathcal{H}(m_0)$  from the dual problem by proving higher regularity of the associated magnetostatic potential  $u_0$ .

The advantage of the dual formulation (1.3) is that the variational principle is strictly convex, entirely local, and does not involve any constraint. Nonetheless we have to take into account that the functional of the dual problem is no longer Fréchet differentiable. Hence, to proceed with a difference quotient method, the functional has to be regularized, but finally we can conclude the desired estimates by a soft  $\Gamma$ -convergence argument.

## 2 A reduced model in thin film micromagnetics

### Notation

We denote by  $x' \in \Omega \subset \mathbb{R}^2$  the tangential components of a position vector  $x = (x', x_3) \in \Omega \times \mathbb{R}$ . We will also attach a prime to differential operators acting on functions defined on  $\mathbb{R}^2$ , e.g.  $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$  or  $\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ .

Since we mainly consider tangential vector fields  $m : \Omega \rightarrow \mathbb{R}^2$  we deviate from the notation in [7], where a prime is attached to tangential vector fields. Instead, we signify a three-dimensional vector field by a fat letter  $\mathbf{m}$ .

### 2.1 The thin film variational principle

We shortly review the main setup for the thin film variational principle, that we are going to consider. For general information on micromagnetic models and their mathematical treatment, we refer to the monograph [10] and the survey [6]. For the rigorous discussion of the underlying  $\Gamma$ -convergence result we refer to [7]. For a bounded base domain  $\Omega \subset \mathbb{R}^2$  of unit area and a thickness parameter  $t > 0$ , let a ferromagnetic film be represented by the cylindrical domain

$$\Omega(t) = \Omega \times (0, t) \subset \mathbb{R}^3 \quad \text{with magnetization } \mathbf{m} = (m, m_3) : \Omega(t) \rightarrow \mathbb{S}^2.$$

We denote by  $d$  the exchange length of the magnetic material. Under the influence of an external magnetic field  $h_t$  which is supposed to be tangential to the film and which has the form  $h_t = t h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , the renormalized micromagnetic energy per unit volume is given by

$$\mathcal{E}_t(\mathbf{m}) = \kappa^2 t \int_{\Omega(t)} |\nabla \mathbf{m}|^2 dx + t^{-2} \int_{\mathbb{R}^3} |\mathcal{H}(\mathbf{m})|^2 dx - 2 \int_{\Omega(t)} h \cdot \mathbf{m} dx,$$

where  $\kappa$  is an aspect ratio  $d/t$  and  $\mathcal{H}(\mathbf{m})$  is determined by the static Maxwell's equations

$$\nabla \cdot [\mathcal{H}(\mathbf{m}) - \mathbf{m}] = 0 \quad \text{and} \quad \nabla \times \mathcal{H}(\mathbf{m}) = 0 \quad \text{in } \mathbb{R}^3.$$

The operator  $\mathbf{m} \mapsto \mathcal{H}(\mathbf{m})$  is just the Helmholtz projection and can be identified with the zero order pseudo-differential operator  $\nabla \Delta^{-1} \nabla \cdot \mathbf{m}$ . For the variational principle

$$\mathcal{E}_t(\mathbf{m}) \rightarrow \min \quad \text{in } \{|\mathbf{m}| = 1 \text{ in } \Omega(t)\}$$

one is interested in a physically relevant thin film limit, featuring aspects of the thin film limit of Gioia and James [9] as well as the large body limit of DeSimone [5] and Tartar [13]. Let us give an informal argument for a such limiting variational principle. For simplicity we will assume that the magnetization field is independent of the cross section, i.e.  $\frac{\partial \mathbf{m}}{\partial x_3} = 0$  in  $\Omega(t)$ . By a scaling argument

for the exchange energy, this is asymptotically true for thin films, i.e. as  $t/d \rightarrow 0$ , see [9] and [7]. Under this assumptions the Fourier transform yields the following formal expansion for the magnetostatic energy

$$2 \int_{\mathbb{R}^3} |\mathcal{H}(\mathbf{m})|^2 dx = t \|m_3\|_{L^2(\mathbb{R}^2)}^2 + t^2 \|\mathcal{H}(m)\|_{\dot{H}^{1/2}(\mathbb{R}^2)}^2 + o(t^2)$$

as  $t$  tends to zero, where  $\mathcal{H}(m)$  denotes the two-dimensional Helmholtz projection of the trivial extension of  $m$  characterized by

$$\nabla' \cdot [\mathcal{H}(m) - m] = 0 \quad \text{and} \quad \nabla^\perp \cdot \mathcal{H}(m) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \quad (2.1)$$

Thus, after renormalization, the penalty on  $|m_3| \neq 0$  enforces asymptotically, as  $t \rightarrow 0$ , the magnetization field to be in-plane. Accordingly, the variational principle reduces to a two-dimensional field theory. This can be made rigorous in terms of variational convergence:

It is shown in [7] that for each sequence  $t \rightarrow 0$  in the regime where  $d = d(t) \rightarrow 0$  such that

$$t \kappa(t)^2 \log(1/t) \rightarrow 0 \quad \text{for} \quad \kappa(t) = \frac{d(t)}{t}$$

the full variational principle  $\Gamma$ -converges to the thin film variational principle on planar vector fields  $m : \Omega \rightarrow \mathbb{R}^2$

$$\mathcal{E}(m) = \frac{1}{4} \|\mathcal{H}(m)\|_{\dot{H}^{1/2}(\mathbb{R}^2)}^2 - \int_{\Omega} h \cdot m dx \rightarrow \min \quad \text{in } \{|m| \leq 1 \text{ in } \Omega\}. \quad (2.2)$$

The underlying topology is induced by the weak convergence

$$\int_0^t \mathbf{m}^{(t)}(\cdot, z) dz \rightharpoonup (m, 0) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^3).$$

Consult the monograph [11] or [2] for the notion and properties of  $\Gamma$ -convergence.

We collect the main features of this  $\Gamma$ -limit:

- (i) The limiting magnetization fields are tangential to the magnetic film.
- (ii) The exchange energy drops out, and this allows for discontinuous minimizers of the reduced variational principle.
- (iii) The magnetostatic energy controls the homogeneous  $H^{1/2}$ -norm of the gradient part of the magnetization and does not allow for normal jumps across a discontinuity line, in particular it enforces tangential magnetization at the boundary.
- (iv) The saturation condition  $|\mathbf{m}| = 1$  is relaxed to the convex constraint that  $|m| \leq 1$  on the sample.

- (v) For an external gradient field  $h = -\nabla' H$ , the functional only depends (in a strictly convex way) on the magnetic surface charge  $\nabla' \cdot m$  induced by the magnetization  $m$ .

Hence the variational principle is convex but highly degenerate and admits many solutions. An open question is whether there exists a saturated minimizer  $m_0$ , i.e. a minimizer which satisfies  $|m_0| = 1$  on the sample. This question is directly connected with our regularity result in view of the notion of viscosity solutions. Indeed, by property (v) the functional allows to add a solenoidal gauge which leaves the energy unaltered. In this context, restoring the saturation condition amounts to the following Dirichlet problem for a stream function  $\psi$

$$|\nabla' \psi + m_0^\perp|^2 = 1 \text{ in } \Omega \text{ with } \psi = 0 \text{ on } \partial\Omega,$$

which is solvable in the sense of viscosity solutions for continuous data, i.e. if there is a continuous minimizer  $m_0$ .

## 2.2 Magnetostatic energy through potentials

It is convenient to describe the magnetostatic energy by a functional of the magnetostatic potential  $u$  which we define via the following variational principle: Given a vector field  $m$  with  $\mathcal{H}(m) \in H^{1/2}(\mathbb{R}^2; \mathbb{R}^2)$  given by (2.1), we consider

$$\mathcal{I}[m](u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \langle m, \nabla' u \rangle \rightarrow \min. \quad (2.3)$$

We denote by  $\mathcal{D}^{1,2}(\mathbb{R}^3)$  the closure of the space of test functions  $C_0^\infty(\mathbb{R}^3)$  with respect to the Dirichlet energy. We infer from the Sobolev inequality that

$$\mathcal{D}^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \int |\nabla u|^2 dx < \infty\} \subset H_{loc}^1(\mathbb{R}^3).$$

**Remark 2.1.** *Note that the functional  $\mathcal{I}[m]$  only depends on the magnetization  $m$  via its Helmholtz projection  $\mathcal{H}(m)$ . Indeed, for  $\mathcal{H}(m) \in H^{1/2}(\mathbb{R}^2; \mathbb{R}^2)$*

$$\langle m, \nabla' u \rangle = \langle \mathcal{H}(m), \nabla' u \rangle \text{ for all } u \in \dot{H}^{1/2}(\mathbb{R}^2).$$

The trace inequality (that can be derived via Fourier transform)

$$\|u\|_{\dot{H}^{1/2}(\mathbb{R}^2)}^2 \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx \text{ for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^3), \quad (2.4)$$

implies that the functional  $\mathcal{I}[m]$  is well defined on  $\mathcal{D}^{1,2}(\mathbb{R}^3)$  and with Young's inequality there holds, for fixed  $m$  the following coercivity property

$$\mathcal{I}[m](u) \geq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \|\mathcal{H}(m)\|_{\dot{H}^{1/2}}^2 \text{ for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^3).$$

We infer from the strict  $\mathcal{D}^{1,2}$ -convexity of the functional  $\mathcal{I}[m]$ , the existence of a unique minimizer for the variational problem (2.3) in the space  $\mathcal{D}^{1,2}(\mathbb{R}^3)$ . The magnetostatic potential  $u$  associated to a magnetization field  $m$  must satisfy the following compatibility condition, that emerges as the associated Euler-Lagrange equation

$$\Delta u = \nabla' \cdot m \, d\mathcal{H}^2 \llcorner \mathbb{R}^2 \quad \text{in } \mathcal{D}'(\mathbb{R}^3), \quad (2.5)$$

where  $\mathcal{H}^2$  denotes the two-dimensional Hausdorff measure. The relation between magnetostatic potentials and the limit of the reduced stray field operator is given by the Fourier representation of the compatibility equation (2.5)

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{|\xi' \cdot \hat{m}(\xi')|^2}{|\xi'|^2 + \eta^2} \, d\eta \, d\xi' = \frac{1}{2} \|\mathcal{H}(m)\|_{\dot{H}^{1/2}}^2. \quad (2.6)$$

Using once more the Fourier representation of the compatibility equation, we see that  $u(\cdot, 0) = \frac{1}{2} \mathcal{R} \cdot m$ , where  $\mathcal{R} = (-\Delta')^{-1/2} \nabla'$  is the (vectorial) Riesz transform. Note that  $\mathcal{R} \cdot m = \mathcal{R} \cdot \mathcal{H}(m)$ . Hence we infer the equivalence

$$\nabla' u = \frac{1}{2} (-\Delta')^{1/2} \mathcal{H}(m) : \mathbb{R}^2 \rightarrow \mathbb{R}^2. \quad (2.7)$$

Consequently we can write the magnetostatic energy as a dual pairing:

$$\frac{1}{2} \|\mathcal{H}(m)\|_{\dot{H}^{1,2}}^2 = \frac{1}{2} \left\langle (-\Delta')^{1/2} \mathcal{H}(m), \mathcal{H}(m) \right\rangle = \langle \nabla' u, \mathcal{H}(m) \rangle. \quad (2.8)$$

Now in view of the variational limit problem for thin films (2.2)

$$\mathcal{E}(m) = \frac{1}{4} \|\mathcal{H}(m)\|_{\dot{H}^{1/2}}^2 + \int_{\Omega} h \cdot m \rightarrow \min \quad \text{for } m \in \mathcal{M}, \quad (2.9)$$

where  $\mathcal{M}$  is the space of admissible magnetizations given by

$$\mathcal{M} = \{m : \Omega \rightarrow \mathbb{R}^2 : |m| \leq 1, \mathcal{H}(m) \in H^{1/2}(\mathbb{R}^2)\},$$

we define equivalently the **primal variational principle** for thin films:

$$\mathcal{E}(m) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \int_{\Omega} h \cdot m \, dx \rightarrow \min \quad \text{for } m \in \mathcal{M} \quad (2.10)$$

where  $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$  satisfies  $\Delta u = \nabla' \cdot m \, d\mathcal{H}^2 \llcorner \mathbb{R}^2$  in  $\mathcal{D}'(\mathbb{R}^3)$ .

We have the following Euler-Lagrange equation as optimality condition.

**Lemma 2.1.** *Any solution  $m_0$  of the above variational principle with associated magnetostatic potential  $u_0$  solves the following variational inequality*

$$\langle \nabla' u_0 - h, m_0 \rangle \leq \langle \nabla' u_0 - h, m \rangle \quad \text{for all } m \in \mathcal{M}.$$

*Proof.* For any  $m$  in the convex space  $\mathcal{M}$  we introduce the admissible family of variations  $m_\varepsilon = (1 - \varepsilon)m_0 + \varepsilon m$ . Carrying out the differentiation for the functional in (2.9) with respect to  $\varepsilon$  gives

$$\left. \frac{d}{d\varepsilon} \mathcal{E}(m_\varepsilon) \right|_{\varepsilon=0} = \mathcal{E}'(m_0) \langle m_0 - m \rangle = \left\langle \frac{1}{2} (-\Delta')^{1/2} \mathcal{H}(m_0) - h, m_0 - m \right\rangle \leq 0,$$

but by (2.7) this is equivalent to  $\langle \nabla' u_0 - h, m_0 - m \rangle \leq 0$ , q.e.d.  $\square$

### 2.3 Dual formulation

In the following we assume  $h$  to be a smooth gradient field. For an admissible pair  $(m, u)$ , i.e.  $m$  and  $u$  satisfy the compatibility equation (2.5), and in view of (2.8) and Remark 2.1, the energy  $\mathcal{E}(m)$  may be written as

$$\mathcal{E}(u, m) = -\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \langle h - \nabla' u, \mathcal{H}(m) \rangle. \quad (2.11)$$

By the trace inequality (2.4), this functional is well defined on the product space  $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{M}$ .

We show that the variational principle (2.10) can be formulated as a saddle point problem over the product space  $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{M}$ . We will infer from the theory of saddle points a dual formulation of the thin film variational principle which only involves the magnetostatic potential. For this purpose we define a functional  $\mathcal{F}$  on the space of magnetostatic potentials by

$$\mathcal{F}(u) = -\inf(\mathcal{E}(u, m) : m \in \mathcal{M}). \quad (2.12)$$

Since  $u \mapsto \mathcal{E}(u, m)$  is concave for each  $m \in \mathcal{M}$ , we infer that  $\mathcal{F}$  is convex, see e.g. [8]. Moreover for a sufficiently regular potential  $u$ , the functional  $\mathcal{F}(u)$  can explicitly be written as

$$\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\Omega} |\nabla' u - h| dx'. \quad (2.13)$$

We call the variational principle which consists in minimizing the functional  $u \mapsto \mathcal{F}(u)$  in the space of magnetostatic potentials  $\mathcal{D}^{1,2}(\mathbb{R}^3)$  the **dual variational principle** with respect to the original (or primal) thin film variational problem (2.10) for magnetizations  $m$ . The connection between the dual and the primal problem is described as follows:

**Lemma 2.2.** *Let  $m_0$  be a solution of the thin film variational principle (2.10), then the associated magnetostatic potential  $u_0$  coincides with the unique minimizer of the functional  $\mathcal{F}$  over  $\mathcal{D}^{1,2}(\mathbb{R}^3)$ . Moreover  $|\mathcal{E}(m_0)| = \mathcal{F}(u_0)$ .*

*Proof.* We need to show that  $(u_0, m_0)$  is a saddle point for the functional

$$(u, m) \mapsto \mathcal{E}(u, m) \quad \text{for } (u, m) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{M}$$

defined in (2.11), that is for all  $m \in \mathcal{M}$  and  $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$  there should hold

$$\mathcal{E}(u, m_0) \leq \mathcal{E}(u_0, m_0) \leq \mathcal{E}(u_0, m). \quad (2.14)$$

Then the saddle point theorem, see [8] chapter 3 or [14] Theorem 2.F, implies in particular that  $u_0$  is a maximizer of the functional

$$u \mapsto \inf(\mathcal{E}(u, m) : m \in \mathcal{M}) \quad \text{for } u \in \mathcal{D}^{1,2}(\mathbb{R}^3),$$

that is in our definition a minimizer of  $\mathcal{F}(u)$ . Moreover the energies agree

$$\mathcal{E}(u_0, m_0) = \sup \left\{ \inf (\mathcal{E}(u, m) : m \in \mathcal{M}) : u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \right\}.$$

Hence by our definition of  $\mathcal{F}$  and the fact that  $\mathcal{E}(u_0, m_0) = \mathcal{E}(m_0)$  for the admissible pair  $(u_0, m_0)$

$$- \inf (\mathcal{E}(m) : m \in \mathcal{M}) = \inf (\mathcal{F}(u) : u \in \mathcal{D}^{1,2}(\mathbb{R}^3)).$$

We check the saddle property (2.14). The first inequality follows by

$$\langle \mathcal{H}(m_0), \nabla' u \rangle = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla u \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_0|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx,$$

where we used the compatibility equation (2.5) and the trace inequality (2.4). Hence

$$\mathcal{E}(u, m_0) \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_0|^2 \, dx - \int_{\Omega} h \cdot m_0 \, dx' = \mathcal{E}(m_0) \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^3),$$

and we conclude since  $\mathcal{E}(u, m)$  and  $\mathcal{E}(m)$  agree on the set of admissible pairs. For the second inequality it is enough to show that  $m_0$  is a minimizer of the linear form

$$m \mapsto - \langle h - \nabla' u_0, \mathcal{H}(m) \rangle$$

which is the same as solving the following variational inequality

$$\langle \nabla' u_0 - h, m_0 \rangle \leq \langle \nabla' u_0 - h, m \rangle \quad \text{for all } m \in \mathcal{M}.$$

But this is true since  $m_0$  is a minimizer of  $m \mapsto \mathcal{E}(m)$  in the convex space  $\mathcal{M}$  and therefore solves the above variational inequality, as shown in Lemma 2.1.  $\square$

We state our final result with a slight regularity improvement for the trace of the magnetostatic potential.

**Proposition 2.1.** *Let  $m_0$  be a minimizer of the reduced thin film variational principle (2.10). Then the associated magnetostatic potential  $u_0 \in \mathcal{D}^{1,2}(\Omega)$  is unique in the class of minimizers with respect to the external gradient field  $h$ , and it is a minimizer for the dual variational principle*

$$\mathcal{F}(u) \rightarrow \min \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3),$$

where  $\mathcal{F}$  is given by (2.13). Moreover  $u_0$  has bounded variation in  $\Omega$ .

*Proof.* We only need to show that  $u|_{\Omega}$  has bounded variation. But by density  $C_0^\infty(\mathbb{R}^3) \subset \mathcal{D}^{1,2}(\mathbb{R}^3)$  we might choose a smooth minimizing sequence  $(u_k)$  for the convex functional (2.12) which admits the form (2.13) for smooth functions. We infer by the uniform  $L^1$ -boundedness of  $(\nabla' u_k|_{\Omega})$  that the weak  $\mathcal{D}^{1,2}$ -limit  $u_0$  has bounded bounded variation in  $\Omega$  and is a minimizer by lower semicontinuity.  $\square$



### 3 The regularity result

The form of the thin film variational principle (2.2) suggests that we cannot hope for a regularity result for every minimizing magnetization field  $m$ , but only for the uniquely determined gradient part, or its divergence, the magnetic charge. Hence the difficulty to prove regularity for the primal problem consists in the incompatibility of the constraint  $|m| \leq 1$  and the Helmholtz projection. The **dual problem** is made up of minimizing the following functional

$$\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \mathcal{G}_0(u) \rightarrow \min \quad (3.1)$$

over the set of potentials  $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$  where the functional  $\mathcal{G}_0$  is given by

$$\mathcal{G}_0(u) = \int_{\Omega} |\nabla'(u + H)| dx' \quad \text{for } \nabla' u \in L^1(\Omega) \quad (3.2)$$

with the potential  $H$  corresponding to a smooth gradient field  $h = -\nabla' H$ . With  $\mathcal{G}_0(u)$  considered as the total variation of the signed measure  $\nabla'(u + H)$  in  $\Omega$  we get a natural extension to functions of bounded variation, cf. e.g. [1].

We endow the space  $\mathcal{D}^{1,2}(\mathbb{R}^3)$  with the topology  $\tau$  induced by the weak  $L^2$ -convergence of the gradient. Note that by the trace theorem  $L^2$ -boundedness of the gradient implies local  $H^{1/2}$ -boundedness of the restriction to  $\{x_3 = 0\}$ . Furthermore by the Sobolev embedding theorem we get local bounds in  $L^4(\mathbb{R}^2)$  for the trace and by the local compactness properties of Bessel potentials local  $L^p$ -compactness is ensured for  $1 \leq p < 4$ . Note that boundedness in  $\mathcal{D}^{1,2}(\mathbb{R}^3)$  does not give any meaning to the gradient  $\nabla' u|_{\Omega}$  as a measurable function nor as a regular signed measure. Therefore it is reasonable to assign the value  $+\infty$  to the functional  $\mathcal{F}(u)$  for  $u|_{\mathbb{R}^2} \notin BV(\Omega)$ , i.e.

$$\mathcal{D}^{1,2}(\mathbb{R}^3) \ni u \mapsto \mathcal{F}(u) \in \mathbb{R} \cup \{+\infty\}.$$

It seems important to mention that the functional  $\mathcal{G}_0(u)$ , incorporating the total variation of  $u$ , cannot be considered as a lower order perturbation of the magnetostatic portion of the energy. In fact by the trace theorem, the Dirichlet energy in  $\mathbb{R}^3$  only controls the  $H^{1/2}$ -norm on  $\mathbb{R}^2$ . But neither  $BV$  embeds into  $H^{1/2}$  nor vice versa. This reflects, in view of the primal problem, that neither the  $H^{1/2}$ -norm of the gradient controls the size, i.e. the constraint  $|m| \leq 1$ , nor can the  $H^{1/2}$ -norm be controlled by a pointwise bound.

#### 3.1 Regularization and functional convergence

Note that the functional  $\mathcal{G}_0(u)$  is not (continuously) differentiable on the space of  $\mathcal{D}^{1,2}(\mathbb{R}^3)$ -restrictions nor on  $BV(\Omega)$ . Therefore we regularize the functional  $\mathcal{G}_0(u)$

and use  $\Gamma$ -convergence theory to infer convergence of minimizers and minimal energies. We define

$$\mathcal{G}_\varepsilon(u) = \int_{\Omega} \sqrt{\varepsilon^2 + |\nabla'(u+H)|^2} dx'. \quad (3.3)$$

Let  $u_\varepsilon$  be a minimizer of the functional

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\varepsilon}{2} \int_{\Omega} |\nabla' u|^2 dx' + \mathcal{G}_\varepsilon(u),$$

which we consider as a functional on  $\mathcal{D}^{1,2}(\mathbb{R})$ . Then we know that a priori  $u_\varepsilon \in H_{loc}^1(\mathbb{R}^3) \cap H^1(\Omega)$  which is enough to proceed with a difference quotient method to show higher regularity. In particular,  $\mathcal{G}_\varepsilon$  is continuously differentiable on  $H^1(\Omega)$  as we will see below. Moreover minimal energies are uniformly bounded, indeed  $\inf \mathcal{F}_\varepsilon \leq \int_{\Omega} (\varepsilon + |\nabla' H|) dx'$ .

**Lemma 3.1.** *Let  $u_\varepsilon$  be the unique minimizer of  $\mathcal{F}_\varepsilon$  in  $\mathcal{D}^{1,2}(\mathbb{R}^2)$ . Then as  $\varepsilon$  tends to zero,  $u_\varepsilon \rightharpoonup u_0$  in  $\mathcal{D}^{1,2}(\mathbb{R}^3)$  where  $u_0$  is the unique minimizer of  $\mathcal{F}$  in  $\mathcal{D}^{1,2}(\mathbb{R}^2)$ . Moreover minimal energies converge  $\mathcal{F}_\varepsilon(u_\varepsilon) \rightarrow \mathcal{F}(u_0)$ .*

*Proof.* Note that by the monotonicity of  $\varepsilon \mapsto \mathcal{F}_\varepsilon(u)$  the sequence of functionals  $(\mathcal{F}_\varepsilon)$   $\Gamma$ -converges to the lower-semicontinuous regularization  $\mathcal{F}_*$  of  $\mathcal{F}$  for any second countable topology  $\tau$  we choose. In particular a sequence of minimizer  $u_\varepsilon$  converge to some function  $u_0$  with respect to  $\tau$ . Then  $\mathcal{F}_\varepsilon(u_\varepsilon) \rightarrow \inf \mathcal{F}$  and  $u_0$  is a minimizer of  $\mathcal{F}_*$ . See e.g. [2], Proposition 2.48 for this result.

Now by the uniform convexity and the uniform  $\mathcal{D}^{1,2}$ -coercivity of the regularized functionals  $\mathcal{F}_\varepsilon$  for  $\varepsilon > 0$  and their pointwise limit  $\mathcal{F}$ , we infer the existence and uniqueness of minimizers  $u_\varepsilon$  and  $u_0$  for the associated variational problems. Moreover we have  $\mathcal{F}_* = \mathcal{F}$  and we conclude the convergence of  $u_\varepsilon$  to  $u_0$  in the weak  $\mathcal{D}^{1,2}$ -sense and the convergence of minimal energies as  $\varepsilon$  tends to zero.  $\square$

### 3.2 Higher Regularity for the tangential gradient

The  $\Gamma$ -convergence result of the previous section allows us to derive a priori bounds for the tangential gradient of the magnetostatic potential  $\nabla' u$  by those associated to the regularized variational principle  $\mathcal{F}_\varepsilon(u) \rightarrow \min$ . We indicate the standard regularization of the modulus function and its gradient by

$$|p|_\varepsilon = \sqrt{\varepsilon^2 + |p|^2} \quad \text{and} \quad \sigma_\varepsilon(p) = \frac{p}{|p|_\varepsilon}.$$

Then for each  $\varepsilon > 0$  and each  $p = p(x') \in L^1(\Omega; \mathbb{R}^2)$  the derivative

$$D\sigma_\varepsilon(p) = \frac{1}{|p|_\varepsilon} \left( \mathbf{1} - \frac{p}{|p|_\varepsilon} \otimes \frac{p}{|p|_\varepsilon} \right) \quad (3.4)$$

is symmetric and non-negative and measurable as a function of  $x \in \Omega$ . For  $\varepsilon = 0$  and generic  $p(x') = \nabla' v(x') \neq 0$  the unit vector  $\sigma(p) = \frac{p}{|p|}$  for  $p = p(x')$  is normal to the level set  $\{v = v(x')\}$ . Moreover,  $\kappa(x') = \nabla' \sigma(p)(x')$  is the curvature matrix at  $v(x')$ .

Now let for any vector field  $p \in L^1(\Omega; \mathbb{R}^2)$ , any unit vector  $\hat{e}$ , and  $0 \leq \lambda \leq 1$

$$p(x', s, \lambda) = \lambda p(x' + s \hat{e}) + (1 - \lambda)p(x').$$

Then  $p(\cdot, s, \lambda) \in L^1(\tilde{\Omega}; \mathbb{R}^2)$  for each subdomain  $\tilde{\Omega} \subset \Omega$  and  $s < \text{dist}(\tilde{\Omega}, \partial\Omega)$ , and

$$A_\varepsilon^s(x') = \int_0^1 D\sigma_\varepsilon(p(x', s, \lambda)) d\lambda, \quad \text{for } x' \in \tilde{\Omega} \quad (3.5)$$

is symmetric and non-negative and measurable as a function on the subdomain  $\tilde{\Omega} \subset \Omega$ . Notice that

$$A_\varepsilon^s \in L^\infty(\tilde{\Omega}; \mathbb{R}^{2 \times 2}) \quad \text{with} \quad |A_\varepsilon^s| \leq c(\varepsilon) \quad \text{for all } s > 0.$$

The definition for  $A_\varepsilon^s$  is motivated by the fact that for a finite difference quotient operator

$$\delta_s p(x') = \frac{1}{s} (p(x' + s \hat{e}) - p(x'))$$

according to a fixed unit vector  $\hat{e}$  and  $\delta_s$  we have

$$\delta_s(\sigma_\varepsilon(p)) = A_\varepsilon^s(x') \delta_s p \quad \text{for all } p \in L^1(\Omega).$$

The formal adjoint  $\delta_s^*$  is given by  $\frac{1}{s} (p(x' - s \hat{e}) - p(x'))$ . Now for  $u \in W^{1,1}(\Omega)$  and a test function  $\phi \in W^{1,1}(\Omega)$ , the variation of  $\mathcal{G}_\varepsilon$  at  $u$  in the direction of  $\phi$  is given by

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{G}_\varepsilon(u + t\phi) = \mathcal{G}'_\varepsilon(u) \langle \phi \rangle = \int_\Omega \sigma_\varepsilon(\nabla'(u + H)) \cdot \nabla' \phi dx'.$$

For  $\eta \in C_0^\infty(\Omega \times \mathbb{R})$  we introduce the test function  $\phi_\varepsilon = \delta_s^*(\eta^2 \delta_s(u_\varepsilon + H)) \in H_0^1(\Omega)$ .

$$\begin{aligned} \mathcal{G}'_\varepsilon(u_\varepsilon) \langle \phi_\varepsilon \rangle &= \int_\Omega \langle \delta_s(\nabla'(u_\varepsilon + H)) | A_\varepsilon^s(x') | \nabla'(\eta^2 \delta_s(u_\varepsilon + H)) \rangle dx' \\ &= \int_\Omega \langle \delta_s(\nabla'(u_\varepsilon + H)) | A_\varepsilon^s(x') | \delta_s(\nabla'(u_\varepsilon + H)) \rangle \eta^2 dx' \\ &\quad + 2 \int_\Omega \langle \delta_s(\nabla'(u_\varepsilon + H)) | A_\varepsilon^s(x') | \nabla' \eta \rangle \delta_s(u_\varepsilon + H) \eta dx'. \end{aligned}$$

Now the second term can be bounded from above using Cauchy-Schwarz and Young's inequality. Hence  $\mathcal{G}'_\varepsilon(u_\varepsilon) \langle \phi_\varepsilon \rangle$  is bounded from below by

$$\begin{aligned} &\frac{1}{2} \int_\Omega \langle \delta_s(\nabla'(u_\varepsilon + H)) | A_\varepsilon^s(x') | \delta_s(\nabla'(u_\varepsilon + H)) \rangle \eta^2 dx' \\ &\quad - 4 \int_\Omega \langle \nabla' \eta | A_\varepsilon^s(x') | \nabla' \eta \rangle |\delta_s(u_\varepsilon + H)|_\varepsilon^2 dx'. \end{aligned}$$

We conclude by the weak formulation of the Euler-Lagrange equations for the functional  $\mathcal{F}_\varepsilon$

$$0 = \int_{\mathbb{R}^3} \nabla u_\varepsilon \cdot \nabla \phi_\varepsilon \, dx + \varepsilon \int_{\Omega} \nabla' u_\varepsilon \cdot \nabla' \phi_\varepsilon \, dx + \mathcal{G}'_\varepsilon(u_\varepsilon) \langle \phi_\varepsilon \rangle$$

using Young's inequality, that for the minimizer  $u_\varepsilon$  following the estimate

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \delta_s u_\varepsilon|^2 \eta^2 \, dx + \frac{\varepsilon}{2} \int_{\Omega} |\nabla' \delta_s u_\varepsilon|^2 \eta^2 \, dx' \\ & + \frac{1}{2} \int_{\Omega} \langle \delta_s(\nabla'(u_\varepsilon + H)) | A_\varepsilon^s(x') | \delta_s(\nabla'(u_\varepsilon + H)) \rangle \eta^2 \, dx' \\ & \leq 2 \int_{\mathbb{R}^3} |\nabla \eta|^2 |\delta_s u_\varepsilon|^2 \, dx + 2\varepsilon \int_{\Omega} |\nabla' \eta|^2 |\delta_s u_\varepsilon|^2 \, dx' \\ & + 4 \int_{\Omega} \langle \nabla' \eta | A_\varepsilon^s(x') | \nabla' \eta \rangle |\delta_s(u_\varepsilon + H)|_\varepsilon^2 \, dx' \leq c(\varepsilon) \sup |\nabla \eta|^2 \mathcal{F}_\varepsilon(u_\varepsilon) \end{aligned}$$

holds true with a constant  $c(\varepsilon)$ . Thus we proved the following a priori regularity result for the approximating potentials  $u_\varepsilon$  which allows us to derive refined bounds for the limit  $u_0$  in a more transparent way.

**Lemma 3.2.** *For finite  $\varepsilon > 0$  we have  $\nabla' u_\varepsilon \in H_{loc}^1(\Omega \times \mathbb{R}) \cap H_{loc}^1(\Omega)$ . Moreover,*

$$0 = \int_{\mathbb{R}^3} \nabla \partial_\alpha u \cdot \nabla \Phi_\alpha \, dx + \varepsilon \int_{\mathbb{R}^2} \nabla' \partial_\alpha u \cdot \nabla' \Phi_\alpha \, dx + \int_{\Omega} \langle \partial_\alpha p_\varepsilon | D\sigma_\varepsilon(p_\varepsilon) | \Phi_\alpha \rangle \eta^2 \, dx'$$

holds true for every testfunction  $\Phi \in C_0^\infty(\Omega \times \mathbb{R}; \mathbb{R}^2)$ .

With  $\Phi_\varepsilon^\alpha = \eta^2 \partial_\alpha u_\varepsilon$  and the notation  $p_\varepsilon = \nabla'(u_\varepsilon + H)$  the analogous estimate

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \nabla' u_\varepsilon|^2 \eta^2 \, dx + \frac{\varepsilon}{2} \int_{\Omega} |\nabla' \nabla' u_\varepsilon|^2 \eta^2 \, dx' \\ & + \frac{1}{2} \sum_{\alpha=1,2} \int_{\Omega} \langle \partial_\alpha p_\varepsilon | D\sigma_\varepsilon(p_\varepsilon) | \partial_\alpha p_\varepsilon \rangle \eta^2 \, dx' \\ & \leq 2 \int_{\mathbb{R}^3} |\nabla \eta|^2 |\nabla' u_\varepsilon|^2 \, dx + 2\varepsilon \int_{\Omega} |\nabla' \eta|^2 |\nabla' u_\varepsilon|^2 \, dx' \\ & + 4 \int_{\Omega} \langle \nabla' \eta | D\sigma_\varepsilon(p_\varepsilon) | \nabla' \eta \rangle |\nabla'(u_\varepsilon + H)|_\varepsilon^2 \, dx'. \end{aligned}$$

Using that  $\langle v | D\sigma_\varepsilon(p) | v \rangle \leq |p|_\varepsilon \leq |v|^2$  for every  $v, p \in \mathbb{R}^2$ , we immediately get the following uniform estimate:

$$\int_{\mathbb{R}^3} |\nabla \nabla' u_\varepsilon|^2 \eta^2 \, dx + \int_{\Omega} \langle \nabla' p_\varepsilon | D\sigma_\varepsilon(p_\varepsilon) | \nabla' p_\varepsilon \rangle \eta^2 \, dx' \leq 8 \sup |\nabla \eta|^2 \mathcal{F}_\varepsilon(u_\varepsilon). \quad (3.6)$$

Taking the limit inferior in (3.6) and taking into account the convergence of minimizers and minimal energies as stated in Lemma 3.1 we have shown:

**Proposition 3.1.** *The solution  $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$  of the dual variational principle has tangential derivatives  $\nabla' u_0 \in H_{loc}^1(\Omega \times \mathbb{R})$  and for every  $\eta \in C_0^\infty(\Omega \times \mathbb{R})$  the following estimate holds true*

$$\int_{\mathbb{R}^3} |\nabla \nabla' u_0|^2 \eta^2 dx \leq 8 \sup |\nabla \eta|^2 \mathcal{F}(u_0).$$

The regularity result above is proved in [7] by a different method. In contrast to our local method on the basis of the dual problem, they used a global method on the basis of the primal problem. The argument relies on the usage of one-parameter subgroups of the conformal diffeomorphisms from  $\Omega$  onto itself to proceed with an abstract difference quotient argument. It includes also the case of finite anisotropy but is restricted to simply connected domains. We conclude this section with some further geometric implication on the level sets of  $u_0 + H$  stemming from the  $BV$ -control under additional assumptions, namely that  $p_\varepsilon = \nabla'(u_\varepsilon + H)$  converges locally uniformly as  $\varepsilon \rightarrow 0$ . First we need the following interpolation inequality:

**Lemma 3.3.** *Let  $p = p_\varepsilon \in L^1 \cap H_{loc}^1(\Omega; \mathbb{R}^2)$  and  $\eta \in C_0^\infty(\Omega)$ . Then we have*

$$\frac{1}{2} \int_{\Omega} |p| |\partial_\alpha \sigma(p)| \eta dx' \leq \int_{\Omega} \langle \partial_\alpha p | D\sigma_\varepsilon(p) | \partial_\alpha p \rangle \eta^2 dx' + \int_{\Omega} |p|_\varepsilon dx'.$$

*Proof.* We have the following pointwise identity

$$|p|_\varepsilon \langle \partial_\alpha p | D\sigma_\varepsilon(p) | \partial_\alpha p \rangle \geq |p| \langle \partial_\alpha p | D\sigma(p) | \partial_\alpha p \rangle = |p|^2 |\partial_\alpha \sigma(p)|^2.$$

On the other hand we get from Young's inequality

$$\frac{|p|^2}{|p|_\varepsilon} |\partial_\alpha \sigma(p)|^2 \eta^2 + |p|_\varepsilon \geq \frac{1}{2} |p| |\partial_\alpha \sigma(p)| \eta$$

and the claim follows after integration.  $\square$

Recall that  $\int_{\Omega} |p_\varepsilon|_\varepsilon = \mathcal{G}_\varepsilon(u_\varepsilon)$ . Thus, in view of Proposition 3.6, the lemma implies

$$\int_{\Omega} |p_\varepsilon| |\nabla' \sigma(p_\varepsilon)| \eta dx' \leq c \left(1 + \sup |\nabla \eta|^2\right) \mathcal{F}_\varepsilon(u_\varepsilon). \quad (3.7)$$

**Proposition 3.2.** *Suppose that  $\nabla' u_\varepsilon \rightarrow \nabla' u_0$  uniformly on  $E_\Lambda \Subset \Omega$  open and  $|\nabla'(u_0 + H)| \geq \Lambda > 0$  on  $E_\Lambda$ . Then  $\sigma(\nabla'(u_0 + H))$  has bounded variation in  $E_\Lambda$  and the following estimate holds true*

$$\int_{E_\Lambda} |\nabla' \sigma(\nabla'(u_0 + H))| \leq \frac{c}{\Lambda} \left(1 + \sup |\nabla \eta|^2\right) \mathcal{F}_\varepsilon(u_\varepsilon).$$

Notice that  $|p_\varepsilon|$  is just the Jacobian of  $u_\varepsilon + H$  and  $\kappa_\varepsilon = \nabla' \sigma(p_\varepsilon)$  the curvature of the level set of  $u_\varepsilon + H$ . Then the coarea formula (cf. e.g. [1]) gives

$$\int_{\Omega} |p_\varepsilon| |\nabla' \sigma(p_\varepsilon)| \eta dx' = \int_{\mathbb{R}} \int_{\{u_\varepsilon + H = \Lambda\}} |\kappa_\varepsilon| \eta d\mathcal{H}^1 d\Lambda. \quad (3.8)$$

### 3.3 Implications for the magnetic field

Now let  $m_0 \in \mathcal{M}$  be a solution of the reduced thin film variational principle (2.10) with associated magnetostatic potential  $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ . In view of Lemma 2.2 and Proposition 3.1 we infer from the trace inequality (2.4) that  $\eta \nabla' u_0 \in H^{1/2}(\mathbb{R}^2, \mathbb{R}^2)$  and by the Sobolev embedding  $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$  that

$$\nabla' u_0 \in L^4_{loc}(\Omega; \mathbb{R}^2).$$

For  $|m| \leq 1$  we infer from the  $L^p$ -boundedness of the Helmholtz transform

$$\|\mathcal{H}(m_0)\|_{L^4} \leq c \|m_0\|_{L^4} \leq c |\Omega|^{1/4}.$$

This implies by Calderón's commutator lemma (cf. e.g. [4, 12])

$$\begin{aligned} \|\eta \mathcal{H}(m_0)\|_{\dot{H}^1_4} &\leq \|\eta \nabla' u_0\|_{L^4} + \| [(-\Delta')^{1/2}, \eta] \mathcal{H}(m_0) \|_{L^4} \\ &\leq c \|\eta \nabla' u_0\|_{H^{1/2}} + c |\Omega|^{1/4} \|\nabla' \eta\|_{L^\infty}. \end{aligned}$$

We conclude by the Sobolev embedding  $H^1_4(\mathbb{R}^2) \hookrightarrow C^{1/2}(\mathbb{R}^2)$  the following regularity result:

**Theorem.** *The uniquely determine gradient part  $\mathcal{H}(m_0)$  of a minimizer  $m_0$  of the reduced thin film variational principle corresponding to a smooth gradient field  $h = -\nabla' H$  is locally Hölder continuous with Hölder exponent  $\alpha = 1/2$ .*

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