DESCRIPTOR SYSTEMS: A GENERAL MATHEMATICAL FRAMEWORK FOR MODELLING, SIMULATION AND CONTROL

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Abstract. Descriptor systems present a general mathematical framework for the modelling, simulation and control of complex dynamical systems arising in many areas of mechanical, electrical and chemical engineering. This paper presents a survey of the current theory of descriptor systems, concerning solvability, stability, model reduction, controllability, observability and optimal control.

Key words. descriptor system, stability, controllability, observability, model reduction, optimal control

AMS subject classifications. 34A09, 93C, 93D20

Notation. We will denote by $\mathbb{R}^{n,m}$ the space of $n \times m$ real matrices. The matrix A^T stands for the transpose of $A \in \mathbb{R}^{n,m}$. The rank of a matrix $A \in \mathbb{R}^{n,m}$ is denoted by $\operatorname{rank} A$. An identity matrix of order n is denoted by I_n or simply by I. We will use $\mathcal{C}^k(\mathbb{I}, \mathbb{R}^n)$ to denote the set of k times continuously differentiable functions from $\mathbb{I} \subseteq \mathbb{R}$ to $\mathbb{R}^n$. Finally, $\|\cdot\|$ denotes the Euclidean vector norm and the spectral matrix norm.

1. Introduction. The mathematical description of the dynamical behavior of complex systems as they arise today in all areas of mechanical, electrical and chemical engineering is typically governed by differential equations, including balance equations, conservation laws, and constraints that couple these parts together [10, 18, 29, 62, 65]. In order to control the dynamical behavior also input and output variables are included. Modern CAD tools such as Matlab/Simulink [49] allow to generate these models automatically. In this paper we only study the time behavior, i.e., we assume that partial differential equations are already semidiscretized in space.

In a general mathematical framework, control systems can be written in the form

$$F(t, x, \dot{x}, u) = 0, \tag{1.1}$$

$$y - G(t, x) = 0, \tag{1.2}$$

where $F : \mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \times \mathbb{D}_u \to \mathbb{R}^l$ and $G : \mathbb{I} \times \mathbb{D}_x \to \mathbb{R}^p$ are continuous functions, $\mathbb{D}_x, \mathbb{D}_{\dot{x}} \subseteq \mathbb{R}^n$ and $\mathbb{D}_u \subseteq \mathbb{R}^m$ are open, $\mathbb{I} = [t_0, t_f] \subseteq \mathbb{R}$, $x : \mathbb{I} \to \mathbb{R}^n$ is a continuously differentiable function and $\dot{x}$ denotes the derivative of x with respect to $t \in \mathbb{I}$. Here, x represents the state, $u : \mathbb{I} \to \mathbb{R}^m$ is the input or control and $y : \mathbb{I} \to \mathbb{R}^p$ is the output of the system. We will allow that the partial derivative of F with respect to $\dot{x}$, denoted by $F_{\dot{x}}$, is rank deficient. Since such systems contain both differential and algebraic equations, they are called *differential-algebraic equations*. In the control community, systems of the form (1.1), (1.2) are known also as *descriptor systems*, *implicit systems* or *singular systems*.

Example 1.1. *As a simple mechanical example, we consider a cart-pendulum system shown in Figure 1.1 that consists of a cart of mass m_1 and a pendulum of*

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length L and of mass m_2. The motion of this system is described by the system in first-order form

$$
\begin{array}{rcl}
\dot{x}_1 & = & x_4, \\
\dot{x}_2 & = & x_5, \\
\dot{x}_3 & = & x_6, \\
m_1\dot{x}_4 & = & -2(x_1 - x_2)\lambda + u, \\
m_2\dot{x}_5 & = & -2(x_2 - x_1)\lambda, \\
m_2\dot{x}_6 & = & -2x_3\lambda - m_2 g, \\
0 & = & (x_2 - x_1)^2 + x_3^2 - L^2,
\end{array}
\tag{1.3}
$$

where x_1 denotes the horizontal position of the cart, x_2 and x_3 are the horizontal and vertical positions of the mass m_2, λ is a Lagrange multiplier and u is an external force acting on the cart. If we are interested in the position of the pendulum only, the output equation has the form $y = [x_2, x_3]^T$. The last equation in (1.3) *describes an algebraic constraint on the variables x_1, x_2 and x_3.*

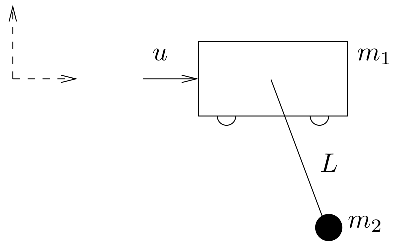

Fig. 1.1. *A cart-pendulum system*

Linearization of (1.1), (1.2) along a nominal trajectory leads to a linear descriptor system with variable coefficients

$$
\begin{array}{rcl}
E(t)\dot{x} & = & A(t)x + B(t)u, \\
y & = & C(t)x,
\end{array}
\tag{1.4}
$$

where $E,\, A : \mathbb{I} \to \mathbb{R}^{l,n}$, $B : \mathbb{I} \to \mathbb{R}^{l,m}$ and $C : \mathbb{I} \to \mathbb{R}^{p,n}$ are continuous matrix-valued functions. If we linearize (1.1), (1.2) along a constant solution, we obtain a linear descriptor system with constant coefficients

$$
\begin{array}{rcl}
E\dot{x} & = & Ax + Bu, \\
y & = & Cx,
\end{array}
\tag{1.5}
$$

where $E,\, A \in \mathbb{R}^{l,n}$, $B \in \mathbb{R}^{l,m}$ and $C \in \mathbb{R}^{p,n}$.

2. General analysis.

2.1. Solvability and index concepts. In this subsection we review some results on the existence and uniqueness of solutions of (1.1).

Definition 2.1. A function $x : \mathbb{I} \to \mathbb{R}^n$ is called a *solution* of system (1.1) if $x \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^n)$ and x satisfies (1.1) pointwise for a given input function u. It is called a *solution of the initial value problem* consisting of (1.1) and

$$
x(t_0) = x_0,
\tag{2.1}
$$

if x is a solution of (1.1) and satisfies (2.1). An initial condition (2.1) is called *consistent* if the corresponding initial value problem has at least one solution.

There also exist weaker solvability concepts, such as impulsive smooth solutions, which can be used to handle inconsistencies or lower smoothness requirements, see, e.g., [37, 62].

In some applications such as robust control it is, furthermore, important to investigate whether the system is solvable for every input function and every initial value that is consistent with this input.

DEFINITION 2.2. A control problem (1.1) is called *consistent* if there exists an input function u for which system (1.1) has a solution x. It is called *regular* (locally with respect to a given pair $(\hat{x}, \hat{u})$ satisfying (1.1)) if it has a unique solution for every sufficiently smooth input function u in a neighborhood of $\hat{u}$ and every initial value in a neighborhood of $\hat{x}(t_0)$ that is consistent for the system with the input function u.

Since for a particular input u, system (1.1) represents a differential-algebraic equation (DAE), it is clear that the solvability theory for control problems is related to that of DAEs. This theory has undergone major changes in the last 20 years, see [10, 28, 37, 62]. Using a behavior approach [33, 59], i.e., combining the variables x and u or even x, u and y into one vector of variables, the theory of descriptor systems can be obtained from the general theory of over- and underdetermined systems of DAEs, see [35, 37, 38]. In many applications, systems are not modelled as input-state systems in the form (1.1), but arise directly in the general form of an over- or underdetermined system

$$F(t, x, \dot{x}) = 0, \tag{2.2}$$

where no a-priori distinction between input and state variables is made. In this case, it is necessary to determine from the model the free variables (control inputs) and which variables can be considered as outputs.

There are many ways to derive the analytical theory and appropriate numerical methods for a general system of the form (2.2). We follow the derivative array approach of [15]. Assuming sufficient smoothness, we generate a nonlinear *derivative array*

$$F_\ell(t, x, \dot{x}, \ldots, x^{(\ell+1)}) = 0, \tag{2.3}$$

which stacks the original equation (2.2) and all its derivatives up to level ℓ in one large system, i.e.,

$$F_\ell(t, x, \dot{x}, \ldots, x^{(\ell+1)}) = \begin{bmatrix} F(t, x, \dot{x}) \\ \frac{d}{dt}F(t, x, \dot{x}) \\ \vdots \\ \frac{d^\ell}{dt^\ell}F(t, x, \dot{x}) \end{bmatrix}.$$

We will denote partial derivatives of F_ℓ with respect to selected variables q from $(t, x, \dot{x}, \ldots, x^{(\ell+1)})$ by $F_{\ell;q}$, e.g.,

$$F_{\ell;x} = \frac{\partial}{\partial x}F_\ell,\ F_{\ell;\dot{x},\ldots,x^{(\ell+1)}} = [\,\frac{\partial}{\partial \dot{x}}F_\ell,\ \ldots,\ \frac{\partial}{\partial x^{(\ell+1)}}F_\ell\,].$$

As basis for the existence of solutions of (2.2) and the construction of numerical methods, we introduce the following hypothesis, see [35, 37].

HYPOTHESIS 1. *Consider a system of nonlinear DAEs* (2.2). *There exist integers* μ, r, a, d, *and* v *such that the set*

$$\mathbb{L}_\mu = \{x_\mu \in \mathbb{I}\times\mathbb{R}^n\times\mathbb{R}^n\times\ldots\times\mathbb{R}^n \ :\ F_\mu(x_\mu) = 0\}$$

associated with (2.3) *and* $\ell = \mu$ *is nonempty, and such that for every point* $x_\mu^0 = (t_0, x_0, \dot{x}_0, \ldots, x_0^{(\mu+1)}) \in \mathbb{L}_\mu$, *where* $x_0^{(j)}$ *denotes an algebraic variable, there exists a sufficiently small neighborhood of* x_μ^0 *in which the following properties hold.*

1. *The set* $\mathbb{L}_\mu \subseteq \mathbb{R}^{(\mu+2)n+1}$ *forms a manifold of dimension* $(\mu+2)n+1-r$.
2. *We have* $\operatorname{rank} F_{\mu;x,\dot{x},\ldots,x^{(\mu+1)}} = r$ *on* $\mathbb{L}_\mu$.
3. *We have*
$$\operatorname{corank} F_{\mu;x,\dot{x},\ldots,x^{(\mu+1)}} - \operatorname{corank} F_{\mu-1;x,\dot{x},\ldots,x^{(\mu)}} = v$$
on $\mathbb{L}_\mu$, *where the corank is the dimension of the corange and* $\operatorname{corank} F_{-1;x} = 0$ *by convention.*
4. *We have* $\operatorname{rank} F_{\mu;\dot{x},\ldots,x^{(\mu+1)}} = r - a$ *on* $\mathbb{L}_\mu$ *and there exist smooth full rank matrix-valued functions* Z_2 *and* T_2 *defined on* $\mathbb{L}_\mu$ *of size* $(\mu+1)l \times a$ *and* $n \times (n-a)$, *respectively, that satisfy*
$$\begin{array}{l} Z_2^T F_{\mu;\dot{x},\ldots,x^{(\mu+1)}} = 0, \\ \operatorname{rank} Z_2^T F_{\mu;x} = a, \quad Z_2^T F_{\mu;x} T_2 = 0 \end{array}$$
on $\mathbb{L}_\mu$.
5. *We have* $\operatorname{rank} F_{\dot{x}} T_2 = d = l - a - v$ *on* $\mathbb{L}_\mu$ *and there exists a smooth full rank matrix-valued function* Z_1 *defined on* $\mathbb{L}_\mu$ *of size* $l \times d$ *such that* $Z_1^T F_{\dot{x}} T_2$ *has full rank.*

The smallest possible μ in Hypothesis 1 is called the *strangeness index* of system (2.2) and also of (1.1). Systems with vanishing strangeness index are called *strangeness-free.* If F is sufficiently smooth and satisfies Hypothesis 1 with μ, r, a, d and v, then every solution of (2.2) solves the reduced system

$$\begin{array}{rcl} F_1(t,x,\dot{x}) & = & 0, \\ F_2(t,x) & = & 0, \end{array} \tag{2.4}$$

with $F_1 : \mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \to \mathbb{R}^d$ and $F_2 : \mathbb{I} \times \mathbb{D}_x \to \mathbb{R}^a$, see [35, 37]. Making use of locally computed projection matrices Z_1, Z_2 and T_2, this system can be constructed locally using the implicit function theorem. It consists of d differential and a algebraic equations and is strangeness-free. However, it may still have undetermined components, since the dimension of x is, in general, larger than $d + a$. By splitting the variables (or coordinate partitioning), such free variables can be interpreted as controls.

Hypothesis 1 directly leads to numerical methods for the solution of general DAEs. Such methods have been implemented in FORTRAN subroutine libraries GELDA [39] and GENDA [41] for linear and nonlinear systems, respectively, and also in a MATLAB DAE Toolbox [40]. A survey on other available software for DAEs can be found in [37].

The strangeness index generalizes the most commonly used concept of the *differentiation index*, e.g., [10] to over- and underdetermined systems (at least in the linear case). Roughly speaking, the differentiation index is the minimum of times that all or part of system (2.2) must be differentiated with respect to t in order to determine $\dot{x}$ as a continuous function of t and x. If the differentiation index is well-defined, then Hypothesis 1 is always satisfied. If both indices are defined, then the strangeness index is zero if the differentiation index is zero, and it is equal to the differentiation index lowered by one, otherwise.

EXAMPLE 2.3. *For a given fixed input function* u, *it is well known that the multibody system* (1.3) *has differentiation index* 3, *see* [18]. *Since Hypothesis* 1 *is satisfied with* $\mu = 2$, $d = 4$, $a = 3$ *and* $v = 0$, *this system has strangeness index* 2.

2.2. Index reduction. It is well known [10, 30, 37] that difficulties arise in the numerical solution of DAEs that are not strangeness-free, i.e., that have differentiation index larger than 1. Furthermore, in the control context, the solution of such systems may require derivatives of the input function which is in practice often only piecewise continuous. Finally, still today, except for special cases, no exact characterization is known of those systems of differentiation index larger than 1, for which the minimum principle of optimal control holds, see [25, 63].

For these reasons, in particular in the control context, one usually performs an index reduction and replaces the higher index system with an equivalent strangeness-free system that has the same solution set. There exist several different techniques for index reduction, see, e.g., [24, 28, 35, 36, 37, 38, 48, 50]. Here we briefly consider only the technique of *index reduction by minimal extension* [36].

As we have noted above, the reduced strangeness-free system (2.4) can be obtained from the derivative array (2.3) by computing the projection matrices Z_1, Z_2 and T_2. Since the derivative array consists of $(\mu+1)l$ equations, where μ is the strangeness index of the system, the problem sizes that can be handled by this approach are limited. However, in many applications, like multibody systems or electrical circuits, the descriptor systems have a special structure that can be used to reduce the computational complexity and memory requirements. If we can identify those equations that are responsible for high index and that have to be differentiated, then we can construct a reduced-size derivative array. We can further decrease the computational effort by introducing a minimal number of new variables, so-called dummy derivatives [50] to obtain an index reduction. We will demonstrate this approach for the multibody system (1.3) of Example 1.1.

EXAMPLE 2.4. *Differentiating the last equation in* (1.3) *twice and eliminating the differentiated variables using other equations, we obtain the equations*

$$0 = (x_2 - x_1)(x_5 - x_4) + x_3 x_6, \tag{2.5}$$

$$\begin{aligned} 0 = {} & 2\lambda\left((m_1+m_2)(x_1 - x_2)^2 + m_1 x_3^2\right) \\ & + m_1 m_2\left(g x_3 - (x_5 - x_4)^2 - x_6^2\right) + m_2(x_2 - x_1)u. \end{aligned} \tag{2.6}$$

A minimally extended strangeness-free system is obtained by putting equations (1.3), (2.5) *and* (2.6) *together and replacing* $\dot{x}_3$ *and* $\dot{x}_6$ *by new variables, say* $\hat{x}_3$ *and* $\hat{x}_6$. *In this way we get a strangeness-free system of the form* (2.4) *given by*

$$\begin{array}{rcl}
\dot{x}_1 & = & x_4, \\
\dot{x}_2 & = & x_5, \\
0 & = & x_6 - \hat{x}_3, \\
0 & = & -2x_3\lambda - m_2 g - m_2\hat{x}_6, \\
m_1\dot{x}_4 & = & -2(x_1 - x_2)\lambda + u, \\
m_2\dot{x}_5 & = & -2(x_2 - x_1)\lambda, \\
0 & = & (x_2 - x_1)^2 + x_3^2 - L^2, \\
0 & = & (x_2 - x_1)(x_5 - x_4) + x_3 x_6, \\
0 & = & 2\lambda\left((m_1+m_2)(x_1 - x_2)^2 + m_1 x_3^2\right) \\
 & & + m_1 m_2\left(g x_3 - (x_5 - x_4)^2 - x_6^2\right) + m_2(x_2 - x_1)u.
\end{array}$$

Looking in detail at this system, we may even omit the third and fourth equations that contain the newly introduced variables, and get a reduced strangeness-free system with the same number of equations and variables as the original system (1.3).

In recent years, the mathematical techniques for index reduction have been well established, so that (at least in theory) it is possible to transform most higher index systems to an equivalent strangeness-free form with the same solution set. Thus, in contrast to classical formulations in minimal coordinates, with this approach the physical interpretation of the variables is not modified and at the same time the resulting system has better properties concerning numerical simulation and control.

However, the computational effort to obtain this reduced strangeness-free formulation may be high if this is done by the available numerical methods. Furthermore, this process of index reduction is based on numerical rank decisions, which is a critical issue in finite precision arithmetic. On the other hand, the analysis of large classes of application problems shows that this computational effort can be significantly reduced by carefully analyzing the structure of the problems [36, 37].

With this analysis being available, however, it would be a much better approach to generate the resulting strangeness-free models in the first place. Thus, we suggest as a *new modelling paradigm* that the providers of automatic modelling tools should use this analysis to generate reduced strangeness-free models right from the beginning. For the example of multibody systems, as in Example 1.1, this would mean that in the process of generating the equations, immediately the first and second derivative of the constraint equations are generated as well, so that the reduced derivative array is available from the modelling tool, see [66]. A similar approach is also available for electrical circuit simulation [3, 4].

In the context of descriptor control systems, there also exists the possibility to achieve index reduction and regularization via feedback control. Transforming a control system via a behavior approach to the form (2.2), we can compute the reduced problem of the form

$$\begin{aligned} F_1(t,x,\dot{x},u) &= 0, \\ F_2(t,x,u) &= 0, \end{aligned} \tag{2.7}$$

that is strangeness-free in the combined vector of variables x and u. However, in general, if one fixes a specific control u, then (2.7) is not necessarily strangeness-free. In this case, using an appropriate feedback $u = K(t,x)$, we can construct a strangeness-free closed-loop problem

$$\begin{aligned} F_1(t,x,\dot{x},K(t,x)) &= 0, \\ F_2(t,x,K(t,x)) &= 0. \end{aligned} \tag{2.8}$$

Since the reduced system is defined locally only, it is even sufficient to use a linear state feedback $u = \tilde{K}x + w$, where the function w has to satisfy appropriate initial conditions so that the closed-loop system becomes strangeness-free, see [35, 37, 38] for details. Under some further assumptions, an index reduction is also possible via output feedback.

2.3. Stability. Although the stability theory for ordinary differential equations has been established at the end of the 19th century by the fundamental work of Lyapunov [44], for DAEs only few stability results exist [28, 45, 46, 54, 55, 67, 70].

Definition 2.5. Let $\mathcal{X}_0$ be a set of consistent initial vectors for the nonlinear system (2.2).

1. A solution $\hat{x}$ of (2.2) is called *stable* in the sense of Lyapunov if
   (i) for all $t_0 \geq 0$ and for all $x_0 \in \mathcal{X}_0$, the initial value problem

$$F(t,x,\dot{x}) = 0, \quad x(t_0) = x_0 \tag{2.9}$$

has a unique solution $x(t,x_0)$ defined on $[0,\infty)$, and

(ii) for every $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\hat{x}(t) - x(t, x_0)\| < \varepsilon$ for all $t \geq t_0$ and for all $x_0 \in \mathcal{X}_0$ with $\|\hat{x}(t_0) - x_0\| < \delta$.

2. A solution $\hat{x}$ of (2.2) is called *asymptotically stable* if it is stable and if there exists $\delta_0 > 0$ such that for the solution $x(t, x_0)$ of (2.9) with $\|\hat{x}(t_0) - x_0\| < \delta_0$, we have that $\lim_{t\to\infty} \|\hat{x}(t) - x(t, x_0)\| = 0$.

Since the stability theory for general nonlinear DAE systems is a difficult open problem, we restrict ourself in the following to linear time-invariant systems of the form (1.5) with square matrices E and A. The stability properties of such systems are well understood and can be characterized in a purely algebraic way using the Weierstraß canonical form of the matrix pencil $\lambda E - A$, see [17, 23, 28].

DEFINITION 2.6. A pencil $\lambda E - A$ with $E, A \in \mathbb{R}^{m\times n}$ is called *regular*, if $m = n$ and $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. Otherwise, the pencil is called *singular*.

Let $\lambda E - A$ be regular. Then there exist nonsingular matrices $W, T \in \mathbb{R}^{n,n}$ such that the pencil

$$\lambda WET - WAT = \lambda \begin{bmatrix} I_{n_f} & \\ & N \end{bmatrix} - \begin{bmatrix} J & \\ & I_{n_\infty} \end{bmatrix}$$

is in *Weierstraß canonical form*. Here J and N are in Jordan canonical form, and N is nilpotent with index of nilpotency ν. The number ν is called the *index* of $\lambda E - A$ and the eigenvalues of J are called *the finite eigenvalues* of $\lambda E - A$. Note that for the linear descriptor system (1.5), the differentiation index is just the index of the pencil $\lambda E - A$. Subspaces $\mathcal{W}, \mathcal{T} \subset \mathbb{R}^n$ are called *left* and *right deflating subspaces* of $\lambda E - A$ if $\dim(\mathcal{W}) = \dim(\mathcal{T})$ and $\mathcal{W} = E\mathcal{T} + A\mathcal{T}$. The matrices

$$P_l = W^{-1} \begin{bmatrix} I_{n_f} & \\ & 0 \end{bmatrix} W, \quad P_r = T \begin{bmatrix} I_{n_f} & \\ & 0 \end{bmatrix} T^{-1}$$

are the *spectral projectors* onto the left and right deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues.

For linear time-invariant systems with a regular pencil, we have the following characterization of stability.

THEOREM 2.7. [28] *Let $\lambda E - A$ be a regular pencil and let $u = 0$.*

1. *The trivial solution $x = 0$ of system* (1.5) *is stable if and only if all the finite eigenvalues of $\lambda E - A$ lie in the closed left half-plane and the eigenvalues on the imaginary axis are simple.*
2. *The trivial solution $x = 0$ of system* (1.5) *is asymptotically stable if and only if all the finite eigenvalues of $\lambda E - A$ lie in the open left half-plane.*

Note that if the trivial solution $x = 0$ of (1.5) with $u = 0$ is (asymptotically) stable and if $P_l Bu \in \mathcal{C}^\nu(\mathbb{I}, \mathbb{R}^n)$, then any solution of the inhomogeneous system (1.5) is also (asymptotically) stable.

EXAMPLE 2.8. *Linearization of the multibody system* (1.3) *along the equilibrium*

$[0, 0, -L, 0, 0, 0, m_2 g/(2L)]^T$ *yields the linear time-invariant system* (1.5) *with*

$$\begin{aligned} E &= \mathrm{diag}(1,\ 1,\ 1,\ m_1,\ m_2,\ m_2,\ 0), \\ A &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -m_2g/L & m_2g/L & 0 & 0 & 0 & 0 & 0 \\ m_2g/L & -m_2g/L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -m_2g/L & 0 & 0 & 0 & 2L \\ 0 & 0 & -2L & 0 & 0 & 0 & 0 \end{bmatrix}, \\ B &= [\,0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0\,]^T, \\ C &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The pencil $\lambda E - A$ has the finite eigenvalues 0, 0 *and* $\pm i\sqrt{(m_1+m_2)g/(m_1 L)}$. *Since the eigenvalue* 0 *is not simple, the linearized system is unstable.*

The asymptotic stability of (1.5) can also be characterized via a projected generalized continuous-time Lyapunov equation

$$E^T X A + A^T X E = -P_r^T Q P_r, \quad X = P_l^T X P_l \tag{2.10}$$

with unknown matrix X.

THEOREM 2.9. [67] *The trivial solution $x = 0$ of system* (1.5) *with $u = 0$ is asymptotically stable if and only if the projected Lyapunov equation* (2.10) *has a unique symmetric, positive semidefinite solution X for every symmetric, positive definite matrix Q.*

Thus, analogous to the case of ordinary differential equations, in theory, we can either use Lyapunov equations or eigenvalues to characterize the (asymptotic) stability of constant coefficient systems. However, the eigenvalues of $\lambda E - A$ may be very ill-conditioned in the sense that they may change largely even for small perturbations in E and/or A. Hence, eigenvalues that are computed numerically in finite precision arithmetic, may not always provide the correct information on the stability of dynamical systems.

EXAMPLE 2.10. *Let $E = I_{20}$ and*

$$A(\varepsilon) = \begin{bmatrix} -1 & 10 & & \\ & \ddots & \ddots & \\ & & \ddots & 10 \\ \varepsilon & & & -1 \end{bmatrix}.$$

All eigenvalues of $\lambda E - A(0)$ are -1 and lie in the open left half-plane. However, for $\varepsilon = 10^{-18}$, the matrix pencil $\lambda E - A(\varepsilon)$ has an eigenvalue $\lambda = \sqrt[20]{10} - 1$ which is in the right half-plane.

As an alternative to the use of eigenvalues in stability analysis, one can employ spectral parameters as in [27, 67]. These parameters not only characterize stability but also the sensitivity of the eigenvalues to perturbations in the matrix or matrix pencils.

Consider a *stability parameter* $\kappa(E, A) = 2\|E\|\|A\|\|X\|$, where X is symmetric, positive semidefinite and satisfies the projected Lyapunov equation (2.10) with $Q = I$.

We set $\kappa(E, A) = \infty$ if (2.10) does not have positive semidefinite solutions. If $\kappa(E, A)$ is finite, then all the finite eigenvalues of $\lambda E - A$ lie in the closed half-plane

$$\left\{ z \in \mathbb{C} \quad : \quad \mathrm{Re}(z) \leq -\frac{\|A\|_2}{\|E\|_2 \kappa(E, A)} \right\}.$$

The parameter $\kappa(E, A)$ can also be used to estimate the solution $x(t, x_0)$ of the consistent initial value problem

$$E\dot{x} = Ax, \qquad x(0) = P_r x_0. \tag{2.11}$$

We have a bound for x as a function of t and x_0 given by

$$\|x(t, x_0)\| \leq c e^{-t\|A\|/(\|E\|\kappa(E,A))} \|P_r x_0\| \tag{2.12}$$

with $c = \sqrt{\kappa(E, A)\|E\|\|(EP_r + A(I - P_r))^{-1}\|}$, see [67]. This estimate shows that the solution of (2.11) decreases exponentially if $\kappa(E, A)$ is bounded.

A fundamental problem in the robustness analysis for descriptor systems is to estimate perturbations in the matrix coefficients that do not change the stability properties of the system. Such perturbations can be described by the *stability radius* defined via

$$\begin{aligned} \mathbf{r}_{\mathcal{V}}(E, A) \quad = \quad \inf \Big\{ & \|[\,\Delta E,\, \Delta A\,]\| \; : \; [\,\Delta E,\, \Delta A\,] \in \mathcal{V} \;\; \text{and} \\ & \exists\, \lambda \in \mathbb{C}^+ \text{ s.t. } \det\big(\lambda(E + \Delta E) - (A + \Delta A)\big) = 0 \Big\}, \end{aligned}$$

where $\mathcal{V} \subset \mathbb{C}^{n,2n}$ is a vector space of admissible perturbations and $\mathbb{C}^+$ denotes the closed right half-plane. The stability radius $\mathbf{r}_{\mathcal{V}}(E, A)$ measures the smallest perturbation in $\mathcal{V}$ such that the perturbed matrix pencil $\lambda(E + \Delta E) - (A + \Delta A)$ is singular or has a finite eigenvalue in the closed right half-plane. The stability radius for standard state space systems with $E = I$ has been first considered in [32] and numerically reliable methods for its computation have been proposed in [12, 31, 60]. Unfortunately, these results are not immediately applicable to descriptor systems, since not only the spectrum but also the index and the regularity may change under perturbations. In general, any small perturbation of a pencil with singular E may yield an unstable system. Therefore, we have to restrict the set of perturbations. This can be done either with structured perturbations that preserve the physical properties of the system or with perturbations that do not change the structure of the deflating subspace corresponding to the infinite eigenvalue of $\lambda E - A$. A lower bound for the stability radius for the pencil $\lambda E - A$, allowing perturbations in A only, is given in [61]. A computable expression for the stability radius for regular pencils of index at most one is studied in [14]. Computationally accessible upper and lower bounds for smallest norm de-regularizing perturbations are discussed in [13].

3. Control theoretic concepts.

3.1. Controllability and observability. In this subsection we discuss controllability and observability of descriptor systems. In contrast to standard state space systems, for descriptor systems there are several different concepts of controllability and observability.

DEFINITION 3.1. Consider a descriptor system (1.1), (1.2).

1. System (1.1), (1.2) is called *completely controllable* (*C-controllable*) if for any given initial state $x_0 \in \mathbb{R}^n$ and final state $x_f \in \mathbb{R}^n$, there exists a control input u that transfers the system from x_0 to x_f in finite time.

2. System (1.1), (1.2) is called *completely observable* (*C-observable*) if the zero output of the descriptor system with $u = 0$ implies that this system has the trivial solution $x = 0$ only.

In general, descriptor systems will not be completely controllable or completely observable, since there exist constraint equations that fix the solution and the output onto a certain manifold. For this reason one also considers the following definition.

DEFINITION 3.2. For system (1.1), a set $\mathcal{R} \subseteq R^n$ is called *reachable* from x_0 if there exists a control input u that transfers the system from x_0 to some $x_f \in \mathcal{R}$ in finite time. System (1.1) is called *controllable within the reachable set* $\mathcal{R}$ (*R-controllable*) if one can reach any state in $\mathcal{R}$ from any consistent initial state x_0.

The corresponding dual concept to controllability within the reachable set needs an appropriate projection to the variables that are associated with the dynamical part of the systems. For general nonlinear descriptor systems this is difficult to describe. For linear time-invariant systems, however, an appropriate dual concept is as follows.

DEFINITION 3.3. System (1.5) is called *observable within the reachable set* (*R-observable*) if the zero output of the descriptor system with $u = 0$ implies that all solutions of this system satisfy $P_r x = 0$, where P_r is the projection onto the right deflating subspace corresponding to the finite eigenvalues of $\lambda E - A$.

Another problem arises for descriptor systems if input functions are used that are only piecewise continuous. If the strangeness index is larger than 0, then, since the solution may depend on the derivative of the input function, it may happen that for such controls no classical solution exists. In the case of linear time-invariant systems, it is possible to characterize the solution set using impulsive smooth distributions [37, 62]. In this situation the following concepts can be introduced.

DEFINITION 3.4. Consider a linear descriptor system (1.5).

1. System (1.5) is called *impulse controllable* (*I-controllable*) if for any given initial state $x_0 \in \mathbb{R}^n$, there exists a state feedback control $u = Kx + w$, such that the pencil $\lambda E - (A + BK)$ of the closed-loop system is regular and of index $\nu \leq 1$.
2. System (1.5) is called *impulse observable* (*I-observable*) if the output of (1.5) is continuous, when we use a step function as input.

For linear time-invariant systems (1.5) these controllability and observability concepts can be characterized algebraically/geometrically in terms of the matrices E, A, B and C, see [11, 16, 17, 71, 72].

THEOREM 3.5.

1. *System* (1.5) *is C-controllable if and only if* $\operatorname{rank}[\alpha E - \beta A,\ B] = n$ *for all* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.
2. *System* (1.5) *is C-observable if and only if* $\operatorname{rank}\left[\alpha E^T - \beta A^T, C^T\right] = n$ *for all* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.
3. *System* (1.5) *is R-controllable if and only if* $\operatorname{rank}[\lambda E - A,\ B] = n$ *for all* $\lambda \in \mathbb{C}$.
4. *System* (1.5) *is R-observable if and only if* $\operatorname{rank}\left[\lambda E^T - A^T,\ C^T\right] = n$ *for all* $\lambda \in \mathbb{C}$.
5. *System* (1.5) *is I-controllable if and only if* $\operatorname{rank}[E,\ AK_E,\ B] = n$, *where the columns of* K_E *span the kernel of* E.
6. *System* (1.5) *is I-observable if and only if* $\operatorname{rank}\left[E^T,\ A^T K_{E^T},\ C^T\right] = n$, *where the columns of* K_{E^T} *span the kernel of* E^T.

One can also show, see [17], that system (1.5) with a regular pencil $\lambda E - A$ is C-controllable if and only if it is R-controllable and $\operatorname{rank}[E,\ B] = n$. Furthermore, system (1.5) is C-observable if and only if it is R-observable and rank $\left[E^T,\ C^T\right] = n$.

Note that the descriptor system (1.5) with a regular pencil $\lambda E - A$ of index $\nu \leq 1$ is I-controllable and I-observable. In view of the general analysis given in Section 2.2, one can show for linear time-invariant systems that the property of the system to be strangeness-free in the behavior setting is equivalent to I-controllability, see [35, 37, 38].

EXAMPLE 3.6. *Consider again the linear system* (1.5) *with E, A, B and C as in Example* 2.8. *Since*

$$\operatorname{rank}\left[E,\ AK_E,\ B\right] = \operatorname{rank}\left[E^T, A^T K_{E^T},\ C^T\right] = 6,$$

this system is neither I-controllable nor I-observable and it is obvious that the system is not C-controllable and C-observable either. On the other hand, we have

$$\operatorname{rank}\left[\lambda E - A,\ B\right] = \operatorname{rank}\left[\lambda E^T - A^T,\ C^T\right] = 7,$$

for all $\lambda \in \mathbb{C}$, i.e., the descriptor system is R-controllable and R-observable.

4. Model order reduction. Modelling of complex physical and technical processes described by partial differential equations, such as fluid flow or elastic mechanical structures, leads after semidiscretization in space to descriptor systems of very large order n, while the number m of inputs and the number p of outputs are typically small compared to n. Despite the ever increasing computational speed, simulation, optimization or real time controller design for such large-scale systems is impossible because of large storage requirements and computation time. A classical way out of this dilemma is the approach of model order reduction, where the original system is approximated by a reduced order system that is in some measure close to the original model. In the linear time invariant case (1.5) this would be a systems of the form

$$\widetilde{E}\,\dot{\widetilde{x}} = \widetilde{A}\,\widetilde{x} + \widetilde{B}\,u, \qquad \widetilde{y} = \widetilde{C}\,\widetilde{x}, \tag{4.1}$$

where $\widetilde{E}$, $\widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$, $\widetilde{C} \in \mathbb{R}^{p,\ell}$ and $\ell \ll n$. Apart from having a much smaller state space dimension, it is also important that the reduced-order model preserves essential properties of the original system, like regularity, stability and passivity, and that the approximation error is small. There exist many different model reduction approaches for standard-state space systems, see [2, 8], but only two of them have been generalized for descriptor systems. These are balanced truncation [52, 53, 58, 68] and moment matching approximation [7, 19, 20, 22]. In this section we briefly describe the general idea of these model reduction methods.

4.1. Balanced truncation. Balanced truncation is one of the most effective and well studied model reduction approaches used in control theory [26, 42, 53]. Consider the descriptor system (1.5) with a regular pencil $\lambda E - A$. Applying the Laplace transform to (1.5) we find

$$\mathbf{y}(s) = C(sE - A)^{-1}B\mathbf{u}(s) + C(sE - A)^{-1}Ex(0),$$

where $\mathbf{u}(s)$ and $\mathbf{y}(s)$ are the Laplace transforms of the input $u(t)$ and the output $y(t)$, respectively. The rational matrix-valued function

$$\mathbf{G}(s) = C(sE - A)^{-1}B \tag{4.2}$$

is called the *transfer function* of (1.5). It describes the input-output relation of (1.5) in frequency domain.

For any rational function $\mathbf{G}$, one can find matrices E, A, B and C such that (4.2) holds [17]. A descriptor system (1.5) with these matrices is called a *realization* of $\mathbf{G}$ and denoted by $\mathbf{G} = [\,E,\, A,\, B,\, C\,]$. The dimension of the matrices E and A defines the order of the realization. The transfer function $\mathbf{G}$ may have many different realizations that may be even of different orders. A realization $[\,E,\, A,\, B,\, C\,]$ of $\mathbf{G}$ is called *minimal* if the dimension of the matrices E and A is as small as possible. However, since $\mathbf{G}$ is invariant under state space transformation, its realization of fixed order is not unique. This means that if $[\,E,\, A,\, B,\, C\,]$ is a realization of $\mathbf{G}$, then $[\,WET,\, WAT,\, WB,\, CT\,]$ is also a realization of $\mathbf{G}$ for any nonsingular matrices W and T.

For a given realization $\mathbf{G} = [\,E,\, A,\, B,\, C\,]$ such that all eigenvalues of $\lambda E - A$ have negative real part, we define the *proper controllability and observability Gramians* $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ as the unique symmetric, positive semidefinite solutions of the *projected generalized continuous-time Lyapunov equations*

$$E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T = -P_lBB^TP_l^T, \qquad P_r\mathcal{G}_{pc}P_r^T = \mathcal{G}_{pc}$$

and

$$E^T\mathcal{G}_{po}A + A^T\mathcal{G}_{po}E = -P_r^TC^TCP_r, \qquad P_l^T\mathcal{G}_{po}P_l = \mathcal{G}_{po}.$$

The *improper controllability and observability Gramians* $\mathcal{G}_{ic}$ and $\mathcal{G}_{io}$ are defined as the unique symmetric, positive semidefinite solutions of the *projected generalized discrete-time Lyapunov equations*

$$A\,\mathcal{G}_{ic}A^T - E\,\mathcal{G}_{ic}E^T = (I - P_l)BB^T(I - P_l)^T, \;\; P_r\mathcal{G}_{ic}P_r^T = 0$$

and

$$A^T\mathcal{G}_{io}A - E^T\mathcal{G}_{io}E = (I - P_r)^TC^TC(I - P_r), \;\; P_l^T\mathcal{G}_{io}P_l = 0.$$

The *proper* and *improper Hankel singular values* of system (1.5) denoted by ς_j and θ_j, respectively, are defined as the square roots of the nonzero eigenvalues of the matrices $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ and $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$. If the realization $\mathbf{G} = [\,E,\, A,\, B,\, C\,]$ is minimal, it has exactly n_f proper and n_∞ improper Hankel singular values, where n_f and n_∞ are the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively.

DEFINITION 4.1. *A minimal realization* $\mathbf{G} = [\,E, A, B, C\,]$ *is called* balanced *if*

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{G}_{ic} = \mathcal{G}_{io} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta \end{bmatrix},$$

where $\Sigma = diag(\varsigma_1, \ldots, \varsigma_{n_f})$ *and* $\Theta = diag(\theta_1, \ldots, \theta_{n_\infty})$.

A minimal realization can always be transformed by a state space transformation into a balanced form [68]. If the realization is not minimal, then it has uncontrollable and unobservable states that can be truncated without changing the input-output relation. Using the input-output energy interpretation of the proper Hankel singular values, see [68], we can also truncate the states of the balanced system that correspond to the small nonzero Hankel singular values. Such states are simultaneously difficult to reach and to observe and, hence, they have a small impact on the energy transfer from input to output. Unfortunately, this does not hold for the improper Hankel singular values. The equations associated with the improper Hankel singular values describe constraints of the system, i.e., they define a manifold in which the solution dynamics takes place. Therefore, a truncation of the states corresponding to the small nonzero improper Hankel singular values may lead to a physically inappropriate approximation.

Note that to perform order reduction we do not need to transform the descriptor system into a balanced form explicitly. Instead, we can combine balancing and truncation by performing the projection

$$\widetilde{E} = W_\ell^T E T_\ell,\ \widetilde{A} = W_\ell^T A T_\ell,\ \widetilde{B} = W_\ell^T B,\ \widetilde{C} = C T_\ell,$$

where the projection matrices $W_\ell, T_\ell \in \mathbb{R}^{n,\ell}$ determine the left and right subspaces associated with dominant proper and nonzero improper Hankel singular values. These matrices can be efficiently computed by a *generalized square root method* described in [52, 68]. One can show that the reduced-order system $\widetilde{\mathbf{G}} = [\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,]$ computed by this method is asymptotically stable, minimal and balanced [52, 68]. Moreover, we have the following upper bound for the $\mathbb{H}_\infty$-norm of the error

$$\begin{array}{rcl} \|\widetilde{\mathbf{G}} - \mathbf{G}\|_{\mathbb{H}_\infty} & := & \sup_{\omega\in\mathbb{R}} \|\widetilde{\mathbf{G}}(i\omega) - \mathbf{G}(i\omega)\| \\ & \le & 2(\varsigma_{\ell_f+1} + \ldots + \varsigma_{n_f}), \end{array}$$

where $\varsigma_{\ell_f+1}, \ldots, \varsigma_{n_f}$ are the truncated proper Hankel singular values. This bound allows an adaptive choice of the state space dimension $\ell = \ell_f + \ell_\infty$ of the reduced model depending on how accurate the approximation is needed. The main difficulty in balanced truncation model reduction for large-scale descriptor systems is that four matrix Lyapunov equations have to be solved. However, recent results on low rank approximations to the solutions of Lyapunov equations [43, 57, 69] make the balanced truncation model reduction approach also feasible for large-scale problems.

4.2. Moment matching. An alternative model reduction approach for descriptor systems is the moment matching approximation which can be formulated as follows. Suppose that $s_0 \in \mathbb{C}$ is not an eigenvalue of the pencil $\lambda E - A$. Then the transfer function $\mathbf{G}(s) = C(sE - A)^{-1}B$ can be expanded into a Laurent series at s_0 as

$$\begin{array}{rcl} \mathbf{G}(s) & = & C\big(I - (s - s_0)(s_0E - A)^{-1}E\big)^{-1}(s_0E - A)^{-1}B \\ & = & M_0 + M_1(s - s_0) + M_2(s - s_0)^2 + \ldots\,, \end{array}$$

where the matrices

$$M_j = -C\left((s_0E - A)^{-1}E\right)^j (s_0E - A)^{-1}B$$

are called the *moments* of the descriptor system (1.5) at s_0. The moment matching approximation problem for the descriptor system (1.5) consists in determining a rational matrix-valued function $\widetilde{\mathbf{G}}(s)$ such that the Laurent series expansion of $\widetilde{\mathbf{G}}(s)$ at s_0 has the form

$$\widetilde{\mathbf{G}}(s) = \widetilde{M}_0 + \widetilde{M}_1(s - s_0) + \widetilde{M}_2(s - s_0)^2 + \ldots\,, \tag{4.3}$$

where the moments $\widetilde{M}_j$ satisfy the moment matching conditions

$$M_j = \widetilde{M}_j, \qquad j = 0, 1, \ldots, k. \tag{4.4}$$

If $s_0 = \infty$, then M_j are the Markov parameters of (1.5) and the corresponding approximation problem is known as *partial realization.* Computation of the partial realization for descriptor systems of a special structure has been considered in [9]. For $s_0 = 0$, the approximation problem (4.3), (4.4) reduces to the *Padé approximation* problem. Efficient algorithms based on Arnoldi and Lanzcos procedures for solving this problem have been presented in [19, 21]. For an arbitrary complex number $s_0 \neq 0$, the moment matching approximation is the problem of *rational interpolation* or *shifted Padé approximation* that has been considered in [7, 19, 20, 22]. Apart from a single interpolation point one can construct a reduced-order system with the transfer function $\widetilde{\mathbf{G}}(s)$ that matches $\mathbf{G}(s)$ at multiple points $\{s_0, s_1, \ldots, s_k\}$. Such an approximation is called a *multi-point Padé approximation* or a *rational interpolant* [1]. It can be computed efficiently for descriptor systems by the rational Krylov subspace method [22, 64] While the moment matching approximation can be efficiently computed also for very large scale problems, stability and passivity are not necessarily preserved in the resulting reduced-order model, so that usually a postprocessing is needed to guarantee these properties.

5. Optimal control. Classical control applications such as stabilization of a system or path following can often be formulated in terms of optimal control problems. However, currently the theoretical basis of optimal control, such as the minimum principle is an open problem for general descriptor systems that are not strangeness-free. Only in very special cases necessary and sufficient conditions like the minimum principle have been derived [25, 51, 56, 63].

Let us consider linear systems with variable coefficients and consider the linear-quadratic optimal control problem of minimizing the cost functional

$$\mathcal{I}(x,u) = \int_{t_0}^{t_f} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

subject to the initial value problem

$$E(t)\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0, \tag{5.1}$$

where $E, A \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{l,n})$, $B \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{l,m})$, $Q \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{n,n})$, $R \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{m,m})$, $S \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{m,n})$ and $Q(t) = Q^T(t)$, $R(t) = R^T(t)$ for all $t \in \mathbb{I} = [t_0, t_f]$. It has been shown in [34, 47] that if the matrix $\begin{bmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{bmatrix}$ is positive definite for all $t \in \mathbb{I}$ and if $x_*(t)$, $\lambda_*(t)$ and $u_*(t)$ satisfy the boundary value problem for the generalized Euler-Lagrange equations

$$\begin{gathered} \mathcal{E}(t) \begin{bmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{u} \end{bmatrix} = \mathcal{A}(t) \begin{bmatrix} x \\ \lambda \\ u \end{bmatrix}, \\ x(t_0) = x_0, \quad E^T(t_f)\lambda(t_f) = 0, \end{gathered} \tag{5.2}$$

where

$$\begin{aligned} \mathcal{E}(t) &= \begin{bmatrix} E(t) & 0 & 0 \\ 0 & -E^T(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathcal{A}(t) &= \begin{bmatrix} A(t) & 0 & B(t) \\ Q(t) & A^T(t) + \dot{E}^T(t) & S(t) \\ S^T(t) & B^T(t) & R(t) \end{bmatrix}, \end{aligned}$$

then $x_*(t)$ and $u_*(t)$ is the optimal solution of the optimal control problem $\min \mathcal{I}(x,u)$ subject to (5.1). Unfortunately, these conditions are, in general, not necessary, since the solution of the boundary value problem (5.2) may not exist or may not be unique. Necessary optimality conditions for descriptor systems of strangeness index zero have been obtained in [5, 6, 34, 51].

6. Conclusion. Descriptor systems present a general framework for the modelling, simulation, control and optimization of complex physical systems. The mathematical analysis and numerical solution methods are well established, see [37] for a recent graduate textbook. However, major problems remain open. These include the stability analysis, the minimum principle, as well as model reduction methods for large-scale nonlinear descriptor systems.

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