THE INFINITE LOCALLY RANDOM GRAPH

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ABSTRACT. We investigate properties of a certain countably infinite graph called the infinite locally random graph, written R_N. The graph R_N arises in the study of models for massive, self-organizing networks like the web graph. We characterize the isomorphism type of R_N as a limit of a random process, and via a domination elimination ordering. We prove that R_N satisfies vertex deletion properties generalizing inexhaustibility. As is the case for the infinite random graph R, R_N has a universal automorphism group and endomorphism monoid. Unlike R, R_N isometrically embeds all finite graphs.

1. INTRODUCTION

Certain countably infinite graphs arise naturally as limits of finite graphs. A well known instance of this is the *infinite random graph*, written R, which may be defined as a limit as follows. Let R_0 be a fixed finite graph. For some $t \geq 0$, assume that R_t is a finite graph containing R_0. For each subset S of $V(R_t)$, add a new node x_S joined only to the nodes of S. We say that x_S *extends* S. The graph R_t along with the new nodes x_S defines the graph R_{t+1}. Let R be the graph with nodes $\bigcup_{t\in\mathbb{N}} V(R_t)$ and edges $\bigcup_{t\in\mathbb{N}} E(R_t)$. We will write $R = \lim_{t\to\infty} R_t$.

The graph R satisfies the *existentially closed* or *e.c.* adjacency property. A graph is e.c. if for all disjoint finite sets of nodes A, B, there is a node $z \notin A \cup B$ joined to each node of A and to no node of B. By a back-and-forth argument, a countably infinite graph is e.c. if and only if it is isomorphic to R (see [9]). The e.c. property therefore supplies a powerful tool for studying R. For example, using the e.c. property one may easily derive that R is *inexhaustible*: for each node x of R, $R - x \cong R$.

1991 *Mathematics Subject Classification.* 05C75, 05C80, 05C90.

Key words and phrases. infinite random graph, locally closed e.c., web graph, domination elimination ordering, isometric embedding, inexhaustible graph, automorphism group, endomorphism monoid.

The second author gratefully acknowledges support from NSERC and MITACS grants.

The graph R arises naturally via the following infinite random process which inspires its name. We add new nodes over countably many discrete time-steps. Fix $p \in (0,1)$. At time $t = 0$ start with any fixed finite graph. At time step $t+1$, add in a new node x_{t+1}. For each of the existing nodes y, add the edge yx_{t+1} independently with probability p. Erdős and Rényi proved in [14] that with probability 1, a limit generated by this random process is isomorphic to R. This instance of a random process with a seemingly deterministic conclusion has made R the centre of much research activity. See the surveys [9, 10] on R for additional background and references.

In real-world, self-organizing networks like the web graph (where nodes correspond to web pages, and edges represent links between pages), each node acts as an independent agent, which will base its decision on how to link to the existing network on local knowledge. As a result, the neighbourhood of a new node will often be an imperfect copy of the neighbourhood of an existing node. Both the *copying models* [1, 18] of the web graph, and the *duplication model* [12] of biological networks incorporate this notion of copying in their design.

Through the study of self-organizing networks, new interesting limit graphs have been recently discovered; see [6]. For a node y of a graph G, define $N(y) = \{x \in V(G) : xy \in E(G)\}$ and $N[y] = N(y) \cup \{y\}$. Fix a finite graph H and let X_0 be isomorphic to H. For some $t \geq 0$, assume that X_t is a finite graph containing X_0. For each node y of X_t, and each subset S of $N(y)$ in $V(R_t)$, add a new node $x_{y,S}$ joined only to the nodes of S. We say that $x_{y,S}$ *copied* from y. The graph X_t along with all the nodes $x_{y,S}$ for all choices of y and S defines X_{t+1}. Let $R_H = \lim_{t\to\infty} X_t$. We define $\uparrow H$ similarly, but replacing $N(y)$ by $N[y]$. Hence, $\uparrow H$ is formed by extending all the subsets of closed neighbour sets.

The graphs R_H and $\uparrow H$ were first studied in [6], which includes the following result. A graph G is *inexhaustible* if for all nodes x of G, $G - x \cong G$. A *ray* is an infinite path that extends indefinitely in one direction; a *double ray* is an infinite path that extends indefinitely in two directions. A *one-way Hamilton path* is a spanning subgraph that is a ray, while a *two-way Hamilton path* is a spanning subgraph that is a double ray.

Theorem 1. (1) *For all finite graphs H and J, $\uparrow H \cong \uparrow J$.*
(2) *For all finite graphs H, R_H and $\uparrow H$ are inexhaustible.*
(3) *For all finite graphs H, R_H and $\uparrow H$ have one- and two-way Hamilton paths.*

By Theorem 1 (1), the initial graph H has no impact on the limit graph $\uparrow H$. Hence, we name this unique isomorphism type the *infinite locally random graph*, written R_N. In contrast, there are $\aleph_0$ many isomorphism types of graphs R_H. For instance, R_{K_2} and R_{C_6} are non-isomorphic.

The graph R_N has other striking properties. For example, for each node x of R_N, the subgraph induced by $N(x)$ is isomorphic to R. This motivates the name of the graph. As R is an induced subgraph of R_N, the graph R_N is $\aleph_0$-*universal*; that is, it embeds all countably infinite graphs as induced subgraphs.

Further, R_N possesses infinitely many pairwise isomorphic connected components. We name the isomorphism type of one of these components $c(R_N)$. We use the notation C_t for the unique nontrivial connected component of X_t in R_N. Hence, $c(R_N) = \lim_{t\to\infty} C_t$, where C_0 is a connected component of H. We may view C_{t+1} as the graph formed by extending only nonempty subsets of closed neighbour sets in C_t. Note that each $v \in V(C_{t+1})\backslash V(C_t)$ is joined to some node in C_t, and $C_0 \cong H$ if H is connected.

The graphs R_H were studied in [6, 7]. The goal of the present article is to present new results on the graph R_N. Section 2 considers various equivalent ways of representing R_N, such as via weak foldings (Theorem 3), and as a limit of a random process (Theorem 4). The graph R_N has interesting properties as a metric space, and these are considered in Section 3. We prove that every finite graph isometrically embeds in R_N; see Theorem 8. In the final section, we consider properties of the automorphism group and endomorphism monoid of R_N. Unlike R, R_N is not homogeneous. Nevertheless, we classify which induced subgraphs G have the property that every isomorphism between copies of G extends to an automorphism of R_N; see Theorem 10. In Theorem 13, we prove that the automorphism group and endomorphism monoid of R_N embed each countable group and monoid, respectively.

All graphs we consider are simple, undirected, and countable. We write $G \leq H$ if G is isomorphic to an induced subgraph of H, and $G \restriction S$ for the subgraph induced by $S \subseteq V(G)$. We write $G \cong H$ if G and H are isomorphic. The complement of a graph G is written $\overline{G}$. The cardinality of $\mathbb{N}$ is denoted by $\aleph_0$, and the cardinality of $\mathbb{R}$ is denoted by $2^{\aleph_0}$.

2. Isomorphic representations of R_N.

By Theorem 1 (1), for all finite graphs H, $\uparrow H \cong R_N$. We will often exploit this fact by choosing an appropriate H. We say that any graph

isomorphic to R_N is an *isomorphic representation* of R_N. Hence, $\uparrow H$ is an isomorphic representation of R_N. In this section, we investigate several distinct isomorphic representations of R_N.

The first isomorphic representation of R_N we consider uses a variant of graph dismantling or folding (see [8, 15]). Suppose that H, J are finite graphs such that H is an induced subgraph of J, and let $v \in V(J)$. Define $H \preccurlyeq_v J$ if there is a node u in $J - v$ such that $N(v) \subseteq N[u]$, and $H = J - v$. We write $H \preccurlyeq J$ if there is a nonnegative integer m, graphs $H_0 = H$, $H_1, \ldots, H_m = J$, and nodes $v_0, \ldots, v_{m-1} \in V(J)$ so that $H_i \preccurlyeq_{v_i} H_{i+1}$ for all $0 \leq i \leq m-1$. We say that the graph J *weakly folds onto* H. For example, $K_1 \preccurlyeq K_n$ for all $n \in \mathbb{N}$, but $K_1 \not\preccurlyeq C_5$. Weak folding is used in the context of domination elimination orderings of graphs; see [11, 13].

We extend weak folding to countable graphs as follows. Let H and J be countable graphs. The relation $H \preccurlyeq_v J$ is defined as in the finite case. Fix I as either $\mathbb{N}$ or one of the sets $\{0, 1, \ldots n\}$, where $n \in \mathbb{N}$. We write $H \preccurlyeq J$ if there exists a sequence of countable graphs $(H_t : t \in I)$ so that $H_0 = H$, $H_t \preccurlyeq_v H_{t+1}$ for all $t \in I$, and $J = \lim_{t\to\infty} H_t$ if $I = \mathbb{N}$, or $J = H_n$ if I is of the form $\{0, 1, \ldots n\}$. For example, $K_1 \preccurlyeq K_{\aleph_0}, \overline{K_{\aleph_0}}$. Note that for all $t > 0$, $H \preccurlyeq X_t$, and $X_t \preccurlyeq X_{t+1}$. Hence, $H \preccurlyeq R_N$, and so R_N weakly folds onto every finite graph by Theorem 1 (1).

A graph G is *locally closed e.c.* if for each node y of G, for each finite $X \subseteq N[y]$, and each finite $Y \subseteq V(G) \setminus X$, there exists a node z not in $\{y\} \cup X \cup Y$ that is joined to X and not to Y. The locally closed e.c. property is therefore a variant of the e.c. property that applies only to sets contained in the closed neighbour set of a node. The graphs R_N and R satisfy the locally closed e.c. property. In contrast to the e.c. property, it was proved in [6] that there are $2^{\aleph_0}$ many non-isomorphic locally closed e.c. countable graphs (in [6], the locally closed e.c. property is referred to as property (A)). The *locally e.c.* property is defined analogously, but replacing $N[y]$ by $N(y)$ (in [6], the locally closed e.c. property is referred to as property (B)).

Before we state an isomorphic representation of R_N via weak folding, we consider one consequence of the locally closed e.c. property. Define $N^c(x) = \{y \in V(R_N) : xy \notin E(R_N)\}$. The graph R has the property that it is locally isomorphic to itself: for all $x \in V(R_N)$, $R_N \restriction N(x) \cong R$. Since $\overline{R} \cong R$, it follows that for all $x \in V(R)$, $R \restriction N^c(x) \cong R$. A graph G is *R-like* if for all $x \in V(G)$, the graphs $G \restriction N^c(x)$ and $G \restriction N(x)$ are isomorphic to R. Other than R itself, few R-like graphs are known. The next theorem demonstrates that many graphs, including R_N, are R-like.

Theorem 2. *If G is locally closed e.c., then G is R-like. In particular, R_N is R-like.*

Proof. By Theorem 2.2 of [6], the locally closed e.c. property implies that for all $y \in V(G)$, $G \restriction N(y) \cong R$. Now fix disjoint finite sets of nodes A, B in $N^c(y)$. Consider A, B as subsets of $\overline{G}$. Then $A, B \subseteq N(y)$ in $\overline{G}$. By the symmetry in the definition of the locally closed e.c. property, $\overline{G}$ is locally closed e.c. Hence, there is a node z of $\overline{G}$ joined to each node of $B \cup \{y\}$ and to no node of A (note that $y \notin A \cup B$). Then in G the node $z \in N^c(y)$ is joined to each node of A and to no node of B. □

A graph H is *finitely approximated* if H weakly folds onto some finite graph. For example, the graphs $\overline{K_{\aleph_0}}$ and $K_{\aleph_0}$ are finitely approximated. By definition, R_N is finitely approximated. The following theorem ties together the relation $\preccurlyeq$, the graph $c(R_N)$, and the locally closed e.c. property.

Theorem 3. *The graph $c(R_N)$ is the unique isomorphism type of the connected countable graphs G that are locally closed e.c. and finitely approximated.*

Proof. Suppose that $G = \lim_{t\to\infty} G_t$, where $G_0 \cong H$ for some finite graph H, and $G_t \preccurlyeq_v G_{t+1}$ for all $t \in \mathbb{N}$. Since G is connected, each $v \in V(G_{t+1})\backslash V(G_t)$ is joined to some node in G_t. By Theorem 1 (1), $R_N \cong \uparrow H$. We use the notation $c(R_N) = \lim_{t\to\infty} C_t$ given in the introduction, where $C_0 \cong H$.

We define an isomorphism $f : c(R_N) \to G$ inductively as follows. Let $f_0 : C_0 \to G_0$ be any fixed isomorphism. As the induction hypothesis, suppose that for a fixed integer $t \geq 0$, there is a finite induced subgraph J_t of G containing G_t along with an isomorphism $f_t : C_t \to J_t$ extending f_0. (Note that we do not claim here that J_t is of the form G_s for some $s \geq 0$.)

Enumerate all pairs (y, S) where y is a node of J_t and $S \subseteq N_{J_t}[y]$ is nonempty as $\{(y_i, S_i) : 1 \leq i \leq k_t\}$. By the locally closed e.c. property for G, there is a node $x_{y_1,S_1} \in V(G)\backslash V(J_t)$ that is joined to S_1 and to no other nodes of J_t. A straightforward (and therefore, omitted) inductive argument supplies, for all $j \in \{1 \dots k_t\}$, a node x_{y_j,S_j} in

$$V(G)\backslash(V(J_t) \cup \{y_1, \dots, y_j\})$$

that is joined to S_j and to no other node of $V(J_t) \cup \{y_1, \dots, y_j\}$. Let

$$T = \{x_{y_i,S_i} : 1 \leq i \leq k_t\} \subseteq V(G).$$

Define J_{t+1} to be the subgraph of G induced by $V(J_t) \cup T$. Note that T is an independent set of nodes. By the definition of C_{t+1}, J_{t+1} is

isomorphic to C_{t+1} by an isomorphism extending f_t. The unique node $v \in V(G_{t+1}) \backslash V(G_t)$ is of the form $x_{y,S}$ for some y and S in G_t. As G_t is an induced subgraph of J_t, we have that $v \in V(J_{t+1})$. Hence, G_{t+1} is contained in J_{t+1}, and the induction step is complete.

Define $f : c(R_N) \to G$ by $f = \bigcup_{t \in \mathbb{N}} f_t$. The map f is an embedding as each map f_t is an isomorphism, and it is onto by construction. Hence, f is an isomorphism. □

We next consider a random process that almost surely generates R_N. For the *closed duplication model* (based on the duplication model of [12]), we add new nodes over a countable number of discrete time-steps. Fix $p \in (0, 1)$. At time $t = 0$ start with any fixed finite graph H. At time step $t + 1$, choose a node u uniformly at random from the nodes of time t. Add a new node x_{t+1}, and for each of the nodes y in $N[u]$, add the edge yx_{t+1} independently with probability p.

The following theorem parallels the Erdős, Rényi result [14] for the infinite random graph.

Theorem 4. *With probability* 1, *a nontrivial connected component of a limit generated by the closed duplication model is isomorphic to* $c(R_N)$.

Proof. By an argument similar to the proof of Theorem 2.1 in [6], with probability 1, C is locally closed e.c. The graph C weakly folds onto H by definition of the model. Hence, by Theorem 3, C is isomorphic to $c(R_N)$. □

For our final isomorphic representation of R_N, we consider graphs of the form $\uparrow H$, where H is countably infinite instead of finite. If we take H in $\uparrow H$ to be countable, then the graph $\uparrow H$ remains countable and locally closed e.c. Finitely approximated graphs always "generate" R_N.

Theorem 5. *If* H *is a finitely approximated countable graph, then* $\uparrow H$ *is an isomorphic representation of* R_N.

Proof. Let G be a finite graph so that $G \preceq H$, and let C be a connected component of $\uparrow H$. Then C weakly folds onto H and hence, C weakly folds onto G. It is not hard to see that C is locally closed e.c. Hence, by Theorem 3, $C \cong c(R_N)$. As $\uparrow H$ has exactly $\aleph_0$ many pairwise isomorphic components, we have that $\uparrow H \cong R_N$. □

Not all countable graphs are finitely approximated. For example, the graph H formed from the infinite one-way path, where each node of the path is attached to a distinct C_5 is not finitely approximated. We do not know if $\uparrow H \cong R_N$.

3. Fractal and metric properties of R_N

Recall from the introduction that a graph G is inexhaustible if for all $x \in V(G)$, $G - x \cong G$. For more on inexhaustible graphs and their properties, see [4]. By Theorem 4.1 of [6], any locally closed e.c. graph (hence, R_N) is inexhaustible. Inexhaustibility is only one type of fractal property a graph may satisfy. For example, the infinite random graph R satisfies the *pigeonhole property*: if T is a set of nodes so that the subgraph induced by T is not isomorphic to R, then $R - T \cong R$. Hence, inexhaustibility is the case where T is finite. The pigeonhole property is sometimes called a *fractal property* of a graph. Other than R, the only other countable isomorphism types of graphs with the pigeonhole property are K_1, $K_{\aleph_0}$, and $\overline{K_{\aleph_0}}$ (see [9]).

For R_N, the situation is more complex. For example, for any node x the subgraph induced by $N(x)$ is isomorphic to R, but $R_N - N(x)$ contains an isolated node and so is not isomorphic to R_N. Despite this example, certain subsets T may be deleted leaving an isomorphic copy of R_N. A subset T of $V(R_N)$ is *robust* if $R_N - T \cong R_N$. As R_N is inexhaustible, each finite subset of R_N is robust.

By Theorem 5, without loss of generality, we assume in what follows that $R_N = \uparrow\overline{K_{\aleph_0}}$ (although we may represent R_N for our purposes by $\uparrow G$, where G is any finitely approximated inexhaustible graph). If t is a positive integer, then a set S of nodes in X_t is called *t-special* if it includes nodes of $V(X_t)\backslash V(X_{t-1})$. We introduce the following notation for subsets of $V(R_N)$. Let S_1 be the set of nodes added to X_0 in time step 1 by extending sets of nodes of X_0. In X_t, let S_t be the set of nodes extending X_{t-1}. Let $S_{1,1} = S_1$. In S_2, there are nodes $S_{2,1} \subseteq S_2$ extending X_0 as S_1 does, and nodes $S_{2,2} \subseteq S_2$ extending 1-special sets of nodes. Note that $S_2 = S_{2,1} \cup S_{2,2}$ and $S_{2,1} \cap S_{2,2} = \emptyset$. In S_t we define a finite sequence $(S_{t,i} : 1 \leq i \leq t)$ of sets of nodes partitioning S_t, with each $S_{t,i}$ consisting of the nodes that extend the $(t-1)$-special sets of X_{t-1}. Note that $S_{t,t}$ is the *only* set extending $(t-1)$-special sets of nodes. If $1 \leq i \leq t-1$, then the nodes in the set $S_{t,i}$ extend the same subsets that $S_{t-1,i}$ does.

We define the t^{th} *column* of R_N, written $\mathrm{Col}(t)$, to be the set $\bigcup_{i \in \mathbb{N}} S_{i,t} \subseteq V(R_N)$. Note that each set $\mathrm{Col}(t)$ is infinite and independent.

A set T of nodes in R_N is *closed* if for all $t \in \mathbb{N}$, $T \cap \mathrm{Col}(t)$ is finite. It is immediate that a subset of a closed set is closed, and that a closed subset of R_N is co-infinite (since each $\mathrm{Col}(t)$ is infinite). For example, a clique is closed, since it meets each $\mathrm{Col}(t)$ in at most one node.

Theorem 6. *If T is a closed subset of R_N, then T is robust.*

Proof. We introduce an isomorphic copy of R_N named R_∞. We define a sequence $(X_{\infty,t} : t \in \mathbb{N})$ of graphs as follows. Let $X_{\infty,0} = X_0$. To define $X_{\infty,1}$, we form disjoint sets of nodes $(S_{\infty,1,i} : i \in \mathbb{N})$, each disjoint from $V(X_{\infty,0})$ and of the same cardinality as $S_{1,1}$, and let

$$V(X_{\infty,1}) = V(X_{\infty,0}) \cup \bigcup_{i \in \mathbb{N}} S_{\infty,1,i}.$$

Now let each $S_{\infty,1,i}$ extend X_0 as $S_{1,1}$ does. More precisely, the subgraph of $X_{\infty,1}$ induced by $V(X_0)$ and $S_{\infty,1,i}$ is isomorphic to X_1. Moreover, $\bigcup_{i \in \mathbb{N}} S_{\infty,1,i}$ is an independent set in $X_{\infty,1}$. It follows from the definition that $X_{\infty,1}$ contains infinitely many subgraphs isomorphic to X_1.

Assume that $X_{\infty,t}$ is defined, countable, and contains infinitely many subgraphs isomorphic to X_t. To define $X_{\infty,t+1}$, form disjoint sets of nodes $(S_{\infty,r,i} : i \geq t)$, each disjoint from $V(X_{\infty,0})$ and of the same cardinality as $S_{t,t}$. Let

$$V(G_{\infty,t}) = V(X_{\infty,t-1}) \cup \bigcup_{i \geq t} S_{\infty,t,i}.$$

Let each of the $S_{\infty,t,i}$ extend one of the subgraphs of $X_{\infty,t}$ isomorphic to X_t as $S_{t,t}$ does. Observe that $\bigcup_{i \geq t} S_{\infty,t,i}$ forms an independent set in $X_{\infty,t+1}$. Clearly, $X_{\infty,t+1}$ contains infinitely many subgraphs isomorphic to X_r. Define

$$R_\infty = \lim_{t \to \infty} X_{\infty,t}.$$

Define the t^{th} *row of* R_∞, written $\mathrm{Row}(t)$, to be the set $\bigcup_{i \in \mathbb{N}} S_{\infty,t,i}$.

Define an isomorphism $f : R_N \to R_\infty$ as follows. The map f sends X_0 to $X_{\infty,0}$ via the identity map. For each $t, i \in \mathbb{N} \backslash \{0\}$, map $\mathrm{Col}(t)$ isomorphically onto $\mathrm{Row}(t)$ with $S_{i,j}$ mapped to $S_{\infty,j,i}$.

Define $T_n = T \cap Col(n)$ for all $n \in \mathbb{N}$. As T is closed, each T_n is finite. Note that in R_∞, $f(T_n)$ is a finite subset of $\mathrm{Row}(n)$. Define $T' = f(T)$. Note that $R_\infty - T'$ is isomorphic to $R_N - T$ via f^{-1}. If we find an isomorphism $g : R_\infty \to R_\infty - T'$, then $f^{-1}gf : R_N \to R_N - T$ is the desired isomorphism.

We construct g by induction as a limit of isomorphisms $g_t : X_{\infty,t} \to X_{\infty,t} - f(T_t)$. Assuming that g_t has been defined, we construct g_{t+1} extending g_t by the back-and-forth method. In the "forward" direction, we must ensure that the domain of g_{t+1} contains $V(X_{\infty,t+1})$; in the "back" direction, we must ensure that the image of g_{t+1} contains $V(X_{\infty,t+1}) \backslash f(T_{t+1})$.

Since $X_0 = X_{\infty,0} \cong \overline{K_{\aleph_0}}$ and $f(T_0)$ in $X_{\infty,0}$ is finite, we may define g_0 to be any isomorphism from $X_{\infty,0}$ to $X_{\infty,0} - f(T_0)$.

For the forward direction, let x be a node in $V(X_{\infty,t+1})\backslash V(X_{\infty,t})$. Hence, x is in some $S_{\infty,t,i}$, and extends some finite set $S \subseteq N[y]$, with y in $V(X_{\infty,t})$. Let $S' = g_t(S)$ and $y' = g_t(y)$. Hence, there is a node x' in $X_{\infty,t+1}$ extending $g_t(S)$ as x extends S. In $X_{\infty,t+1} - f(T_{t+1})$, since T_{t+1} is finite by hypothesis, there are infinitely many nodes x' that extend $g_t(S)$ as x extends S. We may therefore choose any one of these nodes to be the image of x. Going back is straightforward and so we omit the details. □

We may apply Theorem 6 to obtain $2^{\aleph_0}$ many robust subsets of R_N which induce non-isomorphic graphs. Define $\alpha(G)$ to the cardinality of an independent set in G with the largest order.

Corollary 7. (1) *If T has the property that $\alpha(R_N \restriction T)$ is finite, then T is robust.*
(2) *If T is a clique in R_N, then T is robust.*
(3) *R_N is inexhaustible.*

Proof. Items (1) and (2) follow from Theorem 6 since the hypotheses imply that T is closed. Item (3) follows from (2). □

It is not known which sets of nodes T satisfy $R_N - T \cong R_N$. We leave this as an open problem.

The graph R_N displays rich metric properties, unlike R, which is of diameter 2. A graph G *isometrically embeds* in H, written $G \leq_i H$, if there is an embedding of G into H that preserves distances. More precisely, let $d_G(x, y)$ be the distance between x and y if x, y are in the same connected component of G, and ∞ otherwise. We omit reference to G if it is clear from context. An embedding $f : G \to H$ is an *isometric embedding* if for all $x, y \in V(G)$, $d_G(x, y) = d_H(f(x), f(y))$. A graph G is *isometrically constructible* if $G = \lim_{t\to\infty} G_t$, where the sequence $(G_t : t \in \mathbb{N})$ of countable graphs has the property that $G_t \leq_i G_{t+1}$, for all $t \in \mathbb{N}$.

Theorem 8. *The graph R_N is isometrically constructible and isometrically embeds all finitely approximated graphs. In particular, R_N is isometrically embeds all finite graphs.*

Proof. We use the notation $R_H = \lim_{t\to\infty} X_t$ from the Introduction. We prove by induction on $t \in \mathbb{N}$ that $X_t \leq_i X_{t+1}$.

For the base case, fix distinct x, y in $V(X_0)$. Suppose for a contradiction that $d_{X_0}(x, y) > d_{X_1}(x, y) = r$. Hence, in X_1 there is a path P connecting x to y of length r. Then $P \nsubseteq V(X_0)$. In particular, P must contain at least one node of $V(X_1)\backslash V(X_0)$; call such a node *special*. Since $V(X_1)\backslash V(X_0)$ forms an independent set in X_1, the neighbours in

P of a special node are not special. Choose P containing the minimum number of special nodes. Let z be a special node, and let a and b be the neighbours of z in P. Suppose that $z = x_{u,S}$, where u is a node of X_0, and $S \subseteq N(u)$ in $V(X_0)$. Hence, either u is joined to a and b, or u equals one of a or b and is joined to the other node. The latter case gives a contradiction, since then the path formed by deleting z from P connects x and y but is shorter than P.

Therefore, u is joined to a and b. Consider the path P' formed by deleting z from P and adding u. Then P' is a path of length r in X_1 with one less special node than P. This contradiction establishes that $X_0 \leq_i X_1$.

By similar arguments as above, $X_t \leq_i X_{t+1}$ for all $t \geq 1$. Thus, R_N is isometrically constructible.

Now if H is finitely approximated, then $\uparrow H \cong R_N$ by Theorem 5. As $X_0 \cong H$ and $X_0 \leq_i R_N$ (as $X_0 \leq_i X_t$ for all $t > 0$), we have that $H \leq_i R_N$. □

As R_N is $\aleph_0$-universal, it contains $\overline{K_{\aleph_0}}$ as an induced subgraph. By the Axiom of Choice (AC), there exists a *maximal independent set* in R_N: an independent set Y with the property that each node not in Y is joined to some node of Y. In other words, a maximal independent set is an independent dominating set. A direct recursive construction of a maximal independent set in R (without (AC)) was given in [16]. We finish this section with an analogous result for R_N.

Theorem 9. *The graph R_N contains a maximal independent set.*

Proof. Consider the isomorphic representation $\uparrow K_1$ of R_N. Define $Y_0 = V(X_0)$.

We first demonstrate that $c(R_N)$ has a maximal independent set. Assume that Y_t is a maximal independent set in C_t in $c(R_N)$. To form Y_{t+1} proceed as follows. Define

$$\begin{aligned} Y' &= \{x \in (V(C_{t+1}) \cup V(c(R_N)))\backslash V(C_t) : \\ x &= x_{y,S},\ y \in V(C_t),\ S \subseteq N[y] \cap Y_t\}. \end{aligned}$$

Observe that $y \in Y'$ if and only if y is joined to some node of Y_t. Define $Y_{t+1} = Y_t \cup (V(C_{t+1})\backslash Y')$. There are nodes in $V(C_{t+1}) \setminus V(C_t)$ joined to some node of Y_t and no other node of C_t, and there are nodes in $V(C_{t+1})\backslash V(C_t)$ joined to some node in $V(C_t)\backslash Y_t$ and to no other node of C_t. Hence, $Y_t \subsetneqq Y_{t+1} \subsetneqq V(C_{t+1})$.

Define $Y = \bigcup_{t \in \mathbb{N}} V(Y_t)$. Then Y is a recursively defined independent set. If $x \notin Y$, then say $t > 0$ is the first time that x appears. By the

definition of Y_t, x is joined to some node of $Y_{t-1} \subseteq Y$. Hence, Y is a maximal independent set in $c(R_N)$.

Recall that $R_N \cong \bigcup_{i \in \mathbb{N}} c(R_N)_i$, where $c(R_N)_i \cong c(R_N)$ for all $i \in \mathbb{N}$. By the above argument, each $c(R_N)_i$ gives rise to a maximal independent set Y_i. Then $Y = \bigcup_{i \in \mathbb{N}} Y_i$ is a maximal independent set in R_N. □

4. The group and monoid of R_N

For a mapping $f : V(G) \to V(H)$, we will abuse notation and write $f : G \to H$. We denote the image of f as $\mathrm{Im}(f)$ (considered as a subset of $V(H)$), and the domain of f as $\mathrm{Dom}(f)$. A mapping $f : G \to H$ with the property that $xy \in E(G)$ implies that $f(x)f(y) \in E(H)$ is a *homomorphism*. A homomorphism from G to itself is an *endomorphism*. For more on graph homomorphisms, the reader is directed to [15]. We write fg for the composition of two mappings. The monoid of all endomorphisms of G under composition is written $\mathrm{End}(G)$. If $S \subseteq V(G)$, then we write $f \restriction S$ for the restriction of f to S.

An embedding $f : G \to H$ is an injective homomorphism with the property that $xy \notin E(G)$ implies that $f(x)f(y) \notin E(H)$. An *automorphism* is a bijective embedding. The group of all automorphisms of G under composition is written $\mathrm{Aut}(G)$. We write id_G for the identity mapping on G.

The automorphism group and endomorphism monoid of R have been actively studied (see the references in [9] for $\mathrm{Aut}(R)$, and see [2, 3] for more on $\mathrm{End}(R)$). For example, the e.c. property and a back-and-forth argument together imply that R is *homogeneous:* every isomorphism between finite induced subgraphs of R extends to an automorphism. In a certain sense, R is as symmetric as possible. In this section, we focus on properties of the group and monoid of R_N.

Let $G \leq H$, where H is finite. The graph G is *good relative to* H if each isomorphism between two isomorphic copies of G in H extends to an automorphism of H. If H is clear from context, then we omit reference to it. The good induced subgraphs of H are a measure of how much symmetry H possesses. For example, H is homogeneous if and only if each of its finite induced subgraphs are good. A graph is vertex-transitive if K_1 is good.

Let $J = \lim_{t \to \infty} H_t$ be a limit of a countable sequence $\mathcal{C} = (H_t : t \in \mathbb{N})$ of graphs, where $H_t \leq H_{t+1}$ for all $t \in \mathbb{N}$. Define $age_{J,\mathcal{C}} : V(J) \to \mathbb{N}$ by

$$\mathrm{age}_{J,\mathcal{C}}(x) = \begin{cases} t & \text{if } x \in V(H_t) \backslash V(H_{t-1}) \text{ where } t > 0; \\ 0 & \text{else.} \end{cases}$$

We will simply write age(x) if J and $\mathcal{C}$ are clear from context. The age of a finite subset, written age(S), is $\max\{\text{age}(x) : x \in S\}$.

We now characterize the good induced subgraphs of R_N. A node x in a graph G is *universal* if it is joined to all nodes of $V(G)\backslash\{x\}$. Let G contain induced subgraphs H, J. An isomorphism $f : H \to J$ is a *partial isomorphism* of G. We usually omit G when it is clear from context.

Theorem 10. *A finite induced subgraph G of R_N relative to R_N is good if and only if G contains a universal node.*

Proof. For the forward direction, suppose that G has no universal node. Consider the isomorphic representation $\uparrow G$ of R_N. Hence, $X_0 \cong G$.

Claim 1. *There is no node of R_N joined to all of X_0.*

We prove the claim by induction on t to show that no node of X_t is joined to all of X_0. Suppose this holds for X_t. For a contradiction, let z be a node of X_{t+1} joined to all of X_0. Hence, $z \in V(X_{t+1})\backslash V(X_t)$, and so $z = x_{y,S}$ for $y \in V(X_t)$ and $S \subseteq N[y] \cap V(X_t)$. It follows that $V(X_0) \subseteq S$, and so $V(X_0) \subseteq N[y]$. Observe that $y \notin V(X_0)$, for otherwise X_0 would have a universal node. But then y is joined to all of X_0, contradicting induction hypothesis. The proof of the claim follows.

Fix $x \in V(R_N)$ with age(x) > 0. As $R_N \restriction N(x) \cong R$ by Theorem 2, there is an isomorphic copy G' of G in $R_N \restriction N(x)$. As there are infinitely many node-disjoint copies of G in R, without loss of generality, we may assume that $V(G') \neq V(X_0)$. Let f be any isomorphism between G' and X_0. If G were good, then f extends to $F \in \text{Aut}(R_N)$. But then $F(x)$ is joined to all of X_0, contradicting the claim.

For the reverse direction of the theorem, we work with the isomorphic representation $\uparrow G$. Fix an arbitrary copy of G' in R_N. Suppose that age($V(G')$) $= t$ and so $V(G') \subseteq V(X_t)$.

Claim 2. *Each partial isomorphism $f : G' \to X_0$ extends to an automorphism F of R_N.*

This is sufficient to prove the reverse direction of the theorem. To see this, fix another copy G'' of G in R_N, and a partial isomorphism $g : G' \to G''$. There are isomorphisms $g' : G' \to X_0$ and $g'' : G'' \to X_0$ with the property that $g = (g'')^{-1}g'$. By Claim 2, g' extends to some automorphism α' of R_N, and g'' extends to some automorphism α'' of R_N. Hence, $\alpha = (\alpha'')^{-1}\alpha'$ is an automorphism of R_N extending g.

Let z be a universal node of X_0. Hence, in X_1 there is a node z' joined to all of the nodes of X_0.

Claim 3. *Let H be a finite graph containing X_0 as an induced subgraph. Then there is $H' \leq R_N$ and an isomorphism $\beta : H \to H'$ so that $\beta \restriction X_0 = id_{X_0}$.*

We prove Claim 3 by induction on $r = |V(H) \backslash V(G)|$. Our induction hypothesis is that for a given r there is such an H and β as in the statement of the claim, and there is a node $u \in V(R_N)$ joined to all the nodes of H'. The base case follows by taking $H' = X_0$, $\beta = id_{X_0}$, and $u = z'$. Suppose the claim holds for a fixed r. Let H be such that $r + 1 = |V(H) \backslash V(G)|$. Fix $x \in V(H) \backslash V(G)$. We apply the inductive hypothesis to $J = H - x$. Then there is a $J' \leq R_N$, an isomorphism $\beta_1 : J \to J'$ so that $\beta_1 \restriction J' = id_{X_0}$, and a node $u_1 \in V(R_N)$ joined to all the nodes of J'. Suppose that $N_H(x) = S$. There is a node $x' = x_{u_1, \{u_1\} \cup \{\beta_1(S)\}}$ in R_N joined to u_1, and joined to the nodes of J' exactly as x is joined to the nodes of J. Define $H' = R_N \restriction (V(J') \cup \{x'\})$, $\beta : H \to H'$ by

$$\beta(v) = \begin{cases} \beta_1(v) & \text{if } v \neq x; \\ x' & \text{else,} \end{cases}$$

and $u = u_1$. Then β is an isomorphism that is the identity on X_0.

We now prove Claim 2 to finish the proof of the theorem. By Claim 3, there is $X_t' \leq R_N$ and an isomorphism $\beta : X_t \to X_t'$ so that $\beta \restriction X_0 = id_{X_0}$. We work with this isomorphic copy X_t' of X_t in R_N, and for simplicity, discard reference to β.

Let f_0 be any fixed partial isomorphism of X_t to X_t' extending f. We now proceed with a back-and-forth argument to extend f_0 to an automorphism f of R_N. Suppose that $\text{age}(V(X_t')) = m$. We "go back" to find a partial isomorphism f_1 with image X_m. Suppose inductively that all nodes of X_m of age at most $t \geq 1$ are in the image of f_1, where $1 \leq t \leq m - 1$. Let z be a node in $X_m \backslash \text{Im}(f_1)$ with $\text{age}(z) = t + 1$. Then z is of the form $x_{y,S}$, where $\text{age}(y)$, $\text{age}(S) \leq t$. Let z' be the node $x_{f_1^{-1}(y), f_1^{-1}(S)}$. Then the extension of f_1 mapping z' to z is a partial isomorphism.

Let $\text{Dom}(f_1) = J$. Then f_1 is a partial isomorphism mapping J onto X_m. We "go forward" to define f_2 to be a partial isomorphism extending f_1 so that $\text{Dom}(f_1)$ contains $V(X_{t+1})$. Each node $z \in V(X_{t+1}) \backslash \text{Dom}(f_1)$ is of the form $x_{y,S}$, where $\text{age}(y)$, $\text{age}(S) \leq t$. We find an isomorphic image of z analogously as in the case $s = 1$. We similarly define f_3 to be a partial isomorphism extending f_2 containing X_{m+1} in its image.

It is now evident that we may proceed inductively to define f_s a partial isomorphism extending f_0, where $s \geq 1$. If $s > 0$ is even, then

f_s extends f_{s-1} and contains $X_{t+\frac{1}{2}s}$ in its domain. If s is odd, then f_s extends f_{s-1} and contains $X_{t+\lfloor\frac{1}{2}s\rfloor}$ in its image.

The map $F : R_N \to R_N$ defined by $F = \bigcup_{s\in\mathbb{N}} f_s$ is the desired automorphism extending f. □

A graph G is *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on its nodes. A graph is *arc-transitive* if it acts transitively on its arcs. As K_1 and K_2 have universal nodes, Theorem 10 implies the following.

Corollary 11. *The graph R_N is both vertex- and arc-transitive.*

We observe that in contrast to Corollary 11 the graphs R_H described in the introduction are not in general vertex-transitive.

Theorem 12. *The graph R_{K_3} is not vertex-transitive.*

Proof. List the nodes of K_3 at time $t = 0$ as a, b, c. Consider the node $x = x_{a,\{b\}}$ added at $t = 1$. We claim that x is not in a K_3 in R_{K_3}. Hence, R_{K_3} is not vertex-transitive since there is no automorphism sending a to x. We proceed by induction on $t \geq 1$ to prove that x is not in a K_3 in X_t. This clearly holds for $t = 1$. Suppose it is true for a fixed $t \geq 1$, and consider X_{t+1}. For a contradiction, suppose that x is in a K_3 with nodes x, y, z. One of these nodes, say z, must have age $t + 1$. Let z' be the node z copied from at time $t + 1$. Note that $x, y \in N(z) \subseteq N(z')$. Therefore, $z' \neq x, y$. Then x, y, z' forms a K_3 in X_t, which is a contradiction. □

We say that a $\mathrm{Aut}(G)$ is *oligomorphic* if it has only finitely many orbits on n-tuples, for each integer n. Unlike the situation for R, the automorphism group of R_N is not oligomorphic. As R_N isometrically embeds all finite graphs, for each integer r, we may find pairs of nodes x_r, y_r such that $d_{R_N}(x, y) = r$. The pairs $\{(x_r, y_r) : r \in \mathbb{N}\backslash\{0, 1\}\}$ realize $\aleph_0$-many distinct orbits on 2-tuples under the action of $\mathrm{Aut}(R_N)$.

We now prove that $\mathrm{Aut}(R_N)$ and $\mathrm{End}(R_N)$ exhibit universality properties previously observed in [5] for R.

Theorem 13. (1) *The endomorphism group of R_N embeds all countable monoids.*

(2) *The automorphism group of R_N embeds all countable groups.*

Proof. For the proof of (1), we consider the isomorphic representation $\uparrow\overline{K_{\aleph_0}}$ of R_N (note that $\overline{K_{\aleph_0}}$ is finitely approximated; see Theorem 5). If X is a nonempty set, then let $T(X)$ be the monoid of all mappings from X to itself under composition. The endomorphism monoid on $X_0 \cong \overline{K_{\aleph_0}}$ is isomorphic to $T(X)$, where X is any fixed countably

infinite set. As is well-known, each countable monoid embeds in $T(X)$. Hence, the proof of item (1) will follow once we prove that $T(X)$ is isomorphic to a submonoid of $\mathrm{End}(R_N)$.

Let f_0 be an endomorphism of X_0. Assume that f_t is an endomorphism of X_t such that $f_t \restriction X_0 = f_0$. For each node $x_{y,S}$ of $V(X_{t+1})\backslash V(X_t)$, where $y \in V(X_t)$ and S is a finite subset of $N[y] \cap V(X_t)$, define $f_{t+1}(x_{y,S}) = x_{f_t(y),f_t(S)}$. Otherwise, define f_{t+1} to be f_t on X_t. It follows that f_{t+1} is a homomorphism. Hence, $f_{t+1} \in \mathrm{End}(X_{t+1})$ and $f_{t+1} \restriction X_t = f_t$.

Define $F \in \mathrm{End}(R_N)$ by $F = \bigcup_{t\in\mathbb{N}} f_t$. Define $\alpha : \mathrm{End}(X_0) \to \mathrm{End}(R_N)$ by $\alpha(f) = F$. Since $F \restriction X_0 = f$, it follows that α is injective. To prove the $T(X)$ is isomorphic to a submonoid of $\mathrm{End}(R_N)$, it is sufficient to show that α is a monoid homomorphism.

As α clearly preserves the identity endomorphism, we show that for all $x \in V(R_N)$,

$$\alpha(fg)(x) = \alpha(f)\alpha(g)(x). \tag{4.1}$$

Fix $x \in V(R_N)$ with $\mathrm{age}(x) = t$. If $t = 0$, then (4.1) clearly holds. Suppose that (4.1) holds for all nodes x with $\mathrm{age}(x) \leq t$. Let $\mathrm{age}(x) = t + 1$. Then x is of the form $x_{y,S}$, where $y \in V(X_t)$ and S is a finite subset of $N[y]$ in X_t. Now,

$$\begin{aligned} \alpha(fg)(x) &= (fg)_{t+1}(x) \\ &= x_{(fg)_t(y),(fg)_t(S)} \\ &= x_{f_tg_t(y),f_tg_t(S)} \\ &= \alpha(f)\alpha(g)(x), \end{aligned}$$

where the third equality follows by induction hypothesis. Hence. (4.1) holds by induction on t.

The proof of item (2) is similar, and so details are omitted. In this case, we embed $\mathrm{Sym}(X)$ into $\mathrm{Aut}(R_N)$, where $\mathrm{Sym}(X)$ is the symmetric group on a fixed countably infinite set X. The proof then follows by Cayley's theorem. □

Theorem 13 has the following computational consequences. We refer the reader to [17] for any terms not explicitly defined.

Corollary 14. (1) $\mathrm{End}(R_N)$ *does not satisfy any non-trivial semigroup identity.*

(2) $\mathrm{Aut}(R_N)$ *does not satisfy any non-trivial group identity.*

Proof. For (1), since every countable semigroup embeds into $\mathrm{End}(R_N)$ by Theorem 13, so does the free semigroup on a countable set of generators, written $F_S(X)$. If there were an equation $s = t$ in the language of semigroups that is not a consequence of the associative law and that is satisfied by $\mathrm{End}(R_N)$, then $s = t$ would be satisfied by $F_S(X)$, which is a contradiction.

The proof of item (2) is similar, and so is omitted. □

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