Technische Universität Berlin

FAKULTÄT II
MATHEMATIK UND
NATURWISSENSCHAFTEN
Institut für Mathematik

Classes of Cycle Bases

by

Christian Liebchen and Romeo Rizzi

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Abstract

In the last years, new variants of the minimum cycle basis (MCB) problem and new classes of cycle bases have been introduced, as motivated by several applications from disparate areas of scientific and technological inquiries. At present, the complexity status of the MCB problem has been settled only for undirected, directed, and strictly fundamental cycle bases.

In this paper, we offer an unitary classification accommodating these 3 classes and further including the following 4 relevant classes: 2-bases (or planar bases), weakly fundamental cycle bases, totally unimodular cycle bases, and integral cycle bases. The classification is complete in that, for each ordered pair (A, B) of classes considered, we either prove that $A \subseteq B$ holds for every graph or provide a counterexample graph for which $A \not\subseteq B$. The seven notions of cycle bases are distinct (either $A \not\subseteq B$ or $B \not\subseteq A$ is exhibited for each pair (A, B)).

All counterexamples proposed have been designed to be ultimately effective in separating the various algorithmic variants of the MCB problem naturally associated to each one of these seven classes. We even provide a linear time algorithm for computing a minimum 2-basis of a graph. Finally, notice that the resolution of the complexity status of some of the remaining three classes would have an immediate impact on practical applications, as for instance in periodic railway timetabling, only integral cycle bases are of direct use.

1 Introduction

Currently, cycle bases are a hot topic in discrete mathematics. In particular, the minimum cycle basis problem recently has attracted many researchers. To a large extent, this is motivated by the fact that cycle bases serve as input of algorithms to solve several practical applications. Most often, the computation time of the algorithm for the application increases with the weight of the cycle basis that it receives as input. The practical applications arise in electrical engineering (Bollobás [4]), chemistry (Gleiss [9]), and planning of public transportation (Liebchen and Peeters [20]). For further fields of applications, we refer to the numerous references in [3, 6, 8, 14].

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In 2002, Golynski and Horton [10] proposed a speed-up of Horton's classical $O(m^3n)$-algorithm [14]. In 2004, there were two papers that presented a completely different approach to computing a minimum cycle basis of an undirected graph ([3, 17]). The $O(m^2n + mn^2 \log n)$-algorithm of Kavitha et al. constitutes the fastest known deterministic algorithm for general graphs.

Even more recently, the minimum cycle basis problem has also been studied for other classes of cycle bases. Galbiati and Amaldi [8] and Amaldi et al. [1] investigated the minimum strictly fundamental cycle basis problem. For cycle bases of directed graphs, Kavitha and Mehlhorn [16] presented the first polynomial-time algorithm. Liebchen and Rizzi [21] presented a conceptually very simple $\tilde{O}(m^{3.376}n)$ algorithm. This has already been improved to $O(m^3n + m^2n^2 \log n)$ ([12]).

With this paper, we open new lines of research for the minimum cycle basis (MCB) problem. In addition to strictly fundamental cycle bases, undirected cycle bases, and directed cycle bases, we propose to compute an MCB also among the classes of 2-bases (or planar bases), weakly fundamental cycle bases, totally unimodular cycle bases, and integral cycle bases.

We establish several implications between these classes in Sections 3 and 4. Complementary results are given in Section 5, where we provide graphs from which we conclude that no further implications hold in general. Surprisingly, the latter takes more effort and some of these constructions are not trivial. In any case, in each of our examples we could restrain ourselves to graphs that have a unique minimum cycle basis. This way we identify several new variants of the minimum cycle basis problem. More precisely, for general graphs, computing a minimum cycle basis among a certain class of cycle bases is different from computing an MCB among *any* of the other classes. In Section 6 we present a linear time algorithm for computing a minimum 2-basis of a graph.

Notice that the refined classification of cycle bases is of strong relevance for practical applications. For instance, in periodic railway timetabling, only integral cycle bases are applicable ([20, 19]). Unfortunately, for this particular class of cycle bases the complexity of the corresponding MCB problem is still open. It is only known that these do not form a matroid ([21]). Moreover, the only complexity result for a subclass of integral cycle bases is the NP-hardness result for strictly fundamental cycle bases that was obtained by Deo et al. [6].

With weakly fundamental cycle bases and totally unimodular cycle bases we consider two subclasses of integral cycle bases that still generalize strictly fundamental cycle bases. As for any two classes of cycle bases we establish that their corresponding MCB problems differ, any answer to the complexity status of the MCB problem restricted to either weakly fundamental or totally unimodular cycle bases could already be regarded as a better "approximation" of the complexity of the minimum integral cycle basis problem.

2 Notation

In a graph $G = (V, E)$, an *Eulerian subgraph* is a set of edges $F \subseteq E$ such that every node in V is incident with an even number of edges in F. A *circuit* is a non-empty and connected Eulerian subgraph of degree at most 2.

In a digraph $D = (V, A)$, a *circulation* is a weighting $w : A \mapsto \mathbb{Q}$ such that $\sum_{a \in \delta^-(v)} w(a) = \sum_{a \in \delta^+(v)} w(a)$ holds for every node v in V.

Let $D = (V, A)$ denote a directed graph. As usual, we define $n := |V|$ and $m := |A|$. An *oriented circuit* $C = C^+ \dot\cup C^- \subseteq A$ of D consists of *forward arcs* C^+ and *backward arcs* C^-, such that reorienting all arcs in C^- results in a directed cycle in which all arcs point into the same direction. The *incidence vector* γ_C *of an oriented circuit* C is a vector in $\{-1, 0, 1\}^A$ with entry 1 (-1) in component a if a is a forward (backward) arc of C. The *cycle space of* D is the vector subspace $\mathcal{C}_D$ of $\mathbb{Q}^A$ that is generated by the incidence vectors of oriented circuits of D. In practice, the cycle space of D can be regarded as the space of the circulations of D. Recall that the dimension of the cycle space of a connected digraph $D = (V, A)$ is the *cyclomatic number* $\nu := |A| - |V| + 1$ ([4]).

Definition 1 (Directed Cycle Basis) *A* directed cycle basis *of a directed graph D is a set of oriented circuits whose incidence vectors form a basis of $\mathcal{C}_D$.*

For a set of oriented circuits the *cycle matrix* Γ has the incidence vectors of the circuits as its columns.

For a directed graph D, we obtain the *underlying undirected graph* $G(D)$ by removing the directions from the arcs A, i.e. $e = \{i, j\} \in E$ if and only if there exists an arc $a = (i, j) \in A$ or an arc $a = (j, i) \in A$. For a set of arcs $A' \subseteq A$ the *projection onto* $G(D)$ is obtained by removing the directions from the arcs A'. The *cycle space* of an undirected graph $G = (V, E)$ is the vector subspace $\mathcal{C}_G$ of $\mathrm{GF}(2)^E$ that is generated by the incidence vectors of the circuits of G. Here, GF(2) denotes the Galois field over $\{0, 1\}$. In practice, the cycle space of G can be regarded as the space of the Eulerian subgraphs (or $\emptyset$-joins) of G.

Definition 2 (Cycle Basis) *A* cycle basis *of an undirected graph G is a set of circuits of G whose incidence vectors form a basis of $\mathcal{C}_G$.*

Throughout this paper we assume any (directed) graph to be 2-connected. This does not impose a limitation because in general the cycle space of a (directed) graph is the direct sum of the cycle spaces of its 2-connected components.

3 Classes of Cycle Bases

Definition 3 (Undirected Cycle Basis) *A set of oriented circuits of a directed graph D is an* undirected cycle basis *of D, if their projections onto $G(D)$ form a cycle basis of $G(D)$.*

Lemma 4 ([20]) *Every undirected cycle basis of a directed graph D is also a directed cycle basis of D.*

Proof. Let $\{C_1, \dots, C_\nu\}$ be an undirected cycle basis of D. Assume it was not a directed cycle basis of D. Then we had $\lambda_1, \dots \lambda_\nu \in \mathbb{Q}$—at least one being nonzero—such that

$$\lambda_1 \gamma_{C_1} + \cdots + \lambda_\nu \gamma_{C_\nu} = \mathbf{0}. \tag{1}$$

We may assume w.l.o.g. that $\lambda_i \in \mathbb{Z}$ for all $i \in \{1, \dots, \nu\}$ and $\gcd(\{\lambda_i \,|\, \lambda_i \neq 0\}) = 1$.

Now, we take the componentwise projection of (1) onto GF(2). Since we scaled the coefficients λ_i such that at least one is an odd integer, this provides a non-trivial linear combination resulting in the all-zero vector. But this contradicts the fact that $\{C_1, \dots, C_\nu\}$ is an undirected cycle basis. □

Definition 5 (Integral Cycle Basis) *A set $B = \{C_1, \dots, C_\nu\}$ of oriented circuits of a directed graph D is an* integral cycle basis *of D, if every oriented circuit C of D can be written as an integer linear combination of circuits in B, i.e.*

$$\exists \lambda_i \in \mathbb{Z} : \; \gamma_C = \lambda_1 \gamma_{C_1} + \cdots + \lambda_\nu \gamma_{C_\nu}. \tag{2}$$

Definition 6 (Totally Unimodular Cycle Basis) *A directed cycle basis $B = \{C_1, \dots, C_\nu\}$ of a directed graph D is a* totally unimodular cycle basis *of D, if its cycle matrix is totally unimodular.*

The following lemma is a direct consequence of the fact (cfr. [27]) that the polyhedron $\{Ax = b\}$ is integral whenever b is integral and A is totally unimodular.

Lemma 7 *Every totally unimodular cycle basis is an integral cycle basis.*

Theorem 8 *(Theorem 19.3 in [27]) A matrix Γ is totally unimodular, if and only if every subset of its columns can be partitioned into two sets such that the sum of the columns in one set minus the sum of the columns in the other set provides a vector with entries only in $\{-1, 0, +1\}$.*

The following definitions of special classes of cycle bases are formulated in terms of undirected graphs. Nevertheless, they immediately apply to directed graphs, too.

Definition 9 (Weakly Fundamental Cycle Basis (Whitney [31])) *A set $B = \{C_1, \dots, C_\nu\}$ of circuits of an undirected graph G is a* weakly fundamental cycle basis *of G if there exists some permutation σ such that*

$$C_{\sigma(i)} \setminus (C_{\sigma(1)} \cup \cdots \cup C_{\sigma(i-1)}) \neq \emptyset, \; \forall i = 2, \dots, \nu. \tag{3}$$

Lemma 10 *If a cycle basis B of a graph $G = (V, E)$ is not weakly fundamental then there exists an edge $e \in E$ such that it is contained in at least three circuits of B.*

Proof. Let σ denote a permutation that satisfies Condition (3) for the circuits $B_i := \{C_{\sigma(1)}, \ldots, C_{\sigma(i)}\}$, but no circuit $C \in B \setminus B_i$ can extend σ in accordance with Condition (3). In particular,

$$\forall C \in B \setminus B_i,\ \forall e \in C:\ |\{C' \in B \setminus B_i \,|\, e \in C'\}| \geq 2. \tag{4}$$

Assume no edge e in $\bigcup_{C \in B \setminus B_i} C$ is contained in at least three circuits of B. Then Inequality (4) was tight for every edge. But then, $\sum_{C \in B \setminus B_i} C = \mathbf{0}$ (over GF(2)), contradicting the fact that B is a cycle basis of G. □

Definition 11 (2-Basis) *A set B of circuits of an undirected graph G is called a* 2-basis *if B is a cycle basis of G and every edge is contained in at most two circuits of B.*

What we call a 2-basis has also been called a *simple basis* by Diestel [7] or a *planar basis* by Gleiss [9]. The term 2-basis reflects more closely the definition and can e.g. be found in the books [24, 25]. We also borrow from [24] (Theorem 2.4.5) the following statement of MacLane's theorem.

Theorem 12 (MacLane [22]) *A 2-connected graph G has a 2-basis, if and only if it is planar. In this case, any 2-basis of G consists of all facial cycles, except one, of some planar embedding of G.*

Clearly, in the above, we can always assume that the missing facial cycle is the one corresponding to the infinite face, since it is well known that any face of a planar embedding can be made to become the infinite face.

Lemma 13 *Every 2-basis is weakly fundamental.*

Proof. Let $B = \{C_1, \ldots, C_\nu\}$ be a 2-basis of G. Consider a planar embedding of G as from Theorem 12. We can always assume that the missing facial cycle is the one corresponding to the infinite face. We hence define σ iteratively by moving from the boundary of the planar embedding of G towards its "center".

Let $C = \{e_1, \ldots, e_k\}$ be the boundary of the infinite face of G. Denote by C_{e_i} the unique circuit in B that contains $e_i \in C$. In the first iteration, we define

$$C_{\sigma(n)} = C_{e_1},\ \ldots,\ C_{\sigma(n-k+1)} = C_{e_k}.$$

Then, we remove the edges of C from G and proceed in the same way for the 2-connected components of the remaining graph. □

Lemma 14 *Every 2-basis is totally unimodular.*

Proof. Let $B = \{C_1, \ldots, C_\nu\}$ be a 2-basis of G and refer to a planar embedding of G as from Theorem 12 and where the missing facial cycle is the one corresponding to the infinite face. Notice that the cycle matrix of B is the incidence matrix of the dual graph with the column corresponding to the node associated to the infinite face removed. The claim follows from a theorem of Poincaré [26] stating that a $\{0, \pm 1\}$-matrix with at most one $+1$ and at most one -1 in every row is totally unimodular. $\square$

Definition 15 (Strictly Fundamental Cycle Basis) *A set B of circuits of an undirected graph G is a* strictly fundamental cycle basis *of G, if there exists some spanning tree $T \subseteq E$ such that $B = \{C_e \,|\, e \in E \setminus T\}$, where C_e denotes the unique circuit in $T \cup \{e\}$.*

The following well-known inclusion is a direct consequence of the fact (cfr. [27]) that network matrices are totally unimodular.

Lemma 16 *Every strictly fundamental cycle basis is totally unimodular.*

Lemma 17 ([28]) *A cycle basis B is strictly fundamental if and only if*

$$\forall C \in B : \; C \setminus \bigcup_{C' \in B \setminus \{C\}} C' \neq \emptyset. \tag{5}$$

Proof. Assume B is strictly fundamental and more precisely $B = \{C_e \,|\, e \in E \setminus T\}$, where T is a spanning tree and C_e denotes the unique circuit in $T \cup \{e\}$ for every e in $E \setminus T$. Then, for every circuit C_e in B, we immediately have

$$\{e\} \in C_e \setminus \bigcup_{C' \in B \setminus \{C_e\}} C'.$$

Conversely, assume Condition 5 holds for a cycle basis B. For every circuit C in B, let e_C be any edge contained in C but in no other circuit of B. Notice that $T := E \setminus \{e_C \; : C \in B\}$ has precisely $m - (m - n + 1) = n - 1$ edges and is acyclic since no circuit of T can be written as linear combination of circuits in B. Hence T is a spanning tree and the claim follows. $\square$

Corollary 18 *Every strictly fundamental cycle basis is weakly fundamental.*

Proof. By Property (5), we may select *any* permutation σ in order to ensure Condition (3). $\square$

Lemma 19 *The rows and columns of a cycle matrix Γ of a weakly fundamental (strictly fundamental) cycle basis can be permuted such that*

$$\Gamma = \begin{pmatrix} L \\ W \end{pmatrix}, \tag{6}$$

where L is a lower triangular (diagonal, resp.) matrix with all entries of the diagonal in $\{-1, +1\}$. In particular, $\det L = \pm 1$.

Proof. This follows immediately from Definition 9 (Lemma 17, resp.). $\square$

4 Characterizations

We provide characterizations for subclasses of directed cycle bases that are based on their cycle matrices. Hence, we start by giving two algebraic lemmata.

Lemma 20 ([19]) *Consider a connected digraph D, with a directed cycle basis B and the corresponding $m \times \nu$ cycle matrix Γ. A subset of ν rows Γ' of Γ is maximal linearly independent, if and only if they correspond to arcs which form the co-tree arcs of some spanning tree.*

Proof. To prove sufficiency, consider a spanning tree T of D, and let $a_1, \dots, a_\nu$ be the co-tree arcs. Consider the cycle matrix Φ with the incidence vector of the unique circuit in $T \cup \{a_i\}$ in column i. As B is a directed cycle basis, there is a unique matrix $R \in \mathbb{Q}^{\nu \times \nu}$ for combining the circuits of Φ, i.e. $\Gamma R = \Phi$. By construction, the restriction of Φ to the co-tree arcs of T is just the identity matrix. Hence, R is the inverse matrix of Γ'.

Conversely, if the arcs that correspond to the $n-1$ rows which are *not* in Γ' contain a circuit C, take its incidence vector γ_C. As B is a directed cycle basis, we have a unique solution $x_C \neq \mathbf{0}$ to the system $\Gamma x = \gamma_C$. Removing both from Γ and from γ_C the $n-1$ rows that contain C, we obtain $\Gamma x_C = \mathbf{0}$. Therefore, x_C provides a non-trivial linear combination of the zero vector, proving Γ' to be singular. □

Lemma 21 ([19]) *Let Γ be the $m \times \nu$ cycle matrix of some directed cycle basis B. Let A_1 and A_2 be two non-singular $\nu \times \nu$ submatrices of Γ. Then we have $\det A_1 = \pm \det A_2$.*

Proof. By Lemma 20, the ν rows of A_1 are the co-tree arcs $a_1, \dots, a_\nu$ of some spanning tree T. Again, consider the cycle matrix Φ with the incidence vector of the unique circuit in $T \cup \{a_i\}$ in column i. We know that Φ is totally unimodular (Schrijver[27]), and we have $\Phi A_1 = \Gamma$, cf. Berge[2]. Considering only the rows of A_2, we obtain $\Phi' A_1 = A_2$. As $\det \Phi' = \pm 1$, and as the det-function is distributive, we get $\det A_1 = \pm \det A_2$. □

The above lemma allows to define the determinant of a directed cycle basis.

Definition 22 (Determinant of a set of ν oriented circuits) *Let B denote a set of ν oriented circuits in a directed graph D. Consider the matrix Γ with the incidence vectors of B as columns. Let Γ' be the $\nu \times \nu$ submatrix of Γ that arises when deleting the arcs of some spanning tree of D. We define*

$$\det B := |\det \Gamma'|.$$

By the following lemma we are able to extend this definition to undirected graphs.

Lemma 23 *Let B be a set of ν circuits of an undirected graph G. For every orientation D of G and any orientation of the circuits of B, the determinant of the resulting set of ν oriented circuits is the same.*

Proof. All we have to notice is that reorienting an arc of a directed graph is equivalent to multiplying the corresponding row of the cycle matrix by -1. Similarly, reorienting an oriented circuit is equivalent to multiplying the corresponding column of the cycle matrix by -1. Of course, none of these operations changes the absolute value of the determinant of the cycle matrix. $\square$

Definition 24 (Determinant of a set of ν circuits) *The* determinant *of a set B of ν circuits in an undirected graph is the determinant of the ν oriented circuits obtained by any orientation of both the arcs of G and the circuits in B.*

Lemma 25 ([19]) *A set of ν oriented circuits is a directed cycle basis, if and only if its determinant is a positive integer.*

Proof. The cycle matrix of a set of ν oriented circuits is integral. $\square$

Lemma 26 ([19]) *A directed cycle basis has odd determinant, if and only if it is an undirected cycle basis.*

Proof. We may compute the determinant of an integral matrix using Laplacian expansion. This involves only addition, subtraction, and multiplication of pairs of integer numbers. In particular, in order to compute the determinant of an integral matrix, the same elementary operations are conducted in the same order as for computing the determinant of a binary matrix of the same dimension.

Now, let Γ be the cycle matrix of an undirected cycle basis of a directed graph. Let Γ' be the projection of Γ onto $\{0, 1\}$. From the above considerations, we deduce that every intermediate value that appears in performing Laplacian expansion for Γ' is the projection onto $\{0, 1\}$ of the corresponding intermediate value that appears in performing Laplacian expansion for Γ. In particular,

$$\det \Gamma' = 0 \Leftrightarrow \det \Gamma = 2k, \text{ for some } k \in \mathbb{Z}.$$

$\square$

Lemma 27 ([19]) *A directed cycle basis has determinant one, if and only if it is an integral cycle basis.*

Proof. Let B be a directed cycle basis. For the systems of linear equations that we are going to solve, we may consider only ν linear independent rows. By Lemma 20, such rows are the co-tree arcs of some spanning tree T. For this proof, we denote by Γ the non-singular square submatrix of the cycle matrix of B that is induced by the rows that correspond to the elements in $A \setminus T$. The same applies for the incidence vector of an arbitrary oriented circuit C.

B is integral, if and only if for every cycle C of D there exists a vector $x \in \mathbb{Z}^\nu$ such that

$$\Gamma x = \gamma_C, \quad \text{i.e.} \quad x = \Gamma^{-1}\gamma_C. \tag{7}$$

It is known that an integral matrix has determinant ± 1 if and only if its inverse is integral, too (e.g. Theorem 4.3 in [27]). Hence, if B has determinant one, then B is integral because γ_C is always integral. In turn, if B is integral, Equation (7) holds in particular for any $\gamma_C = e_i$. But this implies Γ^{-1} to be integral. □

Corollary 28 ([20]) *Every integral cycle basis is an undirected cycle basis.*

Proof. This follows immediately from Lemma 27 and Lemma 26. □

Corollary 29 *Let B be a totally unimodular cycle basis of a directed graph D. Then every oriented circuit C of D can be written as a $\{-1, 0, +1\}$ linear combination of the circuits in B.*

Proof. By Corollary 28, B is also an undirected cycle basis. Denote by G the underlying undirected graph of D. The absolute value operator $|\cdot|$ can be generalized to vectors in the obvious way. Under this convention, $|\gamma_C|$ is the incidence vector of the projection of C onto G.

As B is an undirected cycle basis, there exists a set I of indices of the set of basic circuits whose GF(2) sum provides $|\gamma_C|$. Consider the columns of the cycle matrix Γ of B indexed by I. We know that there exists at least one partition of I that satisfies the condition of Theorem 8 and provides the vector $y \in \{-1, 0, +1\}^m$.

Notice that $|y|$ is obtained as GF(2) sum of the columns of B indexed by I, whence $|y| = |\gamma_C|$. As C is a circuit, changing the orientation of a proper subset of its edges would leave the cycle space. Hence, $y = \pm\gamma_C$. In case of $y = -\gamma_C$, the unique $\{-1, 0, +1\}$ linear combination of C is precisely the negative of the combination of y. □

Corollary 30 ([20]) *Every weakly fundamental cycle basis is an integral cycle basis.*

Proof. This follows immediately from Lemma 27 and Lemma 19. □

As we can define a spanning tree for every graph, every graph has a strictly fundamental cycle basis. From the inclusions that we identified and that are visualized in Fig. 1, we conclude that every graph has some cycle basis that is part of every class of cycle bases—occasionally with the exception of 2-bases, cf. Theorem 12.

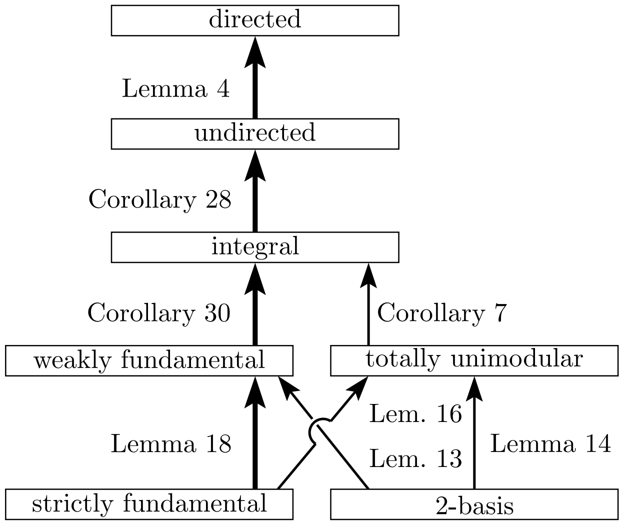


Figure 1: Relationships between classes of directed cycle bases.

5 Examples of Cycle Bases

We present examples that are complementary to the inclusions being visualized in Fig. 1. More specifically, consider a pair (I, J) of subclasses of (directed) cycle bases, for which the arc (I, J) is not in the transitive closure of the graph displayed in Fig. 1. For each of these pairs we provide a (directed) graph together with a (directed) cycle basis B such that $B \in J \setminus I$.

In the next section we summarize some complexity results for the minimum cycle basis problem for several subclasses of directed cycle basis. Consider a subclass $\mathcal{S}$ of directed cycle bases. For a directed graph D with a conservative weight function $w : A \to \mathbb{Q}$, we are looking for a basis B of $\mathcal{C}$ that minimizes

$$\sum_{C \in B} \sum_{a \in C} w_a \tag{8}$$

among the cycle bases of $\mathcal{S}$. If we do not explicitly define a weight function, we assume $w = \mathbf{1}$.

The cycle bases that we present in this section are the unique minimum cycle basis of their graphs. From this, we will conclude that in order to compute a minimum cycle basis among some specific subclass of (directed) cycle bases, there have to be designed tailored algorithms for each of the subclasses.

Example 1

Let C_3 be the simple graph with 3 vertices and 3 edges. Its unique minimum cycle basis is both strictly fundamental and a 2-basis.

Example 2 (2-basis)

The sunflower graph SF(3) in Fig. 2 contains precisely four circuits with three edges. These are independent, whence they constitute its unique minimum cycle basis B. Obviously, B is a 2-basis. And, by Lemma 13, B is also weakly fundamental.

In contrast, B is not strictly fundamental. In particular, the triangle not being incident with the infinite face shares each of its edges with one other triangle. Hence, none of its edges can serve as chord with respect to some spanning tree. This example has been inspired by [15].

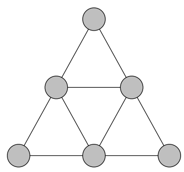


Figure 2: The sunflower graph SF(3) has a unique minimum cycle basis that is a 2-basis.

Example 3 (strictly fundamental cycle basis)

The planar graph G in Fig. 3 has precisely three triangles. These share the edge e. Every spanning tree T with $e \in T$ induces the unique minimum cycle basis B of G. Hence, B is strictly fundamental.

However, B is not a 2-basis. In particular, the edge e is covered by three basic circuits. But this is forbidden in 2-bases.

A planar graph on 8 vertices and 10 edges whose minimum cycle basis is not a 2-basis can already be found in [18].

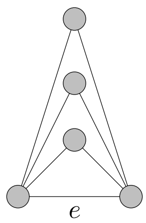


Figure 3: A graph whose unique minimum cycle basis is strictly fundamental but not a 2-basis.

Example 4 (totally unimodular cycle basis)

The graph in Fig. 4 is obtained by putting the graph in Example 3 "on top" of the graph in Example 2. It has six triangles which constitute its unique minimum cycle basis B. By the arguments given in the previous examples, B is neither a 2-basis nor strictly fundamental.

But B is totally unimodular. To that end consider the cycle matrix Γ of B. We use the fact that a matrix M is totally unimodular if and only if the matrix $(e_i|M)$ is totally unimodular, e_i being some unit vector. It can be seen easily, that by recursively eliminating rows and columns of Γ that only contain a unit vector, we eliminate Γ completely. Hence, Γ is totally unimodular.

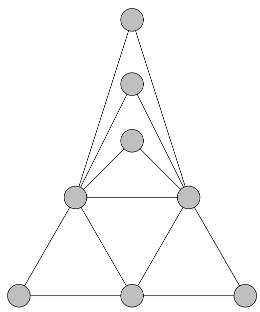


Figure 4: A combination of the graphs of Examples 2 and 3.

Example 5 (totally unimodular cycle basis - not weakly fundamental)

Consider first the graph V_8 in Fig. 5.

Actually, any graph V_{2n} defined by $V(V_{2n}) = \mathbb{N}_{2n}$ and $E(V_{2n}) = \{\{i, i+1\,.mod.\,2n\} : i \in \mathbb{N}_{2n}\} \cup \{\{i, i+n\} : i \in \mathbb{N}_n\}$ could more generally act as basic building block in deriving our example, but the Wagner's graph V_8 is our favorite here since it has a long history in attempts of exiting from the plane.

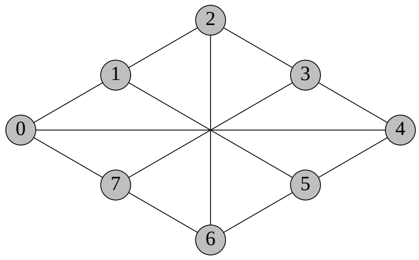


Figure 5: Wagner's graph V_8.

Clearly, $\nu(V_8) = 5$ and the directed circuits $C_1 = 2-3-7-6$, $C_2 = 3-4-0-7$, $C_3 = 4-5-1-0$, $C_4 = 5-6-2-1$, $C_5 = 6-5-4-3-2$ form an undirected cycle basis for V_8 since their independence can be certified by the odd sets $\Sigma_1 = \{\{2,6\},\{5,6\}\}$ which has odd intersection only with C_1 (see Fig. 6 on the left), $\Sigma_2 = \{\{2,6\},\{5,6\},\{3,7\}\}$ which has odd intersection only with C_2 (see Fig. 6 on the right), $\Sigma_3 = \{\{1,5\},\{2,3\},\{2,6\}\}$ which has odd intersection only with C_3 (mirror what for C_2), $\Sigma_4 = \{\{2,3\},\{2,6\}\}$ which has odd intersection only with C_4 (mirror what for C_1), $\Sigma_5 = \{\{2,3\},\{2,6\},\{5,6\}\}$ which has odd intersection only with C_5 (see Fig. 6 in the middle).

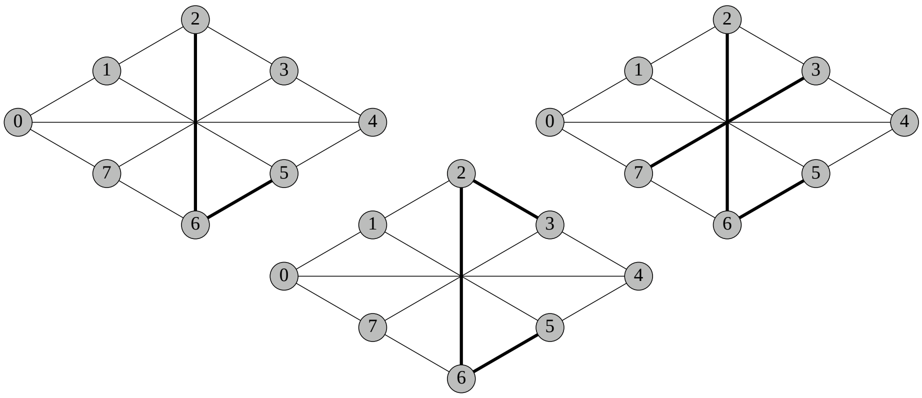


Figure 6: Certificates of independence for C_1 (left), C_2 (right), and C_5 (middle).

When edges $\{6,7\},\{7,0\},\{0,1\},\{1,2\}$ are assigned length 2 and all other edges receive length 1, then all circuits of V_8 have length at least 5, and C_1, C_2, C_3, C_4, C_5 are actually the only 5 circuits of length precisely 5, whence they form a minimum cycle basis.

Consider now the graph G in Fig. 7 obtained by taking six copies of V_8, say on nodes $\{a_0, a_1, \ldots, a_7\}$, $\{b_0, b_1, \ldots, b_7\}$, $\{c_0, c_1, \ldots, c_7\}$, $\{d_0, d_1, \ldots, d_7\}$, $\{e_0, e_1, \ldots, e_7\}$, $\{f_0, f_1, \ldots, f_7\}$, and $\{g_0, g_1, \ldots, g_7\}$ after performing the following node identifications: $g_0 \to f_0 \to e_0 \to d_0 \to c_0 \to b_0 \to a_0$, with

$b_1 \to a_7, b_2 \to a_6,\ c_1 \to b_7, c_2 \to b_6,\ d_1 \to c_7, d_2 \to c_6,\ e_1 \to d_7, e_2 \to d_6,$ $f_1 \to e_7, f_2 \to e_6$, and $g_1 \to e_7, g_2 \to e_6,\ g_7 \to a_1, g_6 \to a_2$.

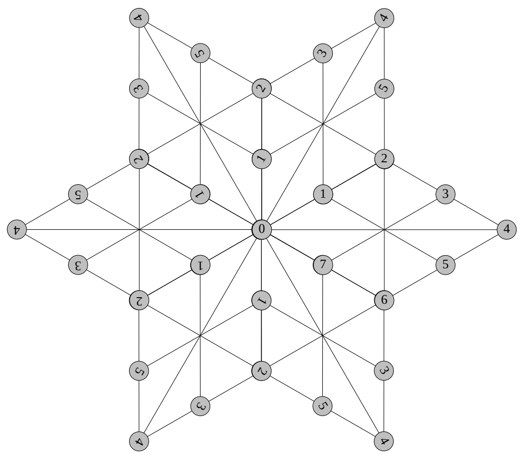


Figure 7: A graph G with a totally unimodular cycle basis which is not weakly fundamental.

Clearly, $n(G) = 31$, $m(G) = 60$, and $\nu(G) = 30$. Moreover, if we take the 5 circuits C_1, C_2, C_3, C_4, C_5 for each one of the 6 copies of V_8 comprising G, we get 30 circuits whose independence can be assessed by taking the six corresponding copies of each of the odd sets $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$. Hence, these 30 circuits form an undirected cycle basis of G. Consider now the weighting introduced above, namely, the 12 edges which are common to two distinct copies of V_8 have length 2 and all other edges have length 1. Under this weighting, no other circuit of G has length smaller than 6, whence these 30 circuits form the unique minimum cycle basis of G. This cycle basis is not weakly fundamental since each edge is contained in at least 2 circuits.

Claim *These* 30 *circuits of* G *constitute the unique minimum cycle basis of* G. *This basis is totally unimodular but not weakly fundamental.*

Proof. Let Γ be the cycle matrix associated to this cycle basis B. It remains to show that Γ is totally unimodular. Assume the contrary and let Γ' be a minimal square submatrix of Γ with $\det \Gamma' \neq 0, \pm 1$. Therefore, by the minimality assumption, each row and each column of Γ' contains at least two non-zero elements. Consider now the graph H with $V(H) = B$ and such that $\{C', C''\} \in E(H)$ if and only if $C', C'' \in B$ and there exists an edge e of G with $e \in (C' \cap C'') \setminus \bigcup_{C \in B \setminus \{C', C''\}} C$. Observe that H is connected. Since each row and each column of Γ' contains at least two non-zero elements, and since H is connected, it follows that Γ' contains all the 30 columns of Γ. Therefore, every submatrix of Γ with less than 30 columns is totally unimodular. We are now in position to derive the total unimodularity of Γ based on Theorem 8. Indeed, let B' be any subset of the columns of Γ. If $|B'| < 30$, then the corresponding matrix is totally unimodular, whence the columns of B' can be partitioned into two sets such that the sum of the columns in one set minus the sum of the columns in the other set provides a vector with entries only in $\{-1, 0, +1\}$. If on the contrary $B' = B$ then we propose to partition B' as $B' = B' \cup \emptyset$. Indeed, each edge of G belongs to at most 3 circuits of B and, under the orientation of the circuits proposed when defining $C_1, \ldots, C_5$ and by the node identifications employed in obtaining G from the 6 copies of V_8, each edge of G gets traversed in each direction. $\square$

Example 6 (weakly fundamental cycle basis)

Consider the orientation of complete bipartite graph $K_{3,3}$ that is displayed in Fig. 8. The oriented cycles $C_1 = (1, 2, 5, 4)$, $C_2 = (1, 2, 3, 6)$, $C_3 = (3, 4, 5, 6)$, and $C_4 = (1, 4, 3, 6)$ constitute a weakly fundamental cycle basis B of $K_{3,3}$. Having noticed that the arcs $(2, 3)$ and $(5, 6)$ are only used by C_2 and C_3,

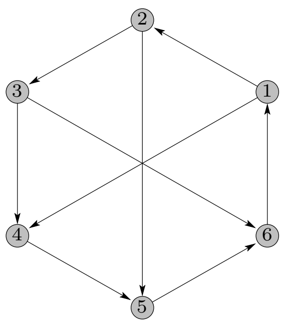


Figure 8: An orientation of the complete bipartite graph $K_{3,3}$.

respectively, one can easily deduce an ordering in accordance with Definition 9.

Consider the submatrix Γ' of the cycle matrix of B that is induced by the arcs $(1, 2)$, $(3, 6)$, and $(4, 5)$ together with the cycles C_1, C_2, and C_3,

$$\Gamma' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Obviously, $\det(\Gamma') = 2$ providing that B is not totally unimodular.

As $K_{3,3}$ has girth four, B is also a minimum cycle basis. To make it the unique minimum cycle basis, we suggest the following weighting:

$$w(a) = \begin{cases} 2, & \text{if } a = (3,6), \\ 4, & \text{if } a = (2,5), \\ 5, & \text{if } a = (2,3) \text{ or } a = (5,6), \text{ and} \\ 3, & \text{otherwise.} \end{cases}$$

Example 7 (integral cycle basis)

When every two vertices of the graph displayed in Fig. 9 and sharing a same label are identified with each other, the resulting simple graph G_{Ch} consists of 17 vertices and 52 edges. This graph has been considered by Champetier [5] who also obtained and proposed it by means of its "planarized" representation in Fig. 9.

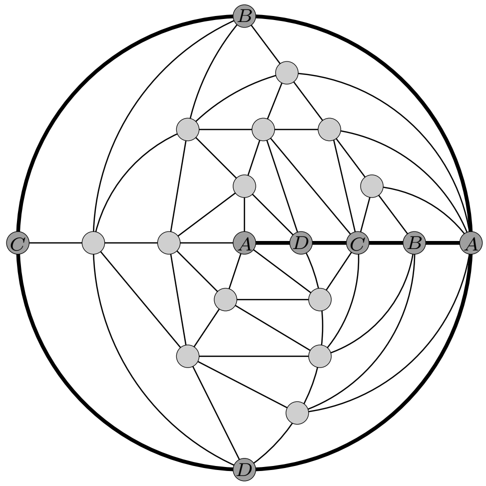


Figure 9: Champetier's graph [5] has a unique minimum cycle basis that is integral but not weakly fundamental.

There are precisely 36 triangles in G_{Ch}: they are the finite faces of the "planarized" visualization of G_{Ch} in Fig. 9.

Claim *The 36 triangles in G_{Ch} constitute the unique minimum cycle basis B of G_{Ch}. B is integral but not weakly fundamental.*

Proof. Consider some orientation D of G_{Ch}. Further, orient all the circuits $C \in B$ clockwise, with respect to Fig. 9. Next, consider the sum C' over $\mathbb{Q}$ of all the triangles, $\gamma_{C'} = \sum_{C \in B} \gamma_C$.

C' is the 4-cycle that links the labeled vertices. In the visualization of Fig. 9, this translates to following the outer bold circuit clockwise, or follow its representation as path from left to right.

Replace C' with any circuit of B to obtain another set B' of ν circuits. On the level of their cycle matrices Γ and Γ', consider the transformation matrix R such that $\Gamma' = \Gamma R$. With $R = [r_1, \ldots, r_\nu]$, we have $r_i = \mathbf{1}$ for some $i \in \{1, \ldots, \nu\}$, and $r_j = e_j$ for all $j \neq i$. Hence, R constitutes a unimodular transformation and thus B and B' have the same determinant.

Observe that the cycle basis B' is weakly fundamental. By Lemma 30 we deduce that B is an integral cycle matrix. But B is *not* weakly fundamental as every arc is part of two or three triangles—this has also been observed in [9]. Finally one can verify that there are no further triangles in G_{Ch}. □

Claim *The cycle matrix Γ of the minimum cycle basis B of Champetier's graph is not totally unimodular.*

Proof. We profit from the characterization of total unimodularity that we quoted in Theorem 8.

Consider the columns of Γ that correspond to the seven triangles that are numbered from 1 to 7 in Fig. 10. Splitting these seven columns of Γ into plus and

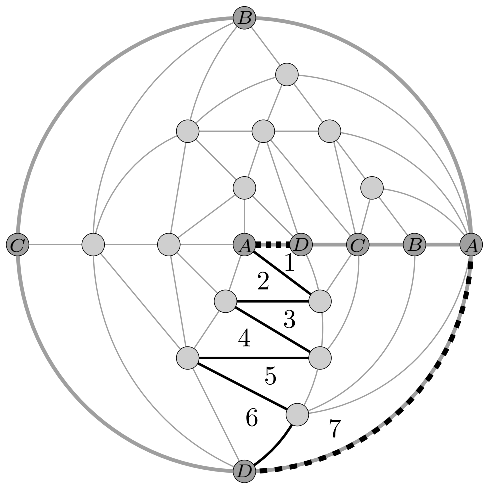


Figure 10: A certificate in Champetier's graph that the unique minimum cycle basis is not totally unimodular.

minus columns corresponds to traversing the seven corresponding basic circuits clockwisely or counter-clockwisely in the visualization of Fig. 10. To prevent that in a row that corresponds to an edge different from $\{C, D\}$ we obtain a value $+2$ or -2, the seven basic circuits have to be oriented either all clockwisely or all counter-clockwisely. But this results in traversing the edge $\{C, D\}$ twice in the same direction which retranslates into a value not in $\{-1, 0, +1\}$ in the row that corresponds to this edge. □

Remark 31 *Leydold and Stadler [18] considered a minimum cycle basis of K_9 that is not weakly fundamental. Liebchen and Peeters [20] provide an even* *smaller example on K_8, and proved this to be node-minimal. However, these cycle bases are not the unique minimum cycle bases for their graphs.*

Example 8 (undirected cycle basis)

Consider the generalized Petersen graph $P_{11,4}$ (cf. Figure 11) with the following weight function

$$w_{ij} = \begin{cases} 4, & \text{if } i \text{ and } j \text{ are outer vertices,} \\ 5, & \text{if } i \text{ and } j \text{ are inner vertices,} \\ 12, & \text{otherwise.} \end{cases}$$

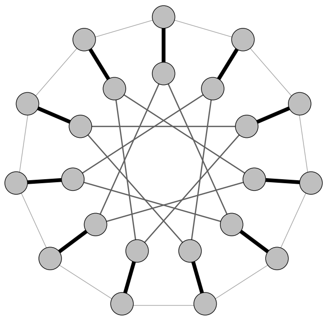


Figure 11: A weighted version of the generalized Petersen Graph $P_{11,4}$ has a unique minium cycle basis that is not integral.

Claim $(P_{11,4}, w)$ *has precisely* 12 *circuits of weight at most* 44*. These constitute the unique minimum cycle basis.*

Proof. We start by computing the dimension of the cycle space of $P_{11,4}$. As we have $n = 22$ and $m = 33$, any cycle basis consists of $\nu = 12$ circuits.

We refer to an edge e with $w_e = 12$ as *spoke*. This enables us to analyze the weight of *any* circuit of $(P_{11,4}, w)$. To that end, notice that the spokes are a cutset that separates the inner vertices of $P_{11,4}$ from its outer vertices. Hence, every circuit contains an even number of spokes.

There are only two circuits that do not contain any spoke: The outer circuit has weight 44 whereas the inner circuit has weight 55. Any circuit with at least four spokes will be too heavy and thus does not need to be investigated in detail.

We discriminate the circuits that contain two spokes according to the number of outer edges they use. As there are always two possible choices for the path through the inner edges, we only consider the shorter one in Table 1. Similarly, the shortest possible circuits use at most $5 = \lfloor \frac{11}{2} \rfloor$ outer edges. In particular,

Number of outer edges	1	2	3	4	5
Number of inner edges	3	5	2	1	4
Weight of the shorter circuit	43	57	46	45	64

Table 1: Weights of the circuits in $(P_{11,4}, w)$ that use two spokes.

we consider the set B that consists of the outer circuit and of the 11 circuits that use precisely one outer edge.

Assume B was not a cycle basis. Then, there exists a non-trivial linear combination of the zero vector, over GF(2). If such a combination made use of any of the 11 circuits that use precisely one outer edge, then it had to use each of these circuits in order to cancel out the spokes. But then, there is no circuit left that may cancel out the inner edges. Hence, B is the unique minimum cycle basis of $(P_{11,4}, w)$. $\square$

Claim *B is not an integral cycle basis.*

Proof. We prove that B is not an integral cycle basis by considering the orientation D of $P_{11,4}$, and of the basic circuits that provides the rational cycle matrix Γ in Equation (9). There, we select all the spokes and 10 of the inner edges as spanning tree T, whose corresponding rows we omit.

$$\Gamma|_{A\setminus T} = \begin{pmatrix} 1 & 1 & & & & & & & & & & \\ 1 & & 1 & & & & & & & & & \\ 1 & & & 1 & & & & & & & & \\ 1 & & & & 1 & & & & & & & \\ 1 & & & & & 1 & & & & & & \\ 1 & & & & & & 1 & & & & & \\ 1 & & & & & & & 1 & & & & \\ 1 & & & & & & & & 1 & & & \\ 1 & & & & & & & & & 1 & & \\ 1 & & & & & & & & & & 1 & \\ 1 & & & & & & & & & & & 1 \\ & 1 & & & 1 & & & & 1 & & & \end{pmatrix}. \tag{9}$$

We compute $\det \Gamma$ by performing Laplacian expansion first on the eight columns that only contain a unit vector. We conclude

$$\begin{aligned} |\det \Gamma| &= |\det(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix})| \\ &= |\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}| \\ &= |1 - (-1) + 1| = 3. \end{aligned}$$

In other words, the unique minimum cycle basis B of $(P_{11,4}, w)$ is not an integral cycle basis. $\square$

Example 9 (directed cycle basis [21])

Consider the generalized Petersen graph $P_{7,2}$ (cf. Figure 12) with the following weight function

$$w_{ij} = \begin{cases} 2, & \text{if } i \text{ and } j \text{ are inner vertices,} \\ 3, & \text{otherwise.} \end{cases}$$

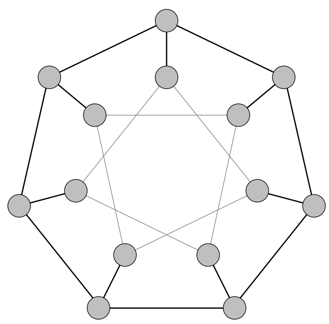


Figure 12: Any orientation of a weighted version of the generalized Petersen Graph $P_{7,2}$ has a unique minimum cycle basis that does not project onto a basis of $P_{7,2}$ [21].

Claim *There are precisely eight circuits in* $(P_{7,2}, w)$ *that have weight at most* 14.

Proof. We borrow the notation of the proof of Claim 5. By the same argument, we only have to consider circuits that contain either zero or two spokes.

Notice that we may assume a circuit that contains two spokes to contain at most three consecutive outer edges and at most three consecutive inner edges, too. Among these circuits, the seven circuits that contain two outer edges and one inner edge have weight 14. The eighth circuit of weight (at most) 14 is the one that precisely contains the seven inner edges. All other circuits have weight at least 15. □

Number of outer edges	1	2	3
Number of inner edges	3	1	2
Weight of the shorter circuit	15	14	19

Table 2: Weights of the circuits in $(P_{7,2}, w)$ that use two spokes.

Claim *The eight circuits in* $(P_{7,2}, w)$ *having weight at most* 14 *do not constitute a cycle basis of* $P_{7,2}$*. In contrast, for every orientation* D *of* $P_{7,2}$*, these circuits are the undirected projections of the unique minimum cycle basis of* D*.*

Proof. The sum over GF(2) of the incidence vectors of the eight circuits of weight 14 is a non-trivial linear combination of the zero vector.

To prove that these eight shortest circuits in $(P_{7,2}, w)$ are the projection of a directed cycle basis of every orientation of $P_{7,2}$, we demonstrate that their determinant is two. To that end, we consider orientations D of $P_{7,2}$ and of the eight circuits that provide the rational cycle matrix Γ in Equation (10). There, we select all the spokes and 6 of the outer edges as spanning tree T, whose corresponding rows we omit.

$$\Gamma|_{A\setminus T} = \begin{pmatrix} 1 & 1 & & & & & & \\ 1 & & 1 & & & & & \\ 1 & & & 1 & & & & \\ 1 & & & & 1 & & & \\ 1 & & & & & 1 & & \\ 1 & & & & & & 1 & \\ 1 & & & & & & & 1 \\ & & & 1 & 1 & & & \end{pmatrix}. \tag{10}$$

Again, we compute $\det \Gamma$ by performing Laplacian expansion first on the five columns that only contain a unit vector. We conclude

$$\begin{aligned} |\det \Gamma| &= |\det(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix})| \\ &= |-2| = 2. \end{aligned}$$

□

Summary

Fig. 13 relates the examples of this section to the lemmata of the previous section. We conclude that there have to be designed tailored algorithms for computing a minimum cycle basis among any of the subclasses of directed cycle bases that we investigated.

6 Complexity of Minimum Cycle Basis Problems

Finally, we list how the different classes of cycle bases behave with respect to the minimum cycle basis problem.

Theorem 32 ([6]) *The minimum strictly fundamental cycle basis problem is NP-hard.*

Theorem 33 *The minimum* 2*-basis problem is solvable in linear time for planar graphs, possibly with weights on the edges.*

Proof. Two planar drawings of the same graph are *equivalent* when the sequence of the edges in clockwise order around each node is the same in both drawings.

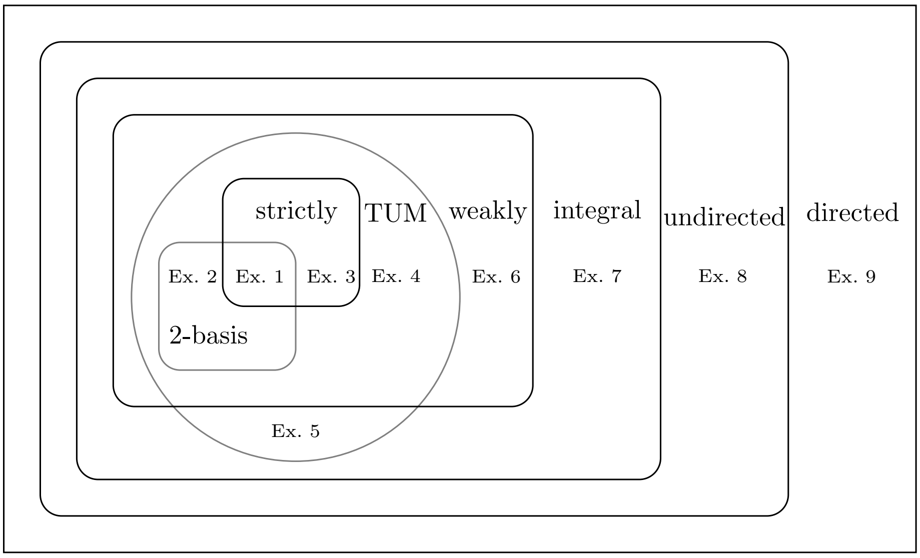


Figure 13: Map of directed cycle bases.

The equivalence classes of planar drawings are called *combinatorial embeddings*. A combinatorial embedding of a planar graph can be found in linear time [25].

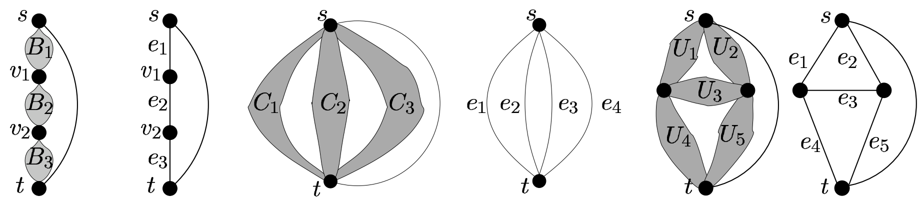
Let us first consider the case when G is 3-connected. If G is 3-connected, then G has a unique combinatorial embedding (Whitney [30]) and hence a unique dual G^* which can also be computed in linear time. By Theorem 12 we only have to identify a face C of G that has maximum weight. Then, a minimum 2-basis of G consists of all the faces of G, except C. Notice that C is also the vertex of G^* maximizing the total weight of the edges incident with it, and hence can be easily found.

When G is not 3-connected then things are less straightforward. By Theorem 12 we are essentially asked to find a maximum weight circuit C of G such that C is the boundary of some face in some planar embedding of G. We call such a circuit C of G a *plain circuit*.

Clearly, when the embedding is not unique, a maximum weight plain circuit might be more elusive. Nevertheless, we can still assume that G is at least 2-connected since the cycle space of a graph is the direct sum of the cycle spaces of its 2-connected components and since the union of a set of 2-bases for the 2-connected components of G would result in a 2-basis for G itself.

Luckily enough, the SPQR-tree data structure is meant to represent all the combinatorial embeddings of a 2-connected planar graph. Gutwenger and Mutzel [11] presented the first linear time algorithm to compute SPQR-trees based on the linear time algorithm for dividing a graph into 3-connected components by Hopcroft and Tarjan [13]. SPQR-trees are a rather complex data structure and discussing them here in full would be out of scope. Actually, we believe that the direct and friendly introduction to SPQR-trees offered in Sections 2.2–2.5 in the thesis of Weiskircher [29] should suffice to our scope. We refer to it and recall here, also rather broadly, the few notions necessary to fully specify our algorithm.

We take for granted that the SPQR-tree of G is a directed tree whose nodes are labeled with a letter in $\{S, P, Q, R\}$. (We are actually referring to what in Weiskircher's thesis is dubbed a Proto-SPQR-tree). Moreover, a subgraph G_v of G is associated to each node v of G. For example, the whole graph G is associated to the root r of T. Also, the nodes labeled with Q are precisely the leaves of T and a different edge of G (comprising its endnodes) is associated to each of them. To each node v of T is also associated a skeleton graph S_v giving a rough representation of the subgraph of G associated to v. Each skeleton graph also contains a further edge e_v called the *virtual edge* which is very relevant in gluing the various pieces of the SPQR-tree representation of G together. For each node v, only S_v is explicitly represented in v whereas G_v can be reconstructed from the pieces of information that are stored in the subtree of T rooted at v. When v is a Q node, then S_v is obtained from G_v by placing the virtual edge e_v in parallel with the unique edge of G_v. Otherwise, when v is not a leaf of T, then the subgraphs associated to the children of v are also subgraphs of G_v and in S_v are represented by virtual edges. In this way, S_v is either a circuit (when v is labeled S), or a bunch of at least three parallel edges (when v is labeled P), or a 3-connected graph (when v is labeled R), or two parallel edges (when v is labeled Q). We refer to Section 2.3 of [29] and



(a) Case S-node

(b) Case P-node

(c) Case R-node

Figure 14: The structure of biconnected graphs and the skeleton of the root of the corresponding (Proto-) SPQR-tree (from [29]).

to Figure 14 for an adequately precise notion of S, P, and R nodes. The main property of T is the following (see Theorem 2.3 in [29]).

> A combinatorial embedding of G uniquely defines a combinatorial embedding of the skeleton of each node of T. Conversely, fixing the combinatorial embedding of the skeleton of each node of T uniquely defines a combinatorial embedding of G.

To compute a maximum weight plain circuit of G we go from the leaves of G up to the root in a way which might resemble dynamic programming. Indeed, for each node v of T we compute two objects:

C_v: the maximum weight plain circuit of G_v not containing the virtual edge of S_v (assuming such a circuit exists);

P_v: the maximum weight plain circuit of G_v containing the virtual edge of S_v.

In the end, where r is the root of T, then the best among C_r and P_r is an optimal plain circuit of G. It remains to specify how these pieces of information can be computed going up the tree towards the root.

One point is that no weight (you might think of a 0 weight) is associated to virtual edges. When v is a Q node, then C_v is left undefined and P_v is the unique cycle in S_v. When v is an S node, and $v_1, v_2, \ldots, v_k$ are the children of v, then C_v is the best among $C_{v_1}, C_{v_2}, \ldots, C_{v_k}$ and $P_v := S_v \Delta P_{v_1} \Delta P_{v_2} \ldots \Delta P_{v_k}$. When v is a P node, and $v_1, v_2, \ldots, v_k$ are the children of v, where we assume w.l.o.g. that $w(P_{v_i}) \geq w(P_{v_{i+1}})$ for $i = 1, 2, \ldots, k-1$, then $P_v := P_{v_1}$ whereas C_v is the best among $P_{v_1} \Delta P_{v_2}$ and the best circuit among $C_{v_1}, C_{v_2}, \ldots, C_{v_k}$. Finally, when v is an R node, and $v_1, v_2, \ldots, v_k$ are the children of v, then, before computing P_v and C_v, we assign to every non-virtual edge e_i of S_v which is not an edge in G the weight of P_{v_i} where v_i is the child of v where the expansion of edge e_i takes place. At this point, in S_v, which is 3-connected, we can compute both the best plain circuit P which contains the virtual edge e_v and the best plain circuit C which does not contain e_v. This can be done in time linear in the size of S_v, essentially as mentioned above. Now, P_v and C_v are obtained from P and C by replacing each non-virtual edge e_i occurring in these objects but which is not an original edge of G with the path obtained from P_{v_i} by removing its virtual edge. In other words, when e_i occurs in P or C it is actually a placemark for a longer path to be expanded (by taking the symmetric difference with P_{v_i}).

We have finished the description of the rules for going up the tree. Each one of these rules, when applied to v, takes linear time in the size of S_v. The sum of all these sizes is linear in the size of G since the SPQR-tree representation of G is known to be linear. □

Theorem 34 ([14]) *The minimum cycle basis problem is polynomially solvable for undirected graphs.*

Theorem 35 ([16, 21]) *The minimum cycle basis problem is polynomially solvable for directed graphs.*

Remark 36 *The asymptotical complexity of both minimizing over weakly fundamental cycle bases and minimizing over integral cycle bases is open for general graphs. Neither of these classes induces a matroid [21].*

Theorem 37 ([18]) *Every minimum cycle basis of a planar graph is weakly fundamental.*

Corollary 38 *The following problems are polynomially solvable for planar graphs:*

1. *the minimum directed cycle basis problem;*
2. *the minimum cycle basis problem;*
3. *the minimum integral cycle basis problem;*
4. *the minimum weakly fundamental cycle basis problem;*
5. *the minimum* 2*-basis problem.*

Proof. This follows from Theorems 33, 34 and 37 in conjunction with Lemmata 28 and 30. □

Proposition 39 ([23]) *Every minimum cycle basis of a graph G is strictly fundamental, if and only if G is planar and none of its duals has a* double claw*, where a double claw between two vertices u and v is a subgraph consisting of three internally node-disjoint paths between u and v.*

7 Conclusions

We present seven classes of cycle bases of directed graphs. A directed graph with a directed cycle basis strictly smaller than any undirected cycle basis showed that undirected cycle basis are a proper subclass of directed cycle basis and that finding minimum cycle basis over the two classes demands for two distinct algorithms. All other classes considered are proper subclasses of undirected cycle basis. We proved all the existing inclusions among these classes and, whenever A and B are two classes with A not included in B, we gave a counterexample digraph whose unique minimum undirected cycle basis was in A but not in B. In conclusion, by means of these eight counterexample digraphs we establish that none of these classes coincide and also that the minimum cycle basis problem has to be treated differently for *any* of these classes. We proposed a linear time algorithm for the minimum 2-basis problem. Our last remark is that, for several of these classes, the complexity status is still open, despite the fact that there are practical applications building just upon these classes.

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