Numerical solution of hybrid differential-algebraic equations

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Abstract

We present the mathematical theory of general over- and underdetermined hybrid (switched) systems of differential-algebraic equations (HDAEs). We give a systematic formulation of HDAEs and discuss existence and uniqueness of solutions, the numerical computation of the switch points and how to perform consistent initialization at switch points. We show how numerical solution methods for DAEs can be adapted for HDAEs and present a comparison of these methods for the real world example of simulating an automatic gearbox.

Keywords: differential-algebraic equation, hybrid system, switched system, multibody system, index reduction, strangeness index, implicit Runge-Kutta method, BDF method.

AMS(MOS) subject classification: 65L80, 93C15, 34H05

Contents

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1 Introduction

A system of differential equations that switches between several modes of operation is called *hybrid system* or *switched system* [2–4, 28, 42, 43, 50]. In this paper, we consider constrained nonlinear dynamical systems that work in different modes that are described by differential-algebraic equations (DAEs).

Such hybrid systems of DAEs *(HDAEs)* connect the continuous dynamics represented by different DAE models with the discontinuous mode changes that are modeled by discrete transition functions. The different modes typically result from a discrete control, where the transitions are reached when certain thresholds are crossed.

It is the aim of this paper to provide the mathematical framework for HDAEs, to discuss the analytical properties and to develop numerical methods that allow the simulation and control of HDAEs.

HDAEs arise in many different application areas, such as robot manipulators, traffic systems, power systems, or biological systems, see e.g. [17] and the references therein. Our motivation arises from the numerical simulation and control of automatic gear boxes, which are modeled by mechanical multibody systems that switch between different modes which describe the different operating conditions. A detailed description of the model of an automatic gear box is given in the Appendix.

To illustrate our notation and concepts, we use a very simple hybrid multibody system of an accelerated pendulum, describing the mathematical model of a rotating pendulum for which the string breaks [24].

Example 1 Consider a pendulum of length l, mass m under the influence of gravity $F_g = -mg$, where $g = 9.81\frac{m}{s^2}$ is the gravitation constant. Furthermore, assume that the pendulum is tangentially accelerated by a linearly increasing force F, see Figure 1.

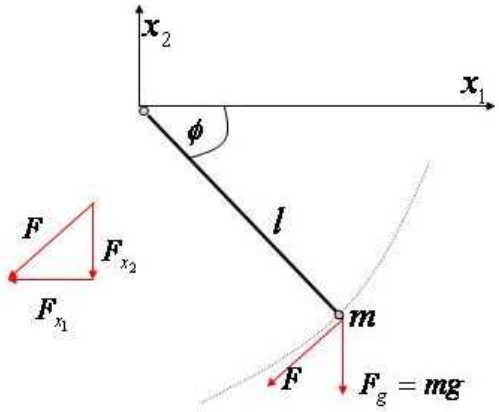

Figure 1: Accelerated pendulum - scheme

Consider first the constrained motion of the pendulum. Using the classical Euler-Lagrange formalism [19, 23] in Cartesian coordinates $[x_1, x_2]^T = [x, y]^T$ and velocities $[x_3, x_4]^T = [\dot{x}, \dot{y}]^T$ and the acceleration forces $F(x_1, x_2) = F_{x_1}(x_1) + F_{x_2}(x_2)$, one obtains the following system of DAEs.

$$\begin{aligned} \dot{x}_1 &= x_3, && \text{(1a)} \\ \dot{x}_2 &= x_4, && \text{(1b)} \\ m\dot{x}_3 &= -2x_1\lambda + F_{x_1}, && \text{(1c)} \\ m\dot{x}_4 &= -mg - 2x_2\lambda + F_{x_2}, && \text{(1d)} \\ 0 &= x_1^2 + x_2^2 - l^2. && \text{(1e)} \end{aligned}$$

Now suppose that when a certain centrifugal force $F_{c_{\max}}$ is reached, the system changes from a pendulum to a flying mass point, i.e. the rope or rod is cut. In this case the system is not constrained anymore and the equations of motion are given by

$$\begin{aligned} \dot{x}_1 &= x_3, && \text{(2a)} \\ \dot{x}_2 &= x_4, && \text{(2b)} \\ m\dot{x}_3 &= 0, && \text{(2c)} \\ m\dot{x}_4 &= -mg. && \text{(2d)} \end{aligned}$$

If we consider the complete system, then it consists of two (operating) modes and it switches once between (1) and (2). A typical task would be to determine the switching point, to simulate the movement of the mass point, or to control the switching and the successive flight.

There exist many numerical solution methods and software for DAEs [8, 18, 19, 23, 38, 40], but most of these cannot be directly applied to HDAEs and also the analysis of HDAEs has not yet been investigated in detail, see [3, 4, 28, 42, 43]

for such approaches. In practical applications the model is usually simulated in one of the modes, the switching points are determined and then the system is simulated in the next mode. For the active control of hybrid systems, however, this mode-by-mode approach is typically not applicable, since the control may influence the switching and, hence, the hybrid system has to be considered as a whole.

It is therefore necessary to study HDAEs based on a general formulation that enables a simulation and control of the system with all its possible modes together. There exist different approaches of an abstract modeling with hybrid systems, e.g. in [2, 3, 28, 42, 43]. But until now no formulation has established itself as a standard. Furthermore, the mentioned approaches are formulated from a view point of computer science and control theory. There, the general idea is based on concepts of discrete state machines, to which dynamic components are added, leading to so called *hybrid automata*. These formulations serve to give predictions about reachability of certain states or about the order of modes [42], but they do not contain a mathematical framework to analyze the dynamical behavior of the complete system. In contrast to this, we base the abstract modeling concept on the general theory of DAEs.

The paper is organized as follows. In Section 2 we give a formal definition of HDAEs. In Section 3 we briefly discuss DAEs and the strangeness index concept. This concept is applied in Section 4 to derive a strangeness-free formulation of HDAEs. Index reduction and numerical integration of HDAEs is discussed in Section 5 and a numerical comparison of different solution methods on the basis of automatic gearbox model is presented in Section 6.

2 Formulation of HDAEs

We examine hybrid systems that are composed of different systems of DAEs and transition conditions between these DAEs that are considered in the general form

$$F(t, x, \dot{x}) = 0, \tag{3}$$

with $F \in C(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}}, \mathbb{R}^m)$, $\mathbb{I} \subseteq \mathbb{R}$ (compact) interval, $\mathbb{D}_x, \mathbb{D}_{\dot{x}} \subseteq \mathbb{R}^n$ open. Such systems include, in particular, nonlinear control problems

$$\begin{aligned} F(t, \xi, w, \dot{\xi}) &= 0, \\ y &= G(t, \xi). \end{aligned} \tag{4}$$

Here $\xi \in \mathbb{R}^{n_\xi}$ is the state, $w \in \mathbb{R}^{n_w}$ the input, and $y \in \mathbb{R}^{n_y}$ the output of the system. Control systems of the form (4) can be rewritten in the form (3) by combining the vector functions as

$$x = \begin{bmatrix} \xi \\ w \\ y \end{bmatrix} \in \mathbb{R}^n,$$

with $n = n_\xi + n_w + n_y$, see [34].

The formulation (4) in the form (3) can be viewed as representing the system via a behavior approach [44]. However, very often systems do not arise or are not directly modeled as input-state-output systems in the form (4) but arise in the general form (3) and it is an extra task to determine an input-state-formulation from these general variables, see [26, 27].

Definition 2 (Mode of a HDAE) *In a HDAE we will have several DAEs that we call* modes. *Let $F^i(t, x, \dot{x}) = 0$, $i \in \{1, \ldots, N_F\}$ with $F^i : D_i \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$, be the DAE that describes the dynamics of the hybrid system in mode M_i. Here D_i is a union of intervals $\mathbb{I}_k \subset \mathbb{R}_+$, in which mode M_i describes the system dynamics. The HDAE is said to be* in mode M_i *if $t \in D_i$.*

A certain mode M_i describes the system dynamics in different phases of the process in an interval $\mathbb{I}_j$, e.g. if mode M_i is active in intervals $\mathbb{I}_1$ and $\mathbb{I}_3$, then the domain D_i of mode M_i contains both these intervals. The intervals $\mathbb{I}_j$ are defined as half open intervals $[\alpha_j, \beta_j)$ and it is assumed that $\bigcup_j D_j = \mathbb{I}$. Furthermore, the system is not allowed to be in more than one mode at any time, i.e. $\bigcap_j D_j = \emptyset$. The closure of the set D_j is denoted by $\overline{D_j}$.

For a certain time $\hat{t} \in D_\ell$, the *state of a HDAE* is given by the solution of the DAE at $\hat{t}$, i.e. it satisfies the DAE $F^\ell(\hat{t}, x_\ell(\hat{t}), \dot{x}_\ell(\hat{t})) = 0$.

While a classical DAE has only one mode M in which it lives exclusively, HDAEs switch between different modes on the basis of *switching or transition conditions* that are realized through threshold values.

Definition 3 (Transition condition) *For a mode M_i of a HDAE a* transition condition *S^i is defined as:*

$$S^i(t, x_i, \dot{x}_i) \quad = \quad \begin{bmatrix} S^i_1 \\ \vdots \\ S^i_{N_{S_i}} \end{bmatrix} (t, x_i, \dot{x}_i). \tag{5}$$

If $S^i_j(\hat{t}, x, \dot{x}) > 0$ for all j, then the integration process continues in the same mode with the same DAE F^i. If there exist an integer j and a time $\hat{t}$, such that $S^i_j(\hat{t}, x, \dot{x}) \leq 0$, then the system is switched to another mode M_k where the integer k that determines the new mode is determined by a trigger function *S^i_j.*

The trigger functions S^i_j are disjointly defined. This means that at no point $(t, x_i, \dot{x}_i)$ more than one trigger function S^i_j have a sign change. This is relatively easy to realize combinatorially and is outlined in the next paragraph. In the following $S^i \leq 0$ means that for exactly one j the condition $S^i_j \leq 0$ holds and for all $\ell \neq j$ the other trigger functions satisfy $S^i_\ell > 0$. Then a crossing of a threshold in mode M_i can be characterized by a sign change, and initiates a mode change to mode M_k.

To guarantee that these sign changes lead to a unique sequence of modes or in other words that for all points $(t, x_i, \dot{x}_i)$ at most one trigger function has a sign change, we use the following procedure.

Suppose that a transition condition S^i is not unique. Let S^i_j, S^i_k be two conditions that are negative at the same time $\hat{t}$, i.e. there exists a $\hat{t} \in D_i$, where

$S_j^i(\hat{t}) \leq 0$ and $S_k^i(\hat{t}) \leq 0$ holds. Assume w.l.o.g. that $j < k$. Then we modify the transition condition $\hat{S}^i$ to

$$\hat{S}^i = \left(S_1^i, \ldots, S_j^i \backslash S_k^i, \ldots, S_k^i \backslash S_j^i, \ldots, S_{N_{S_i}}^i, S_j^i \wedge S_k^i\right),$$

where $S_j^i \backslash S_k^i$ denotes the logical difference between two trigger functions and $S_j^i \wedge S_k^i$ the intersection of these. Proceeding in this way for all non-unique situations, the system, respectively its transition condition, can be preprocessed so that a unique mode sequence can be guaranteed.

Remark 4 Theoretically, for a HDAE with N_F modes M_i there are $\frac{N_F!}{(N_F-2)!}$ possible transitions. But out of these, usually only a small portion have a real physical meaning. Therefore, it is important that the definition of the transition conditions contains as much system information as possible, because otherwise the majority of these conditions may in fact formulate only combinatorial but not physically relevant possibilities. These considerations become especially important at implementation time, where by considering only physically relevant switches the programming effort can be drastically reduced.

The transition conditions are associated with a table-like *allocation function* $H(i,j)$, which for a mode M_i and a transition condition $S_j^i \leq 0$ assigns the successor mode k, with $k \neq i$.

$$H(i,j) = k, \text{ for } S_j^i \leq 0, \quad i,k \in \{1,\ldots,N_F\},\ j \in \{1,\ldots,N_{S_i}\}. \tag{6}$$

Furthermore, since in each mode we want the solution of the DAE to exist, consistent initial conditions are needed at all possible mode changes, see [8]. In order to guarantee this, a HDAE must contain *transition functions* T_j^i that determine consistent initial values for the integration in mode M_j coming from mode M_i.

Definition 5 (Transition function) *Let x_i be the state of a HDAE in mode M_i in an integration interval $\mathbb{I}_k \in D_i$ and let $x_i(\beta_k)$ be the smooth extension of x_i to the interval border β_k. The* transition function *T_j^i translates x_i to the initial condition of mode M_j at the time $\alpha_{k+1} = \beta_k$ of the integration interval $\mathbb{I}_{k+1} \in D_j$ by computing initial conditions for the DAE F^j, i.e.*

$$\begin{aligned} &T_j^i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_j}, \\ &\text{with } T_j^i(x_i(\beta_k)) = x_j^{\alpha_{k+1}} \text{ at transition time } \beta_k = \alpha_{k+1} \in \overline{D_i} \cap D_j. \end{aligned} \tag{7}$$

The new initial value vector obtained by the transition function may contain variables of the predecessor mode. The remaining variables must be determined from the last computed state.

After these preparations we are able to formally define HDAEs.

Definition 6 (HDAE) *Let $\mathbb{I} \subset \mathbb{R}$ with $\mathbb{I} = \bigcup_{i=1}^{N_\mathbb{I}} \mathbb{I}_i$, $N_\mathbb{I} \in \mathbb{N}$ define an integration interval with subintervals $\mathbb{I}_i = [\alpha_i, \beta_i)$ for $i = 1, \ldots, N_\mathbb{I} - 1$ and $\mathbb{I}_{N_\mathbb{I}} = [\alpha_{N_\mathbb{I}}, \beta_{N_\mathbb{I}}]$. Suppose further, that $\beta_i = \alpha_{i+1}$ for all $i = 1, \ldots, N_\mathbb{I} - 1$*

and $\alpha_i < \beta_i$ for all $i = 1, \ldots, N_{\mathbb{I}}$. The number of subintervals $N_{\mathbb{I}}$ is not known a priori, but is assumed to be finite.

A hybrid system of differential-algebraic equations (HDAE) *F is defined as a set of N_F systems of differential-algebraic equations*

$$F^\ell(t, x_\ell, \dot{x}_\ell) = 0, \quad F^\ell : D_\ell \times \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \to \mathbb{R}^{m_\ell}, \ \ell = 1, \ldots, N_F, \tag{8}$$

with sufficiently often differentiable functions F^ℓ. Each DAE F^ℓ is only defined on the union D_ℓ of certain integration intervals $\mathbb{I}_i$. For each interval $\mathbb{I}_i$ we assume that consistent initial values $x_\ell^{\alpha_i}$ for the currently valid mode M_ℓ exist, such that the solution in $\mathbb{I}_i$ exist and, furthermore, we assume that transition conditions $S^l(t, x_\ell, \dot{x}_\ell)$, defined as in (5) *are available. If these conditions cause a mode change to mode M_k, then the successor mode is determined by the allocation function H, as defined in* (6)*, and we assume that new consistent initial values for the next mode may be computed via the transition function T_k^l in* (7)*.*

Since the intervals $\mathbb{I}_i$ are defined as half open intervals $[\alpha_i, \beta_i)$, for $i = 1, \ldots, N_{\mathbb{I}} - 1$, the HDAE can be seen as one system of DAEs with possible discontinuities in a finite number of points given by the interval boundaries α_i, $i = 1, \ldots, N_{\mathbb{I}} - 1$.

Remark 7 HDAEs that have been defined in this way have no limitations with respect to the systems of DAEs $F^i = 0$ in the different modes, as long as these are solvable in the interval set D_i, but there are restrictions concerning the transition processes. The number of mode changes must be finite, as otherwise no reasonable numerical integration is possible. Furthermore, we assume that the integration intervals have a nonzero measure. This means that an instantaneous multiple mode change is not possible.

Remark 8 In order to include descriptor control systems and standard DAEs in the same framework we have only required that the solution in the subintervals D_ℓ exists but not that it is unique. In a numerical solution and in the active control of HDAES, however, a procedure that assigns the free variables (controls) in a unique way has to be provided.

Example 9 Consider again the accelerated pendulum of Example 1. Obviously this is a HDAE with two different modes, where only one mode change is possible. The exact computation of the switching point as well as the simulation of the movement on the circular manifold in the pendulum phase heavily influences the simulation of the flying phase. Considering the equations of motion in the two modes, according to Definition 6, the model contains the following parts. The variables in mode M_i are summarized by z_i. If mode M_1 is defined as the pendulum phase modeled by (1) and in the flying phase by (2) then the HDAE is given by:

$$\begin{array}{lrcl} \text{Pendulum phase:} & M_1 : F^1(t, z_1, \dot{z}_1) & = & 0, \quad \text{with } F^1 : D_1 \times \mathbb{R}^5 \times \mathbb{R}^5 \longrightarrow \mathbb{R}^5, \\ \text{Flying phase:} & M_2 : F^2(t, z_2, \dot{z}_2) & = & 0, \quad \text{with } F^2 : D_2 \times \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4. \end{array}$$

The integration interval is $\mathbb{I} = D_1 \cup D_2$. The only possible mode change from M_1 to M_2 is described by:

- a transition condition $S^1(t, z_1, \dot{z}_1) = F_{c_{\max}}(t, z_1, \dot{z}_1)$,
- with allocation function $H(1,1) = 2$,
- and transition function

$$\begin{aligned} T_2^1(z_{1,1}(\beta_1), \ldots, z_{1,5}(\beta_1)) &= \left[\begin{array}{ccc} z_{2,1}(\alpha_2) & \ldots & z_{2,4}(\alpha_2) \end{array}\right]^T \\ &= \left[\begin{array}{ccc} z_{1,1}(\beta_1) & \ldots & z_{1,4}(\beta_1) \end{array}\right]^T, \end{aligned}$$

where $z_1 = [z_{1,1}, \ldots, z_{1,5}]^T, z_2 = [z_{2,1}, \ldots, z_{2,4}]^T$ and the mode change occurs from M_1 to M_2 with $D_1 = [\alpha_1, \beta_1), D_2 = [\alpha_2, \beta_2]$. Furthermore, $z_1(\beta_1)$ is the smooth extension of the solution in D_1 to the interval border.

So far we have not discussed the different types of DAEs that describe the dynamics in the different subintervals D_j. This topic is briefly discussed in the following section.

3 DAEs and the strangeness index

To describe the HDAE, we consider in each mode nonlinear DAEs of the form

$$F(t, x, \dot{x}) = 0 \tag{9}$$

where F is a sufficiently smooth vector valued function $F \in C(\mathbb{I} \times \mathbb{R}^n) \times \mathbb{R}^n), \mathbb{R}^m)$.

The analytical theory for differential-algebraic equations has been developed quite far in the last two decades, and also a wide variety of numerical methods is available, see [8, 12, 13, 21, 29–31, 33, 36, 38, 45–47]. Furthermore, the theory and also numerical techniques have been extended to control problems [10, 15, 16, 26, 27, 34, 37, 48].

As in [33], we introduce a nonlinear derivative array, see also [11, 14], of the form

$$F_k(t, x, \dot{x}, \ldots, x^{(k+1)}) = 0, \tag{10}$$

which stacks the original equation and all its derivatives up to level k in one large system, i. e.,

$$F_k(t, x, \dot{x}, \ldots, x^{(k+1)}) = \left[\begin{array}{c} F(t, x, \dot{x}) \\ \frac{d}{dt} F(t, x, \dot{x}) \\ \vdots \\ \frac{d^k}{dt^k} F(t, x, \dot{x}) \end{array}\right]. \tag{11}$$

Partial derivatives of F_k with respect to selected variables p from $(t, x, \dot{x}, \ldots, x^{(k+1)})$ are denoted by $F_{k;p}$, e. g.,

$$F_{k;x} = \frac{\partial}{\partial x} F_k, \quad F_{k;\dot{x},\ldots,x^{(k+1)}} = [\, \frac{\partial}{\partial \dot{x}} F_k \;\; \ldots \;\; \frac{\partial}{\partial x^{(k+1)}} F_k \,].$$

To obtain an existence and uniqueness result for general DAEs, the following hypothesis was introduced in [34].

Hypothesis 10 *Consider the general system of nonlinear DAEs (9). There exist integers μ, r, a, d, and v such that the solution set*

$$\mathbb{L}_\mu = \{(t, x, \dot{x}, \ldots, x^{(\mu+1)}) \in \mathbb{R}^{(\mu+2)n+1} \mid F_\mu(t, x, \dot{x}, \ldots, x^{(\mu+1)}) = 0\}$$

is not empty, and the following properties hold:

1. *The set $\mathbb{L}_\mu \subseteq \mathbb{R}^{(\mu+2)n+1}$ forms a manifold of dimension $(\mu+2)n+1-r$.*
2. *We have*
$$\operatorname{rank} F_{\mu;x,\dot{x},\ldots,x^{(\mu+1)}} = r \tag{12}$$
on $\mathbb{L}_\mu$.
3. *We have*
$$\operatorname{corank} F_{\mu;x,\dot{x},\ldots,x^{(\mu+1)}} - \operatorname{corank} F_{\mu-1;x,\dot{x},\ldots,x^{(\mu)}} = v \tag{13}$$
on $\mathbb{L}_\mu$. (The corank is the dimension of the corange and we use the convention that $\operatorname{corank} F_{-1;x} = 0$.)
4. *We have*
$$\operatorname{rank} F_{\mu;\dot{x},\ldots,x^{(\mu+1)}} = r - a \tag{14}$$
on $\mathbb{L}_\mu$ such that there are smooth full rank matrix functions Z_2 and T_2 defined on $\mathbb{L}_\mu$ of size $((\mu+1)m, a)$ and $(n, n-a)$, respectively, satisfying
$$Z_2^T F_{\mu;\dot{x},\ldots,x^{(\mu+1)}} = 0, \quad \operatorname{rank} Z_2^T F_{\mu;x} = a, \quad Z_2^T F_{\mu;x} T_2 = 0 \tag{15}$$
on $\mathbb{L}_\mu$.
5. *We have*
$$\operatorname{rank} F_{\dot{x}} T_2 = d = m - a - v \tag{16}$$
on $\mathbb{L}_\mu$.

A DAE that satisfies Hypothesis 10 with $n = m = d + a$ is called *regular*.

Remark 11 It is not a simple task to verify the conditions in Hypothesis 10 in the context of finite precision arithmetic. Within the limits of numerical procedures to determine the rank of a matrix (for example using singular value decompositions) the conditions of Hypothesis 10 can be checked locally by determining ranks and nullspaces of the Jacobians.

Note that this approach allows redundancies, under-determinedness and it is not required that ranks are constant in a neighborhood of the solution in the whole space but only on a submanifold.

As in [31, 33, 34], the smallest possible μ in Hypothesis 10 is called the *strangeness-index* of (9). Systems with vanishing strangeness-index are called *strangeness-free*. The strangeness-index generalizes the differentiation-index [8] to over- and underdetermined systems, see [33], and in the case that the

differentiation-index is defined, both indices are 0 for ordinary differential equations, while the strangeness-index is one less than the differentiation-index otherwise.

In has been shown in [34] that Hypothesis 10 implies (locally) the existence of a reduced system (in the original variables) of the form

$$\begin{array}{ll} \text{(a)} & \hat{F}(t, x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) = 0, \\ \text{(b)} & x_3 - \mathcal{R}(t, x_1, x_2) = 0, \end{array} \tag{17}$$

with $\hat{F} = Z_1^T F$. An initial condition is consistent, if it satisfies the algebraic equation $x_3 - \mathcal{R}(t, x_1, x_2) = 0$. Eliminating x_3 and $\dot{x}_3$ in (17a) with the help of (17b) and its derivative, this leads to a system

$$\begin{aligned} &\hat{F}(t, x_1, x_2, \mathcal{R}(t, x_1, x_2), \dot{x}_1, \dot{x}_2, \mathcal{R}_t(t, x_1, x_2) \\ &+\mathcal{R}_{x_1}(t, x_1, x_2)\dot{x}_1 + \mathcal{R}_{x_2}(t, x_1, x_2)\dot{x}_2) = 0. \end{aligned}$$

By part 5 of Hypothesis 10 we may assume w.l.o.g. that this system can (locally via the implicit function theorem) be solved for $\dot{x}_1$, leading to a system of the form

$$\begin{aligned} \dot{x}_1 &= \mathcal{L}(t, x_1, x_2, \dot{x}_2), \\ x_3 &= \mathcal{R}(t, x_1, x_2). \end{aligned} \tag{18}$$

Obviously, in this system $x_2 \in C^1(\mathbb{I}, \mathbb{R}^u)$, with $u = n - d - a$, can be chosen arbitrarily (at least when staying in the domain of definition of $\mathcal{R}$ and $\mathcal{L}$). When x_2 has been chosen, then the resulting system has (locally) a unique solution for x_1 and x_3, provided that a consistent initial condition is given.

Remark 12 The system in the form (18) can be interpreted as a control system, in which x_3 plays the role of an output and x_2 can be chosen freely, i.e. may be considered as input.

Theorem 13 *[34] Let F in (9) be sufficiently smooth and satisfy Hypothesis 10 with μ, r, a, d, v and $u = n - d - a$. Then every solution of (9) also solves the reduced problems (17) and (18) consisting of d differential and a algebraic equations.*

Remark 14 In the reduced systems (17) and (18) we have not used the quantity v. This quantity measures the number of equations in the original system that give rise to trivial equations $0 = 0$, i. e., it counts the number of redundancies in the system. Together with a and d it gives a complete classification of the m equations into d differential equations, a algebraic equations and v trivial equations. Of course, trivial equations can be simply removed without altering the solution set.

In Table 1 we summarize the notation that we use for HDAEs.

The simulation and control of HDAEs poses several difficulties when compared to the simulation and control of standard DAEs. The reduction to a strangeness-free system must be realized just as for conventional DAEs to be able to apply specially suited numerical methods for the integration process.

Notation	*Meaning*
F	hybrid systems of DAEs
$F^\ell = F^\ell(t, x_\ell, \dot{x}_\ell)$	DAE in mode M_ℓ
M_ℓ	Mode l of the HDAE F, where the DAE F^ℓ describes the dynamics
$x_\ell^{\alpha_i}$	Initial conditions for interval $\mathbb{I}_i$ in mode M_ℓ
$1 \times n_\ell \times n_\ell$	Dimension of the space of definition of F^ℓ
m_ℓ	Dimension of the space of values of F^ℓ
F_k	Derivative array of HDAE F, where $k \in \mathbb{N}_+^{N_F}$
$F_{k_\ell}^\ell$	Derivative array of DAE F^ℓ in mode ℓ up to order k_ℓ
$\mathbb{I} = \bigcup_k \mathbb{I}_k = \bigcup_i D_i$	Integration interval
$\mathbb{I}_k = [\alpha_k, \beta_k)$	k-th integration interval, simulated in mode M_i, $x_i(\beta_k)$ is the solution smoothly extended to β_k. The last interval $\mathbb{I}_{N_\mathbb{I}} = [\alpha_{N_\mathbb{I}}, \beta_{N_\mathbb{I}}]$ is assumed closed.
$D_\ell = \bigcup_{k \in \{1,\ldots,N_I\}} I_k$	Domain of mode M_ℓ,
$S^\ell = (S_1^\ell, \ldots, S_{N_{S_\ell}}^\ell)^T$	Transition conditions for mode M_ℓ
$H(l, j)$	allocation function that determines the successor mode M_ℓ, where $S_j^\ell \leq 0$,
T_j^i	Transition function, that transfers the state at the mode change from mode M_i to mode M_j
N_F	Number of modes of the HDAE F
$N_\mathbb{I}$	Number of integration intervals
N_{S_ℓ}	Number of transition conditions for mode M_ℓ
$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{N_F})^T \in \mathbb{N}_+^{N_F}$	vector of strangeness indices for F,
$\mathbf{a} = (a_1, \ldots, a_{N_F})^T \in \mathbb{N}_+^{N_F}$	vector of number of algebraic equations for F,
$\mathbf{d} = (d_1, \ldots, d_{N_F})^T \in \mathbb{N}_+^{N_F}$	vector of number of differential equations for F,
$\mathbf{v} = (v_1, \ldots, v_{N_F})^T \in \mathbb{N}_+^{N_F}$	vector of number of vanishing equations for F,
$\mathbf{u} = (u_1, \ldots, u_{N_F})^T \in \mathbb{N}_+^{N_F}$	vector of number of free variables (controls) for F,
μ_{max}	maximal strangeness index, $\mu_{max} = \max_{i=1,\ldots,N_F} \mu_i$.

Table 1: Variables and functions for HDAEs

But in hybrid systems this must possibly be done very often on (possibly) short intervals, for every different DAEs associated with a different mode. Furthermore, the characteristic quantities μ, d, a, u, v may change in every mode, so that it is possible that in some mode the systems behaves like a control system and in others not.

For HDAEs, the values of the states $x(\hat{t})$ at times $\hat{t}$ of mode changes must be determined very accurately, since the states of the integration process in one mode form the basis for the computation of initial values for the successor mode.

It also has to be assured that one obtains a reasonable mode sequence, which means that a cyclic change between different modes must be prevented. This special phenomena of hybrid systems, the so called *chattering*, may arise for example, when nearly equal thresholds for mode changes of different modes are

given and the system starts to oscillate around these. These oscillations may be real in the physical model but also may arise due to errors in the numerical method.

Definition 15 (Numerical chattering) *An HDAE F as in* (8)*,* chatters numerically*, if in the integration process a repeated p-cyclic sequence of mode changes occurs through the modes $M_{s_1}, \ldots, M_{s_p}$ with $s_i \in \{1, \ldots, N_F\}$ on intervals of length $L \leq L_{min}$, where L_{min} is the length of the smallest interval $\mathbb{I}_j$.*

An example of numerical chattering is discussed in section 6.

There are several possibilities to prevent numerical chattering, and this should preferable already be done in the modeling phase. One possibility is the introduction of hystereses such that the integration of each mode is done in an interval of a length bounded from below [2,3,28,50]. A hysteresis approach is a good choice when the danger of numerical chattering exists between two modes M_i, M_j and these originally had transition conditions that contained thresholds that differ only by their sign $S_k^i = -S_\ell^j$. There, a hysteresis can be easily realized by adding $\epsilon > 0$ to the transition conditions $\hat{S}_k^i = \hat{S}_k^i + \epsilon,\ \hat{S}_\ell^j = -\hat{S}_k^i + \epsilon$. But to ensure that the minimal interval length for the hybrid system is kept, when introducing such a retarded chattering between the modes $M_i,\ M_j$ is hard to achieve.

For independent transition conditions between two modes $M_i,\ M_j$ or a numerical chattering between more than two modes the introduction of a hysteresis is not that intuitive. Furthermore, the correct switching points may be important for the system behavior. In this case a hysteresis should not be artificially introduced. Under these circumstances a more sophisticated modeling that includes these effects may be the only solution.

In our numerical method, we determine and observe the mode transitions in detail. In the case of numerical chattering the numerical method sets flags or even stops to allow the user to apply necessary system changes.

4 Strangeness-free formulation of HDAEs

We have developed a formulation of HDAEs that can be used to describe the dynamics in the different modes. In this section we will describe the prerequisites for a reasonable numerical integration of such systems. For this, several concepts of the theory of DAEs [34] will be generalized to hybrid systems.

In the following, we consider HDAEs F that satisfy Hypothesis 10 in every mode F^ℓ, and for which the dimensions m_ℓ, n_ℓ and the integers $\mu_\ell, r_\ell, d_\ell, a_\ell, u_\ell, v_\ell$ are constant in each mode but may be different for different modes. If the assumption that $m_\ell, n_\ell, \mu_\ell, r_\ell, d_\ell, a_\ell, u_\ell, v_\ell$ are constant in mode M_ℓ is violated, then, as a long as it is violated only at a finite number of points, for the analysis we may just introduce new modes and further switching points. Thus, to assure a reasonable framework for the numerical integration of HDAEs we introduce the following Hypothesis.

Hypothesis 16 *In each mode M_ℓ, and each domain D_ℓ, $\ell \in \{1, \ldots, N_F\}$ the DAE F^ℓ satisfies Hypothesis 10 with constant characteristic values $\mu_\ell, a_\ell, d_\ell, u_\ell, v_\ell$.*

If Hypothesis 16 holds, then the values $\mu_\ell, a_\ell, d_\ell, u_\ell, v_\ell$ are well-defined for each domain D_ℓ and therefore for each integration interval $\mathbb{I}_i$. Hypothesis 16 implies that for every time instance $t \in D_\ell$, projection matrices $Z_{\ell,1}, Z_{\ell,2}, T_{\ell,2}$ can be computed which allow to construct a strangeness-free system with the same solution set. By solving these transformed systems, the solution of the hybrid system (8) can be computed for every time t.

Under Hypothesis 16, vectors of characteristic values $\boldsymbol{\mu} \in \mathbb{N}_+^{N_F}$, $\mathbf{a} \in \mathbb{N}_+^{N_F}$, $\mathbf{d} \in \mathbb{N}_+^{N_F}$, $\mathbf{u} \in \mathbb{N}_+^{N_F}$ and $\mathbf{v} \in \mathbb{N}_+^{N_F}$ can be defined for a hybrid system, where the ℓ-th element of each vector contains the corresponding invariant for the mode M_ℓ.

Another important assumption for the numerical simulation is to require consistency for the initial values at the different transitions from one mode to another.

Hypothesis 17 *For the hybrid system F as in* (8)*, in all integration intervals $\mathbb{I}_i$, $i = 1, \ldots, N_\mathbb{I}$, the initial values $x_\ell^{\alpha_i}$ that are obtained by the transition function at a mode change are consistent with the integrated DAE F^ℓ in the current mode M_ℓ.*

If an initial condition after a mode change is not consistent, then it is possible to determine consistent initial conditions using the local representation of the solution manifold. Different approaches for consistent initialization have been implemented in the codes GELDA and GENDA, see [38–40].

Definition 18

(i) A hybrid system F as in (8) *is called* regular *if for each mode M_ℓ, $\ell = 1, \ldots, N_F$, the corresponding DAE F^ℓ is regular.*

(ii) For a hybrid system F, as in (8)*, which satisfies the Hypotheses 16 and 17, the* maximal strangeness index μ_{max} *is defined as*

$$\mu_{max} = \max_{\ell=1,\ldots,N_F} \{\mu_\ell\}, \tag{19}$$

where μ_ℓ is the strangeness index of F^ℓ on D_ℓ.

(iii) A hybrid system F is called strangeness-free*, if $\mu_{max} = 0$.*

5 Numerical integration of HDAEs

The numerical integration (or control) of HDAEs can be realized conceptually like the numerical integration (or control) of general DAEs, see [32, 34, 38–40] or by introducing appropriate new variables, as in [35]. For general DAEs the numerical integration is realized by generating, locally in each integration step,

projectors $Z_{\ell,1}, Z_{\ell,2}$ that would lead to a strangeness-free formulation in the mode M_ℓ, if they were actually applied, see [40].

This integration (or computation of controls) is continued until one of the transition conditions (5) gives a zero crossing, i.e. a crossing of a threshold occurs. After determining this crossing point within a certain error tolerance, the system is transferred to the next mode with help of the transition function (7), and the numerical integration (or computation of controls) is continued in the new mode. However, in order to deal with realistic hybrid systems, we allow the possibility that in every mode the system may have different characteristic values, and, in particular, we allow that the number v_ℓ of redundant equations and the number u_ℓ of free variables (controls) may be different in each mode.

In the following, we consider the index reduction process and the integration of regular HDAEs in more detail. For standard DAEs, the strangeness-index represents an invariant that characterizes the potential difficulties in the numerical solution of such systems, [32,34]. For HDAEs F as in (8), the strangeness-index is considered separately in each mode. Accordingly, a strangeness-free system with the same solution can be extracted in each mode and thus also for the complete hybrid system F. In this way HDAEs with a given maximal strangeness index μ_{max} can be treated, if the Hypotheses 10, 16 and 17 are fulfilled.

Analogous to handling regular nonlinear DAEs, one needs the corresponding derivative arrays, which must be computed for each mode of the hybrid system. This *derivative array F^ℓ of level j_ℓ in mode M_ℓ* is defined for $\ell = 1, \ldots, N_F$ via

$$F^\ell_{j_\ell}(t, x_\ell, \dot{x}_\ell, \ldots, x_\ell^{(j_\ell+1)}) = 0, \tag{20}$$

with

$$F^\ell_{j_\ell}(t, x_i, \dot{x}_i, \ldots, x_\ell^{(j_i+1)}) = \begin{bmatrix} F^\ell(t, x_\ell, \dot{x}_\ell) \\ \frac{d}{dt}F^\ell(t, x_\ell, \dot{x}_\ell) \\ \vdots \\ \frac{d^{j_\ell}}{dt^{j_\ell}}F^\ell(t, x_\ell, \dot{x}_\ell) \end{bmatrix}.$$

As in [34], the Jacobians for the derivative array F_{j_ℓ} of a hybrid system F are defined via the Jacobians for the different modes M_ℓ.

The projectors $Z_{\ell,1}, T_{\ell,2}, Z_{\ell,2}$ are defined according to Hypothesis 10 for each mode M_ℓ.

To discuss existence and uniqueness of solutions for a hybrid system F, the *solution set of the derivative array $F_{\boldsymbol{\mu}}$* has to be defined, where $\boldsymbol{\mu} \in \mathbb{N}_+^{N_F}$, $\mu_\ell \in \mathbb{N}$ for $\ell = 1, \ldots, N_F$. This solution set has the form:

$$\mathbb{L}_{\boldsymbol{\mu}} = \left\{ \begin{array}{ll} (t, x_\ell, \dot{x}_\ell, \ldots, x_\ell^{(\mu_\ell+1)}) \in \mathbb{R}^{(\mu_\ell+2)n_\ell+1} & \mid F^\ell_{\mu_\ell}(t, x_\ell, \dot{x}_\ell, \ldots, x_\ell^{(\mu_\ell+1)}) = 0, \\ \text{for } \ell = 1, \ldots, N_F. & \end{array} \right\}. \tag{21}$$

It was shown in [34], that Hypothesis 10 is invariant under a number of equivalence transformations for nonlinear differential-algebraic equations. Under the given assumptions, this result holds in each mode of a hybrid system.

The steps to determine the strangeness-free form of a nonlinear DAE can be carried out independently in each different mode. For the derivative array (20),

in each integration interval $\mathbb{I}_\ell$ for every mode M_ℓ, the initial values from $\mathbb{L}_{\boldsymbol{\mu}}$ must be chosen in a consistent way, so that the solution in each of the modes exists, and is unique if $u_\ell = 0$.

Furthermore, in each mode M_ℓ one defines the functions $\hat{F}_2^\ell = Z_{\ell,2}^T F_{\mu_\ell}^\ell$ and $\hat{F}_1^\ell = Z_1^T F^\ell$.

Then analogously to the construction for nonlinear DAEs in [33] the variables may be split as $x_\ell = (x_{\ell,1}, x_{\ell,2}, x_{\ell,3})$ and with $\hat{F}_2^\ell, \hat{F}_1^\ell$ we (locally) obtain functions $\mathcal{R}_\ell(t, x_{\ell,1}, x_{\ell,2})$ and $\mathcal{L}_\ell(t, x_{\ell,1}, x_{\ell,2})$ as in (18) such that for all $\ell = 1, \ldots, N_F$ the hybrid system in mode M_ℓ has the solution x_ℓ which satisfies an equivalent strangeness-free system of the form

$$\dot{x}_{\ell,1} = \mathcal{L}_\ell(t, x_{\ell,1}, x_{\ell,2}), \quad x_{\ell,3} = \mathcal{R}_\ell(t, x_{\ell,1}, x_{\ell,2}). \tag{22}$$

Note that if $u_\ell = 0$, then the free variables (controls) $x_{\ell,2}$ do not occur.

Thus, if there exists a complete set of consistent initial values $z_{\ell,\mu_\ell}^{\alpha_\ell} \in \mathbb{L}_{\boldsymbol{\mu}}$ at each mode change, then from the derivative array $F_{\mu_\ell}^\ell$, we locally obtain a strangeness-free system of the form (22) and thus, connecting all these systems together, a strangeness-free hybrid system of the form

$$\dot{x}_1 = \mathcal{L}(t, x_1, x_2), \quad x_3 = \mathcal{R}(t, x_1, x_2). \tag{23}$$

It should be noted, however, that this choice of a hybrid system depends strongly on the choice and the consistency of the initial values in each mode. It is clear that if we want a continuous solution of the hybrid system, then the transition function from one mode to the next must guarantee this. However, if the number of equations, or the number of free variables changes at a mode change, then this condition may be difficult to realize. In particular, we may face the situation that after a mode change the solution is not unique.

Theorem 19 *Let F be a regular hybrid system as in* (8) *and suppose that F is sufficiently often differentiable and satisfies the Hypotheses 10, 16 and 17, with vectors of characteristic values $\boldsymbol{\mu}, \mathbf{a}, \mathbf{d}, \mathbf{u}$ and $\mathbf{v} = 0$. Then every sufficiently smooth solution of* (8) *is a solution of* (22) *with vectors of characteristic values $\mathbf{a}$ and $\mathbf{d}$.*

If at each mode change the transition function T is such that the resulting initial condition is consistent, then in each mode there exist (control) functions so that for this control function the solution exists and is unique.

Moreover, if also $\mathbf{u} = 0$, and if at each mode change the transition function T is such that the resulting initial condition is consistent, then the solution exists and is unique.

Proof. If $x_\ell^\star$, $\ell = 1, \ldots, N_F$ describes a sufficiently smooth solution of F determined from the DAEs F_ℓ in the modes M_ℓ, then it also solves the reduced strangeness-free systems of DAEs (22), since for all $\ell = 1, \ldots, N_F$, and all $t \in D_\ell$

$$(t, x_\ell^\star(t), \dot{x}_\ell^\star(t), \ldots, \left(\tfrac{d}{dt}\right)^{\mu_\ell+1} x_\ell^\star(t)) \in \mathbb{L}_{\mu_\ell}^\ell.$$

Obviously, if there are no free solution components and the initial conditions are consistent then in each mode the solution exists and is unique. □

For ordinary differential equations (ODEs) with switching conditions and discontinuous right hand side, a so called *switch algorithm* was formulated in [19]. We generalize this procedure here for hybrid systems of DAEs as in (8). Let us first discuss the case of regular hybrid systems of DAEs. The time stepping procedure in the numerical integration in each mode M_ℓ can be achieved via any method for strangeness-free systems, e.g. by the version of a BDF method suggested in [33] as it is implemented in the code GENDA, see [39, 40].

Let D_h denote a backward difference (BDF) operator, see [8], which is of the form

$$D_h x_i^\ell = \frac{1}{h} \sum_{j=0}^{k} \alpha_j x_{i-j}^\ell \tag{24}$$

where h is the stepsize, x_i^ℓ denotes a numerical approximation to $x_\ell(t_i)$, and α_l are the coefficients of the k-step BDF method. The only property of D_h that we need is that it is linear with respect to the actual approximation x_i^ℓ with nonzero derivative α_0/h.

To proceed from t_{i-1} to $t_i = t_{i-1} + h$ then one solves the nonlinear system

$$\begin{array}{ll} \text{(a)} & F_{\mu_\ell}(t_i, x_i^\ell, w_i^\ell) = 0, \\ \text{(b)} & \tilde{Z}_1^T F(t_i, x_i^\ell, D_h x_i^\ell) = 0 \end{array} \tag{25}$$

for (x_i^ℓ, w_i^ℓ) and takes the part x_i^ℓ as approximation to $x^\ell(t_1)$. Here $\tilde{Z}_1$ denotes some fixed approximation (with orthonormal columns) to Z_1 at the desired solution.

Analogously, the time stepping procedure could be based on Runge-Kutta methods [22] instead of the above mentioned BDF-methods. For an s-stage Runge-Kutta-method the differential operator D_h would be replaced by

$$D_h x_i^\ell = x_{i-1}^\ell + h \sum_{j=0}^{s} \beta_j \hat{X}_{i-1,j}^\ell, \tag{26}$$

where

$$\hat{X}_{i-1,j}^\ell = f(t_{i-1} + \gamma_j h, X_{i-1,j}^\ell)$$

and the $X_{i-1,j}^\ell$ represent internal stages given by

$$X_{i-1,j}^\ell = x_{i-1}^\ell + h \sum_{k=1}^{s} \alpha_{j,k} \hat{X}_{i-1,k}^\ell, \qquad \text{for } j = 1, \ldots, s.$$

The nonlinear system (25) then can be solved as described above.

On the basis of these time stepping procedures we then have the following switch algorithm.

Algorithm 20 (Switch algorithm for regular HDAEs) *Let F be a given regular HDAE. Then it can be integrated as follows.*

1. *After each time step of the numerical integration in mode M_ℓ, (which consists in solving the nonlinear system (25)) check, whether discontinuities or crossing of thresholds occurred, by testing of the transition conditions $S^i(t, x_\ell, \dot{x}_\ell)$ for sign changes. Also, as in the code GENDA of [39, 40], check the characteristic values μ_ℓ, a_ℓ, d_ℓ, u_ℓ, v_ℓ.*

2. *If no sign change has occurred and the characteristic values have remained constant, then continue the integration normally. Otherwise localize the switch point exactly via the stepsize controller of the integrator as a root of a transition condition.*

3. *If none of the characteristic values has changed, then determine the successor mode M_j with the allocation function H from* (6).

4. *With the help of the transfer function T_j^i compute the new initial values for the mode M_j. Switch to the DAE F^j.*

5. *Check the initial values for consistency, if necessary, determine new consistent initial conditions, for example as in [9, 38, 40].*

6. *Restart the integration with consistent initial values from the time point of the mode change.*

7. *If one or more of the characteristic values has changed, then it strongly depends on this change how one proceeds, see the discussion below. However, since we have assumed regularity, the solution in the next mode will be unique, once consistent initial conditions have been determined.*

Thus, besides the pure integration process that may include an index reduction for simulating the hybrid system, we need a root finding procedure to determine the switch points, a process to compute consistent initial values in case of a mode change and, furthermore, an efficient organization of the different systems of DAEs F_ℓ. As methods for the determination of switch points we may use any root finding method like bisection, the secant method or the Newton-Raphson-method [24, 52].

So far we have discussed regular systems for which the characteristic values are constant in each mode. But in HDAEs it may happen that the characteristic values change.

Example 21 (Example 1 continued) Consider again the example of the accelerated pendulum. There, the number of equations and characteristic values in both modes are different.

Before the switch in the pendulum mode M_1 we have $\mu_1 = 2$, $d_1 = 2$ and $a_1 = 3$, but after the switch in the flying mode M_2 we have $\mu_2 = 0$, $d_2 = 4$ and $a_2 = 0$.

The transition condition on the accelerating force must be checked in each integration step and if a sign change occurs, then the exact switching point must be determined by one of the above mentioned methods. In this case in the second phase the transfer of the conditions is easily obtained by extending the values from mode M_1 to the interval boundary, and using them as initial values in mode M_2.

But it may also happen that a system has free components (controls) before a switch and is regular after the switch or vice versa, or that the number of controls changes. Consider the following example.

Example 22 A possible extension to the pendulum example could be flight control of the flying mass point by some control u that results in forces in horizontal and vertical direction to land at a predefined target point $(\hat{x_1}, \hat{x_2})$. The control u could represent a thrust accelerating or decelerating with certain limitations. In such a case, a third mode M_3 of a flying phase with additional thrust $f_\star(u)$ would be defined as follows.

$$\begin{aligned} \dot{x}_1 &= x_3, && (27a)\\ \dot{x}_2 &= x_4, && (27b)\\ m\dot{x}_3 &= f_x(u), && (27c)\\ m\dot{x}_4 &= -mg + f_\star(u). && (27d) \end{aligned}$$

The new free component u can then be determined by some control algorithm resulting in additional equations that together form the equations of mode M_3.

As a typical application of HDAEs the theory developed above may be applied to hybrid multibody systems. In this case the hybrid system contains N_F multibody systems, modeled with the DAEs $F^\ell,\ i = 1, \ldots, N_F$, and it switches between these at discrete times during a simulation. When the described theory is restricted to multibody systems, then it simplifies especially for the mode transitions.

As discussed for example in [23, 49], multibody systems can be modeled in different ways. We use in each mode an equation of the form $0 = F^\ell(p_\ell, v_\ell, \lambda_\ell)$ given by

$$\begin{aligned} 0 &= -\dot{p_\ell} + v_\ell, && n_{p_\ell} \text{ equations}\\ 0 &= -R_\ell \dot{v_\ell} + f_\ell(p_\ell, v_\ell) - g_{i,p_\ell}^T(p_\ell)\lambda_\ell, && n_{p_\ell} \text{ equations} \qquad (28)\\ 0 &= g_\ell(p_\ell), && n_{\lambda_\ell} \text{ equations}, \end{aligned}$$

where p, v, λ are the position, velocity constraints and Lagrange multipliers, respectively, R^ℓ is the mass matrix, and f, g are external forces and algebraic constraints, respectively. The index ℓ indicates the mode M_ℓ.

The maximal strangeness-index μ_{max} of a hybrid multibody systems depends on the formulation of the different models F^ℓ of the constrained multibody systems. If in each mode M_ℓ the DAE represents a multibody system, then $\mu_{max} = 2$, see [6].

When treating hybrid multibody systems, the transfer functions (7) T_j^i for mode changes turn out to be quite simple. If the hybrid multibody system F is modeled throughout all modes M_ℓ by the same position variables p, see example 1, then the DAEs $F^\ell = F^\ell(p, v, \lambda_\ell)$ just differ in the number of Lagrange multipliers λ_ℓ, depending on the number of holonomic algebraic constraints $g_\ell(p)$. In this case it follows that via the transition functions T_j^i only new initial values for additional new Lagrange multipliers λ_ℓ must be computed.

More sophisticated multibody formalisms such as in [6, 49], however, in general may lead to more complicated transition functions.

In this section we have shown how methods for the integration of DAEs such as the BDF and Runge-Kutta methods can be adapted to HDAEs. In the next section we demonstrate this approach for the hybrid system of an automatic gearbox.

6 Example - automatic gearbox

This section illustrates the presented concepts via the model of an electronically controlled five gear automatic gearbox originating in the research department of DaimlerChrysler RIC/EK in Berlin. There are three planetary gear sets which can realize different transmission ratios. To adjust and switch between these gears, three clutches and brakes are closed or opened, whereas two freewheels additionally allow to block the shafts in one direction [1].

Figure 2 gives a schematic overview of the structure of the gearbox and the variable numbering. For details see [24].

The system represents a hybrid system as the different gears as well as the gear shifting phases define different modes. When operating without failure, the brakes and clutches are closed, opened, or slipping and the freewheels may be in closed or free position. Depending on the state of these switching components the number of variables and equations change.

All components are modeled separately, as in [24] according to the Lagrange formulism, the shafts, the gear sets, the engine and load, the clutches, brakes and freewheels. The explicit model is presented in the appendix.

Each component was modeled in various different ways to compare simulation results for different aspects of the model. For the simulation of each component, all the algebraic constraints were used together with their first and second derivative to generate different formulations of different strangeness-index, i.e. models of strangeness-index $\mu = 2$ like general multibody systems with holonomic constraints on position level (EoM), of strangeness-index $\mu = 1$ like multibody systems with constraints on velocity level (EoM_2), the Gear-Gupta-Leimkuhler form (GGL) [20] or the particular form that is necessary to apply the solver MEXAX [41]. Furthermore, also strangeness-free multibody systems with constraints on the acceleration level, the Baumgarte stabilization [7] as well as the minimally extended formulation of [35] were implemented. See also [51] for a detailed comparison of these different formulations.

Depending on the formulation, the model in the original form

$$\begin{aligned} \dot{p} &= v && \text{(29a)} \\ R\dot{v} &= f(p,v) - g_p^T(p)\lambda && \text{(29b)} \\ 0 &= g(p) && \text{(29c)} \end{aligned}$$

contains between 70 and 100 variables in the different modes. The variables and parameters in (29) are denoted as in (28).

Figure 2 gives a schematic overview of the structure of the gearbox and the variable numbering. For details see the Appendix

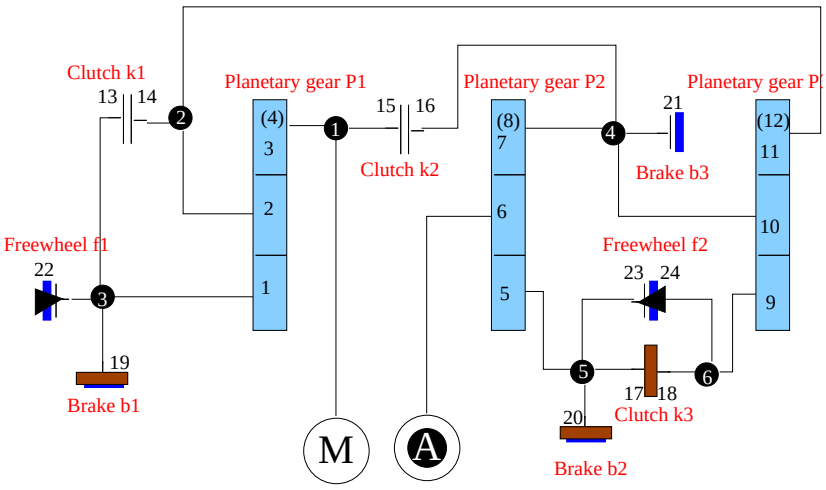

Figure 2: Schematic overview of the gearbox structure

It was shown in [24] that for this real world example, chattering occurs if the freewheels are contained in the different modes of the model. For the following simulation results, the freewheels will be omitted as they have only a supporting function when the gearbox is running.

The automatic gearbox was simulated with various standard integrators for DAEs that were modified to be usable for HDAEs. For comparison, the results of a DaimlerChrysler internal simulation, realized with the in-house solver ASIM were used. For this reason, parameters and simulation settings were adapted to the ones used in this simulation with ASIM. Further information about the solver ASIM as well as detailed information on the simulation settings can be found in [24].

The model together with these settings were simulated in different formulations and with different integrators. In the first formulation all the constraints (29c) of the model were differentiated until the underlying ordinary differential equation was determined. As the gearbox was modeled as pure multibody system with holonomic constraints on position level, we get three analytically equivalent model representations - two with nonholonomic constraints on velocity and acceleration level and one system that is just a system of ordinary differential equations. The more the holonomic constraints are differentiated and substituted, the more stable and easier the simulation is, as the strangeness index decreases. But at the same time the omitted holonomic constraints are more and more violated, see [24] and also [51].

We also simulated more sophisticated model representations that have smaller strangeness index and better satisfaction of the constraints. These are the Gear-Gupta-Leimkuhler form, an overdetermined form with all constraints and its derivatives, the minimal extension to strangeness-free form [35] and the complete derivative array.

As integrators we used the codes RADAU5, ODASSL, DASSL, MEXAX and DGENDA, each for that model representation that it is constructed for. As these solvers were designed to solve plain DAEs, they were enhanced for solving HDAEs. This enhancement was needed for the computation of the

mode transitions and the evaluation of new initial values at every possible mode change. The main challenge, however, was to find the switching points by finding roots of the switching conditions of the actual mode. Here, bisection on the step size was used as simplest possibility. Another sometimes even easier possibility is to use a continuous approximation of the solution achieved from the last steps, which in the case of RADAU5 is an extra feature of the solver.

The main goal of the following simulations is to demonstrate the results for HDAEs with small maximal strangeness index. Therefore, the simulation and modeling techniques for hybrid multibody systems applied to the automatic gearbox were more central than exact parameter studies. As an example we compare the input and output angular velocity for the simulation of the minimal extension strangeness-free form of the gearbox model using the code DASSL in Figure 3 with those of the code ASIM. This minimal extension form includes all the constraint equations as well as all the dynamic equations and therefore all the necessary information to obtain accurate results.

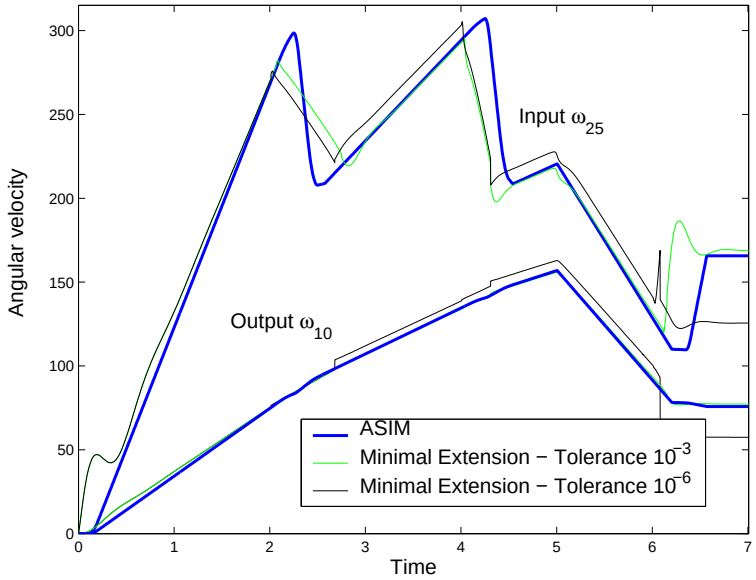

Figure 3: Simulation of the minimal extension form with DASSL

The results, in particular the difference to the ASIM simulation shows that ASIM did not capture all the details and that the solution behavior may change drastically after the switches when inappropriate solvers and tolerances are used.

In the following comparison the main criteria for evaluating the simulation results were the simulation time and the deviation from the solution manifold spanned by the holonomic constraints and their derivatives. For this the magnitude of the residual between solution and constraints was calculated. The results are summarized in Table 2 which uses the following notation. "EoM" stands for a hybrid system with holonomic constraints for each DAE, "EoM2(D)" for a hybrid system with constraints on velocity level for each DAE with D as damping parameter and GGL denotes the Gear-Gupta-Leimkuhler-

form of the hybrid system, i.e. for the DAE in each mode.

In the columns with Res_i, $i = 1, 2, 3$, the magnitude of the residual in the constraints on position, velocity and acceleration level are given. The bracketed value gives the magnitude of the residual in the switching points if significantly different. These indicate an insufficient determination of the switching points. If these residuals are too large then they must be corrected by smaller tolerances and more advanced strategies to find these points.

Integrator	*Form*	*Tolerance*	*Time*	Res_1	Res_2	Res_3
RADAU5	EoM	-6	0.9	-7	-5	-3 (2)
		-3	0.7	-5	-4 (-2)	-1 (4)
		-1	0.4	-4	-3 (-1)	1 (8)
	EoM2	-6	0.8	-8	-6 (-1)	-3 (-2)
		-3	1.5	-6	-5 (-1)	-1 (1)
		-1	0.7	-4	-4 (-2)	-1 (2)
	GGL	-6	1.1	-7	-6	-3 (-1)
		-3	1.0	-5	-4	-1 (1)
		-1	0.7	-4	-3	1 (2)
ODASSL	Over-determined form	-6	4.8	-8	-8	-8
		-4	1.3	-8	-8	-8
DASSL	Minimal extension	-6	5.1	-9	-8	-5 (-1)
		-3	0.5	-7	-6	-2 (1)
MEXAX	EoM2(20)	-3	64.2	-11	-11	8
	EoM2(30)	-3	121.3	-11	-11	8
	EoM2(50)	-3	140.8	-12	-11	7

Table 2: Simulation results

Table 2 confirms the theoretical results for HDAEs and for the different model representations achieved in section 3 and in [24]. The simulations with MEXAX with strong damping significantly for example need a lot more computing time and although they fulfil constraints on position and velocity level very well, the constraints on acceleration level are neglected completely.

The more sophisticated model representations, i.e. the minimal extended and the overdetermined form, guarantee that the residuals in all constraints are satisfied to high order of magnitude with the cost of a slight increas in computation time and higher modeling effort.

More comparisons and details can be found in [24].

7 Summary

We have shown how the theory for general systems of nonlinear differential algebraic equations as it was derived in [33, 34] can be extended to hybrid systems of DAEs in a straightforward way, thus showing the high flexibility of this approach. We have applied this approach to the model of a real world example given by an automatic gearbox. We have demonstrated that this approach can

be used in combination with different solution methods to obtain numerical simulation results that also work well at switching points.

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A Automatic gearbox: Model

In this appendix we present the modeling details for the example of an automatic gearbox (in the following called NAG) that was used for numerical comparison in section 6.

The gearbox is basically modeled as a multibody system which is then transformed into various formulations for the comparison of different solvers for hybrid DAEs.

The main components of the automatic gearbox are planetary gearsets, clutches, brakes, shafts, torque input and output. The system was modeled according to the documentation of the NAG from DaimlerChrysler [1]. The hybrid character of this system arises from system changes when the gear is shifted by means of clutches and brakes. In this way, different parts of the gearbox are included or excluded from the system dynamics.

The electro-hydraulic control of clutches and brakes was not modeled but given by characteristics of pressures. A refinement of the model to include these componets will make this a control system.

A.1 Model components

The usual approach to describe multibody systems in descriptor form is the Lagrange formalism, where the model equations in descriptor form are determined from energy balances [25].

Engine and Load The engine was not modeled separately but an input torque characteristic $M_M(t)$ and engine inertia J_M were used.

Analogously, the load or gearbox output and the Kardan-shaft were simplified to the torque characteristic $M_B(t)$ and inertia J_B.

Shafts Shafts connect components of the gearbox and transfer torques between those. They contain certain stiffness and damping characteristics summarized in the stiffness constant D and the damping constant C.

If more components must be connected, an auxiliary shaft node is introduced that has transmission rate one and a torque equilibrium describes the torque flow. These auxiliary modes were modeled mass- and lossless.

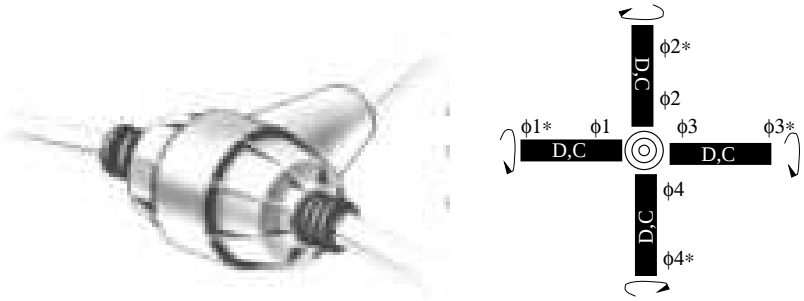
For n shafts acting at a shaft node, the equations for the angle φ and the angular velocities ω are

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\sum_{i=1}^{n} \left(D_i(\varphi - \varphi_i) + C_i(\omega - \omega_i) \right) \end{bmatrix},$$

where φ_i, ω_i represents the ith shaft at the opposite end of the shaft. To obtain a regular mass matrix the equations are transformed to:

$$\dot{\varphi} = -\frac{1}{\sum_{j=1}^{n} C_j} \left(\sum_{i=1}^{n} D_i(\varphi - \varphi_i) \right) + \sum_{i=1}^{n} \omega_i.$$

In Figure 4(b) a shaft node is schematically displayed.



(a) Shaft node (b) Scheme

Figure 4: Scheme of an shaft node

Planetary gearset Planetary gearsets are used in automatic gearboxes to switch between different gears and transmissions without interrupting the tractive force from engine to the wheels. A planetary gearset has at least two degrees of freedom and it can realize different transmissions with different activations of clutches and brakes.

A basic planetary gearset consists of a sun wheel, a carrier with three planets and a ring wheel.

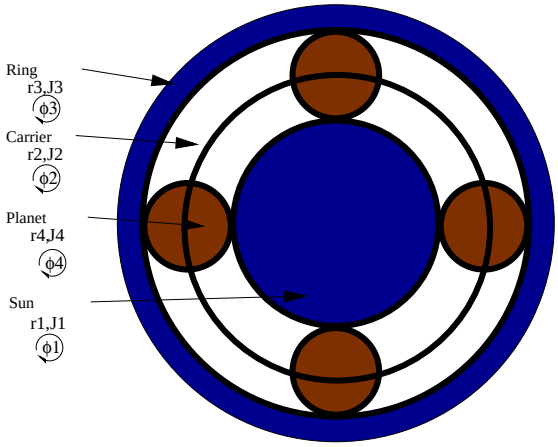


Figure 5: Scheme of a planetary gearset

To achieve a deterministic transmission, one wheel must be blocked or two wheels must be coupled. To describe the four rotations the following notation was chosen: the angle φ_1 of the sun, the angle φ_2 of the carrier, the angle φ_3 of the ring and the angle φ_4 of the planets. The radii r_i, inertias J_i, and torques M_i are denoted correspondingly. The angle at the opposite end of the connected shaft is denoted by $\varphi_{i\star}$.

The attached torques M_i are

$$M_i := -D(\varphi_i - \varphi_{i\star}) - C(\omega_i - \omega_{i\star}), \quad \text{where } i = 1, 2, 3.$$

The geometry of a planetary gearset leads to the following holonomic constraints g on position level [5, 24].

$$g(\varphi) := \begin{bmatrix} 2r_2\varphi_2 - r_3\varphi_3 - r_1\varphi_1 \\ 2r_4\varphi_4 - r_3\varphi_3 + r_1\varphi_1 \end{bmatrix}.$$

In these constraints the vector φ contains the variables that $g^\star$ depends on, where $g^\star$ stands for the different constraints.

The constraints on velocity level then are obtained by differentiation and the substitution $\omega_i = \dot{\varphi}_i$

$$g^I(\varphi) := \begin{bmatrix} 2r_2\omega_2 - r_3\omega_3 - r_1\omega_1 \\ 2r_4\omega_4 - r_3\omega_3 + r_1\omega_1 \end{bmatrix}$$

and on acceleration level correspondingly by twice differentiating

$$\begin{aligned} & g^{II}(\varphi) \\ := \; & \begin{bmatrix} \frac{2r_2}{J_2}(-M_2 - 2r_2\lambda_1) - \frac{r_3}{J_3}(-M_3 + r_3\lambda_1 + r_3\lambda_2) - \frac{r_1}{J_1}(-M_1 + r_1\lambda_1 - r_1\lambda_2) \\ \frac{2r_4}{J_4}(-2r_4\lambda_2) - \frac{r_3}{J_3}(-M_3 + r_3\lambda_1 + r_3\lambda_2) + \frac{r_1}{J_1}(-M_1 + r_1\lambda_1 - r_1\lambda_2) \end{bmatrix}. \end{aligned}$$

The dynamics of a basic planetary gearset is then described by the system

$$\begin{aligned} \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & J_4 \end{bmatrix} \begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \\ \ddot{\varphi}_3 \\ \ddot{\varphi}_4 \end{bmatrix} &= \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ 0 \end{bmatrix} - \begin{bmatrix} -r_1 & r_1 \\ 2r_2 & 0 \\ -r_3 & -r_3 \\ 0 & 2r_4 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \quad (30) \\ 0 &= g^\star(\varphi) \end{aligned}$$

where $g^\star$ may be any of the constraints given above.

Clutches and brakes Clutches are used to connect or disconnect two shafts of a gear box. To do this, the angular velocities have to be equalized which is done by a pressure p on the clutch. The pressure p results in some friction torque M_R that can be modeled by a friction model such as

$$M_R := c_f \cdot \begin{cases} p(\dot{\varphi}_1 - \dot{\varphi}_2) & \text{linear damping,} \\ \begin{cases} p & \dot{\varphi}_1 > \dot{\varphi}_2 \\ -p & \dot{\varphi}_1 < \dot{\varphi}_2 \\ 0 & \dot{\varphi}_1 = \dot{\varphi}_2 \end{cases} & \text{constant damping.} \end{cases} \quad (31)$$

The constant c_f, describing properties of the clutch, gives the relation between damping torque and pressure force. Each clutch disc has some inertia J_i and its dynamical behavior is described by the angle φ_i, the angular velocity $\dot{\varphi}_i$ and the angular acceleration $\ddot{\varphi}_i$. The attached shafts give the torques M_i.

The clutches have two operating modes, "Open/Sliding" and "Closed" and make the model a hybrid system. Both modes must be modeled with different systems of DAEs where the transition condition for the "Closed" mode is $|M_R| > |M_1 + M_2| \;\wedge\; \omega_1 = \omega_2$.

If we again let the variables at the opposite end of the shafts be denoted by $i\star$, then the torques M_i are given by

$$\begin{aligned} M_1 &:= -D(\varphi_1 - \varphi_{1\star}) - C(\omega_1 - \omega_{1\star}), \\ M_2 &:= -D(\varphi_2 - \varphi_{2\star}) - C(\omega_2 - \omega_{2\star}). \end{aligned}$$

Closed clutch If the clutched is closed with same angular velocities on both sides and a sufficiently big friction torque M_R, then the friction torque M_R just appears in the transition condition of the mode but does not influence the dynamics.

The holonomic constraints on the angles of the clutch discs are:

$$g(\varphi) := \varphi_1 - \varphi_2 + \triangle(\varphi_2(t_k) - \varphi_1(t_k)),$$

where t_k is the clutch closing time. The constraints on velocity level, therefore, are

$$g^I(\varphi) := \omega_1 - \omega_2$$

and on acceleration level

$$g^{II}(\varphi) := \frac{1}{J_1}(M_1 - \lambda) - \frac{1}{J_2}(M_2 + \lambda).$$

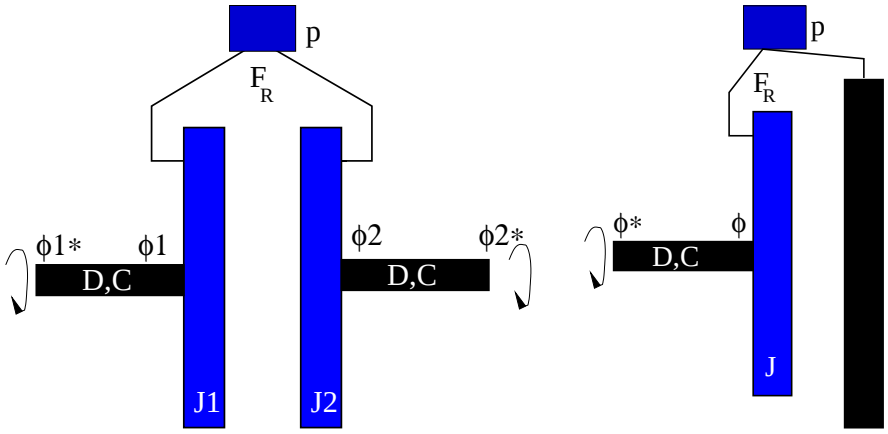
The dynamics of a clutch in closed mode is, thus, described by the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & J_1 & 0 & 0 \\ 0 & 0 & 0 & J_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ M_1 \\ M_2 \\ g(\varphi) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \lambda.$$

Open/Sliding clutch If a clutch is not closed, then the shafts are turning with different angular velocities or the friction moment is to small to hold the acting forces. In this mode no constraint exists but the friction torque M_R itself is influencing the dynamics.

These can be modeled by a system of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ M_1 - M_R \\ M_2 + M_R \end{bmatrix}.$$



(a) Clutch (b) Brake

Figure 6: Scheme of clutch and brake

Brakes are more or less special clutches, where the second disc is fastened at the housing. The equations of motion then simplify as the rotational variables at the housing side of the brake can be neglected.

Again there are the two modes "Sliding/Open" and "Closed" but the transition condition in the closed mode changes to $|M_R| > |M| \ \wedge \ \omega = 0$.

With a similar notation as for the clutch we obtain the following DAEs for both modes. The torques acting from the shafts are given by

$$M := -D(\varphi - \varphi_\star) - C(\omega - \omega_\star). \tag{32}$$

Closed brake The holonomic constraint is

$$g(\varphi) := \varphi + \varphi(t_k),$$

where t_k is the clutch closing time. The constraints on velocity level then are

$$g^I(\varphi) := \omega$$

and on acceleration level

$$g^{II}(\varphi) := \frac{1}{J}(M - \lambda).$$

The dynamics is then described by the system

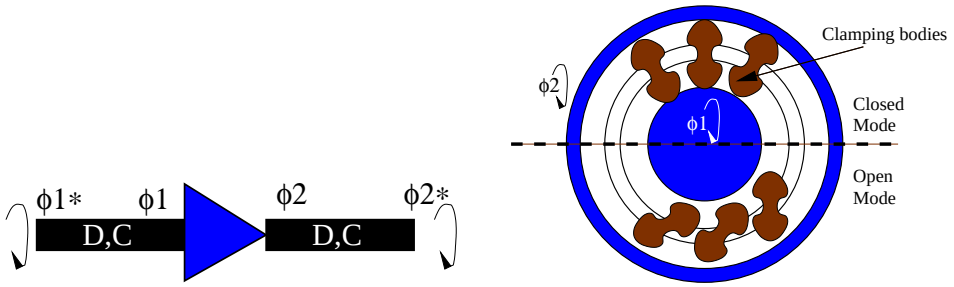
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\omega} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \omega \\ M \\ g(\varphi) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \lambda.$$

Sliding/Open brake The dynamics of a sliding or open brake are analogously described by the system

$$\begin{bmatrix} 1 & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ M - M_R \end{bmatrix}.$$

Free wheels For optimization reasons and to block the gearbox in certain modes in one direction, free wheels are built into the gearbox. A free wheel connects two shafts like a clutch. In one direction it blocks the motion, whereas the other the shafts can move freely.

A scheme of functionality and structure is given in Figure 7.



(a) Scheme (b) Functionality

Figure 7: Schemes of a free wheel

The free wheels are modeled without inertias and are denoted similarly to the clutches. The moments from the attached shafts then are

$$\begin{aligned} M_1 &:= -D(\varphi_1 - \varphi_{1\star}) - C(\omega_1 - \omega_{1\star}), \\ M_2 &:= -D(\varphi_2 - \varphi_{2\star}) - C(\omega_2 - \omega_{2\star}). \end{aligned}$$

Closed free wheel In this case both shafts turn with equal angular velocity but the transition condition is $\omega_1 - \omega_2 > 0$.

The holonomic constraint is

$$g(\varphi) := \varphi_1 - \varphi_2 + \triangle(\varphi_2(t_k) - \varphi_1(t_k)),$$

with t_k as closing time. The constraints on velocity level then are

$$g^I(\varphi) := \omega_1 - \omega_2$$

and on acceleration level:

$$g^{II}(\varphi) := \lambda \left(\frac{1}{J_1} - \frac{1}{J_2} \right).$$

The dynamics is described by the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ M_1 \\ M_2 \\ g(\varphi) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \lambda,$$

or in a more compact form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -\frac{D}{C}(\varphi_1 - \varphi_{1\star}) + \omega_{1\star} - \frac{1}{C}\lambda \\ -\frac{D}{C}(\varphi_2 - \varphi_{2\star}) + \omega_{2\star} + \frac{1}{C}\lambda \\ g(\varphi) \end{bmatrix}.$$

Open free wheel If the condition $\omega_1 - \omega_2 < 0$ holds, then both shafts may turn freely and no dynamic interaction or constraints hold. The dynamics then can be described by the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ M_1 \\ M_2 \end{bmatrix},$$

which can be transformed to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \begin{bmatrix} -\frac{D}{C}(\varphi_1 - \varphi_{1\star}) + \omega_{1\star} \\ -\frac{D}{C}(\varphi_2 - \varphi_{2\star}) + \omega_{2\star} \end{bmatrix}.$$

A.2 Hybrid multibody system automatic gearbox

This section summarizes the overall structure of the model of the automatic gearbox NAG as a hybrid multibody system. To do this, the components of the previous section are connected and complemented by inputs and outputs.

Different combinations of the different states of clutches, brakes and freewheels determine gears, gear changes, failure, safety or undetermined modes. For simplicity we will just discuss the first gears and the interconnecting gear changes here. A gear represents a certain transmission from input to output revolutions and torques which decrease the more the higher the gear, to allow higher velocities of the car.

In Figure 2 a scheme of the NAG mechanics is depicted, where M and A represent engine and gearbox output, respectively. There are 6 shaft nodes that distribute the torques, 30 angles φ_i denoted as the corresponding components and the planetary gear sets with sun $P_j.s$, carrier $P_j.t$, ring $P_j.h$ and planets $P_j.p$ with angles denoted by $\varphi_i, \varphi_{i+1}, \varphi_{i+2}, \varphi_{i+3}$ for $i = 1, 5, 9$, respectively.

Time dependent inputs to the mechanical system are engine torque, car brake torque and the pressures at gearbox brakes and clutches.

The system dimension differs between the modes and the multibody modeling technique but is on the order of 80. In the following sections these modes and the used modeling techniques are presented. The freewheels f_1, f_2 were not regarded in the simulations with its different modes as they serve mainly for safety functionality [1] but they are kept always open. Furthermore, the simulation of the freewheels bears the problem of chattering and must be treated carefully as was discussed in detail in [24].

Modes As mentioned before, the gearbox may be operated in different gears that are roughly speaking represented by different modes of the hybrid system.

These and the modes from switching phases between the gears are given here, but of course one can refine the model to include many more operating modes.

The modes arising from the different gears are easily determined. They are depicted in Table 3. Some of the additional modes arising from the switching phases are depicted in Table 4. In the following the modes M_ℓ are denoted according to this table. The notation of gears represents either a gear i or a shift from a gear i to j denoted by ij. The total number of physically possible modes of the system depends on the variety of direct mode shifts.

It is an important observation from Table 4 that if the model contains switching freewheels, most modes corresponding to the different gears or the transient phases allow both modes of the freewheel. Together with the natural modeling as done in [24], this is the reason for the possible occurence of chattering which will disturb the simulation.

Gear	*Transmission*	b_1	b_2	b_3	k_1	k_2	k_3	f_1	f_2
1	3.59	•	•				•	•	•
2	2.19		•		•		•		•
3	1.41		•		•	•			
4	1				•	•	•		
5	0.83	•				•	•	•	
N	–	•					•		
R_1	-3.16	•		•			•	•	
R_2	-1.93			•	•		•		

Table 3: Gears and transmissions of the NAG [1].

Mode	*Gear*	b_1	b_2	b_3	k_1	k_2	k_3	f_1	f_2
1	1/12	•	•				•	•/–	•/–
12	12		•				•	•/–	•/–
21	21		•				•	•/–	•/–
2	2/21/23		•		•		•		•/–
23	23		•		•				•/–
32	32		•		•				•/–
3	3/32/34		•		•	•			
34	34				•	•			
43	43				•	•			
4	4/45/43				•	•	•		
45	45					•	•		
54	54					•	•		
5	5/54	•				•	•	•/–	
N	N	•					•		
R_1	R_1	•		•			•	•/–	
R_2	R_2			•	•		•		

Table 4: Some modes of the NAG

As an example, in the following the gear shift form gear one to gear two

and its mode changes will be described.

Example 23 (Gear shift) Consider the system in mode M_1 and gear one. At a certain time $\hat{t}_1$, the control input gives the signal to change to gear two. Therefore, the pressure at brake b_1 falls and at clutch k_1 rises. Although the gear shift 12 is started, no mode change occurs until brake b_1 starts slipping. Say at the time $\hat{t}_2$ the brake b_1 starts slipping and, therefore, still in the gear shift 12 we proceed to mode M_{12}. With further rising pressure in clutch k_1, it will close at a time $\hat{t}_3$ and finally the gear shift ends in gear two and mode M_2. It is assumed that clutch k_1 will not close before brake b_1 starts slipping as this would represent a failure mode.

To illustrate the functionality of a gearbox we will take a closer look at the first gear, i.e., at mode M_1.

Example 24 (First Gear) To start a car the gearbox is set to the first gear, where it transmits with a big transmission the engine torque and revolution to the output shaft and overcomes the big inertia of the car.

To achieve this transmission rate, all planetary gear sets are involved. Brake b_1 fixes sun $P_1.s$ and the engine revolutions are transmitted from the ring $P_1.h$ to the carrier $P_1.t$ directly. As the clutches k_1, k_2 are open, this torque flows to ring $P_3.h$. The suns $P_2.s, P_3.s$ are connected by clutch k_3 and by brake b_2 fixed to the housing. Therefore, the torque of ring $P_3.h$ is transmitted to carrier $P_3.t$, from there to ring $P_2.h$ and finally to carrier $P_2.t$, whose dynamics are equal to those of the output shaft. Thus, the output shaft is turning in positive direction with reduced revolutions [1].

The first gear is used until a control signal starts a gear shift, for example to switch to gear two as described above. This control signal depends on many parameters as engine revolutions, vehicle velocity or driving strategy.

A.3 Model representations

There exist several modeling techniques for multibody systems with differential algebraic equations, [24, 51]. These result in different model representations with analytically equal solution but different numerical characteristics. For completeness the model equations for each of the treated modeling technics are given below.

To obtain the set of equations for the model of the NAG, the dynamics of all components must be combined for each mode as listed in Table 4.

As this hybrid system of an automatic gearbox gets its hybridicity mainly through additional or removed Lagrange multipliers, only mode M_1 is given as complete DAE system, whereas the others are represented by the differences to this mode.

In the same manner, further model representations are characterized only by its differences to the first modeling technique using holonomic constraints, also called constraints on position level.

A.3.1 Equations of motion with holonomic constraints (EoM)

The dynamics is described by the descriptor forms of the components combined as stated above for the different modes. The systems are given in residual form for the modes $M_1, M_{12}, M_{21}, M_2, M_{23}, M_{32}$ and M_3.

Mode M_1 The vector to describe a state of the NAG in mode M_1 with constraints on position level is

$$x = \begin{bmatrix} \varphi \\ \omega \\ \lambda \end{bmatrix} \in \mathbb{R}^{69},$$

with

$$\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{30} \end{bmatrix}, \ \omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_{30} \end{bmatrix}, \ \lambda = \begin{bmatrix} \lambda_{P_1.1} \\ \lambda_{P_1.2} \\ \lambda_{P_2.1} \\ \lambda_{P_2.2} \\ \lambda_{P_3.1} \\ \lambda_{P_3.2} \\ \hat{\lambda} \end{bmatrix}, \ \hat{\lambda} = \begin{bmatrix} \lambda_{k_3} \\ \lambda_{b_1} \\ \lambda_{b_2} \end{bmatrix}. \tag{33}$$

The first 30 equations are first-order differential equations

$$0 = -\dot{\varphi}_i + \omega_i \quad i = 1, \ldots, 30 \tag{34}$$

The equations for the planetary gear sets P_1, P_2, P_3 are

$$P_1 \begin{cases} 0 &= -J_{P_1.s}\dot{\omega}_1 + r_{P_1.s}\lambda_{P_1.1} - r_{P_1.s}\lambda_{P_1.2} - D(\varphi_1 - \varphi_{27}) - C(\omega_1 - \omega_{27}) \\ 0 &= -J_{P_1.t}\dot{\omega}_2 - 2r_{P_1.t}\lambda_{P_1.1} - D(\varphi_2 - \varphi_{26}) - C(\omega_2 - \omega_{26}) \\ 0 &= -J_{P_1.h}\dot{\omega}_3 + r_{P_1.h}\lambda_{P_1.1} + r_{P_1.h}\lambda_{P_1.2} - D(\varphi_3 - \varphi_{25}) - C(\omega_3 - \omega_{25}) \\ 0 &= -J_{P_1.p}\dot{\omega}_4 - 2r_{P_1.p}\lambda_{P_1.2} \end{cases} \tag{35a}$$

$$P_2 \begin{cases} 0 &= -J_{P_2.s}\dot{\omega}_5 + r_{P_2.s}\lambda_{P_2.1} - r_{P_2.s}\lambda_{P_2.2} - D(\varphi_5 - \varphi_{29}) - C(\omega_5 - \omega_{29}) \\ 0 &= -J_{P_2.t}\dot{\omega}_6 - 2r_{P_2.t}\lambda_{P_2.1} - M_B(t), \quad \text{with brake torque } M_B(t) \\ 0 &= -J_{P_2.h}\dot{\omega}_7 + r_{P_2.h}\lambda_{P_2.1} + r_{P_2.h}\lambda_{P_2.2} - D(\varphi_7 - \varphi_{28}) - C(\omega_7 - \omega_{28}) \\ 0 &= -J_{P_2.p}\dot{\omega}_8 - 2r_{P_2.p}\lambda_{P_2.2} \end{cases} \tag{35b}$$

$$P_3 \begin{cases} 0 &= -J_{P_3.s}\dot{\omega}_9 + r_{P_3.s}\lambda_{P_3.1} - r_{P_3.s}\lambda_{P_3.2} - D(\varphi_9 - \varphi_{30}) - C(\omega_9 - \omega_{30}) \\ 0 &= -J_{P_3.t}\dot{\omega}_{10} - 2r_{P_3.t}\lambda_{P_3.1} - D(\varphi_{10} - \varphi_{28}) - C(\omega_{10} - \omega_{28}) \\ 0 &= -J_{P_3.h}\dot{\omega}_{11} + r_{P_3.h}\lambda_{P_3.1} + r_{P_3.h}\lambda_{P_3.2} - D(\varphi_{11} - \varphi_{26}) - C(\omega_{11} - \omega_{26}) \\ 0 &= -J_{P_3.p}\dot{\omega}_{12} - 2r_{P_3.p}\lambda_{P_3.2} \end{cases} \tag{35c}$$

The clutches k_1, k_2, k_3, brakes b_1, b_2, b_3, and the free wheels in their open mode f_1, f_2 are given by

$$k_1 \quad \begin{cases} 0 &= -J_{k_1}\dot{\omega}_{13} - D(\varphi_{13} - \varphi_{27}) - C(\omega_{13} - \omega_{27}) - M_{R,k_1} \\ 0 &= -J_{k_1}\dot{\omega}_{14} - D(\varphi_{14} - \varphi_{26}) - C(\omega_{14} - \omega_{26}) + M_{R,k_1} \end{cases} \tag{36a}$$

$$k_2 \quad \begin{cases} 0 &= -J_{k_2}\dot{\omega}_{15} - D(\varphi_{15} - \varphi_{25}) - C(\omega_{15} - \omega_{25}) - M_{R,k_2} \\ 0 &= -J_{k_2}\dot{\omega}_{16} - D(\varphi_{16} - \varphi_{28}) - C(\omega_{16} - \omega_{28}) + M_{R,k_2} \end{cases} \tag{36b}$$

$$k_3 \quad \begin{cases} 0 &= -J_{k_3}\dot{\omega}_{17} - D(\varphi_{17} - \varphi_{29}) - C(\omega_{17} - \omega_{29}) - \lambda_{k_3} \\ 0 &= -J_{k_3}\dot{\omega}_{18} - D(\varphi_{18} - \varphi_{30}) - C(\omega_{18} - \omega_{30}) + \lambda_{k_3} \end{cases} \tag{36c}$$

$$b_1 \quad 0 = -J_{b_1}\dot{\omega}_{19} - D(\varphi_{19} - \varphi_{27}) - C(\omega_{19} - \omega_{27}) - \lambda_{b_1} \tag{36d}$$

$$b_2 \quad 0 = -J_{b_2}\dot{\omega}_{20} - D(\varphi_{20} - \varphi_{29}) - C(\omega_{20} - \omega_{29}) - \lambda_{b_2} \tag{36e}$$

$$b_3 \quad 0 = -J_{b_3}\dot{\omega}_{21} - D(\varphi_{21} - \varphi_{28}) - C(\omega_{21} - \omega_{28}) - M_{R,b_3} \tag{36f}$$

$$f_1 \quad 0 = -D(\varphi_{22} - \varphi_{27}) - C(\omega_{22} - \omega_{27}) \tag{36g}$$

$$f_2 \quad \begin{cases} 0 &= -D(\varphi_{23} - \varphi_{29}) - C(\omega_{23} - \omega_{29}) \\ 0 &= -D(\varphi_{24} - \varphi_{30}) - C(\omega_{24} - \omega_{30}), \end{cases} \tag{36h}$$

where friction torques $M_{R,k_1}, M_{R,k_2}, M_{R,b_3}$ in the normal mode are zero. For simplicty of presentation, we have have omitted the dependence on the torque parameters. For example $M_{R,k_1}(t, \omega_{13}, \omega_{14})$ is given as M_{R,k_1}.

The six shaft nodes interconnecting the components are modeled by

$$\begin{aligned} 0 &= -D(\varphi_{25} - \varphi_3) - C(\omega_{25} - \omega_3) - D(\varphi_{25} - \varphi_{15}) - C(\omega_{25} - \omega_{15}) \\ &\quad + M_M(t), \quad \text{with engine torque } M_M(t) \end{aligned} \tag{37a}$$

$$\begin{aligned} 0 &= -D(\varphi_{26} - \varphi_2) - C(\omega_{26} - \omega_2) - D(\varphi_{26} - \varphi_{11}) - C(\omega_{26} - \omega_{11}) \\ &\quad - D(\varphi_{26} - \varphi_{14}) - C(\omega_{26} - \omega_{14}) \end{aligned} \tag{37b}$$

$$\begin{aligned} 0 &= -D(\varphi_{27} - \varphi_1) - C(\omega_{27} - \omega_1) - D(\varphi_{27} - \varphi_{13}) - C(\omega_{27} - \omega_{13}) \\ &\quad - D(\varphi_{27} - \varphi_{19}) - C(\omega_{27} - \omega_{19}) - D(\varphi_{27} - \varphi_{22}) - C(\omega_{27} - \omega_{22}) \end{aligned} \tag{37c}$$

$$\begin{aligned} 0 &= -D(\varphi_{28} - \varphi_7) - C(\omega_{28} - \omega_7) - D(\varphi_{28} - \varphi_{10}) - C(\omega_{28} - \omega_{10}) \\ &\quad - D(\varphi_{28} - \varphi_{16}) - C(\omega_{28} - \omega_{16}) - D(\varphi_{28} - \varphi_{21}) - C(\omega_{28} - \omega_{21}) \end{aligned} \tag{37d}$$

$$\begin{aligned} 0 &= -D(\varphi_{29} - \varphi_5) - C(\omega_{29} - \omega_5) - D(\varphi_{29} - \varphi_{17}) - C(\omega_{29} - \omega_{17}) \\ &\quad - D(\varphi_{29} - \varphi_{20}) - C(\omega_{29} - \omega_{20}) - D(\varphi_{29} - \varphi_{23}) - C(\omega_{29} - \omega_{23}) \end{aligned} \tag{37e}$$

$$\begin{aligned} 0 &= -D(\varphi_{30} - \varphi_9) - C(\omega_{30} - \omega_9) - D(\varphi_{30} - \varphi_{18}) - C(\omega_{30} - \omega_{18}) \\ &\quad - D(\varphi_{30} - \varphi_{24}) - C(\omega_{30} - \omega_{24}). \end{aligned} \tag{37f}$$

The constraints on position level in mode M_1 are

$$\begin{aligned}
0 &= 2r_{P_1.t}\varphi_2 - r_{P_1.s}\varphi_1 - r_{P_1.h}\varphi_3 && \text{(38a)}\\
0 &= 2r_{P_1.p}\varphi_4 + r_{P_1.s}\varphi_1 - r_{P_1.h}\varphi_3 && \text{(38b)}\\
0 &= 2r_{P_2.t}\varphi_6 - r_{P_2.s}\varphi_5 - r_{P_2.h}\varphi_7 && \text{(38c)}\\
0 &= 2r_{P_2.p}\varphi_8 + r_{P_2.s}\varphi_5 - r_{P_2.h}\varphi_7 && \text{(38d)}\\
0 &= 2r_{P_3.t}\varphi_{10} - r_{P_3.s}\varphi_9 - r_{P_3.h}\varphi_{11} && \text{(38e)}\\
0 &= 2r_{P_3.p}\varphi_{12} + r_{P_3.s}\varphi_9 - r_{P_3.h}\varphi_{11} && \text{(38f)}\\
0 &= \varphi_{17} - \varphi_{18} - \Delta_{k_3} && \text{(38g)}\\
0 &= \varphi_{19} - \Delta_{b_1} && \text{(38h)}\\
0 &= \varphi_{20} - \Delta_{b_2}, && \text{(38i)}
\end{aligned}$$

Modes M_{12} and M_{21} In mode M_{12} and M_{21} the friction torques M_{R,k_1}, M_{R,b_1} are developing differently but otherwise the dynamics can be described by the same equations. In comparison to mode M_1, (36d) is changed to

$$0 = -J_{b_1}\dot{\omega}_{19} - D(\varphi_{19} - \varphi_{27}) - C(\omega_{19} - \omega_{27}) - M_{R,b_1}$$

In addition, equation (38h) and the corresponding Lagrange multiplier λ_{b_1} do not occur.

The state vector, therefore, is (33) $x = [\varphi^T, \omega^T, \lambda^T]^T \in \mathbb{R}^{68}$ with $\hat{\lambda} = [\lambda_{k_3}^T, \lambda_{b_2}^T]^T$.

Mode M_2

In mode M_2 the state vector is $x = [\varphi^T, \omega^T, \lambda^T]^T \in \mathbb{R}^{69}$ with $\hat{\lambda} = [\lambda_{k_1}^T, \lambda_{k_3}^T, \lambda_{b_2}^T]^T$. The new Lagrange multiplier λ_{k_1} is coupled to clutch k_1. Correspondingly, the equations changed in comparison to M_{12}, M_{21} are (36a), which must be changed to

$$\begin{aligned}
0 &= -J_{k_1}\dot{\omega}_{13} - D(\varphi_{13} - \varphi_{27}) - C(\omega_{13} - \omega_{27}) - \lambda_{k_1},\\
0 &= -J_{k_1}\dot{\omega}_{14} - D(\varphi_{14} - \varphi_{26}) - C(\omega_{14} - \omega_{26}) + \lambda_{k_1}.
\end{aligned}$$

The new constraint then is

$$0 = \varphi_{13} - \varphi_{14} - \Delta_{k_1}. \qquad (39)$$

Mode M_{23} and M_{32}

As the modes M_{12}, M_{21}, the modes M_{23}, M_{32} contain the same system of DAEs except for certain temporal development of some friction torques, which are denoted by M_{R,k_2}, M_{R,k_3}.

The changes to mode M_2 affect (36c) which is given by

$$\begin{aligned}
0 &= -J_{k_3}\dot{\omega}_{17} - D(\varphi_{17} - \varphi_{29}) - C(\omega_{17} - \omega_{29}) - M_{R,k_3},\\
0 &= -J_{k_3}\dot{\omega}_{18} - D(\varphi_{18} - \varphi_{30}) - C(\omega_{18} - \omega_{30}) + M_{R,k_3}.
\end{aligned}$$

Furthermore, equation (38g) and its Lagrange multiplier λ_{k_3} are omitted. The state vector now is (33), $x = [\varphi^T, \omega^T, \lambda^T]^T \in \mathbb{R}^{68}$ with $\hat{\lambda} = [\lambda_{k_1}^T, \lambda_{b_2}^T]^T$.

Mode M_3

In mode M_3 we have the state vector $x = [\varphi^T, \omega^T, \lambda^T]^T \in \mathbb{R}^{69}$ with $\hat{\lambda} = [\lambda_{k_1}^T, \lambda_{k_2}^T, \lambda_{b_2}^T]^T$. The new Lagrange multiplier λ_{k_2} is bound to clutch k_2. According to this, the equations (36b) are changed, which is the only difference to modes M_{23}, M_{32}

$$\begin{aligned} 0 &= -J_{k_2}\dot{\omega}_{15} - D(\varphi_{15} - \varphi_{25}) - C(\omega_{15} - \omega_{25}) - \lambda_{k_2}, \\ 0 &= -J_{k_2}\dot{\omega}_{16} - D(\varphi_{16} - \varphi_{28}) - C(\omega_{16} - \omega_{28}) + \lambda_{k_2}. \end{aligned}$$

The new constraint then is

$$0 = \varphi_{15} - \varphi_{16} - \Delta_{k_2}. \tag{40}$$

A.3.2 Dynamical systems with constraints on vleocity or acceleration level (EoM$_2$,EoM$_1$)

Instead of modeling multibody systems with holonomic constraints, it is also possible to replace them by their derivatives.

As the strangeness index changes through these differentiations, the DAE system will be denoted by EoM_2 and EoM_1, respectively.

The DAE structure in all modes stays the same, so just the derivatives of equations (38), (39) and (40) are given here.

Constraints on velocity level (EoM$_2$) For the EoM$_2$ formulation of the NAG dynamics, the holonomic constraints (38) are replaced by the following derivatives

$$\begin{aligned} 0 &= 2r_{P_1.t}\omega_2 - r_{P_1.s}\omega_1 - r_{P_1.h}\omega_3 && \text{(41a)} \\ 0 &= 2r_{P_1.p}\omega_4 + r_{P_1.s}\omega_1 - r_{P_1.h}\omega_3 && \text{(41b)} \\ 0 &= 2r_{P_2.t}\omega_6 - r_{P_2.s}\omega_5 - r_{P_2.h}\omega_7 && \text{(41c)} \\ 0 &= 2r_{P_2.p}\omega_8 + r_{P_2.s}\omega_5 - r_{P_2.h}\omega_7 && \text{(41d)} \\ 0 &= 2r_{P_3.t}\omega_{10} - r_{P_3.s}\omega_9 - r_{P_3.h}\omega_{11} && \text{(41e)} \\ 0 &= 2r_{P_3.p}\omega_{12} + r_{P_3.s}\omega_9 - r_{P_3.h}\omega_{11} && \text{(41f)} \\ 0 &= \omega_{17} - \omega_{18} && \text{(41g)} \\ 0 &= \omega_{19} && \text{(41h)} \\ 0 &= \omega_{20}. && \text{(41i)} \end{aligned}$$

Furthermore, in the modes M_2, M_3, the constraints (39) and (40) must be replaced by

$$\begin{aligned} 0 &= \omega_{13} - \omega_{14}, && \text{(42a)} \\ 0 &= \omega_{15} - \omega_{16}. && \text{(42b)} \end{aligned}$$

Constraints on acceleration level (EoM$_1$) In the EoM$_1$ formulation, the second derivatives replace the holonomic constraints from A.3.1.

The variables $\dot{\omega}_i$ in the derivatives actually can be expressed directly via (35) and (36).The second derivatives of (38) are

$$\begin{aligned}
0 &= 2r_{P_1.t}\dot{\omega}_2 - r_{P_1.s}\dot{\omega}_1 - r_{P_1.h}\dot{\omega}_3 && \text{(43a)}\\
0 &= 2r_{P_1.p}\dot{\omega}_4 + r_{P_1.s}\dot{\omega}_1 - r_{P_1.h}\dot{\omega}_3 && \text{(43b)}\\
0 &= 2r_{P_2.t}\dot{\omega}_6 - r_{P_2.s}\dot{\omega}_5 - r_{P_2.h}\dot{\omega}_7 && \text{(43c)}\\
0 &= 2r_{P_2.p}\dot{\omega}_8 + r_{P_2.s}\dot{\omega}_5 - r_{P_2.h}\dot{\omega}_7 && \text{(43d)}\\
0 &= 2r_{P_3.t}\dot{\omega}_{10} - r_{P_3.s}\dot{\omega}_9 - r_{P_3.h}\dot{\omega}_{11} && \text{(43e)}\\
0 &= 2r_{P_3.p}\dot{\omega}_{12} + r_{P_3.s}\dot{\omega}_9 - r_{P_3.h}\dot{\omega}_{11} && \text{(43f)}\\
0 &= \dot{\omega}_{17} - \dot{\omega}_{18} && \text{(43g)}\\
0 &= \dot{\omega}_{19} && \text{(43h)}\\
0 &= \dot{\omega}_{20}. && \text{(43i)}
\end{aligned}$$

In mode M_2, M_3 we need to replace again (39) and (40) by

$$\begin{aligned}
0 &= \dot{\omega}_{13} - \dot{\omega}_{14}, && \text{(44a)}\\
0 &= \dot{\omega}_{15} - \dot{\omega}_{16}. && \text{(44b)}
\end{aligned}$$

A.3.3 Overdeterminded strangeness-free form

As described in [19], a numerically more stable form for modeling multibody systems is the overdetermined form. It is strangeness-free and contains the holonomic constraints and their first and second derivatives.

Therefore, the constraints and its derivatives are added to the equations of motion as given in the previous sections. The resulting system of DAEs in mode M_1 has 87 equations in 69 variables.

A.3.4 Gear-Gupta-Leimkuhler form (GGL)

The Gear-Gupta-Leimkuhler form, see [20], combines holonomic and constraints on vleocity level. It does not result in an overdetermined system, because the constraints on velocity level are added to the equations of motion via further Lagrange multipliers η.

This results in a bigger state vector. For example in mode M_1, the new state x has the following form

$$x = \begin{bmatrix} \varphi \\ \omega \\ \lambda \\ \eta \end{bmatrix} \in \mathbb{R}^{78},\ \varphi, \omega \text{ as before, } \kappa = \begin{bmatrix} \kappa_{P_1.1} \\ \kappa_{P_1.2} \\ \kappa_{P_2.1} \\ \kappa_{P_2.2} \\ \kappa_{P_3.1} \\ \kappa_{P_3.2} \\ \hat{\kappa} \end{bmatrix},\ \hat{\kappa} = \begin{bmatrix} \kappa_{k_3} \\ \kappa_{b_1} \\ \kappa_{b_2} \end{bmatrix} \text{ with } \kappa \in \{\lambda, \eta\}. \tag{45}$$

The DAE in mode M_1 accordingly contains the already given equations (35), (36), (37), (38) and (41) together with equations (34) modified to the GGL form.

The equations (34) change for those components that are constrained. Again as an example, the modified equations for mode M_1 are listed. In all the other modes the equations must be adjusted accordingly

$$\begin{aligned}
0 &= -\dot{\varphi}_1 + \dot{\omega}_1 + (r_{P_1.s}\eta_{P_1.1} - r_{P_1.s}\eta_{P_1.2})/J_{P_1.s} \\
0 &= -\dot{\varphi}_2 + \dot{\omega}_2 - 2r_{P_1.t}\eta_{P_1.1}/J_{P_1.t} \\
0 &= -\dot{\varphi}_3 + \dot{\omega}_3 + (r_{P_1.h}\eta_{P_1.1} + r_{P_1.h}\eta_{P_1.2})/J_{P_1.h} \\
0 &= -\dot{\varphi}_4 + \dot{\omega}_4 - 2r_{P_1.p}\eta_{P_1.2}/J_{P_1.p} \\
0 &= -\dot{\varphi}_5 + \dot{\omega}_5 + (r_{P_2.s}\eta_{P_2.1} - r_{P_2.s}\eta_{P_2.2})/J_{P_2.s} \\
0 &= -\dot{\varphi}_6 + \dot{\omega}_6 - 2r_{P_2.t}\eta_{P_2.1}/J_{P_2.t} \\
0 &= -\dot{\varphi}_7 + \dot{\omega}_7 + (r_{P_2.h}\eta_{P_2.1} + r_{P_2.h}\eta_{P_2.2})/J_{P_2.h} \\
0 &= -\dot{\varphi}_8 + \dot{\omega}_8 - 2r_{P_2.p}\eta_{P_2.2}/J_{P_2.p} \\
0 &= -\dot{\varphi}_9 + \dot{\omega}_9 + (r_{P_3.s}\eta_{P_3.1} - r_{P_3.s}\eta_{P_3.1})/J_{P_3.s} \\
0 &= -\dot{\varphi}_{10} + \dot{\omega}_{10} - 2r_{P_3,t}\eta_{P_3.1}/J_{P_3.t} \\
0 &= -\dot{\varphi}_{11} + \dot{\omega}_{11} + (r_{P_3.h}\eta_{P_3.1} + r_{P_3.h}\eta_{P_3.2})/J_{P_3.h} \\
0 &= -\dot{\varphi}_{12} + \dot{\omega}_{12} - 2r_{P_3,p}\eta_{P_3.2}/J_{P_3.p} \\
0 &= -\dot{\varphi}_{17} + \dot{\omega}_{17} - \eta_{k_3}/J_{k_3} \\
0 &= -\dot{\varphi}_{18} + \dot{\omega}_{18} + \eta_{k_3}/J_{k_3} \\
0 &= -\dot{\varphi}_{19} + \dot{\omega}_{19} - \eta_{b_1}/J_{b_1} \\
0 &= -\dot{\varphi}_{21} + \dot{\omega}_{21} - \eta_{b_2}/J_{b_2}.
\end{aligned}$$

A.3.5 Minimal extension to a strangeness-free form

To achieve a minimal extension to a strangeness free model as described in [35], some transformations in the different modes are necessary. As the minimal extensions differ from mode to mode they will be given for each mode seperately.

The models for the free wheels without mode changes as treated here and the shaft nodes must be given in a form with regular mass matrix. Therefore, the angles $\varphi_{22}, \varphi_{23}, \varphi_{24}$ describe the free wheels, but the corresponding angular velocities ω_i are calculated directly via $\omega_i = f_i(t, x, \dot{x})$ and $\dot{\varphi}_i = f_i(t, x, \dot{x})$, $i = 22, 23, 24$. Similarly, this is done for the angles of the shaft nodes $\varphi_{25}, \ldots, \varphi_{30}$.

Minimal extensions for a model are not unique. For the NAG we have chosen the one that was easiest to implement. As before, the model equations of mode M_1 are given in detail, whereas in the further modes just the differences are presented. Again the same scheme as in A.3.1 is used.

Mode M_1 In mode M_1 there are nine constraints (38) resulting in an extension of nine new variables. We have chosen v_i, w_i, as:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} := \begin{bmatrix} \dot{\varphi}_2 \\ \dot{\varphi}_4 \\ \dot{\varphi}_6 \\ \dot{\varphi}_8 \\ \dot{\varphi}_{10} \\ \dot{\varphi}_{12} \\ \dot{\varphi}_{17} \\ \dot{\varphi}_{19} \\ \dot{\varphi}_{20} \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix} := \begin{bmatrix} \dot{\omega}_2 \\ \dot{\omega}_4 \\ \dot{\omega}_6 \\ \dot{\omega}_8 \\ \dot{\omega}_{10} \\ \dot{\omega}_{12} \\ \dot{\omega}_{17} \\ \dot{\omega}_{19} \\ \dot{\omega}_{20} \end{bmatrix}. \tag{46}$$

Of the first 30 equations (34) of the EoM form, those containing the new variables (46) must be changed. This results in

$$0 = -v_i + \omega_j, \quad i = 1, \ldots, 9$$

for the pairs

$$(i, j) \in \{(1,2), (2,4), (3,6), (4,8), (5,10), (6,12), (7,17), (8,19), (9,20)\}.$$

Additionally the equations for free wheels and shaft nodes must be adjusted,

$$0 = -\dot{\varphi}_{22} + \omega_{27} - \frac{D}{C}(\varphi_{22} - \varphi_{27}) \tag{47a}$$

$$0 = -\dot{\varphi}_{23} + \omega_{29} - \frac{D}{C}(\varphi_{23} - \varphi_{29}) \tag{47b}$$

$$0 = -\dot{\varphi}_{24} + \omega_{30} - \frac{D}{C}(\varphi_{24} - \varphi_{30}) \tag{47c}$$

$$0 = -\dot{\varphi}_{25} + \frac{1}{2}(\omega_3 + \omega_{15}) + \frac{D}{2C}\left(\frac{M_M(t)}{D} - 2\varphi_{25} + \varphi_3 + \varphi_{15}\right) \tag{47d}$$

$$0 = -\dot{\varphi}_{26} + \frac{1}{3}(\omega_2 + \omega_{11} + \omega_{14}) - \frac{D}{3C}(3\varphi_{26} - \varphi_2 - \varphi_{11} - \varphi_{14}) \tag{47e}$$

$$0 = -\dot{\varphi}_{27} + \frac{1}{4}(\omega_1 + \omega_{13} + \omega_{19} + \omega_{22}) - \frac{D}{4C}(4\varphi_{27} - \varphi_1 - \varphi_{13} - \varphi_{19} - \varphi_{22}) \tag{47f}$$

$$0 = -\dot{\varphi}_{28} + \frac{1}{4}(\omega_7 + \omega_{10} + \omega_{16} + \omega_{21}) - \frac{D}{4C}(4\varphi_{28} - \varphi_7 - \varphi_{10} - \varphi_{16} - \varphi_{21}) \tag{47g}$$

$$0 = -\dot{\varphi}_{29} + \frac{1}{4}(\omega_5 + \omega_{17} + \omega_{20} + \omega_{23}) - \frac{D}{4C}(4\varphi_{29} - \varphi_5 - \varphi_{17} - \varphi_{20} - \varphi_{23}) \tag{47h}$$

$$0 = -\dot{\varphi}_{30} + \frac{1}{3}(\omega_9 + \omega_{18} + \omega_{24}) - \frac{D}{3C}(3\varphi_{30} - \varphi_9 - \varphi_{18} - \varphi_{24}), \tag{47i}$$

with $\omega_{22}, \ldots, \omega_{30}$ computed as stated above.

Similarily, the equations for angular velocities for planetary gear sets (35), clutches and brakes (36a) to (36f) must be adapted. Again, just those containing new variables are changed. For the planetary gear sets these are

$$
\begin{array}{ll}
P_1 & \left\{ \begin{array}{rcl} 0 &=& -J_{P_1.t} w_1 - 2r_{P_1.t}\lambda_{P_1.1} - D(\varphi_2 - \varphi_{26}) - C(\omega_2 - \omega_{26}) \\ 0 &=& -J_{P_1.p} w_2 - 2r_{P_1.p}\lambda_{P_1.2} \end{array} \right. \\
P_2 & \left\{ \begin{array}{rcl} 0 &=& -J_{P_2.t} w_3 - 2r_{P_2.t}\lambda_{P_2.1} - M_B(t) \quad \text{with brake torque } M_B(t) \\ 0 &=& -J_{P_2.p} w_4 - 2r_{P_2.p}\lambda_{P_2.2} \end{array} \right. \\
P_3 & \left\{ \begin{array}{rcl} 0 &=& -J_{P_3.t} w_5 - 2r_{P_3.t}\lambda_{P_3.1} - D(\varphi_{10} - \varphi_{28}) - C(\omega_{10} - \omega_{28}) \\ 0 &=& -J_{P_3.p} w_6 - 2r_{P_3.p}\lambda_{P_3.2}, \end{array} \right.
\end{array}
$$

and for the clutches k_1, k_2, k_3 and brakes b_1, b_2, b_3

$$
\begin{array}{lrcl}
k_3 & 0 &=& -J_{k_3} w_7 - D(\varphi_{17} - \varphi_{29}) - C(\omega_{17} - \omega_{29}) - \lambda_{k_3} \\
b_1 & 0 &=& -J_{b_1} w_8 - D(\varphi_{19} - \varphi_{27}) - C(\omega_{19} - \omega_{27}) - \lambda_{b_1} \\
b_2 & 0 &=& -J_{b_2} w_9 - D(\varphi_{20} - \varphi_{29}) - C(\omega_{20} - \omega_{29}) - \lambda_{b_2} \quad .
\end{array}
$$

The equations (36g) to (36h) and (37) no longer belong to the system.

The minimal extension contains all possible constraints, i.e. which are added without changes to the system. In mode M_1 we add (38), (41) and (43).

The state vector x for the minimal extension in mode M_1 according to (33) then is

$$
x = \begin{bmatrix} \varphi \\ \omega \\ \lambda \\ v \\ w \end{bmatrix} \in \mathbb{R}^{78}, \ \varphi, \lambda \text{ as before and } \omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_{21} \end{bmatrix}, \ v = \begin{bmatrix} v_1 \\ \vdots \\ v_9 \end{bmatrix}, \ w = \begin{bmatrix} w_1 \\ \vdots \\ w_9 \end{bmatrix}.
$$

Modes M_{12} and M_{21} As for the EoM form, in comparison to mode M_1, equation (48) changes to

$$
0 = -J_{b_1}\dot{\omega}_{19} - D(\varphi_{19} - \varphi_{27}) - C(\omega_{19} - \omega_{27}) - M_{R,b_1}
$$

and additionally we have the differential equation

$$
0 = -\dot{\varphi}_{19} + \omega_{19}.
$$

As before, the constraint (38h) is omitted and also the Lagrange multiplier λ_{b_1}. Therefore the corresponding derivatives (41h), (43h) are not part of the model either and the resulting minimal extension has the form

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} := \begin{bmatrix} \dot{\varphi}_2 \\ \dot{\varphi}_4 \\ \dot{\varphi}_6 \\ \dot{\varphi}_8 \\ \dot{\varphi}_{10} \\ \dot{\varphi}_{12} \\ \dot{\varphi}_{17} \\ \dot{\varphi}_{20} \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} := \begin{bmatrix} \dot{\omega}_2 \\ \dot{\omega}_4 \\ \dot{\omega}_6 \\ \dot{\omega}_8 \\ \dot{\omega}_{10} \\ \dot{\omega}_{12} \\ \dot{\omega}_{17} \\ \dot{\omega}_{20} \end{bmatrix}.
$$

Here the numbering of the newly introduced variables changes but it will not be given here explicitly. Then we have a new state vector $x \in \mathbb{R}^{75}$.

Mode M_2 In mode M_2 the state vector has dimension $x \in \mathbb{R}^{78}$. Again the Lagrange multiplier λ_{k_1} and the constraint on clutch k_1 belong to the system. In comparison to the modes M_{12}, M_{21} the equations (36a) must be changed to

$$\begin{aligned} 0 &= -J_{k_1} w_7 - D(\varphi_{13} - \varphi_{27}) - C(\omega_{13} - \omega_{27}) - \lambda_{k_1}, \\ 0 &= -J_{k_1} \dot{\omega}_{14} - D(\varphi_{14} - \varphi_{26}) - C(\omega_{14} - \omega_{26}) + \lambda_{k_1}. \end{aligned}$$

The new constraint is

$$0 = \varphi_{13} - \varphi_{14} - \Delta_{k_1},$$

and with correspondingly minimally extended variables we get

$$0 = -v_7 + \omega_{13}.$$

The minimal extension for mode M_2, omitting again renumbering of variables, then is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} := \begin{bmatrix} \dot{\varphi}_2 \\ \dot{\varphi}_4 \\ \dot{\varphi}_6 \\ \dot{\varphi}_8 \\ \dot{\varphi}_{10} \\ \dot{\varphi}_{12} \\ \dot{\varphi}_{13} \\ \dot{\varphi}_{17} \\ \dot{\varphi}_{20} \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix} := \begin{bmatrix} \dot{\omega}_2 \\ \dot{\omega}_4 \\ \dot{\omega}_6 \\ \dot{\omega}_8 \\ \dot{\omega}_{10} \\ \dot{\omega}_{12} \\ \dot{\varphi}_{13} \\ \dot{\omega}_{17} \\ \dot{\omega}_{20} \end{bmatrix}.$$

Modes M_{23}, M_{32} As before, compared to mode M_2 the equations (48), (36c) change to

$$\begin{aligned} 0 &= -J_{k_3} \dot{\omega}_{17} - D(\varphi_{17} - \varphi_{29}) - C(\omega_{17} - \omega_{29}) - M_{R,k_3}, \\ 0 &= -J_{k_3} \dot{\omega}_{18} - D(\varphi_{18} - \varphi_{30}) - C(\omega_{18} - \omega_{30}) + M_{R,k_3}, \end{aligned}$$

and we have again the differential equation

$$0 = -\dot{\varphi}_{17} + \omega_{17}.$$

The constraint (38g) and its Lagrange multiplier λ_{k_3} are omitted and also its derivatives (41g), (43g). The minimal extension then has the form

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} := \begin{bmatrix} \dot{\varphi}_2 \\ \dot{\varphi}_4 \\ \dot{\varphi}_6 \\ \dot{\varphi}_8 \\ \dot{\varphi}_{10} \\ \dot{\varphi}_{12} \\ \dot{\varphi}_{13} \\ \dot{\varphi}_{20} \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix} := \begin{bmatrix} \dot{\omega}_2 \\ \dot{\omega}_4 \\ \dot{\omega}_6 \\ \dot{\omega}_8 \\ \dot{\omega}_{10} \\ \dot{\omega}_{12} \\ \dot{\omega}_{13} \\ \dot{\omega}_{20} \end{bmatrix}.$$

The state vector x in these modes is $x \in \mathbb{R}^{75}$.

Mode M_3 In mode M_3 the state vector is of dimension $x \in \mathbb{R}^{78}$. The Lagrange multiplier λ_{k_2} returns to the DAE system and connects the constraint of clutch k_2. In comparison to the modes M_{23}, M_{32} the equations (36b) are changed to

$$\begin{aligned} 0 &= -J_{k_2} w_8 - D(\varphi_{15} - \varphi_{25}) - C(\omega_{15} - \omega_{25}) - \lambda_{k_2}, \\ 0 &= -J_{k_2} \dot{\omega}_{16} - D(\varphi_{16} - \varphi_{28}) - C(\omega_{16} - \omega_{28}) + \lambda_{k_2}, \end{aligned}$$

the new constraint is added

$$0 = \varphi_{15} - \varphi_{16} - \Delta_{k_2},$$

and finally with newly minimal extended variables we get

$$0 = -v_8 + \omega_{15}.$$

The minimal extension in mode M_3 without explicitly stating the renumbering then is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} := \begin{bmatrix} \dot{\varphi}_2 \\ \dot{\varphi}_4 \\ \dot{\varphi}_6 \\ \dot{\varphi}_8 \\ \dot{\varphi}_{10} \\ \dot{\varphi}_{12} \\ \dot{\varphi}_{13} \\ \dot{\varphi}_{15} \\ \dot{\varphi}_{20} \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix} := \begin{bmatrix} \dot{\omega}_2 \\ \dot{\omega}_4 \\ \dot{\omega}_6 \\ \dot{\omega}_8 \\ \dot{\omega}_{10} \\ \dot{\omega}_{12} \\ \dot{\varphi}_{13} \\ \dot{\omega}_{15} \\ \dot{\omega}_{20} \end{bmatrix}.$$

A.4 Mode transition conditions for the NAG

Joining the models for each mode and the connection between these will result in the hybrid system of the gearbox NAG. These connections consist of mode transition conditions and transition functions to continue the solution in the following mode.

The mode transition conditions for the mode changes occuring in our simulations are given in Table 5.

For the hybrid multibody system NAG, as described in [24], the mode transfer functions are quite easy as mainly new Lagrange multipliers λ_i must be introduced or omitted. The Gear-Gupta-Leimkuhler form additionally needs initial values for the η_i coupling the constraints on velocity level to the DAE system. Similarily, for the minimal extension the extended variables must be initialized for each mode.

In the following, the initial values for the Lagrange multipliers λ are given, where the switching time is $\hat{t}$. In the Gear-Gupta-Leimkuhler form the η_i are trivially initially zero and the initial values for the minimal extension are obtained trivially as well and not given here.

In mode M_1, the Lagrange multiplier λ_{b_1} on the constraint for brake b_1 is initialized by $\lambda_{b_1} = M_{R,b_1}(\hat{t})$.

In mode M_2 the Lagrange multiplier λ_{k_1} on the constraint for clutch k_1 is added, when coming from mode M_1. Its initial value is $\lambda_{k_1} = M_{R,k_1}(\hat{t})$.

From the other direction, to mode M_2 from M_3, a Lagrange multiplier λ_{k_3} for the constraint on clutch k_3 is initialized by $\lambda_{k_3} = M_{R,k_3}(\hat{t})$.

In mode M_3 the Lagrange multiplier λ_{k_2} on the constraint on clutch k_2 is finally initialized by $\lambda_{k_2} = M_{R,k_2}(\hat{t})$.

Switch	Mode transition condition and meaning
$M_1 \to M_{12}$	$S_1^1(t, x_1, \dot{x}_1) = M_{R,b_1} - (D(\varphi_{19} - \varphi_{27}) + C(\omega_{19} - \omega_{27}))$ Meaning: Opening of brake b_1
$M_{12} \to M_2$	$S_1^{12}(t, x_{12}, \dot{x}_{12}) = \omega_{13} - \omega_{14}$ if $M_{R,k_1} > D(\varphi_{13} - \varphi_{27} + \varphi_{14} - \varphi_{26}) + C(\omega_{13} - \omega_{27} + \omega_{14} - \omega_{26})$ Meaning: Closing of clutch k_1
$M_2 \to M_{23}$	$S_1^2(t, x_2, \dot{x}_2) = M_{R,k_3} - (D(\varphi_{17} - \varphi_{29} + \varphi_{18} - \varphi_{30}) + C(\omega_{17} - \omega_{29} + \omega_{18} - \omega_{30}))$ Meaning: Opening of clutch k_3
$M_2 \to M_{21}$	$S_2^2(t, x_2, \dot{x}_2) = M_{R,k_1} - (D(\varphi_{13} - \varphi_{27} + \varphi_{14} - \varphi_{26}) + C(\omega_{13} - \omega_{27} + \omega_{14} - \omega_{26}))$ Meaning: Opening of clutch k_1
$M_{23} \to M_3$	$S_1^{23}(t, x_{23}, \dot{x}_{23}) = \omega_{15} - \omega_{16}$ if $M_{R,k_2} > D(\varphi_{15} - \varphi_{25} + \varphi_{16} - \varphi_{28}) + C(\omega_{15} - \omega_{25} + \omega_{16} - \omega_{28})$ Meaning: Closing of clutch k_2
$M_3 \to M_{34}$	$S_1^3(t, x_3, \dot{x}_3) = M_{R,b_2} - (D(\varphi_{20} - \varphi_{29}) + C(\omega_{20} - \omega_{29}))$ Meaning: Opening of brake b_2
$M_3 \to M_{32}$	$S_2^3(t, x_3, \dot{x}_3) = M_{R,k_2} - (D(\varphi_{15} - \varphi_{25} + \varphi_{16} - \varphi_{28}) + C(\omega_{15} - \omega_{25} + \omega_{16} - \omega_{28}))$ Meaning: Opening of clutch k_2
$M_{21} \to M_1$	$S_1^{21}(t, x_{21}, \dot{x}_{21}) = \omega_{19}$ if $M_{R,b_1} > D(\varphi_{19} - \varphi_{27}) + C(\omega_{19} - \omega_{27})$ Meaning: Closing of brake b_1

Table 5: Mode transition condition for the simulation of the NAG

A.5 Simulation settings

For comparison reasons a standard parameter and input specification, based on a DaimlerChrysler simulation with the in-house code ASIM, was used for the simulation of the NAG with different hybrid system models and different solvers.

Simulation procedure

- The time interval $[0, 7]$ was used.
- The start of the simulation was done in gear 1 and mode M_1 at time $t_0 = 0$ at rest. All angles, velocities and accelerations were initialized to zero.

- At time $t_1 = 2$, externally a switch to gear 2 is started. When gear 2 and, therefore, mode M_2 is reached, it is integrated until time $t_2 = 4$ At this time it is switched to gear 3. There in gear 3 and mode M_3 the integration is continued until time $t_3 = 6$, where is finally switched down to gear 2.
- Engine torque and brake torque are fixed according to the specifications of the ASIM simulation.

Pressure and interfaces to the hydraulics For the brakes and clutches a fixed loading and opening interval was set as follows

Component	*Parameter*	*Value*
Clutch	Loading time	0.2
	Opening time	0.1
Brake	Loading time	0.2
	Opening time	0.1

Table 6: Loading and opening times for hydraulic components of the NAG

In Figure 8 loading and opening of clutch k_2 for the mode chain $M_2, M_{23}, M_3, M_{32}, M_2$ is presented.

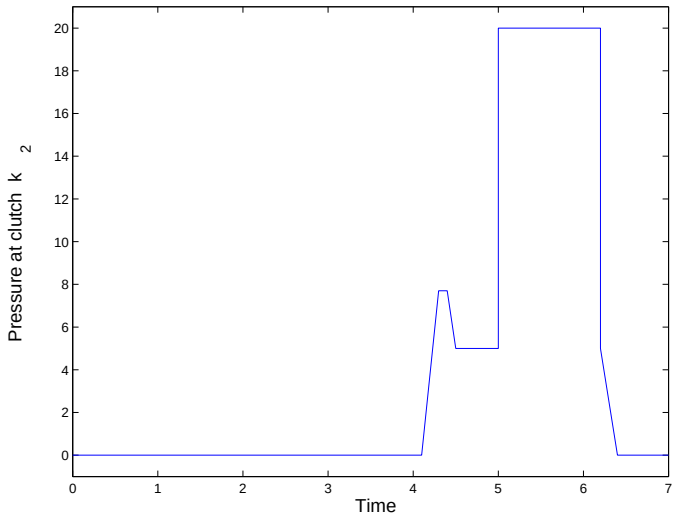


Figure 8: Pressure at clutch k_2

The friction torque on the clutches and brakes is modeled by linear and constant damping as

$$\begin{aligned} M_R(t, \omega_1, \omega_2) &= C_1 F_R(t, \omega_1, \omega_2) \\ &= C_1 p(t) \left(\frac{2\theta(\omega_1 - \omega_2)}{1 + (\theta(\omega_1 - \omega_2))^2} + \frac{2}{\pi} \text{atan}\left(\theta(\omega_1 - \omega_2)\right) \right). \end{aligned}$$

For a nonzero difference of angular velocities, this means an almost constant torque $M_R = \pm C_1 p(t)$ with component dependent constant C_1. If a zero cross-

ing occurs, then the duration of the change of the friction torque M_R depends on the parameter θ.

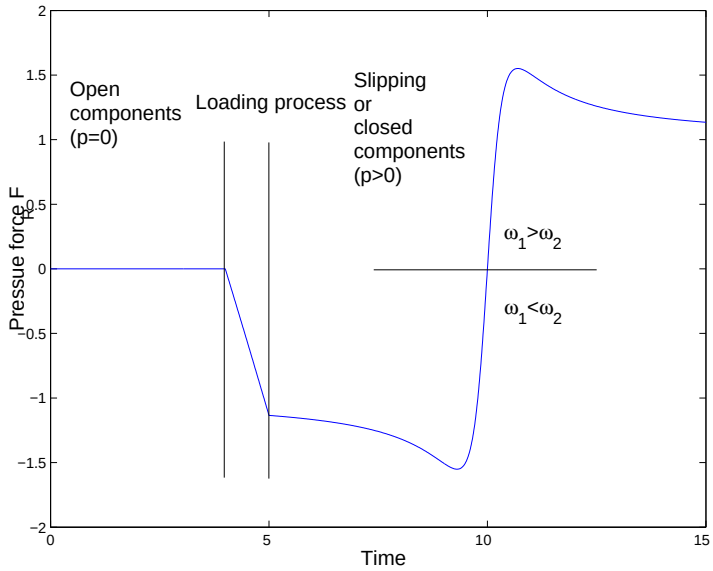


Figure 9: Friction torque on clutches and brakes

In Figure 9 the pressure evolution is given

$$p(t) = \begin{cases} 0 & \text{for } t < 4 \\ t-4 & \text{for } 4 \leq t \leq 5 \\ 1 & \text{for } t > 5 \end{cases},$$

where a zero crossing of ω_1, ω_2 occurs at $t = 10$ as an example for a loading of a clutch.