

# Robust Utility Maximization for Complete and Incomplete Market Models

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## Abstract

We investigate the problem of maximizing the robust utility functional  $\inf_{Q \in \mathcal{Q}} E_{Qu}(X)$ . We give the dual characterization for its solution for both a complete and an incomplete market model. To this end, we introduce the new notion of reverse  $f$ -projections and use techniques developed for  $f$ -divergences. This is a suitable tool to reduce the robust problem to the classical problem of utility maximization under a certain measure: the reverse  $f$ -projection. Furthermore, we give the dual characterization for a closely related problem, the minimization of expenditures given a minimum level of expected utility in a robust setting and for an incomplete market.

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## 1 Introduction

A standard problem in the theory of incomplete financial markets consists in maximizing the utility of payoffs. But how do we measure utility?

The usual approach goes back to von Neumann-Morgenstern and Savage. It provides conditions on an investor's preferences which guarantee that the utility of a contingent claim is given by the expectation  $E_Q u(X)$  for some measure  $Q$  and a utility function  $u$ . The problem of utility maximization with the initial endowment  $x$  can then be formulated as

$$\text{Maximize } E_Q[u(X)] \text{ over all } X \text{ with } \sup_{P \in \mathcal{P}_Q} E_P X \leq x \quad (1)$$

for some set of reasonable equivalent local martingale measures  $\mathcal{P}_Q$ . This problem is well understood, in particular due to articles by Kramkov and Schachermayer [13] and Goll and Rüschemdorf [10].

However, both from a normative and a descriptive point of view, there are good reasons to consider alternative utility functionals. In 1989, Gilboa and Schmeidler [9] proposed a more flexible set of axioms for preference orders on payoff profiles. It led to a numerical representation by a robust utility functional of the form

$$U(X) := \inf_{Q \in \mathcal{Q}} E_Q u(X)$$

for some set of subjective measures  $\mathcal{Q}$ . This approach covers the uncertainty about the probabilities of market events: The agent has in mind a whole set of possible probability distributions and takes a worst case approach in evaluating the expected utility of a payoff. For an overview and more details on such robust representations of preferences, see Föllmer and Schied [6].

In this article we deal with the *robust* utility maximization problem

$$\text{Maximize } \inf_{Q \in \mathcal{Q}} E_Q[u(X)] \text{ over all } X \text{ with } \sup_{P \in \tilde{\mathcal{P}}} E_P X \leq x \quad (2)$$

for some convex set  $\tilde{\mathcal{P}}$  of reasonable equivalent local martingale measures. We do not require the payoffs to be obtainable from dynamic trading in the underlying assets. However, Goll and Rüschemdorf [10] showed that the optimal claim for (1) is replicable by trading in these assets. Furthermore they show that the solution to (1) coincides with the solution to the problem of maximizing portfolios that can be obtained from dynamic trading in the underlying assets (see

[10], Theorem 5.1). It follows immediately from the form of the optimal payoff in our Theorem 2 that these results are carried forward to the problem of robust utility maximization.

In 2002, Baudoin [2] solved Problem (2) for a complete market model of “weak information”, which means that  $\mathcal{Q}_P$  is the set of measures under which some given random variable has a specific law. In 2003, Schied [19] solved it with methods from robust statistics, again for a complete market model and under the condition that there exists a “least favorable measure” which is for example the case if  $\mathcal{Q}$  is the core of a convex capacity.

Using a similar approach as Goll and Rüschendorf [10] we give a dual characterization for the general solution to (2) for both a complete and an incomplete market model. Furthermore, we give a dual characterization for a closely related problem, the minimization of expenditures given a minimum level  $w$  of robust expected utility:

$$\text{Minimize } \sup_{P \in \tilde{\mathcal{P}}} E_P[X] \text{ over all } X \text{ with } \inf_{Q \in \mathcal{Q}} E_Q u(X) \geq w. \quad (3)$$

The main idea is to identify a measure  $Q^*$  such that the robust utility maximization problem (2) is equivalent to the standard problem (1) corresponding to  $Q^*$ . Goll and Rüschendorf [10] solve the standard problem by means of its dual problem, the minimization of the  $f$ -divergence

$$f(P|Q) := E_Q f \left( \frac{dP}{dQ} \right)$$

over the set of equivalent local martingale measures. In our approach we turn things round: For a given equivalent local martingale measure  $P$ , we minimize the  $f$ -divergence  $f(P|Q)$  over the set  $\mathcal{Q}$ . This minimizing measure  $Q_P$ , which we call the *reverse  $f$ -projection*, has the property that for a complete market, problem (1) with  $Q = Q_P$  is equivalent to our problem (2). With the aid of a characterization of the reverse  $f$ -projections  $Q_P$  it is then possible to give the dual characterization of the solution to the robust utility maximization problem for both a complete and an incomplete market model.

The paper is organized as follows. After giving some definitions in Section 2 we introduce reverse  $f$ -projections in Section 3 and identify those measures  $Q_P$  under which for a complete market, problem (1) is equivalent to (2). In Sections 4 and 5 we give the dual characterizations for a complete and for an incomplete market model. Here, we combine the idea of reverse  $f$ -projections with techniques developed in [10] by Goll and Rüschendorf. In Section 6 we discuss uniqueness and existence of the robust  $f$ -projection. In order to illustrate our approach we

discuss a specific diffusion model in Section 7. In Section 8 we show how the introduction of reverse  $f$ -projections allows us to easily solve a related problem: the minimization of expenditures given a minimum level of robust expected utility.

## 2 Preliminaries

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, Q_0)$  be a filtered probability space where  $\Omega$  is Polish and  $\mathcal{F} = \mathcal{F}_T$  is the Borel  $\sigma$ -field. For a  $\mathbb{R}^d$ -valued semimartingale  $S$ , let  $\mathcal{P} \neq \emptyset$  be the set of all equivalent local martingale measures for  $S$ . Let us also fix a set  $\mathcal{Q}$ , the *set of subjective measures*, which are equivalent to  $Q_0$ . We assume that  $\mathcal{Q}$  is convex and compact with respect to the weak topology for measures.

**Definition 1** *Let  $P \in \mathcal{P}$ ,  $Q \in \mathcal{Q}$ , and let  $f : (0, \infty) \rightarrow \mathbb{R}$  be strictly convex. The  $f$ -divergence of  $P$  with respect to  $Q$  is defined as*

$$f(P|Q) := \int f\left(\frac{dP}{dQ}\right) dQ \in (-\infty, \infty].$$

$P_Q \in \mathcal{P}$  is called the  $f$ -projection of  $Q$  on  $\mathcal{P}$  if it minimizes the  $f$ -divergence:

$$f(P_Q|Q) = f(\mathcal{P}|Q) := \inf_{P \in \mathcal{P}} f(P|Q).$$

$P^* \in \mathcal{P}$  is called the robust  $f$ -projection of  $\mathcal{Q}$  on  $\mathcal{P}$  if it minimizes the robust  $f$ -divergence  $\inf_{Q \in \mathcal{Q}} f(P|Q)$ :

$$\inf_{Q \in \mathcal{Q}} f(P^*|Q) = f(\mathcal{P}|\mathcal{Q}) := \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} f(P|Q).$$

**Example 1** *For  $f(x) = x^p$  ( $p < 0$  or  $p > 1$ ), we obtain the  $p$ -distance  $E_Q \left[ \left( \frac{dP}{dQ} \right)^p \right]$ , for  $f(x) = x \log x$ , the relative entropy  $E_Q \left[ \frac{dP}{dQ} \log \left( \frac{dP}{dQ} \right) \right] = E_P \left[ \log \left( \frac{dP}{dQ} \right) \right]$ , and for  $f(x) = -\log x$ , the reverse relative entropy  $E_Q \left[ -\log \left( \frac{dP}{dQ} \right) \right]$ .*

The following basic result about  $f$ -projections was proved by Rüschemdorf [18]:

**Theorem 1 ([18], Thm. 5)** *Let  $f$  be differentiable,  $Q \in \mathcal{Q}$ , and  $P_Q \in \mathcal{P}$  with  $f(P_Q|Q) < \infty$ . Then  $P_Q$  is the  $f$ -projection of  $Q$  on  $\mathcal{P}$  if and only if*

$$\int f' \left( \frac{dP_Q}{dQ} \right) dP_Q \leq \int f' \left( \frac{dP_Q}{dQ} \right) dP \text{ for all } P \in \mathcal{P} \text{ with } f(P|Q) < \infty.$$

As Goll and Rüschendorf remark in [10] the condition  $f'(\frac{dP_Q}{dQ}) \in L^1(P_Q)$  assumed in [18] is not used in the proof of this theorem.

In this article, a *utility function* is defined as a function  $u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  which is strictly increasing, strictly concave, continuously differentiable in  $\text{dom}(u) := \{x \in \mathbb{R} : u(x) > -\infty\}$ , and satisfies

$$u'(\infty) = \lim_{x \rightarrow \infty} u'(x) = 0, \quad (\text{U1})$$

$$u'(\bar{x}) = \lim_{x \rightarrow \bar{x}} u'(x) = \infty \quad (\text{U2})$$

for  $\bar{x} := \inf\{x \in \mathbb{R} : u(x) > -\infty\}$ .

Let  $I := (u')^{-1}$ . The convex conjugate function  $v : \mathbb{R}_+ \rightarrow \mathbb{R}$  of a utility function  $u$  is defined by

$$v(y) := \sup_{x \in \mathbb{R}} (u(x) - xy) = u(I(y)) - yI(y).$$

Let us first formulate the problem of robust utility maximization for a complete market model. For an incomplete market model, this formulation will be part of our results. For  $P \in \mathcal{P}$ , we want to maximize

$$U(X) := \inf_{Q \in \mathcal{Q}_P} E_Q u(X)$$

where  $\mathcal{Q}_P \subseteq \mathcal{Q}$  is some set of “reasonable” subjective measures. We will consider the set

$$\mathcal{X}_P(x) := \{X : X \in L^1(P), E_P X \leq x, \text{ and } E_Q[u(X)^-] < \infty \forall Q \in \mathcal{Q}_P\}$$

of contingent claims that are affordable under  $P$ . Hence, the problem of utility maximization in a complete market can be formulated as

$$\text{Maximize } \inf_{Q \in \mathcal{Q}_P} E_Q u(X) \text{ over all } X \in \mathcal{X}_P(x).$$

Our aim is to find a characterization of the solution to this problem for complete and incomplete market models.

### 3 Reverse $f$ -Projections

In this section we fix an equivalent local martingale measure  $P \in \mathcal{P}$ . We then want to characterize the measure  $Q_P$  that minimizes the  $f$ -divergence  $f(P|Q)$  over all  $Q \in \mathcal{Q}$ .

For a strictly convex, differentiable function  $f : (0, \infty) \rightarrow \mathbb{R}$ , we define

$$\hat{f}(x) := xf\left(\frac{1}{x}\right). \quad (4)$$

**Lemma 1**  $\hat{f} : (0, \infty) \rightarrow \mathbb{R}$  is also a strictly convex, differentiable function, and we have  $\hat{\hat{f}} = f$ .

*Proof* Differentiability and the relation  $\hat{\hat{f}} = f$  are obvious. For  $\alpha \in (0, 1)$  and  $x, y > 0, x \neq y$ , we define

$$\gamma := \frac{\alpha x}{\alpha x + (1 - \alpha)y}.$$

Then we have  $0 < \gamma < 1$ , and

$$\begin{aligned} \hat{f}(\alpha x + (1 - \alpha)y) &= (\alpha x + (1 - \alpha)y) f\left(\frac{1}{\alpha x + (1 - \alpha)y}\right) \\ &= (\alpha x + (1 - \alpha)y) f\left(\gamma \frac{1}{x} + (1 - \gamma) \frac{1}{y}\right) \\ &< (\alpha x + (1 - \alpha)y) \left(\gamma f\left(\frac{1}{x}\right) + (1 - \gamma) f\left(\frac{1}{y}\right)\right) \\ &= \alpha x f\left(\frac{1}{x}\right) + (1 - \alpha)y f\left(\frac{1}{y}\right) \\ &= \alpha \hat{f}(x) + (1 - \alpha) \hat{f}(y). \end{aligned}$$

Thus,  $\hat{f}$  is strictly convex. □

For  $Q \in \mathcal{Q}$ , we have the following relation between  $f$  and  $\hat{f}$ -divergences:

$$\begin{aligned} f(P|Q) &:= E_Q \left[ f\left(\frac{dP}{dQ}\right) \right] = E_P \left[ \frac{dQ}{dP} f\left(\frac{dP}{dQ}\right) \right] \\ &= E_P \left[ \hat{f}\left(\frac{dQ}{dP}\right) \right] =: \hat{f}(Q|P). \end{aligned}$$

Hence, the  $f$ -divergence of  $P$  with respect to  $Q$  is equal to the  $\hat{f}$ -divergence of  $Q$  with respect to  $P$ . This symmetry was already observed by Liese and Vajda [14] in Theorem 1.13.

**Definition 2** Let  $Q_P \in \mathcal{Q}$  satisfy

$$f(P|Q_P) = f(P|\mathcal{Q}) := \inf_{Q \in \mathcal{Q}} f(P|Q).$$

Then  $Q_P$  is called the reverse  $f$ -projection of  $P$  on  $\mathcal{Q}$ .

Of course, the reverse  $f$ -projection coincides with the  $\hat{f}$ -projection. Since  $\mathcal{Q}$  is weakly compact, by Liese and Vajda [14], Proposition 8.4, the reverse  $f$ -projection always exists.

**Example 2** For  $f(x) = x^p$  ( $p < 0$  or  $p > 1$ ), we have  $\hat{f}(x) = x^{1-p}$ . Hence the class of  $p$ -distances is invariant under this transformation. For  $f(x) = x \log x$ , we have  $\hat{f}(x) = -\log x$ . Thus, the relative entropy is transformed into the reverse relative entropy and vice versa.

Applying Theorem 1 to the  $\hat{f}$ -projection we see that a measure  $Q_P \in \mathcal{Q}$  with  $\hat{f}(Q_P|P) < \infty$  is the reverse  $f$ -projection of  $P$  on  $\mathcal{Q}$  if and only if

$$\int \hat{f}'\left(\frac{dQ_P}{dP}\right) dQ_P \leq \int \hat{f}'\left(\frac{dQ}{dP}\right) dQ \quad \forall Q \in \mathcal{Q} \text{ with } \hat{f}(Q|P) < \infty. \quad (5)$$

Let  $u$  be a utility function as defined in Section 2, and let  $v$  denote its convex conjugate function. For  $\lambda > 0$ , we define  $v_\lambda$  and  $\hat{v}_\lambda$  by  $v_\lambda(x) := v(\lambda x)$ , resp.  $\hat{v}_\lambda(x) := xv_\lambda(1/x)$ . We want to characterize the reverse  $v_\lambda$ -projection  $Q_P$ . To this end, observe that

$$(\hat{v}_\lambda)'(x) = v\left(\frac{\lambda}{x}\right) - \frac{\lambda}{x}v'\left(\frac{\lambda}{x}\right) = v\left(\frac{\lambda}{x}\right) + \frac{\lambda}{x}I\left(\frac{\lambda}{x}\right) = u\left(I\left(\frac{\lambda}{x}\right)\right).$$

Applying (5) to  $\hat{f} = \hat{v}_\lambda$  now leads to

**Proposition 1** *Let  $\lambda > 0$  and  $Q_P(\lambda) \in \mathcal{Q}_P(\lambda) := \{Q \in \mathcal{Q} : v_\lambda(P|Q) < \infty\}$ . Then  $Q_P(\lambda)$  is the reverse  $v_\lambda$ -projection if and only if the following holds:*

$$E_{Q_P(\lambda)} u\left(I\left(\lambda \frac{dP}{dQ_P(\lambda)}\right)\right) = \inf_{Q \in \mathcal{Q}_P(\lambda)} E_Q u\left(I\left(\lambda \frac{dP}{dQ_P(\lambda)}\right)\right). \quad (6)$$

With this result we can now address the problem of robust utility maximization.

## 4 Duality Results for a Complete Market Model

In this section let us consider a market with a unique equivalent local martingale measure  $P$  which means that the market is complete. Denote by  $Q_P(\lambda)$  the reverse  $v_\lambda$ -projection of  $P$  on  $\mathcal{Q}$  and  $\mathcal{Q}_P(\lambda) := \{Q \in \mathcal{Q} : v_\lambda(P|Q) < \infty\}$ . We will need the following two assumptions:

$$v_\mu(P|Q_P(\lambda)) < \infty \text{ for all } \lambda, \mu > 0, \quad (A1)$$

and

$$E_Q u\left(I\left(\lambda \frac{dP}{dQ_P(\lambda)}\right)\right)^- < \infty \text{ for all } Q \in \mathcal{Q}_P(\lambda) \text{ for all } \lambda > 0. \quad (A2)$$

The following lemma provides some technicalities.

**Lemma 2** *Assume that (A1) holds.*

(i) For  $\lambda > 0$ ,

$$H(\lambda) := \inf_{Q \in \mathcal{Q}} v_\lambda(P|Q)$$

is a finite, convex function.

(ii) Let  $Q \in \mathcal{Q}$ . If  $v_\lambda(P|Q) < \infty$  for all  $\lambda > 0$ , then  $I\left(\lambda \frac{dP}{dQ}\right) \in L^1(P)$  for all  $\lambda > 0$ .

*Proof* (i) The convexity of  $H(\lambda)$  can be shown with exactly the same argument as in the proof of Theorem 2. Finiteness follows from (A1).

(ii) Let us define the function  $g(x) := v(x) - v(1) - v'(1)(x - 1)$  which is convex and positive due to the convexity of  $v$ . Furthermore, we have  $I(x) = -v'(x) = -g'(x) - v'(1)$ , and hence  $I\left(\lambda \frac{dP}{dQ}\right) \in L^1(P)$  if and only if  $g'\left(\lambda \frac{dP}{dQ}\right) \in L^1(P)$ .

Since  $g$  is convex we have for  $0 < y < x$ ,

$$g(x) - g(x - y) \leq yg'(x) \leq g(x + y) - g(x).$$

Therefore,

$$y|g'(x)| \leq \max\{g(x + y), g(x - y)\} - g(x) \leq g(x + y) + g(x - y) - g(x)$$

If we set  $x := \lambda \frac{dP}{dQ}$  and  $y := \mu \frac{dP}{dQ}$  for  $0 < \mu < \lambda$ , then we get

$$\mu E_P \left| g' \left( \lambda \frac{dP}{dQ} \right) \right| \leq E_Q g \left( (\lambda + \mu) \frac{dP}{dQ} \right) + E_Q g \left( (\lambda - \mu) \frac{dP}{dQ} \right) - E_Q g \left( \lambda \frac{dP}{dQ} \right).$$

Since  $E_Q g \left( \lambda \frac{dP}{dQ} \right) > -\infty$  for all  $\lambda > 0$ , and since  $E_Q g \left( \tilde{\lambda} \frac{dP}{dQ} \right) = E_Q v \left( \tilde{\lambda} \frac{dP}{dQ} \right) - v(1) - v'(1)(\tilde{\lambda} - 1) < \infty$  for all  $\tilde{\lambda} > 0$  by assumption, we have proved (ii).  $\square$

For  $x > \bar{x}$ , we define

$$V_P(x) := \inf_{\lambda > 0} \left\{ \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda \frac{dP}{dQ} \right) + \lambda x \right\}.$$

$V_P$  as the concave conjugate to  $H$  is a concave function of  $x$ , and we denote by  $\partial V_P(x)$  the superdifferential of  $V_P$  in  $x$ .

Now we are ready to prove the main result of this section.

**Proposition 2** *Assume that (A1) and (A2) hold and let  $x > \bar{x}$ .*

(i) *We have the following equivalence:*

$$\lambda \in \partial V_P(x) \iff x = E_P I \left( \lambda \frac{dP}{dQ_P(\lambda)} \right).$$



(ii) Let  $\lambda_P(x) \in \partial V_P(x)$ ,  $\mathcal{Q}_P := \mathcal{Q}_P(\lambda_P(x))$ , and denote by  $Q_P$  the reverse  $v_{\lambda_P(x)}$ -projection of  $P$  on  $\mathcal{Q}$ . Then

$$\begin{aligned}
\sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X) &= \inf_{Q \in \mathcal{Q}_P} \sup_{X \in \mathcal{X}_P(x)} E_Q u(X) \\
&= \sup_{X \in \mathcal{X}_P(x)} E_{Q_P} u(X) \\
&= \inf_{\lambda > 0} \left\{ \inf_{Q \in \mathcal{Q}} v_\lambda(P|Q) + \lambda x \right\} \\
&= \inf_{Q \in \mathcal{Q}} v_{\lambda_P(x)}(P|Q) + \lambda_P(x)x \\
&= E_{Q_P} u \left( I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \right) \\
&= \inf_{Q \in \mathcal{Q}_P} E_Q u \left( I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \right).
\end{aligned}$$

**Remark 1** The proposition shows that  $I \left( \lambda_P(x) \frac{dP}{dQ_P} \right)$  can be interpreted as the optimal claim that is affordable under  $P$ . Under  $Q_P$  the expected utility of this claim is minimal. It may therefore be considered as a worst case measure for the robust utility maximization. Furthermore, it follows from the second equality that the maximization of the robust utility functional is equivalent to the standard problem of utility maximization under  $Q_P$ . However, in general  $Q_P$  differs for different  $P$ .

*Proof First step.* By Lemma 2(i) the function  $H(\lambda) := \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda \frac{dP}{dQ} \right)$  is convex. By Rockafellar [17], Theorem 23.5,  $\inf_{\lambda > 0} (H(\lambda) + \lambda x)$  achieves its infimum in  $\lambda = \lambda_P(x)$  if and only if  $-x \in \partial H(\lambda_P(x))$  which is by [17], Theorem 7.4 and Corollary 23.5.1, equivalent to  $\lambda_P(x) \in \partial V_P(x)$ . In this case, we have

$$\begin{aligned}
V_P(x) &= \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda_P(x) \frac{dP}{dQ} \right) + \lambda_P(x)x \\
&= E_{Q_P} v \left( \lambda_P(x) \frac{dP}{dQ_P} \right) + \lambda_P(x)x \\
&= \inf_{\lambda > 0} \left\{ E_{Q_P} v \left( \lambda \frac{dP}{dQ_P} \right) + \lambda x \right\}.
\end{aligned}$$

*Second step.* For all  $X \in L^1(P)$  with  $E_P X \leq x$  and for all  $\lambda > 0$ , we have

$$\begin{aligned}
E_{Q_P} u(X) &\leq E_{Q_P} u(X) + \lambda(x - E_P X) \\
&\leq E_{Q_P} v \left( \lambda \frac{dP}{dQ_P} \right) + \lambda x \\
&= E_{Q_P} u \left( I \left( \lambda \frac{dP}{dQ_P} \right) \right) + \lambda \left( x - E_P I \left( \lambda \frac{dP}{dQ_P} \right) \right)
\end{aligned} \tag{7}$$

where the second inequality and the equality follow from the definition of  $v$ . Due to Assumptions (U1) and (U2) the function  $I$  is decreasing with range  $(\bar{x}, \infty)$ . By Lemma 2(ii) and Assumption (A1),  $I \left( \lambda \frac{dP}{dQ_P} \right) \in L^1(P)$  for all  $\lambda > 0$ . Therefore, the function  $g(\lambda) := E_P I \left( \lambda \frac{dP}{dQ_P} \right)$  is continuous and decreasing with range  $(\bar{x}, \infty)$ . Hence, for every  $x > \bar{x}$ , there exists  $\lambda_P(x) > 0$  such that  $x = g(\lambda_P(x))$ . Thus, the above inequalities hold as equalities with  $\lambda = \lambda_P(x)$  if and only if  $x = E_P I \left( \lambda_P(x) \frac{dP}{dQ_P} \right)$  and  $X = I \left( \lambda_P(x) \frac{dP}{dQ_P} \right)$ . In this case, we have

$$\begin{aligned}
\sup_{X \in \mathcal{X}_P(x)} E_{Q_P} u(X) &= \inf_{\lambda > 0} \left\{ E_{Q_P} v \left( \lambda \frac{dP}{dQ_P} \right) + \lambda x \right\} \\
&= E_{Q_P} v \left( \lambda_P(x) \frac{dP}{dQ_P} \right) + \lambda_P(x) x \\
&= E_{Q_P} u \left( I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \right).
\end{aligned} \tag{8}$$

Thus, by our results from the first step we have proved (i).

*Third step.* We know from Proposition 1 that

$$E_{Q_P} u \left( I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \right) = \inf_{Q \in \mathcal{Q}_P} E_Q u \left( I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \right).$$

Putting everything together we now have

$$\begin{aligned}
\sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X) &\leq \inf_{Q \in \mathcal{Q}_P} \sup_{X \in \mathcal{X}_P(x)} E_Q u(X) \\
&\leq \sup_{X \in \mathcal{X}_P(x)} E_{Q_P} u(X) \\
&= \inf_{\lambda > 0} \left\{ \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda \frac{dP}{dQ} \right) + \lambda x \right\} \\
&= \inf_{\lambda > 0} \left\{ E_{Q_P} v \left( \lambda \frac{dP}{dQ_P} \right) + \lambda x \right\} \\
&= \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda_P(x) \frac{dP}{dQ} \right) + \lambda_P(x) x \\
&= E_{Q_P} u \left( I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \right) \\
&= \inf_{Q \in \mathcal{Q}_P} E_Q u \left( I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \right).
\end{aligned}$$

Now Assumption (A2) guarantees that  $I \left( \lambda_P(x) \frac{dP}{dQ_P} \right) \in \mathcal{X}_P(x)$ , which completes the proof.

□

**Remark 2** *Using methods from robust statistics Schied [19] obtains the corresponding result in the form of Kramkov and Schachermayer [13] for the complete market case under the condition that there exists a measure  $Q_P$  which is the reverse  $f$ -projection for every convex function  $f$ .*

For obtaining a nicer interpretation of our results, we state the following

**Lemma 3** *Assume that*

$$I \left( \lambda \frac{dP}{dQ} \right) \in L^1(P) \text{ for all } Q \in \mathcal{Q} \text{ and all } \lambda > 0. \quad (\text{A3})$$

*Then we have for  $\lambda > 0$  and  $x > \bar{x}$ ,*

$$\begin{aligned}
\mathcal{Q}_P(\lambda) &:= \{Q \in \mathcal{Q} : v_\lambda(P|Q) < \infty\} \\
&= \{Q \in \mathcal{Q} : \sup_{X \in \mathcal{X}_P(x)} E_Q u(X) < \infty\},
\end{aligned}$$

*and  $\mathcal{Q}_P(\lambda)$  is independent of  $\lambda$  and  $x$ .*

*Proof* Under Assumption (A3), (7) and (8) in the proof of the last proposition hold with the corresponding  $\lambda_P(x)$  for all  $Q \in \mathcal{Q}$  (instead of  $Q_P$ ). Therefore, we get the following implications for  $Q \in \mathcal{Q}$ :

1. If  $\sup_{X \in \mathcal{X}_P(x)} E_Q u(X) < \infty$  for some  $x > \bar{x}$ , then by (8)  $v_\lambda(P|Q) < \infty$  for some  $\lambda > 0$ .
2. If  $v_\lambda(P|Q) < \infty$  for some  $\lambda > 0$ , then by (7)  $\sup_{X \in \mathcal{X}_P(x)} E_Q u(X) < \infty$  for all  $x > \bar{x}$ .
3. If  $\sup_{X \in \mathcal{X}_P(x)} E_Q u(X) < \infty$  for all  $x > \bar{x}$ , then, since for all  $\lambda > 0$ , we have  $E_P I\left(\lambda \frac{dP}{dQ}\right) = x \in (\bar{x}, \infty)$  and hence,  $\lambda = \lambda_P(x)$  for some  $x > \bar{x}$ , we now get from (8) that  $v_\lambda(P|Q) < \infty$  for all  $\lambda > 0$ .
4. From  $v_\lambda(P|Q) < \infty$  for all  $\lambda > 0$  follows, of course, that  $\sup_{X \in \mathcal{X}_P(x)} E_Q u(X) < \infty$  for some  $x > \bar{x}$ .

Thus, we have proved the lemma. □

**Remark 3** (i) If (A3) holds, then the representation of the set  $\mathcal{Q}_P(\lambda)$  in Lemma 3 leads to the following interpretation of the utility maximization problem: The agent considers only those subjective measures  $Q$  that generate a finite maximum expected utility  $\sup_{X \in \mathcal{X}_P(x)} E_Q u(X)$  and therefore wants to maximize  $\inf_{Q \in \mathcal{Q}_P} E_Q u(X)$  and not  $\inf_{Q \in \mathcal{Q}} E_Q u(X)$ .

(ii) Condition (A3) is always satisfied for the logarithmic utility function. Indeed, for  $u(x) = \log x$ , we have  $I(x) = 1/x$  and hence,  $E_P I\left(\lambda \frac{dP}{dQ}\right) = 1/\lambda$ .

For the exponential utility function,  $u(x) = 1 - e^{-x}$ , we have  $I(x) = -\log x$  and  $v(x) := 1 - x + x \log x$  and hence,  $E_P I\left(\lambda \frac{dP}{dQ}\right) = -\log \lambda - v(P|Q)$ . Thus, (A3) is satisfied if and only if  $v(P|Q) < \infty$  for all  $Q \in \mathcal{Q}$ .

For the power utility function with  $u(x) = \frac{x^p}{p}$ ,  $p \in (0, 1)$ , we have  $\bar{x} = 0$ ,  $I(x) = x^{\frac{1}{p-1}}$ , and  $v(x) = \frac{1-p}{p} x^{\frac{p}{p-1}}$ . Hence,  $E_P I\left(\lambda \frac{dP}{dQ}\right) = \lambda^{\frac{1}{1-p}} \frac{p}{1-p} v(P|Q)$ , and we see that (A3) is satisfied if and only if  $v(P|Q) < \infty$  for all  $Q \in \mathcal{Q}$ .

So in the latter two cases, we could replace  $\mathcal{Q}$  by the convex subset  $\{Q \in \mathcal{Q} : v(P|Q) < \infty\}$ . Then, for the common utility functions, we may always interpret the set  $\mathcal{Q}_P(\lambda)$  as in Lemma 3.

(iii) As Goll and Rüschendorf [10] remark, the  $v_\lambda$ -projections are independent of  $\lambda$  for the latter three utility functions. It is obvious from the expressions in (ii) that also the reverse  $v_\lambda$ -projections are independent of  $\lambda$  in these cases.

## 5 Duality Results for an Incomplete Market Model

In this section we assume that the market model is incomplete. This means that, instead of a single measure  $P$ , we have a whole set  $\mathcal{P}$  of equivalent local martingale measures. We start

with a definition of a minimax measure in our robust setting. We define

$$U_P(x) := \sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X)$$

where  $\mathcal{Q}_P := \mathcal{Q}_P(\lambda_P(x))$  as in Proposition 2.

**Definition 3** A measure  $P^* = P^*(x) \in \mathcal{P}$  is called a robust minimax measure for  $x$  if

$$U_{P^*}(x) = U(x) := \inf_{P \in \mathcal{P}} \sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X).$$

**Theorem 2** Let  $x > \bar{x}$  and assume that (A1) and (A2) hold for all  $P \in \mathcal{P}$ .

(i) If  $\lambda(x) \in \partial U(x)$ , then

$$U(x) := \inf_{P \in \mathcal{P}} \sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X) = v_{\lambda(x)}(\mathcal{P}|\mathcal{Q}) + \lambda(x)x.$$

(ii)  $P^* \in \mathcal{P}$  is a robust  $v_{\lambda(x)}$ -projection on  $\mathcal{P}$  for some  $\lambda(x) \in \partial U(x)$  if and only if  $P^*$  is a robust minimax measure.

(iii) If a robust minimax measure  $P^*$  exists, then we have  $E_{P^*} I\left(\lambda(x) \frac{dP^*}{dQ^*}\right) = x$ , and the solution to the robust utility maximization problem is

$$X^* := I\left(\lambda(x) \frac{dP^*}{dQ^*}\right)$$

where  $Q^* := Q_{P^*}$  is the reverse  $v_{\lambda(x)}$ -projection of  $P^*$ .

(iv) If a robust minimax measure  $P^*$  exists, then

$$\inf_{P \in \mathcal{P}} \sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X) = \sup \left\{ \inf_{Q \in \mathcal{Q}_{P^*}} E_Q u(X) : \sup_{P \in \mathcal{P}_{Q^*}} E_P X \leq x \right\}$$

where

$$\mathcal{P}_{Q^*} := \{P \in \mathcal{P} : v_{\lambda(x)}(P|Q^*) < \infty\}$$

and

$$\mathcal{Q}_{P^*} := \{Q \in \mathcal{Q} : v_{\lambda(x)}(P^*|Q) < \infty\}.$$

If furthermore, Assumption (A3) holds for all  $P \in \mathcal{P}$ , then

$$\mathcal{P}_{Q^*} = \{P \in \mathcal{P} : \sup_{X \in \mathcal{X}_P(x)} E_{Q^*} u(X) < \infty\}$$

and

$$\mathcal{Q}_{P^*} = \{Q \in \mathcal{Q} : \sup_{X \in \mathcal{X}_{P^*}(x)} E_Q u(X) < \infty\},$$

and the two sets are independent of  $\lambda$  and  $x$ .

**Remark 4** (i)  $Q^*$  is the measure that minimizes  $\inf_{P \in \mathcal{P}} v_{\lambda(x)}(P|Q)$  over all  $Q \in \mathcal{Q}$ . Maximizing the utility functional  $E_{Q^*} u(X)$  is equivalent to maximizing  $\inf_{Q \in \mathcal{Q}_{P^*}} E_Q u(X)$ . By (ii) the incomplete market case is reducible to the complete market case under the measure  $P^*$ . Hence, the robust utility maximization problem in an incomplete market can be reduced to the standard problem under  $Q^*$  in a complete market with the equivalent local martingale measure  $P^*$ .

(ii) The representation of the utility maximization problem in (iv) shows that under Condition (A3), if we price contingent claims under the robust  $v_{\lambda(x)}$ -projection  $P^*$  we make sure that the optimal claim is affordable under all measures that are contained in the set  $\mathcal{P}_{Q^*}$ , i.e., that generate a finite maximum expected utility under  $Q^*$ .

*Proof* (i) From Proposition 2 we get

$$\begin{aligned} U(x) &:= \inf_{P \in \mathcal{P}} \sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q X \\ &= \inf_{P \in \mathcal{P}} \inf_{\lambda > 0} \left\{ \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda \frac{dP}{dQ} \right) + \lambda x \right\} \\ &= \inf_{\lambda > 0} \left\{ \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda \frac{dP}{dQ} \right) + \lambda x \right\} \\ &= \inf_{\lambda > 0} \{v_\lambda(\mathcal{P}|\mathcal{Q}) + \lambda x\}. \end{aligned}$$

Let us define

$$H(\lambda) := v_\lambda(\mathcal{P}|\mathcal{Q}).$$

We now want to show that  $H$  is convex. To this end, let  $\epsilon > 0$  be fixed and choose  $\lambda_1, \lambda_2 > 0$ ,  $P_1, P_2 \in \mathcal{P}$ , and  $Q_1, Q_2 \in \mathcal{Q}$  such that

$$H(\lambda_i) + \epsilon \geq E_{Q_i} v \left( \lambda_i \frac{dP_i}{dQ_i} \right)$$

for  $i = 1, 2$ . For  $\gamma \in (0, 1)$ , define  $\tilde{Q} := \gamma Q_1 + (1 - \gamma) Q_2 \in \mathcal{Q}$ . Then

$$\frac{dQ_1}{d\tilde{Q}} = \left( \gamma + (1 - \gamma) \frac{dQ_2}{dQ_1} \right)^{-1},$$

$$\frac{dQ_2}{d\tilde{Q}} = \left( 1 - \gamma + \gamma \frac{dQ_1}{dQ_2} \right)^{-1},$$

and

$$\gamma \frac{dQ_1}{d\tilde{Q}} + (1 - \gamma) \frac{dQ_2}{d\tilde{Q}} = 1.$$

Therefore,

$$\begin{aligned} & \gamma H(\lambda_1) + (1 - \gamma)H(\lambda_2) + 2\epsilon \\ & \geq \gamma E_{Q_1} v \left( \lambda_1 \frac{dP_1}{dQ_1} \right) + (1 - \gamma) E_{Q_2} v \left( \lambda_2 \frac{dP_2}{dQ_2} \right) \\ & = E_{\tilde{Q}} \left[ \gamma \frac{dQ_1}{d\tilde{Q}} v \left( \lambda_1 \frac{dP_1}{dQ_1} \right) + (1 - \gamma) \frac{dQ_2}{d\tilde{Q}} v \left( \lambda_2 \frac{dP_2}{dQ_2} \right) \right] \\ & \stackrel{v \text{ convex}}{\geq} E_{\tilde{Q}} v \left( \lambda_1 \gamma \frac{dQ_1}{d\tilde{Q}} \frac{dP_1}{dQ_1} + \lambda_2 (1 - \gamma) \frac{dQ_2}{d\tilde{Q}} \frac{dP_2}{dQ_2} \right) \\ & = E_{\tilde{Q}} v \left( \lambda_1 \gamma \frac{dP_1}{d\tilde{Q}} + \lambda_2 (1 - \gamma) \frac{dP_2}{d\tilde{Q}} \right) \\ & \stackrel{\mathcal{P} \text{ convex}}{\geq} \inf_{P \in \mathcal{P}} E_{\tilde{Q}} v \left( (\lambda_1 \gamma + \lambda_2 (1 - \gamma)) \frac{dP}{d\tilde{Q}} \right) \\ & \geq \inf_{Q \in \mathcal{Q}} \inf_{P \in \mathcal{P}} E_Q v \left( (\lambda_1 \gamma + \lambda_2 (1 - \gamma)) \frac{dP}{dQ} \right) \\ & = H(\gamma \lambda_1 + (1 - \gamma) \lambda_2). \end{aligned}$$

Statement (i) now follows from results due to Rockafellar: By [17], Theorem 23.5,  $\inf_{\lambda > 0} \{H(\lambda) + \lambda x\}$  achieves its infimum in  $\lambda = \lambda(x)$  if and only if  $-x \in \partial H(\lambda(x))$  which is by [17], Theorem 7.4 and Corollary 23.5.1, equivalent to  $\lambda(x) \in \partial U(x)$ .

(ii) By Proposition 2 we have for all  $P \in \mathcal{P}$ ,

$$\inf_{\lambda > 0} \left\{ \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda \frac{dP}{dQ} \right) + \lambda x \right\} = \sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X).$$

Thus, the robust minimax measure coincides with the robust  $v_{\lambda(x)}$ -projection where  $\lambda(x) \in \partial U(x)$  minimizes  $H(\lambda) + \lambda x = v_{\lambda}(\mathcal{P}|\mathcal{Q}) + \lambda x$ .

Statement (iii) now follows from Proposition 2.

To prove (iv) we apply Theorem 1:  $P^*$  is an  $\inf_{Q \in \mathcal{Q}} v_{\lambda(x)}$ -projection if and only if

$$\int I \left( \lambda(x) \frac{dP^*}{dQ^*} \right) (dP^* - dP) \geq 0 \quad \forall P \in \mathcal{P} \text{ with } v_{\lambda(x)}(P|Q^*) < \infty.$$

(Recall that  $v'(x) = -I(x)$ .) By (ii) and Proposition 2(i)  $E_{P^*}I\left(\lambda(x)\frac{dP^*}{dQ^*}\right) = x$ . Thus, we have

$$E_P I\left(\lambda(x)\frac{dP^*}{dQ^*}\right) \leq x \quad \forall P \in \mathcal{P} \text{ with } v_{\lambda(x)}(P|Q^*) < \infty.$$

Since  $I\left(\lambda(x)\frac{dP^*}{dQ^*}\right)$  is the optimal claim it follows that

$$\inf_{P \in \mathcal{P}} \sup_{X \in \mathcal{X}_P(x)} \inf_{Q \in \mathcal{Q}_P} E_Q u(X) = \sup \left\{ \inf_{Q \in \mathcal{Q}_{P^*}} E_Q u(X) : \sup_{P \in \mathcal{P}_{Q^*}} E_P X \leq x \right\}$$

where

$$\mathcal{P}_{Q^*} := \{P \in \mathcal{P} : v_{\lambda(x)}(P|Q^*) < \infty\}$$

and

$$\mathcal{Q}_{P^*} := \{Q \in \mathcal{Q} : v_{\lambda(x)}(P^*|Q) < \infty\}.$$

The interpretation of the set  $\mathcal{P}_{Q^*}$  follows from the inequalities in the proof of Lemma 2 in exactly the same way as the interpretation of the set  $\mathcal{Q}_P$ .  $\square$

**Remark 5** *We did not use the fact that  $\mathcal{P}$  is the set of equivalent local martingale measures. Hence,  $\mathcal{P}$  could be any set of equivalent measures.*

## 6 Existence and Uniqueness of the Robust Minimax Measure

In this section we want to discuss the existence of the robust minimax measure or  $v$ -projection and show that there is at most one such measure.

Let us start with the problem of uniqueness. In order to keep notations simpler, we will replace  $f$  by  $v$  and consider  $v$ -projections in this section. If  $v$  is strictly convex, then so is  $\hat{v}$ . Hence, for every  $Q \in \mathcal{Q}$  there is at most one  $v$ -projection  $P_Q$  of  $Q$  on  $\mathcal{P}$ , and for every  $P \in \mathcal{P}$  there is at most one reverse  $v$ -projection  $Q_P$  of  $P$  on  $\mathcal{Q}$ . For the robust  $v$ -projection, this does not necessarily hold true. However, in our setting where  $\mathcal{Q}$  is weakly compact by Liese and Vajda [14], Proposition 8.4, there exists a reverse  $v$ -projection  $Q_P \in \mathcal{Q}$  to every  $P \in \mathcal{P}$ , and we get

**Proposition 3** *If a robust  $v$ -projection  $P^* \in \mathcal{P}$  exists, let  $Q^*$  be its reverse  $v$ -projection. Then the density  $\frac{dP^*}{dQ^*}$  is  $Q_0$ -almost surely unique.*



*Proof* Assume that  $P_1$  and  $P_2 \in \mathcal{P}$  are two robust  $v$ -projections with reverse  $v$ -projections  $Q_{P_1}$  and  $Q_{P_2}$ . Then we have

$$\begin{aligned} \gamma \inf_{Q \in \mathcal{Q}} v(P_1|Q) + (1 - \gamma) \inf_{Q \in \mathcal{Q}} v(P_2|Q) &= \inf_{Q \in \mathcal{Q}} v(P_1|Q) \\ &\leq \inf_{Q \in \mathcal{Q}} v(\gamma P_1 + (1 - \gamma)P_2|Q). \end{aligned} \quad (9)$$

On the other hand we get with the same arguments and definition of  $\tilde{Q}$  as in the proof of Theorem 2

$$\begin{aligned} \gamma \inf_{Q \in \mathcal{Q}} v(P_1|Q) + (1 - \gamma) \inf_{Q \in \mathcal{Q}} v(P_2|Q) &= \gamma v(P_1|Q_{P_1}) + (1 - \gamma)v(P_2|Q_{P_2}) \\ &\geq E_{\tilde{Q}} v \left( \gamma \frac{dP_1}{d\tilde{Q}} + (1 - \gamma) \frac{dP_2}{d\tilde{Q}} \right) \\ &\geq \inf_{Q \in \mathcal{Q}} v(\gamma P_1 + (1 - \gamma)P_2|Q). \end{aligned}$$

The first inequality holds as equality if and only if  $\frac{dP_1}{dQ_{P_1}} = \frac{dP_2}{dQ_{P_2}} \tilde{Q}$ - and hence also  $Q_0$ -almost surely. Thus, by (9) the density is  $Q_0$ -almost surely unique.  $\square$

Let us now turn to the problem of existence. As shown by Kramkov and Schachermayer [13], the robust minimax measure  $P^*$  does not necessarily exist even in the classical setting with  $\mathcal{Q} = \{Q\}$ . In one of their counterexamples, they show that even for a bounded price process the infimum of  $v(P|Q)$  may not be attained in  $\mathcal{P}$  if the utility function is logarithmic ([13], Example 5.1 BIS). On the other hand, they show in the classical setting  $\mathcal{Q} = \{Q\}$  that even if there is no minimax measure, a solution of the utility maximization problem may still exist. Let  $\mathcal{Y} := \{Y \geq 0 : Y_0 = 1 \text{ and } XY \text{ is a supermartingale for all } X \geq 0 \text{ that are stochastic integrals of the underlying semimartingale}\}$  and let  $\bar{x} = 0$ . Kramkov and Schachermayer [13] prove that a solution  $X_Q^*$  to the utility maximization problem exists if the asymptotic elasticity of the utility function,  $\limsup_{x \rightarrow \infty} xu'(x)/u(x)$ , is strictly less than one. Furthermore, they show that in this case we have  $V(\lambda) := \inf_{Y \in \mathcal{Y}} E_Q v(\lambda Y) = \inf_{P \in \mathcal{P}} E_Q v \left( \lambda \frac{dP}{dQ} \right)$ , but there does not necessarily exist a  $v_\lambda$ -projection. Schied and Wu [20] extend these results to the robust setting and obtain existence of the solution to the robust utility maximization problem under the condition that the asymptotic elasticity is strictly less than one.

In our paper however, the focus is on the representation of the optimal claim in terms of martingale measures. In this section we will specify conditions which guarantee that the infimum of  $\inf_{Q \in \mathcal{Q}} v(P|Q)$  over  $\mathcal{P}$  is indeed attained by some measure  $P \in \mathcal{P}$ .

For a certain class of utility functions, existence results for the classical case  $\mathcal{Q} = \{Q\}$  can

be found in the literature. In 1975, Csiszár [1] showed existence for the case where the set of local equivalent martingale measures  $\mathcal{P}$  is closed in variation and  $v(x) = x \log x$  which is equivalent to  $u(x) = -e^{-x}$ . In 1987, Liese and Vajda [14] considered the case where  $\mathcal{P}$  is closed in variation and  $\lim_{x \rightarrow \infty} v(x)/x = \infty$ . In 2000, Frittelli [8] proved existence for the case where the semimartingale is locally bounded and  $u(x) = -e^{-x}$ . As he remarks, if the semimartingale is locally bounded, then the set  $\mathcal{P}$  is closed in variation. In 2002, Bellini and Frittelli [3] showed existence under the assumption that the semimartingale is locally bounded and the domain of the utility function is  $\mathbb{R}$ . The last condition means that  $\bar{x} = -\infty$ , and it is easily checked that this is equivalent to the condition  $\lim_{x \rightarrow \infty} v(x)/x = \infty$ .

Here we will consider the existence problem for robust  $v$ -projections. Let  $\mathcal{M}_1(\Omega)$  be the set of all probability measures on  $(\Omega, \mathcal{F})$ . In the general setting where  $\mathcal{Q}$  does not only consist of one measure, we obtain the following result:

**Proposition 4** *Let  $g : [0, \infty) \rightarrow \mathbb{R}$  satisfy  $\lim_{x \rightarrow \infty} g(x)/x = \infty$ . Assume that  $\mathcal{P}$  is closed in variation and that there exist  $Q_0 \in \mathcal{Q}$  and constants  $c_0, c_1 > 0$  such that for  $P \in \mathcal{P}$ ,*

$$\inf_{Q \in \mathcal{Q}} v(P|Q) \leq c_0 \quad \implies \quad g(P|Q_0) \leq c_1. \quad (10)$$

*Then there exists a robust  $v$ -projection  $P^*$  on  $\mathcal{P}$ .*

*Proof* We will first show that under Condition (10), the closure of the set

$$\mathcal{P}_0 := \left\{ P \in \mathcal{P} : \inf_{Q \in \mathcal{Q}} v(P|Q) \leq c_0 \right\} \quad (11)$$

is compact in the weak topology for measures. Since  $E_{Q_0} g \left( \frac{dP}{dQ_0} \right) \leq c_1$  if  $\inf_{Q \in \mathcal{Q}} v(P|Q) \leq c_0$ , the set  $\mathcal{K} := \left\{ \frac{dP}{dQ_0} : P \in \mathcal{P}_0 \right\}$  is uniformly integrable due to the de la Vallée-Poussin criterion. Hence, by Dunford and Schwartz [4], Corollary IV.8.11,  $\mathcal{K}$  is weakly sequentially compact on  $L^1(Q_0)$ . By [4], Theorem V.6.1, the weak closure of the set  $\mathcal{K}$  is weakly compact. By [4], Theorem V.3.13, it is contained in the convex and strongly closed set of densities of measures is  $\mathcal{P}$  with respect to  $Q_0$ . Hence, the closure  $\bar{\mathcal{P}}_0$  of  $\mathcal{P}_0$  is weakly compact in the topology on  $\mathcal{M}_1(\Omega)$  that corresponds to the weak topology on  $L^1(Q_0)$ , and it is contained in  $\mathcal{P}$ . Since this topology is stronger than the weak topology for measures (cf. Liese and Vajda [14], Lemma 1.46),  $\bar{\mathcal{P}}_0$  is compact with respect to the weak topology for measures.

Now, by [14], Theorem 1.47,  $v(\cdot|\cdot)$  is lower semicontinuous on the space  $\mathcal{M}_1(\Omega) \times \mathcal{M}_1(\Omega)$  endowed with the weak product topology. Since by Tychonov's theorem  $\bar{\mathcal{P}}_0 \times \mathcal{Q}$  is weakly compact,  $v$  achieves its infimum  $(P^*, Q^*)$  on  $\bar{\mathcal{P}}_0 \times \mathcal{Q} \subseteq \mathcal{P} \times \mathcal{Q}$ .  $\square$

Let us illustrate the application of Proposition 4 in the case where  $\mathcal{Q}$  is a weakly compact set such that

$$\sup_{Q \in \mathcal{Q}} E_{Q_0} \left[ \left( \frac{dQ}{dQ_0} \right)^\beta \right] < \infty \quad (12)$$

for some  $\beta > 1$ , and where

$$\lim_{x \rightarrow \infty} \frac{v(x)}{x^p} = \infty \quad (13)$$

for some  $p > 1$ . The following lemma shows that Condition (10) is satisfied with  $g(x) = x^\alpha$  for some  $\alpha > 1$ . Recall that the reverse  $v$ -projection exists for every  $P \in \mathcal{P}$  due to Liese and Vajda [14] because  $\mathcal{Q}$  is weakly compact.

**Lemma 4** *Under Assumptions (12) and (13), there is an  $\alpha > 1$  such that*

$$\sup \left\{ E_{Q_0} \left[ \left( \frac{dP}{dQ_0} \right)^\alpha \right] : v(P|Q_P) = \inf_{Q \in \mathcal{Q}} v(P|Q) \leq c \right\} < \infty$$

for any constant  $c > 0$ .

*Proof* Define  $\alpha := \frac{\beta p}{p-1+\beta}$ . Then  $\alpha > 1$  and  $\beta = 1 + \frac{(\alpha-1)p}{p-\alpha}$ . Due to Assumption (13) there are constants  $a_0 \in \mathbb{R}$ ,  $a_1 > 0$  such that  $v(x) \geq a_0 + a_1 x^p$  for  $x \geq 0$ . Let  $\tilde{p} := p/\alpha$  and  $\tilde{q} = p/(p-\alpha)$ . Then  $1/\tilde{p} + 1/\tilde{q} = 1$ , and by Hölder's inequality we have

$$\begin{aligned} E_{Q_0} \left[ \left( \frac{dP}{dQ_0} \right)^\alpha \right] &= E_{Q_P} \left[ \left( \frac{dP}{dQ_P} \right)^\alpha \left( \frac{dQ_P}{dQ_0} \right)^{\alpha-1} \right] \\ &\leq E_{Q_P} \left[ \left( \frac{dP}{dQ_P} \right)^{\alpha \tilde{p}} \right]^{1/\tilde{p}} E_{Q_P} \left[ \left( \frac{dQ_P}{dQ_0} \right)^{(\alpha-1)\tilde{q}} \right]^{1/\tilde{q}} \\ &= E_{Q_P} \left[ \left( \frac{dP}{dQ_P} \right)^p \right]^{\alpha/p} E_{Q_0} \left[ \left( \frac{dQ_P}{dQ_0} \right)^\beta \right]^{(p-\alpha)/p} \\ &\leq \left( \frac{1}{a_1} v(P|Q_P) - \frac{a_0}{a_1} \right)^{\alpha/p} E_{Q_0} \left[ \left( \frac{dQ_P}{dQ_0} \right)^\beta \right]^{(p-\alpha)/p} \\ &\leq \left( \frac{c - a_0}{a_1} \right)^{\alpha/p} c_1^{(p-\alpha)/p} \end{aligned}$$

if  $v(P|Q_P) \leq c$  and  $c_1 := \sup_{Q \in \mathcal{Q}} E_{Q_0} \left[ \left( \frac{dQ}{dQ_0} \right)^\beta \right]$ . □

**Remark 6** *Similar to the equivalence of the two conditions  $\bar{x} = -\infty$  and  $\lim_{x \rightarrow \infty} v(x)/x = \infty$  above, (13) is equivalent to the condition that the utility function  $u$  satisfies  $u(x) \geq -c|x|^{\frac{p}{p-1}-\epsilon}$  for some  $\epsilon > 0$  and for  $x < 0$ . In fact, sufficiency of the condition on  $u$  follows from the estimate*

$$\begin{aligned} \liminf_{x \rightarrow \infty} \frac{v(x)}{x^p} &= \liminf_{x \rightarrow \infty} \sup_{y \in \mathbb{R}} \left( \frac{u(y)}{x^p} - yx^{1-p} \right) \\ &= \liminf_{x \rightarrow \infty} \sup_{y \in \mathbb{R}} \left( \frac{u(yx)}{x^{p/(p-1)}} - y \right) \\ &\geq \sup_{y \in \mathbb{R}} \left( \liminf_{x \rightarrow \infty} \frac{u(yx)}{x^{p/(p-1)}} - y \right) \\ &= \sup_{y \in \mathbb{R}} (-y) = \infty. \end{aligned}$$

*The necessity can be shown in a similar way. Hence, the utility function must decrease slower than some power function. An example for such a utility function is*

$$u(x) = \begin{cases} \frac{1}{q}(x+c)^q - \frac{c^q}{q} & \text{for } x \geq 0 \\ -q(c^{-q} - x)^{1/q} + \frac{q}{c} & \text{for } x < 0 \end{cases}$$

*for some constant  $c > 0$  and  $0 < q < 1$ .*

The following general result is shown in Föllmer and Gundel [5]. It includes all utility functions that are defined on the whole real line, thus also the case of exponential utility functions. The proof is more involved; instead of Hölder's inequality it uses Young's inequality for certain Orlicz spaces.

**Theorem 3** *Assume that  $\mathcal{P}$  is closed in variation, that*

$$\lim_{x \rightarrow \infty} \frac{v(x)}{x} = \infty,$$

*and that*

$$\left\{ \frac{dQ}{dQ_0} : Q \in \mathcal{Q} \right\}$$

*is weakly compact in  $L^1(Q_0)$  for some measure  $Q_0 \in \mathcal{Q}$ . Then there exists a robust  $v$ -projection  $P^*$  on  $\mathcal{P}$ .*

**Remark 7** *In the classical setting with  $\mathcal{Q} = \{Q\}$ , Condition (10) is trivially satisfied for  $g = v$  and hence by Lemma 4, the  $v$ -projection exists if  $\lim_{x \rightarrow \infty} v(x)/x = \infty$ . This is also shown by Liese and Vajda [14] in Proposition 8.5. Theorem 3 provides the natural extension to the robust setting.*

## 7 An Example

There are, in certain situations, means of determining the  $f$ -projection  $P_Q$  explicitly. This suggests the following method for finding the robust  $f$ -projection  $P^*$ : First calculate  $P_Q$  for each  $Q$  and then find the pair  $(P_Q, Q)$  that has the smallest  $f$ -divergence. Here, we want to give an example for this approach.

In diffusion models for financial markets it is feasible to estimate the volatility of assets using historical data. However, estimations of the drift are much less reliable. Let us consider an example of a model in which the volatility and the structure of the drift are known, but there is uncertainty about the size of the drift.

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, Q_0)$  be a two-dimensional Wiener space on which we are given two independent Brownian motions  $B = (B_t)_{0 \leq t \leq T}$  and  $W = (W_t)_{0 \leq t \leq T}$  with  $B_0 = W_0 = 0$ . We assume that  $\mathcal{F} = \mathcal{F}_T$  and that  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is the smallest filtration that contains the filtration which is generated by the two Brownian motions and that satisfies the usual condition (see [12]). The price process of an asset is modelled by

$$dS_t = S_t(\sigma_t dB_t + \mu_t dt) \quad (0 \leq t \leq T).$$

Finding equivalent local martingale measures for this model is equivalent to determining them for the model

$$\tilde{S}_t := B_t + \int_0^t \alpha_s ds \quad (0 \leq t \leq T)$$

with  $\alpha = \mu/\sigma$ . We assume that the process  $\alpha = (\alpha_t)_{0 \leq t \leq T}$  is  $B$ -integrable and predictable with respect to the filtration  $(\mathcal{F}_t^W)_{0 \leq t \leq T}$  that is generated by  $W$ .

For some interval  $[b_1, b_2] \subseteq \mathbb{R}_+$ , we define  $\mathcal{Q}$  as the set of measures under which  $S$  has a drift of  $b\mu$ , or  $\tilde{S}$  has a drift of  $b\alpha$  for some  $b \in [b_1, b_2]$ , i.e.,

$$\mathcal{Q} := \left\{ Q_b : \frac{dQ_b}{dQ_0} = \mathcal{E} \left( (b-1) \int_0^T \alpha_s dB_s \right) \text{ for some } b \in [b_1, b_2] \right\},$$

where  $\mathcal{E}$  is the Itô exponential

$$\mathcal{E} \left( (b-1) \int_0^T \alpha_s dB_s \right) = \exp \left( (b-1) \int_0^T \alpha_s dB_s - \frac{(b-1)^2}{2} \int_0^T \alpha_s^2 ds \right).$$

We are considering the three utility functions  $\log x$ ,  $-e^{-x}$ , and  $x^p$  for  $0 < p < 1$ . To solve the dual problems we have to deal with the  $f$ -divergences  $-\log x$ ,  $x \log x$ , and  $x^q$  where  $q := \frac{p}{p-1} < 0$ . We assume that suitable integrability conditions are satisfied for each of the

utility functions such that the densities in the following define equivalent measures and hence, the  $f$ -projections exist.

- (i)  $f(x) = -\log x$ . For each  $Q_b \in \mathcal{Q}$ , the  $f$ -projection  $P_{Q_b}$  has the density

$$\frac{dP_{Q_b}}{dQ_b} = \mathcal{E} \left( - \int_0^T b \alpha_s dB_s^{(b)} \right),$$

i.e.,  $P_{Q_b}$  coincides with the minimal martingale measure (see Föllmer and Schweizer [7]).  $B^{(b)}$  is the Brownian motion under the measure  $Q_b$ . This result was proved by Schweizer [21] for general  $\alpha$ . The  $f$ -divergence becomes

$$\begin{aligned} f(P_{Q_b}|Q_b) &= E_{Q_b} \left[ \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right] \\ &= E_{Q_0} \left[ E_{Q_0} \left[ \mathcal{E} \left( (b-1) \int_0^T \alpha_s dB_s \right) \middle| \mathcal{F}_T^W \right] \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right] \\ &= \frac{b^2}{2} E_{Q_0} \left[ \int_0^T \alpha_s^2 ds \right]. \end{aligned}$$

The second equality holds due to the  $\mathcal{F}_T^W$ -measurability of  $\int_0^T \alpha_s^2 ds$  and the last equality holds because  $E[\mathcal{E}((b-1) \int_0^T \alpha_s dB_s) | \mathcal{F}_T^W] = 1$  due to the independence of  $B$  from  $\mathcal{F}_T^W$ .

- (ii)  $f(x) = x \log x$ . For each  $Q_b \in \mathcal{Q}$ , the  $f$ -projection  $P_{Q_b}$  has the density

$$\frac{dP_{Q_b}}{dQ_b} = C_b \exp \left( - \int_0^T b \alpha_s dS_s \right)$$

with  $C_b := E_{Q_b} \left[ \exp \left( - \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right) \right]^{-1}$ . This result is due to Grandits and Rheinländer [11]. We have

$$\begin{aligned} f(P_{Q_b}|Q_b) &= -\log \left( E_{Q_b} \left[ \exp \left( - \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right) \right] \right) \\ &= -\log \left( E_{Q_0} \left[ \exp \left( - \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right) \right] \right). \end{aligned}$$

The second equality holds for the same reasons as above.

- (iii)  $f(x) = x^q$ . For each  $Q_b \in \mathcal{Q}$ , the  $f$ -projection  $P_{Q_b}$  has the density

$$\frac{dP_{Q_b}}{dQ_b} = C_b \exp \left( - \int_0^T b \alpha_s dS_s - \frac{q-1}{2} b^2 \int_0^T \alpha_s^2 ds \right)$$

with  $C_b := E_{Q_b} \left[ \exp \left( -q \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right) \right]^{-1}$ . This result was also shown in [11]. We have

$$\begin{aligned} f(P_{Q_b}|Q_b) &= \left( E_{Q_b} \left[ \exp \left( -q \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right) \right] \right)^{(1-q)} \\ &= \left( E_{Q_0} \left[ \exp \left( -q \frac{b^2}{2} \int_0^T \alpha_s^2 ds \right) \right] \right)^{(1-q)}. \end{aligned}$$

Now we see that in any case, the pair of measures that generates the smallest  $f$ -divergence is the one with  $b = b_1$ . In this model,  $Q^* := Q_{b_1}$  is the measure that is closest to a martingale measure in the sense that it has the smallest drift.

Further interesting examples can be found in the paper [19] by Schied.

## 8 Expenditure Minimization

A problem that is closely related to the one of utility maximization is the minimization of expenditures given the agent has a minimum level  $w$  of expected utility. That is, given her subjective probability measure  $Q$  and the equivalent local martingale measure  $P$ , she wants to solve the problem

$$\text{Minimize } E_P Y \text{ under the constraint } E_Q u(Y) \geq w. \quad (14)$$

The key idea for solving this is to define the *reverse utility function*  $\hat{u}$  by the concave conjugate of  $\hat{v}$ :

$$\hat{u}(x) := \inf_{y>0} \{ \hat{v}(y) + xy \} \quad (15)$$

for  $-x \in \mathcal{U} := (\inf_x u(x), \sup_x u(x))$  and to apply Theorem 2 to these transforms. We have

$$\begin{aligned} \hat{I} \left( \frac{1}{x} \right) &:= -\hat{v}' \left( \frac{1}{x} \right) = (\hat{u}')^{-1} \left( \frac{1}{x} \right) = -u(I(x)), \\ \hat{u}(x) &= -u^{-1}(-x), \text{ and} \end{aligned} \quad (16)$$

$$\hat{u} \left( \hat{I} \left( \frac{1}{x} \right) \right) = -I(x).$$

We can replace  $Y$  in (14) by  $-\hat{u}(X)$  and hence  $u(Y)$  by  $-X$  to see that, by interchanging the roles of the sets  $\mathcal{Q}$  and  $\mathcal{P}$ , we can apply Theorem 2 to the transforms  $\hat{u}$  and  $\hat{v}$  to solve the expenditure minimization problem.

Note that  $\hat{u}$  is also a utility function as defined in Section 2 with

$$\hat{u}'(x) \rightarrow 0 \text{ as } x \rightarrow -\inf_{x \in \mathbb{R}} u(x)$$

and

$$\hat{u}'(x) \rightarrow \infty \text{ as } x \rightarrow -\sup_{x \in \mathbb{R}} u(x).$$

**Remark 8** *It might not be possible to define the function  $\hat{u}$  for arbitrarily large  $x$ . This is, for example, the case if  $u(x) = x^p$  for  $0 < p < 1$ , where  $\mathcal{U} = (0, \infty)$ . But as long as the latter two conditions on  $\hat{u}'$  are satisfied, the range of  $\hat{I}$  is  $-\mathcal{U}$ . This is sufficient to guarantee the existence of  $\lambda_P(x)$  in the second step of the proof of Proposition 2, the only point where Conditions (U1) and (U2) were used. So this proof still works and hence, Theorem 2 is valid for the utility function  $\hat{u}$ .*

We need to assume that (A1), (A2), and (A3) hold for the transforms  $\hat{u}$ ,  $\hat{I}$ , and  $\hat{v}$  with the roles of  $\mathcal{P}$  and  $\mathcal{Q}$  interchanged. Define  $\mathcal{P}_Q(\lambda) := \{P \in \mathcal{P} : v_\lambda(P|Q) < \infty\}$  and assume that the  $v_\lambda$ -projection  $P_Q(\lambda)$  of  $Q$  on  $\mathcal{P}$  exists for every  $Q \in \mathcal{Q}$  and  $\lambda > 0$ . Replacing  $\hat{I}$  and  $\hat{u} \circ \hat{I}$  by the terms in (16) and  $\hat{v}$  by  $xv(1/x)$  this leads to the following assumptions for all  $Q \in \mathcal{Q}$ ,

$$v_\mu(P_Q(\lambda)|Q) < \infty \text{ for all } \lambda, \mu > 0, \quad (\text{A4})$$

$$E_P I \left( \lambda \frac{dP_Q(\lambda)}{dQ} \right)^+ < \infty \text{ for all } P \in \mathcal{P}_Q(\lambda) \text{ for all } \lambda > 0, \quad (\text{A5})$$

and

$$u \left( I \left( \lambda \frac{dP}{dQ} \right) \right) \in L^1(Q) \text{ for all } P \in \mathcal{P} \text{ and all } \lambda > 0. \quad (\text{A6})$$

The last assumption is, as (A3), only needed for reasons of economical interpretation in the following. But since this is our aim in this section, we will assume that (A6) holds.

We set

$$\hat{V}_Q(w) := -\inf_{\lambda > 0} \left\{ \inf_{P \in \mathcal{P}} E_P \hat{v} \left( \lambda \frac{dQ}{dP} \right) - \lambda w \right\}.$$

For  $\hat{\lambda}_Q(w) \in \partial \hat{V}_Q(w)$ , we define  $\mathcal{P}_Q := \mathcal{P}_Q(\hat{\lambda}_Q(w))$ ,

$$\mathcal{Y}_Q(w) := \{Y : u(Y) \in L^1(Q), E_Q u(Y) \geq w, E_P Y^+ < \infty \forall P \in \mathcal{P}_Q\},$$

and

$$\hat{U}(w) := \sup_{Q \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}_Q(w)} \sup_{P \in \mathcal{P}_Q} E_P Y.$$

Our aim is to find a contingent claim  $Y^*$  that achieves this infimum.



**Proposition 5** *Let Assumptions (A4), (A5), and (A6) hold and let  $w \in \mathcal{U}$  and  $\hat{\lambda}(w) \in \partial \hat{U}(w)$ . We assume the existence of a measure  $\hat{P}^* \in \mathcal{P}$  that minimizes  $\inf_{Q \in \mathcal{Q}} v_{\hat{\lambda}(w)}(P|Q)$  over all  $P \in \mathcal{P}$ . We denote by  $\hat{Q}^* \in \mathcal{Q}$  the measure that minimizes  $v_{\hat{\lambda}(w)}(\hat{P}^*|Q)$  over all  $Q \in \mathcal{Q}$ .*

(i) *We have the following representation of the expenditure minimization problem in an incomplete market:*

$$\begin{aligned} \hat{U}(w) &:= \sup_{Q \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}_Q(w)} \sup_{P \in \mathcal{P}_Q} E_P Y \\ &= \inf \left\{ \sup_{P \in \mathcal{P}_{\hat{Q}^*}} E_P[Y] : \inf_{Q \in \mathcal{Q}_{\hat{P}^*}} E_Q u(Y) \geq w \right\} \\ &= - \left\{ \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_P \hat{v} \left( \hat{\lambda}(w) \frac{dQ}{dP} \right) - \hat{\lambda}(w) w \right\} \end{aligned}$$

where

$$\mathcal{Q}_{\hat{P}^*} = \{Q \in \mathcal{Q} : \sup_{E_Q u(Y) \geq w} E_{\hat{P}^*} Y > -\infty\}$$

and

$$\mathcal{P}_{\hat{Q}^*} = \{P \in \mathcal{P} : \sup_{E_{\hat{Q}^*} u(Y) \geq w} E_P Y > -\infty\}.$$

(ii) *The solution to the expenditure minimization problem in an incomplete market is given by*

$$Y^* = I \left( \frac{1}{\hat{\lambda}(w)} \frac{d\hat{P}^*}{d\hat{Q}^*} \right).$$

(iii) *Let  $I \left( \lambda(x) \frac{dP^*}{dQ^*} \right)$  be the solution to the robust utility maximization problem as in Theorem 2 where  $\lambda(x) \in \partial U(x)$ ,  $P^*$  is the robust  $v_{\lambda(x)}$ -projection, and  $Q^*$  is the reverse  $v_{\lambda(x)}$ -projection of  $P^*$ . If  $w = E_{Q^*} u \left( I \left( \lambda(x) \frac{dP^*}{dQ^*} \right) \right)$ , then  $I \left( \lambda(x) \frac{dP^*}{dQ^*} \right)$  is also the solution to the expenditure minimization problem.*

(iv) *Let  $I \left( \frac{1}{\hat{\lambda}(w)} \frac{d\hat{P}^*}{d\hat{Q}^*} \right)$  be the solution to the expenditure minimization problem as in (ii). If  $x = -E_{\hat{P}^*} \hat{u} \left( \hat{I} \left( \hat{\lambda}(w) \frac{d\hat{Q}^*}{d\hat{P}^*} \right) \right) = E_{\hat{P}^*} I \left( \frac{1}{\hat{\lambda}(w)} \frac{d\hat{P}^*}{d\hat{Q}^*} \right)$ , then  $I \left( \frac{1}{\hat{\lambda}(w)} \frac{d\hat{P}^*}{d\hat{Q}^*} \right)$  is also the solution to the robust utility maximization problem.*

**Remark 9** (i) *If  $\hat{P}^*$  exists, then the measure  $\hat{Q}^*$  always exists since  $\mathcal{Q}$  is weakly compact.*

(ii) *The middle term in (i) has a nice interpretation: The agent wants to minimize her costs  $\sup_{P \in \mathcal{P}_{\hat{Q}^*}} E_P Y$  under the condition that her utility measured by the robust utility functional is  $w$  at least.*

(iii) The last two statements of this proposition that describe the relationship between the problems of utility maximization and expenditure minimization are a well-known result for the case of non-random payoffs (see, e.g., Mas-Colell et al. [15], Prop. 3.E.1).

*Proof* (i) Since we want to apply Theorem 2 we set  $y := -w$  and define

$$\tilde{U}(y) := -\hat{U}(-y) = \inf_{Q \in \mathcal{Q}} \sup_{Y \in \mathcal{Y}_Q(-y)} \inf_{P \in \mathcal{P}_Q} E_P[-Y]$$

Then  $\partial \hat{U}(w) = \partial \tilde{U}(y)$ . Hence, instead of minimizing  $E_P Y$  we will consider the equivalent problem of maximizing  $E_P[-Y]$ . In order to avoid a too complicated description we write the constraint simply as  $E_Q u(Y) \geq w$  instead of  $Y \in \mathcal{Y}_Q(w)$ .

Define  $X := -u(Y)$  and replace  $Y$  by  $u^{-1}(-X) = -\hat{u}(X)$  in the equation above. We get

$$\tilde{U}(y) = \inf_{Q \in \mathcal{Q}} \sup_{E_Q X \leq y} \inf_{P \in \mathcal{P}_Q} E_P \hat{u}(X).$$

Now we can apply Theorem 2 to obtain

$$\begin{aligned} \tilde{U}(y) &= \inf_{\lambda > 0} \left\{ \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_P \hat{v} \left( \lambda \frac{dQ}{dP} \right) + \lambda y \right\} \\ &= \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_P \hat{v} \left( \hat{\lambda}(-y) \frac{dQ}{dP} \right) + \hat{\lambda}(-y) y \\ &= \sup \left\{ \inf_{P \in \mathcal{P}_{\hat{Q}^*}} E_P \hat{u}(X) : \sup_{Q \in \mathcal{Q}_{\hat{P}^*}} E_Q X \leq y \right\} \end{aligned}$$

where

$$\begin{aligned} \mathcal{Q}_{\hat{P}^*} &:= \{Q \in \mathcal{Q} : \hat{v}_{\hat{\lambda}(-y)}(Q|\hat{P}^*) < \infty\} \\ &= \{Q \in \mathcal{Q} : \sup_{E_Q X \leq y} E_{\hat{P}^*} \hat{u}(X) < \infty\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_{\hat{Q}^*} &:= \{P \in \mathcal{P} : \hat{v}_{\hat{\lambda}(-y)}(\hat{Q}^*|P) < \infty\} \\ &= \{P \in \mathcal{P} : \sup_{E_{\hat{Q}^*} X \leq y} E_P \hat{u}(X) < \infty\} \end{aligned}$$

and  $\hat{\lambda}(-y) \in \partial \tilde{U}(y)$ . Replacing  $-y$  by  $w$ ,  $X$  by  $-u(Y)$ , and  $\hat{u}(X)$  by  $-Y$  completes the proof of (i).

(ii) We get from Theorem 2 that for

$$X^* := \hat{I} \left( \hat{\lambda}(w) \frac{d\hat{Q}^*}{d\hat{P}^*} \right),$$

we have

$$\sup \left\{ \inf_{P \in \mathcal{P}_{\hat{Q}^*}} E_P \hat{u}(X) : \sup_{Q \in \mathcal{Q}_{\hat{P}^*}} E_Q X \leq y \right\} = \inf_{P \in \mathcal{P}_{\hat{Q}^*}} E_P \hat{u}(X^*).$$

Hence, it follows from the proof of (i) that

$$Y^* := u^{-1}(-X^*) = -\hat{u}(X^*) = I \left( \frac{1}{\hat{\lambda}(w)} \frac{d\hat{P}^*}{d\hat{Q}^*} \right)$$

is the solution to the expenditure minimization problem. The last equality follows from  $\hat{u}(\hat{I}(1/y)) = -I(y)$ .

(iii) Let now  $I \left( \lambda(x) \frac{dP^*}{dQ^*} \right)$  be the solution to the robust utility maximization problem. We want to show that for  $w = -y = E_{Q_{P^*}} u \left( I \left( \lambda(x) \frac{dP^*}{dQ^*} \right) \right)$ , we have

$$\inf_{\lambda > 0} \left\{ \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_P \hat{v} \left( \lambda \frac{dQ}{dP} \right) + \lambda y \right\} = \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_P \hat{v} \left( \frac{1}{\lambda(x)} \frac{dQ}{dP} \right) + \frac{y}{\lambda(x)}. \quad (17)$$

Then, according to (i) and (ii),  $I \left( \lambda(x) \frac{dP^*}{dQ^*} \right)$  would also be the solution to the problem of expenditure minimization with the minimum utility level  $w$ .

We define the two convex functions

$$H(\lambda) := \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda \frac{dP}{dQ} \right)$$

and

$$\hat{H}(\lambda) := \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_P \hat{v} \left( \lambda \frac{dQ}{dP} \right).$$

Then  $H(\lambda) = \lambda \hat{H}(1/\lambda)$ .

By Rockafellar [17], Theorem 23.5, Equation (17) is equivalent to  $-y \in \partial(\hat{H}(1/\lambda(x)))$  which is equivalent to

$$\frac{y}{\lambda(x)} + \hat{H} \left( \frac{1}{\lambda(x)} \right) \leq \frac{y}{\lambda} + \hat{H} \left( \frac{1}{\lambda} \right) \quad \forall \lambda > 0. \quad (18)$$

So we will now show that (18) holds. With the definition of  $U$  as in Section 5 we have by [17], Theorem 7.4 and Corollary 23.5.1, that  $\lambda(x) \in \partial U(x)$  if and only if  $-x \in \partial H(\lambda(x))$ .

This is equivalent to

$$x\lambda(x) + H(\lambda(x)) \leq x\lambda + H(\lambda) \quad \forall \lambda > 0. \quad (19)$$

By Theorem 2(iii) we have  $x = E_{P^*} I \left( \lambda(x) \frac{dP^*}{dQ^*} \right)$ . Hence

$$\begin{aligned} y = -w &= -E_{Q^*} u \left( I \left( \lambda(x) \frac{dP^*}{dQ^*} \right) \right) \\ &= -E_{Q^*} v \left( \lambda(x) \frac{dP^*}{dQ^*} \right) - \lambda(x) E_{P^*} I \left( \lambda(x) \frac{dP^*}{dQ^*} \right) \\ &= - \inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} E_Q v \left( \lambda(x) \frac{dP}{dQ} \right) - \lambda(x)x \\ &= -H(\lambda(x)) - \lambda(x)x. \end{aligned}$$

Replacing  $x$  in (19) by  $(-H(\lambda(x)) - y)/\lambda(x)$  leads to (18) which completes the proof.

(iv) now follows from (iii) by interchanging  $u$ ,  $I$ , and  $v$  and its transforms and  $\mathcal{Q}$  and  $\mathcal{P}$ .  $\square$

## References

- [1] Csiszár, I.: I-Divergence Geometry of Probability Distributions and Minimization Problems. *Annals of Probability*, Vol.3, No.1, 146-158 (1975).
- [2] Baudoin, F.: Conditioned Stochastic Differential Equations: Theory, Examples and Application to Finance. *Stochastic Processes and their Applications* 100, 109-145 (2002).
- [3] Bellini, F., Frittelli, M.: On the Existence of Minimax Martingale Measures. *Mathematical Finance* 12, No. 1, 1-21 (2002).
- [4] Dunford, N., Schwartz, J. T.: *Linear Operators. Part 1: General Theory*. New York: Interscience Publishers 1958.
- [5] Föllmer, H., Gundel, A.: On the Existence of Robust Projections in the Class of Martingale Measures. Preprint, Humboldt-Universität zu Berlin (2004).
- [6] Föllmer, H., Schied, A.: *Stochastic Finance (Studies in Mathematics 27)*. Berlin, New York: De Gruyter 2002.
- [7] Föllmer, H., Schweizer, M.: Hedging of Contingent Claims Under Incomplete Information. In: Davis, M., Elliott, R. (eds.): *Applied Stochastic Analysis*. London: Stochastic Monographs 5, 389-414 (1991).

- [8] Frittelli, M.: The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets. *Mathematical Finance*, Vol. 10, No. 1, 39-52 (2000).
- [9] Gilboa, I., Schmeidler, D.: Maxmin Expected Utility with Non-Unique Prior. *Journal of Mathematical Economics* 18, 141-153 (1989).
- [10] Goll, Th., Rüschemdorf, L.: Minimax and Minimal Distance Martingale Measures and Their Relationship to Portfolio Optimization. *Finance and Stochastics* 5, 557-581 (2001).
- [11] Grandits, P., Rheinländer, T.: On the Minimal Entropy Martingale Measure. *Annals of Probability*, Vol. 30, No. 3, 1003-1038 (2002).
- [12] Karatzas, I., Shreve, S.: *Brownian Motion and Stochastic Calculus* (Graduate Texts in Mathematics 113). Second edition, New York, Berlin, Heidelberg: Springer 1991.
- [13] Kramkov, D., Schachermayer, W.: The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets. *Annals of Applied Probability* 9, No. 3, 904-950 (1999).
- [14] Liese, F., Vajda, I.: *Convex Statistical Distances*. Leipzig: Teubner 1987.
- [15] Mas-Colell, A., Whinston, M. D., Green, J. R.: *Microeconomic Theory*. New York: Oxford University Press 1995.
- [16] Neveu, J.: *Martingales à Temps Discret*. Paris: Masson et Cie 1972.
- [17] Rockafellar, R. T.: *Convex Analysis*. Princeton, NJ: Princeton University Press 1970.
- [18] Rüschemdorf, L.: On the Minimum Discrimination Information Theorem. *Statistics & Decisions*, Supplement Issue No. 1, 263-283 (1984).
- [19] Schied, A.: *Optimal Investments for Robust Utility Functionals in Complete Market Models*. Preprint, Technische Universität Berlin (2003).
- [20] Schied, A., Wu, C.-T.: *Duality Theory for Robust Utility Maximization in Incomplete Market Models*. Preprint, Technische Universität Berlin (2004).
- [21] Schweizer, M.: A Minimality Property of the Minimal Martingale Measure. *Statist. Probab. Lett.* 42, 27-31 (1999).