STRUCTURED EIGENVALUE CONDITION NUMBERS

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Abstract. This paper investigates the effect of structure-preserving perturbations on the eigenvalues of linearly and nonlinearly structured eigenvalue problems. Particular attention is paid to structures that form Jordan algebras, Lie algebras, and automorphism groups of a scalar product. Bounds and computable expressions for structured eigenvalue condition numbers are derived for these classes of matrices, which include complex symmetric, pseudo symmetric, persymmetric, skew-symmetric, Hamiltonian, symplectic, and orthogonal matrices. In particular we show that under mild assumptions on the scalar product, the structured and unstructured eigenvalue condition numbers are equal for structures in Jordan algebras. For Lie algebras, the effect on the condition number of incorporating structure varies greatly with the structure. We identify Lie algebras for which structure does not affect the eigenvalue condition number.

Key words. Structured eigenvalue problem, condition number, Jordan algebra, Lie algebra, automorphism group, symplectic, perplectic, pseudo-orthogonal, pseudo-unitary, complex symmetric, persymmetric, perskew-symmetric, Hamiltonian, skew-Hamiltonian, structure preservation.

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1. Introduction.

There is a growing interest in structured perturbation analysis due to the substantial development of algorithms for structured problems. When these algorithms preserve structure (see for example [2], [4], [13], and the literature cited therein) it is often appropriate to consider condition numbers that measure the sensitivity to structured perturbations. In this paper we investigate the effect of structure-preserving perturbations on linearly and nonlinearly structured eigenvalue problems.

Suppose that $\mathbb{S}$ is a class of structured matrices and define the (absolute) *structured condition number* of a simple eigenvalue λ of $A \in \mathbb{S}$ by

$$\kappa(A,\lambda;\mathbb{S}) = \lim_{\epsilon\to 0}\sup\Big\{\frac{|\widehat{\lambda}-\lambda|}{\epsilon} : \widehat{\lambda}\in \mathrm{Sp}(A+E),\ A+E\in\mathbb{S},\ \|E\|\le\epsilon\Big\}, \tag{1.1}$$

where $\mathrm{Sp}(A+E)$ denotes the spectrum of $A+E$ and $\|\cdot\|$ is an arbitrary matrix norm. Let x and y be the normalized right and left eigenvectors associated with λ, i.e.,

$$Ax = \lambda x, \quad y^*A = \lambda y^*, \quad \|x\|_2 = \|y\|_2 = 1. \tag{1.2}$$

Moreover, let $\kappa(A,\lambda) \equiv \kappa(A,\lambda;\mathbb{C}^{n\times n})$ denote the standard unstructured eigenvalue condition number, where n is the dimension of A. Clearly,

$$\kappa(A,\lambda;\mathbb{S}) \le \kappa(A,\lambda).$$

If this inequality is not always close to being attained then $\kappa(A,\lambda)$ may severely overestimate the worst case effect of structured perturbations. Note that the standard eigenvalue condition number allows complex perturbations even if A is real. Our

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definition in (1.1) automatically forces the perturbations to be real when A is real and $\mathbb{S} \subset \mathbb{R}^{n\times n}$.

In this paper we consider the case where $\mathbb{S}$ is a smooth manifold. This covers linear structures and some nonlinear structures, such as orthogonal, unitary and symplectic structures. We show that for such $\mathbb{S}$, the structured problem in (1.1) simplifies to a linearly constrained optimization problem. We obtain an explicit expression for $\kappa(A, \lambda; \mathbb{S})$, thereby extending Higham and Higham's work [11] for linear structures in $\mathbb{C}^{n\times n}$.

Associated with a scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$ are three important classes of structured matrices: an automorphism group, a Lie algebra, and a Jordan algebra. We specialize our results to each of these three classes, starting with the linear structures. We show that under mild assumptions on the scalar product, the structured and unstructured eigenvalue condition numbers are equal for structures in Jordan algebras. For example, this equality holds for real and complex symmetric matrices, pseudo-symmetric, persymmetric, Hermitian, and J-Hermitian matrices. For Lie algebras, the effect on the condition number of incorporating structure varies greatly with the structure. We identify Lie algebras for which structure does not affect the eigenvalue condition number, such as skew-Hermitian structures, and Lie algebras for which the ratio between the unstructured and structured eigenvalue condition number can be large, such as skew-symmetric or perskew-symmetric structures. Our treatment extends and unifies recent work on these classes of matrices by Graillat [9] and Rump [18].

Finally we show how to compute structured eigenvalue condition numbers when $\mathbb{S}$ is the automorphism group of a scalar product. This includes the classes of unitary, complex orthogonal, and symplectic matrices. We provide bounds for the ratio between the structured and unstructured condition number. In particular we show that for unitary matrices this ratio is always equal to 1. This latter result also holds for orthogonal matrices with one exception: when λ is real, the structured eigenvalue condition number is zero.

Note that for $\lambda \neq 0$ a relative condition number, on both data and output spaces, can also be defined, which is just $\kappa(A, \lambda; \mathbb{S})\|A\|/|\lambda|$. Our results comparing the structured and unstructured absolute condition numbers clearly apply without change to the relative condition numbers.

The rest of this paper is organized as follows. Section 2 provides the definition and a computable expression for the structured eigenvalue condition number of a nonlinearly structured matrix. In Section 3, we introduce the scalar products and the associated structures to be considered. Firstly, we treat linear structures (Jordan and Lie algebras) in Section 4 and investigate the corresponding structured condition numbers. Nonlinear structures (automorphism groups) are discussed in Section 5.

2. Structured condition number. It is well known that simple eigenvalues $\lambda \in \mathrm{Sp}(A)$ depend analytically on the entries of A in a sufficiently small open neighborhood $\mathcal{B}_A$ of A [19]. To be more specific, there exists a uniquely defined analytic function $f_\lambda : \mathcal{B}_A \to \mathbb{C}$ so that $\lambda = f_\lambda(A)$ and $\widehat{\lambda} = f_\lambda(A+E)$ is an eigenvalue of $A+E$ for every $A+E \in \mathcal{B}_A$. Moreover, one has the expansion

$$\hat{\lambda} = \lambda + \frac{1}{|y^*x|} y^* E x + O(\|E\|^2). \tag{2.1}$$

Combined with (1.1) this yields

$$\kappa(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|} \lim_{\epsilon\to 0} \sup\Big\{ \frac{|y^*Ex|}{\epsilon} : \ A+E\in\mathbb{S},\ \|E\|\le\epsilon \Big\}. \tag{2.2}$$

The difficulty in obtaining an explicit expression for the supremum in (2.2) depends on the nature of $\mathbb{S}$ and the matrix norm $\|\cdot\|$. For example, when $\|\cdot\|$ is the Frobenius norm or the matrix 2-norm and for unstructured perturbations (i.e., $\mathbb{S}=\mathbb{C}^{n\times n}$), the supremum in (2.2) is attained by $E=\epsilon yx^*$, which implies the well known formula [21]

$$\kappa_\nu(A,\lambda) = 1/|y^*x|, \quad \nu = 2, F.$$

Note that $\kappa_\nu(A,\lambda)\ge 1$ always, but $\kappa_\nu(A,\lambda;\mathbb{S})$ can be less than 1 for $\nu=2,F$.

When $\mathbb{S}$ is a smooth manifold (see [12] for an introduction to smooth manifolds), the task of computing the supremum (2.2) simplifies to a linearly constrained optimization problem.

THEOREM 2.1. *Let λ be a simple eigenvalue of $A\in\mathbb{S}$, where $\mathbb{S}$ is a smooth real or complex manifold. Then the structured condition number for λ with respect to $\mathbb{S}$ is given by*

$$\kappa(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|} \sup\left\{ |y^*Ex| : \ E\in T_A\mathbb{S},\ \|E\|=1 \right\}, \tag{2.3}$$

where $T_A\mathbb{S}$ is the tangent space at A.

Proof. Let $E\in T_A\mathbb{S}$ with $\|E\|=1$. Then there is a smooth curve $G_E:(-\epsilon,\epsilon)\to\mathbb{K}^{n\times n}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) satisfying $G_E(0)=0$, $G_E'(0)=E$ and $A+G_E(t)\in\mathbb{S}$ for all t. We have

$$\lim_{t\to 0}\frac{G_E(t)}{\|G_E(t)\|} = \lim_{t\to 0}\frac{Et+O(|t|^2)}{\|Et+O(|t|^2)\|} = E.$$

Hence, using the expansion (2.1),

$$\lim_{t\to 0}\frac{|\lambda(A+G_E(t))-\lambda(A)|}{\|G_E(t)\|} = \lim_{t\to 0}\frac{1}{|y^*x|}\cdot\frac{\left|y^*G_E(t)x+O(\|G_E(t)\|^2)\right|}{\|G_E(t)\|} = \frac{|y^*Ex|}{|y^*x|}.$$

This implies $\kappa(A,\lambda;\mathbb{S})\ge\eta$, where η denotes the right-hand side of (2.3). Equality holds since the curves G_E form a covering of an open neighborhood of $A\in\mathbb{S}$. ◻

It is convenient to introduce the notation

$$\phi(x,y;\mathbb{S}) = \sup\Big\{ |y^*Ex| : \ E\in\mathbb{S},\ \|E\|=1 \Big\} \tag{2.4}$$

so that (2.3) can be rewritten as

$$\kappa(A,\lambda;\mathbb{S}) = \phi(x,y;T_A\mathbb{S})/|y^*x|. \tag{2.5}$$

In a similar way to [20], an explicit expression for $\kappa(A,\lambda;\mathbb{S})$ can be obtained if one further assumes that the matrix norm $\|\cdot\|$ under consideration is the Frobenius norm $\|\cdot\|_F$. Let us rewrite

$$y^*Ex = \operatorname{vec}(y^*Ex) = (x^T\otimes y^*)\operatorname{vec}(E) = (\overline{x}\otimes y)^*\operatorname{vec}(E),$$

where $\otimes$ denotes the Kronecker product and vec denotes the operator that stacks the columns of a matrix into one long vector [8, p. 180]. Note that $T_A\mathbb{S}$ is a linear vector space of dimension $m \leq n^2$. Hence, there is an $n^2 \times m$ matrix B such that for every $E \in T_A\mathbb{S}$ there exists a uniquely defined parameter vector p with

$$\mathrm{vec}(E) = Bp, \quad \|E\|_F = \|p\|_2. \tag{2.6}$$

Any matrix B satisfying these properties is called a *pattern matrix* for $T_A\mathbb{S}$, see also [10], [20], and [6]. The relationships in (2.6) together with (2.4) yield

$$\phi_F(x, y; T_A\mathbb{S}) = \sup \left\{ |(\bar{x} \otimes y)^* Bp| : \ \|p\|_2 = 1, \ p \in \mathbb{K}^m \right\}, \tag{2.7}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will use the subscripts F and 2 to refer to the use of the Frobenius and matrix 2-norm in (2.4).

When $\mathbb{K} = \mathbb{C}$ the supremum is taken over all $p \in \mathbb{C}^m$ and consequently, from (2.5),

$$\kappa_F(A, \lambda; \mathbb{S}) = \frac{1}{|y^* x|} \|(\bar{x} \otimes y)^* B\|_2. \tag{2.8}$$

Complications arise if $\mathbb{K} = \mathbb{R}$ but λ is a complex eigenvalue or B is a complex matrix. In this case, the supremum is also taken over all $p \in \mathbb{R}^m$ but $(\bar{x} \otimes y)^* B$ may be a complex vector. In a similar way as in [5] for the standard eigenvalue condition number we can show that the real structured eigenvalue condition number is within a small factor of the complex one in (2.8). To be more specific,

$$\frac{1}{\sqrt{2}|y^* x|} \|(\bar{x} \otimes y)^* B\|_2 \leq \kappa_F(A, \lambda; \mathbb{S}) \leq \frac{1}{|y^* x|} \|(\bar{x} \otimes y)^* B\|_2, \tag{2.9}$$

see also [9], [18]. To obtain an exact expression for the real structured eigenvalue condition number, let us consider the relation

$$|(\bar{x} \otimes y)^* Bp|^2 = \left| \mathrm{Re}\left((\bar{x} \otimes y)^* B\right) p \right|^2 + \left| \mathrm{Im}\left((\bar{x} \otimes y)^* B\right) p \right|^2,$$

which together with (2.7) implies

$$\kappa_F(A, \lambda; \mathbb{S}) = \frac{1}{|y^* x|} \left\| \begin{bmatrix} \mathrm{Re}\left((\bar{x} \otimes y)^* B\right) \\ \mathrm{Im}\left((\bar{x} \otimes y)^* B\right) \end{bmatrix} \right\|_2. \tag{2.10}$$

For a real pattern matrix B, this formula can be rewritten as

$$\kappa_F(A, \lambda; \mathbb{S}) = \frac{1}{|y^* x|} \|[x_R \otimes y_R + x_I \otimes y_I, \ x_R \otimes y_I - x_I \otimes y_R]^T B\|_2, \tag{2.11}$$

where $x = x_R + \imath x_I$ and $y = y_R + \imath y_I$ with $x_R, x_I, y_R, y_I \in \mathbb{R}^n$. If additionally λ is real, we can choose x and y real and (2.11) reduces to (2.8).

The difficulty in computing (2.8), (2.10) or (2.11) lies in characterizing the tangent space $T_A\mathbb{S}$ and building the pattern matrix B. We show in section 5 how these tasks can be achieved when $\mathbb{S}$ is an automorphism group.

It is difficult to compare the explicit formula for $\kappa_F(A, \lambda; \mathbb{S})$ in (2.8) or (2.10) to that of the standard condition number $\kappa_F(A, \lambda) = 1/|y^* x|$ unless $\mathbb{S}$ has some special structure. Noschese and Pasquini [17] show that for perturbations having an assigned zero structure (or sparsity pattern), (2.8) reduces to

$$\kappa_F(A, \lambda; \mathbb{S}) = \|(yx^*)|_{\mathbb{S}}\|_F / |y^* x|,$$

TABLE 3.1

A sampling of structured matrices associated with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{M}}$, where M is the matrix defining the scalar product.

Space	M	**Lie Group** $\mathbb{G}=\{G : G^\star = G^{-1}\}$	**Jordan Algebra** $\mathbb{J}=\{S : S^\star = S\}$	**Lie Algebra** $\mathbb{L}=\{K : K^\star = -K\}$
		Bilinear forms		
$\mathbb{R}^n$	I	Real orthogonals	Symmetrics	Skew-symmetrics
$\mathbb{C}^n$	I	Complex orthogonals	Complex symmetrics	Cplx skew-symmetrics
$\mathbb{R}^n$	$\Sigma_{p,q}$	Pseudo-orthogonals[a]	Pseudo symmetrics	Pseudo skew-symmetrics
$\mathbb{C}^n$	$\Sigma_{p,q}$	Cplx pseudo-orthogonals	Cplx pseudo-symm.	Cplx pseudo-skew-symm.
$\mathbb{R}^n$	R	Real perplectics	Persymmetrics	Perskew-symmetrics
$\mathbb{R}^{2n}$	J	Real symplectics	Skew-Hamiltonians	Hamiltonians
$\mathbb{C}^{2n}$	J	Complex symplectics	Cplx J-skew-symm.	Complex J-symmetrics
		Sesquilinear forms		
$\mathbb{C}^n$	I	Unitaries	Hermitian	Skew-Hermitian
$\mathbb{C}^n$	$\Sigma_{p,q}$	Pseudo-unitaries[b]	Pseudo Hermitian	Pseudo skew-Hermitian
$\mathbb{C}^{2n}$	J	Conjugate symplectics	J-skew-Hermitian	J-Hermitian

Here, $R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$ and $\Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \in \mathbb{R}^{n\times n}$ are symmetric and $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is skew-symmetric.

[a]Physicists refer to the pseudo-orthogonal group with $\Sigma_{3,1} = \begin{bmatrix} I_3 & \\ & -1 \end{bmatrix}$ as the Lorentz group.

[b]Pseudo-unitary matrices are sometimes called $\Sigma_{p,q}$-unitaries, or hypernormal matrices in the signal processing literature.

where $(yx^*)|_{\mathbb{S}}$ means the restriction of the rank-one matrix yx^* to the sparsity structure of $\mathbb{S}$. For example if the perturbation is upper triangular then $(yx^*)|_{\mathbb{S}}$ is the upper triangular part of yx^*.

Starting from (2.5) we compare in sections 4 and 5 the structured condition number to the unstructured one for structured matrices belonging to the Jordan algebra, Lie algebra, or automorphism group of a scalar product.

3. Structured matrices in scalar product spaces. We now introduce the basic definitions and properties of scalar products and the structured classes of matrices associated with them.

The term *scalar product* will be used to refer to any nondegenerate bilinear or sesquilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{K}^n$. Here $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. It is well known that any real or complex bilinear form $\langle\cdot,\cdot\rangle$ has a unique matrix representation given by $\langle\cdot,\cdot\rangle = x^T My$, while a sesquilinear form can be represented by $\langle\cdot,\cdot\rangle = x^* My$, where the matrix M is nonsingular. We will denote $\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ as needed.

A bilinear form is symmetric if $\langle x,y\rangle = \langle y,x\rangle$, and skew-symmetric if $\langle x,y\rangle = -\langle y,x\rangle$. Hence for a symmetric form $M = M^T$ and for a skew-symmetric form $M = -M^T$. A sesquilinear form is Hermitian if $\langle x,y\rangle = \overline{\langle y,x\rangle}$ and skew-Hermitian if $\langle x,y\rangle = -\overline{\langle y,x\rangle}$. The matrices associated with such forms are Hermitian and skew-Hermitian, respectively.

To each scalar product there corresponds a notion of *adjoint*, generalizing the idea of transpose ^{T} and conjugate transpose *, that is, for any matrix $A \in \mathbb{K}^{n\times n}$ there is a unique adjoint $A^\star$ with respect to the form defined by $\langle Ax, y\rangle_{\mathrm{M}} = \langle x, A^\star y\rangle_{\mathrm{M}}$ for all x and y in $\mathbb{K}^n$. In matrix terms the adjoint is given by

$$A^\star = \begin{cases} M^{-1}A^TM & \text{for bilinear forms,} \\ M^{-1}A^*M & \text{for sesquilinear forms.} \end{cases}$$

It is well known [1] that the set of self-adjoint matrices

$$\mathbb{J} = \left\{S \in \mathbb{K}^{n\times n} : \langle Sx, y\rangle_{\mathrm{M}} = \langle x, Sy\rangle_{\mathrm{M}}\right\} = \left\{S \in \mathbb{K}^{n\times n} : S^\star = S\right\}$$

forms a Jordan algebra, while the set of skew-adjoint matrices

$$\mathbb{L} = \left\{L \in \mathbb{K}^{n\times n} : \langle Lx, y\rangle_{\mathrm{M}} = -\langle x, Ly\rangle_{\mathrm{M}}\right\} = \left\{L \in \mathbb{K}^{n\times n} : L^\star = -L\right\}$$

forms a Lie algebra. The sets $\mathbb{L}$ and $\mathbb{J}$ are linear subspaces, but they are not closed under multiplication. A third class of matrices associated with $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ are those preserving the form, i.e.,

$$\mathbb{G} = \left\{G \in \mathbb{K}^{n\times n} : \langle Gx, Gy\rangle_{\mathrm{M}} = \langle x, y\rangle_{\mathrm{M}}\right\} = \{G \in \mathbb{K}^{n\times n} : \; G^\star = G^{-1}\}.$$

They form a Lie group under multiplication. We refer to $\mathbb{G}$ as an automorphism group. Table 3.1 shows a sample of well-known structured matrices in $\mathbb{L}$, $\mathbb{J}$, or $\mathbb{G}$ associated with some scalar products. In the rest of this paper we concentrate on structures belonging to at least one of these three classes.

The eigenvalues of matrices in $\mathbb{J}$, $\mathbb{L}$ and $\mathbb{G}$ have interesting pairing properties as shown by the following theorem.

Theorem 3.1. [15, Thm. 7.2, Thm. 7.6] *Let $A \in \mathbb{L}$ or $A \in \mathbb{J}$. Then the eigenvalues of A occur in pairs as shown below, with the same Jordan structure for each eigenvalue in a pair.*

	Bilinear	*Sesquilinear*
$A \in \mathbb{J}$	*"no pairing"*	$\lambda, \overline{\lambda}$
$A \in \mathbb{L}$	$\lambda, -\lambda$	$\lambda, -\overline{\lambda}$
$A \in \mathbb{G}$	$\lambda, 1/\lambda$	$\lambda, 1/\overline{\lambda}$

There is no eigenvalue structure property that holds for Jordan algebras of all bilinear forms. However for certain special classes of $\mathbb{J}$ there may be additional structure in the eigenvalues. For example, it is known that the eigenvalues of any real or complex skew-Hamiltonian matrix all have even multiplicity [7]. More generally we have the following result.

Proposition 3.2. *[15, Prop. 7.7] Let $\mathbb{J}$ be the Jordan algebra of any skew-symmetric bilinear form on $\mathbb{K}^n$. Then for any $A \in \mathbb{J}$, the eigenvalues of A all have even multiplicity. Furthermore, all Jordan blocks of a fixed size appear an even number of times.*

Hence we will not consider matrices in these algebras since they cannot have simple eigenvalues.

There are two important classes of scalar products termed unitary and orthosymmetric [15]. The scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is *unitary* if αM is unitary for some $\alpha > 0$. A scalar product is said to be *orthosymmetric* if

$$M = \begin{cases} \pm M^T \ \text{(bilinear forms),} \\ \beta M^*, \ |\beta| = 1, \ \text{(sesquilinear forms).} \end{cases}$$

Note that important classes of structured matrices arise in this context as witnessed by the entries in Table 3.1, all of which have unitary and orthosymmetric scalar product with $\alpha = 1$ and $\beta = 1$. The unitary property will be needed when measuring structured perturbations with a unitarily invariant norm. We will need the orthosymmetry property to show that left multiplication by M maps $\mathbb{L}$ and $\mathbb{J}$ to the sets Skew($\mathbb{K}$) and Sym($\mathbb{K}$) for bilinear forms and to a scalar multiple of Herm($\mathbb{C}$) for sesquilinear forms (see Lemma 4.1), where

$$\mathrm{Skew}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : A^T = -A\}, \quad \mathrm{Sym}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : A^T = A\}$$

are the sets of symmetric and skew-symmetric matrices on $\mathbb{K}^{n\times n}$ and Herm($\mathbb{C}$) is the set of Hermitian matrices.

4. Jordan and Lie algebras. Let $\mathbb{S}$ be the Jordan algebra or Lie algebra of a scalar product on $\mathbb{K}^n$. Since $\mathbb{S}$ is a linear subspace of $\mathbb{K}^{n\times n}$, the tangent space at $A \in \mathbb{S}$ is $\mathbb{S}$ itself. Hence (2.5) becomes

$$\kappa(A, \lambda; \mathbb{S}) = \phi(x, y; \mathbb{S})/|y^* x|.$$

The analysis of the structured eigenvalue condition number $\kappa(A, \lambda; \mathbb{S})$ is reduced to the study of the five structures Sym($\mathbb{K}$), Skew($\mathbb{K}$) ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and Herm($\mathbb{C}$) if one further assumes that the scalar product is both orthosymmetric and unitary. This is shown in the next lemma.

LEMMA 4.1. *Let $\mathbb{S}$ be the Jordan algebra or Lie algebra of a scalar product. which is both unitary and orthosymmetric, i.e., αM is unitary for some $\alpha > 0$ and, for bilinear forms, $M^T = \epsilon M$ with $\epsilon = \pm 1$. Let $E \in \mathbb{S}$ so that $E^\star = \delta E$ with $\delta = \pm 1$. Then, for any unitarily invariant norm,*

$$\phi(x, y; \mathbb{S}) = \phi(x, \widetilde{y}; \widetilde{\mathbb{S}}),$$

where $\widetilde{y} = \alpha M y$ and, for bilinear forms on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$),

$$\widetilde{\mathbb{S}} = \begin{cases} \mathrm{Sym}(\mathbb{K}) & \textit{if } \delta = \epsilon, \\ \mathrm{Skew}(\mathbb{K}) & \textit{if } \delta \neq \epsilon, \end{cases} \tag{4.1}$$

and for sesquilinear forms on $\mathbb{C}^n$,

$$\widetilde{\mathbb{S}} = \begin{cases} \mathrm{Herm}(\mathbb{C}) & \textit{if } \delta = +1, \\ \imath\, \mathrm{Herm}(\mathbb{C}) & \textit{if } \delta = -1. \end{cases} \tag{4.2}$$

Proof. For bilinear forms, orthosymmetry implies that $M^T = \epsilon M$, $\epsilon = \pm 1$. Then $E \in \mathbb{S} \iff E^\star = M^{-1} E^T M = \delta E \iff ME = \delta E^T M$, and so $(ME)^T = \delta (E^T M)^T = \epsilon\delta(ME)$. This shows that $M \cdot \mathbb{S} = \widetilde{\mathbb{S}}$ with $\widetilde{\mathbb{S}}$ as in (4.1).

For sesquilinear forms, orthosymmetry means that $M^* = \gamma M$ for some $|\gamma| = 1$. Choosing β so that $\beta^2 = \gamma$, an argument similar to the one for bilinear forms shows that for any $E \in \mathbb{S}$, $\beta M E$ is either Hermitian or skew-Hermitian. But the set of all skew-Hermitian matrices is $i \cdot \mathrm{Herm}(\mathbb{C})$. Hence $\beta M \cdot \mathbb{S} = \widetilde{\mathbb{S}}$ with $\widetilde{\mathbb{S}}$ as in (4.2).

Since αM is unitary,

$$\begin{aligned} \phi(x, y; \mathbb{S}) &= \sup\{|y^* E x| : \ E \in \mathbb{S}, \ \|E\| = 1\} \\ &= \sup\{|y^* (\alpha M)^* F x| : \ F \in \beta M \cdot \mathbb{S}, \ \|F\| = 1\} \\ &= \phi(x, \widetilde{y}, \widetilde{\mathbb{S}}), \end{aligned}$$

where $F = \alpha\beta ME$ with $\beta = 1$ for bilinear forms and $|\beta| = 1$ for sesquilinear forms. $\square$

Lemma 4.1 is a key result for comparing $\kappa(A, \lambda; \mathbb{S})$ to $\kappa(A, \lambda)$.

4.1. Jordan algebras. Graillat [9] and Rump [18] show that for the structures symmetric, complex symmetric, persymmetric, complex persymmetric and Hermitian, the structured and unstructured eigenvalue condition numbers are equal for the 2-norm. These are examples of Jordan algebras (see Table 3.1). The next theorem extends these results to all Jordan algebras of a unitary and orthosymmetric scalar product. Unlike the proofs in [9] and [18], our unifying proof does need to consider each Jordan algebra individually.

THEOREM 4.2. *Let λ be a simple eigenvalue of $A \in \mathbb{J}$, where $\mathbb{J}$ is the Jordan algebra of an orthosymmetric and unitary scalar product $\langle\cdot,\cdot\rangle_M$ on $\mathbb{K}^n$. Then, for the 2-norm,*

$$\kappa_2(A, \lambda; \mathbb{J}) = \kappa_2(A, \lambda).$$

Proof. Since $\langle\cdot,\cdot\rangle_M$ is unitary, αM is unitary for some $\alpha > 0$. Let x and y be as in (1.2) and define $\widetilde{y} = \alpha M y$.

For bilinear forms, orthosymmetry means that $M = \pm M^T$. We do not need to consider the case $M = -M^T$ since from Proposition 3.2 the eigenvalues of matrices in Jordan algebras of skew-symmetric bilinear forms all have even multiplicity. Now, if $M = M^T$ then from Lemma 4.1 we have that

$$\phi_2(x, y; \mathbb{J}) = \phi_2(x, \widetilde{y}; \mathrm{Sym}(\mathbb{K})). \tag{4.3}$$

But $A \in \mathbb{J}$ implies $A = A^\star = M^{-1}A^TM$ and

$$Ax = \lambda x \Longleftrightarrow \overline{x}^*A^T = \lambda\overline{x}^* \Longleftrightarrow \overline{x}^*MA = \lambda\overline{x}^*M$$

so that we can take $y = \alpha M^*\overline{x}$ as a normalized left eigenvector for A associated to λ. Rump [18, Lem. 2.5] shows that there exists $\widetilde{E} \in \mathrm{Sym}(\mathbb{R}) \subset \mathrm{Sym}(\mathbb{C})$ such that $\widetilde{E}x = \mu\bar{x}$ with $|\mu| = 1$ and $\|\widetilde{E}\|_2 = 1$. Hence

$$|\widetilde{y}^*\widetilde{E}x| = |y^*(\alpha M)^{-1}\bar{x}| = |x^T(\alpha M)(\alpha M)^{-1}\bar{x}| = 1,$$

so that $\phi_2(x, \widetilde{y}; \mathrm{Sym}(\mathbb{K})) = 1$ and therefore $\kappa_2(A, \lambda; \mathbb{J}) = \kappa_2(A, \lambda)$ using (2.5) and (4.3).

For sesquilinear forms, using Lemma 4.1 we have

$$\phi_2(x, y; \mathbb{J}) = \phi_2(x, \widetilde{y}; \mathrm{Herm}(\mathbb{C})). \tag{4.4}$$

Hence to show that $\phi_2(x, y; \mathbb{J}) = 1$ we need to find $\widetilde{E} \in \mathrm{Herm}(\mathbb{C})$ such that $|\widetilde{y}^*\widetilde{E}x| = 1$. Define $\mu = (\widetilde{y}^*x)/|\widetilde{y}^*x|$ which has unit modulus. If $x \neq \mu\widetilde{y}$, there exists a unique unitary reflector $\widetilde{E}$ such that $\widetilde{E}x = \mu\widetilde{y}$ and $\|\widetilde{E}\|_2 = 1$ [14, Thm. 8.6]. Furthermore $\widetilde{E} = I + \xi uu^*$, where $u = \mu\widetilde{y} - x$ and $\xi = 1/(u^*x) \in \mathbb{R} \setminus \{0\}$ so that $\widetilde{E} \in \mathrm{Herm}(\mathbb{C})$. Hence $|\widetilde{y}^*\widetilde{E}x| = |\widetilde{y}^*\widetilde{y}| = 1$. The equality $\kappa_2(A, \lambda; \mathbb{J}) = \kappa_2(A, \lambda)$ holds using (2.5) and (4.4). $\square$

For Jordan algebras $\mathbb{J}$ of sesquilinear forms, eigenvalues come in pairs λ and $\bar{\lambda}$ and if λ is simple so is $\bar{\lambda}$ (see Theorem 3.1). For unitary scalar products, αM is unitary for some $\alpha > 0$, and, if x and y are normalized right and left eigenvectors associated with λ then αMy and αMx are normalized right and left eigenvectors associated with $\bar{\lambda}$. Hence, $|(\alpha Mx)^*(\alpha My)| = |x^*y|$ so that

$$\kappa(A, \lambda; \mathbb{J}) = \kappa(A, \bar{\lambda}; \mathbb{J}).$$

4.2. Lie algebras. We show that, with the exception of symmetric bilinear forms, incorporating structure does not affect the eigenvalue condition number for matrices in Lie algebras of scalar products that are both orthosymmetric and unitary. These include as special cases the skew-symmetric, complex skew-symmetric, and skew-Hermitian matrices considered by Rump [18].

THEOREM 4.3. *Let λ be a simple eigenvalue of $A \in \mathbb{L}$, where $\mathbb{L}$ is the Lie algebra of an orthosymmetric and unitary scalar product $\langle \cdot, \cdot \rangle_M$ on $\mathbb{C}^n$.*

- *For symmetric bilinear forms,*

$$\begin{aligned} 1 &\le \kappa_2(A, \lambda; \mathbb{L}) \le \kappa_2(A, \lambda), && \lambda \neq 0, \\ 0 &= \kappa(A, 0; \mathbb{L}) < \kappa(A, 0) = \frac{1}{|x^T M x|} && \textit{(for any norm in (2.2)).} \end{aligned}$$

- *For skew-symmetric bilinear forms,*

$$\frac{\kappa_\nu(A, \lambda)}{\sqrt{2}} \le \kappa_\nu(A, \lambda; \mathbb{L}) \le \kappa_\nu(A, \lambda), \quad \nu = 2, F.$$

- *For sesquilinear forms,*

$$\kappa_2(A, \lambda; \mathbb{L}) = \kappa_2(A, \lambda).$$

Proof. Since $\langle \cdot, \cdot \rangle_M$ is unitary, αM is unitary for some $\alpha > 0$. Let x and y be as in (1.2) and define $\widetilde{y} = \alpha My$.

For bilinear forms, orthosymmetry implies $M = \pm M^T$.

(i) If $M = M^T$ (symmetric bilinear form) then using Lemma 4.1 we have

$$\phi_2(x, y; \mathbb{L}) = \phi_2(x, \widetilde{y}; \mathrm{Skew}(\mathbb{C})). \tag{4.5}$$

A necessary and sufficient condition for the existence of $\widetilde{E} \in \mathrm{Skew}(\mathbb{C})$ such that $\widetilde{E}x = b$ for some $b \in \mathbb{C}^n$ is that $b^T x = 0$ [16, Thm. 3.2]. We cannot choose $b = \widetilde{y}$ since $\widetilde{y}^T x \neq 0$ in general. Hence, $\phi_2(x, \widetilde{y}; \mathrm{Skew}(\mathbb{C})) \le 1$. However, for $A \in \mathbb{L}$, $A^\star = -A$ and

$$\lambda \langle x, x \rangle_M = \langle \lambda x, x \rangle_M = \langle Ax, x \rangle_M = \langle x, A^\star x \rangle_M = -\langle x, Ax \rangle_M = -\lambda \langle x, x \rangle_M$$

so that if $\lambda \neq 0$, $\langle x, x \rangle_M = (Mx)^T x = 0$. Take $b = \alpha Mx$ of unit 2-norm so that $b^T x = 0 = b^* \bar{x}$. There exists Q unitary such that $[\bar{x}, b] = Q[e_1, -e_2]$, where e_k denotes the kth column of the identity matrix. Then $\widetilde{E} = Q(e_1 e_2^T - e_2 e_1^T)Q^T \in \mathrm{Skew}(\mathbb{C})$ satisfies $\widetilde{E}x = b$ and $\|\widetilde{E}\|_2 = 1$. Hence

$$|\widetilde{y}^* \widetilde{E} x| = |y^*(\alpha M)^*(\alpha M)x| = |y^* x|$$

and therefore $\phi_2(x, \widetilde{y}; \mathrm{Skew}(\mathbb{C})) \ge |y^* x|$ which implies $\kappa_2(A, \lambda; \mathbb{L}) \ge 1$ using (2.5) and (4.5). Now if $\lambda = 0$ then since $A \in \mathbb{L}$,

$$Ax = 0 \iff -A^\star x = 0 \iff M^{-1} A^T M x = 0 \iff (Mx)^T A = 0$$

so that we can take $y = \overline{Mx}$ as a left eigenvector of $\lambda = 0$. Hence for any $E \in \mathbb{L}$,

$$y^* E x = x^T (ME) x = (x^T (ME) x)^T = -x^T (ME) x = 0$$

and therefore $\kappa(A, 0; \mathbb{L}) = 0$.

(ii) If $M = -M^T$ (skew-symmetric bilinear form) then using Lemma 4.1 we have

$$\phi_\nu(x, y; \mathbb{L}) = \phi_\nu(x, \widetilde{y}; \mathrm{Sym}(\mathbb{C})), \quad \nu = 2, F.$$

It is not difficult to show that $S = \widetilde{y}x^* + \bar{x}\widetilde{y}^T - (\widetilde{y}^T x)\bar{x}x^* \in \mathrm{Sym}(\mathbb{C})$ satisfies $Sx = \widetilde{y}$ with $\|S\|_2 \le \|S\|_F = \sqrt{2 - |\widetilde{y}^T x|^2} \le \sqrt{2}$. Taking $\widetilde{E} = S/\sqrt{2}$ gives

$$|\widetilde{y}^* \widetilde{E} x| = |\widetilde{y}^* \widetilde{y}|/\sqrt{2} = 1/\sqrt{2}$$

and therefore $\phi_\nu(x, y; \mathbb{L}) = \phi_\nu(x, \widetilde{y}; \mathrm{Sym}(\mathbb{C})) \ge 1/\sqrt{2}$. The lower bound follows using (2.5).

For sesquilinear forms, Lemma 4.1 implies

$$\phi_2(x, y; \mathbb{L}) = \phi_2(x, \widetilde{y}; \imath\, \mathrm{Herm}(\mathbb{C})).$$

As in the proof of Theorem 4.2, there exists $\widetilde{E} \in \mathrm{Herm}(\mathbb{C})$ such that $\widetilde{E}x = \mu\widetilde{y}$ with $|\mu| = 1$ and $\|\widetilde{E}\|_2 = 1$. Hence $|\widetilde{y}^*(i\widetilde{E})x| = |\widetilde{y}^*\widetilde{y}| = 1$ which implies $\phi_2(x, y; \mathbb{L}) = \phi_2(x, \widetilde{y}; \imath\, \mathrm{Herm}(\mathbb{C})) = 1$. The result follows from (2.5). $\square$

Note that Theorem 4.3 deals with complex perturbations only. However, for real bilinear forms the results still hold when λ is real. For complex λ, the lower bounds need to be multiplied by $1/\sqrt{2}$ in view of (2.9).

For the special case where $M = I$, i.e., when $\mathbb{L}$ is the set of complex skew-symmetric matrices, Rump [18] exhibits a 3×3 example showing that the ratio $\kappa_2(A, \lambda; \mathbb{L})/\kappa_2(A, \lambda)$ for $\lambda \ne 0$ can be arbitrarily small. Does this hold for all Lie algebras of symmetric bilinear forms on $\mathbb{K}^n$? The numerical experiments we ran on perskew-symmetric matrices and on pseudo-skew-symmetric matrices using (2.8) and (2.10) seem to confirm it. However it is difficult to give a general and unifying proof of this fact without considering specific algebras.

From Theorem 3.1 we know that eigenvalues of matrices in $\mathbb{L}$ come in pairs $\lambda, -\lambda$ for bilinear forms and $\lambda, -\bar{\lambda}$ for sesquilinear forms and that if λ is simple so is $-\lambda$ (or $-\bar{\lambda}$). We can show that for unitary scalar products,

$$\kappa(A, \lambda; \mathbb{L}) = \begin{cases} \kappa(A, -\lambda; \mathbb{L}) & \text{(bilinear forms)}, \\ \kappa(A, -\bar{\lambda}; \mathbb{L}) & \text{(sesquilinear forms)}. \end{cases}$$

5. Automorphism groups. We now consider structured condition numbers for automorphism groups $\mathbb{G}$ associated with the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$,

$$\mathbb{G} = \{A \in \mathbb{K}^{n \times n} : A^\star = A^{-1}\}.$$

This includes the groups of symplectic matrices ($M = J$), real and complex orthogonal matrices ($M = I$), as well as Lorentz transformations ($M = \mathrm{diag}(1, 1, 1, -1)$). We first show how to compute $\kappa_F(A, \lambda; \mathbb{G})$ in (2.8) and (2.10), then consider properties of the structured condition number, and finally provide lower bounds for $\kappa_2(A, \lambda; \mathbb{G})$.

5.1. Computation of $\kappa_F(A,\lambda;\mathbb{G})$. An automorphism group $\mathbb{G}$ forms a smooth manifold. The Jacobian of the function

$$F(A) = \begin{cases} A^T MA - M \text{ (bilinear forms)} \\ A^* MA - M \text{ (sesquilinear forms)} \end{cases}$$

at $A \in \mathbb{K}^{n\times n}$ can be represented as the linear function

$$J_A(X) = \begin{cases} A^T MX + X^T MA \text{ (bilinear forms)} \\ A^* MX + X^* MA \text{ (sesquilinear forms).} \end{cases}$$

The tangent space $T_A\mathbb{G}$ at $A \in \mathbb{G}$ coincides with the kernel of this Jacobian,

$$T_A\mathbb{G} = \{X: \ J_A(X) = 0\} = \{AH: \ H^\star = -H\} = A\cdot\mathbb{L}, \tag{5.1}$$

where $\mathbb{L}$ is the Lie algebra of $\langle\cdot,\cdot\rangle_{\mathrm{M}}$.

As the Lie algebra $\mathbb{L}$ in (5.1) is independent of A, it is often simple to explicitly construct a pattern matrix L such that for every $H \in \mathbb{L}$ there exists a uniquely defined parameter vector q with

$$\mathrm{vec}(H) = Lq, \quad \|H\|_F = \|q\|_2.$$

To obtain a pattern matrix B for $A\cdot\mathbb{L}$ in the sense of (2.6), we can compute a QR decomposition $(I\otimes A)L = BR$, where the columns of B form an orthonormal basis for the space spanned by the columns of L, and R is an upper triangular matrix. Hence,

$$\mathrm{vec}(AH) = (I\otimes A)\,\mathrm{vec}(H) = (I\otimes A)Lq = Bp,$$

where $p = Rq$, and $\|AH\|_F = \|\,\mathrm{vec}(AH)\|_2 = \|p\|_2$.

According to (2.8) we have

$$\kappa_F(A,\lambda;\mathbb{G}) = \frac{1}{|y^*x|}\|(\overline{x}\otimes y)^*B\|_2 = \frac{|\lambda|}{|y^*x|}\|(\overline{x}\otimes y)^*LR^{-1}\|_2 \tag{5.2}$$

if $\mathbb{K} = \mathbb{C}$ or if $\mathbb{K} = \mathbb{R}$ with λ real. Otherwise, when $\mathbb{K} = \mathbb{R}$ and λ is complex or, when B is complex, (2.10) implies

$$\kappa_F(A,\lambda;\mathbb{G}) = \frac{1}{|y^*x|}\left\| \begin{bmatrix} \mathrm{Re}\left(\lambda(\bar{x}\otimes y)^*LR^{-1}\right) \\ \mathrm{Im}\left(\lambda(\bar{x}\otimes y)^*LR^{-1}\right) \end{bmatrix} \right\|_2. \tag{5.3}$$

The construction of pattern matrices is made easier when $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is orthosymmetric. For bilinear forms, using (4.1) we have that for $H \in \mathbb{L}$, $\widetilde{H} := MH \in \widetilde{\mathbb{L}}$ where

$$\widetilde{\mathbb{L}} = \begin{cases} \mathrm{Skew}(\mathbb{K}) \text{ if } M = M^T, \\ \mathrm{Sym}(\mathbb{K}) \text{ if } M = -M^T. \end{cases}$$

For sesquilinear forms, orthosymmetry implies that $M^* = \gamma M$ for some $|\gamma| = 1$, and from (4.2) we have that for $H \in \mathbb{L}$, $\widetilde{H} := \beta MH \in \widetilde{\mathbb{L}}$ where $\widetilde{\mathbb{L}} = \mathrm{Herm}(\mathbb{C})$ and β is such that $\beta^2 = \gamma$. Let $\widetilde{L}$ be a pattern matrix for $\widetilde{H} \in \widetilde{\mathbb{L}}$, i.e., $\mathrm{vec}(\widetilde{H}) = \widetilde{L}\widetilde{q}$ with $\|\widetilde{q}\|_2 = \|\widetilde{H}\|_F$. There are only three different types of pattern matrices and they are easy to construct. For example, for $n = 2$,

- if $\widetilde{\mathbb{L}} = \mathrm{Sym}(\mathbb{K})$, $\widetilde{q} \in \mathbb{K}^3$, and

$$\widetilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

- if $\widetilde{\mathbb{L}} = \mathrm{Skew}(\mathbb{K})$, $\widetilde{q} \in \mathbb{K}^1$, and

$$\widetilde{L} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix},$$

- if $\widetilde{\mathbb{L}} = \mathrm{Herm}(\mathbb{C})$, $\widetilde{q} \in \mathbb{R}^4$, and

$$\widetilde{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that for Hermitian structures, $\widetilde{L}$ is complex but the vector of parameters is real since we need to force the diagonal entries of $\widetilde{H}$ to be real. The pattern matrix $\widetilde{B}$ for AH is then obtained via a QR decomposition $(I \otimes AM^{-1})\widetilde{L} = \widetilde{B}\widetilde{R}$. For orthosymmetric complex bilinear forms (2.8) becomes

$$\kappa_F(A, \lambda; \mathbb{G}) = \frac{|\lambda|}{|y^*x|} \|(x^T \otimes y^* M^{-1})\widetilde{L}\widetilde{R}^{-1}\|_2. \tag{5.4}$$

and for orthosymmetric sesquilinear forms or orthosymmetric real bilinear forms, (2.10) becomes

$$\kappa_F(A, \lambda; \mathbb{G}) = \frac{1}{|y^*x|} \left\| \begin{bmatrix} \mathrm{Re}\left(\lambda(x^T \otimes y^* M^{-1})\widetilde{L}\widetilde{R}^{-1}\right) \\ \mathrm{Im}\left(\lambda(x^T \otimes y^* M^{-1})\widetilde{L}\widetilde{R}^{-1}\right) \end{bmatrix} \right\|_2.$$

5.2. Properties of $\kappa_F(A, \lambda; \mathbb{S})$. The eigenvalues of $A \in \mathbb{G}$ come in pairs λ and $1/\lambda$ for bilinear forms, and in pairs λ and $1/\overline{\lambda}$ for sesquilinear forms. In both cases these pairs have the same Jordan structure, and hence the same algebraic and geometric multiplicities (see Theorem 3.1). Hence if λ is simple so is $1/\lambda$ or $1/\bar{\lambda}$. For unitary scalar products, there are interesting relations between the structured condition numbers of these eigenvalue pairings.

THEOREM 5.1. *Let λ be a simple eigenvalue of $A \in \mathbb{G}$, where $\mathbb{G}$ is the automorphism group of a unitary scalar product on $\mathbb{K}^n$. Then for the (absolute) unstructured eigenvalue condition number,*

$$\kappa(A, \lambda) = \begin{cases} \kappa(A, 1/\lambda) & (\textit{bilinear forms}), \\ \kappa(A, 1/\bar{\lambda}) & (\textit{sesquilinear forms}), \end{cases}$$

whereas for the (absolute) structured eigenvalue condition number,

$$\kappa(A, \lambda; \mathbb{G}) = \begin{cases} |\lambda|^2\, \kappa(A, 1/\lambda; \mathbb{G}) & (\textit{bilinear forms}), \\ |\lambda|^2\, \kappa(A, 1/\bar{\lambda}; \mathbb{G}) & (\textit{sesquilinear forms}). \end{cases}$$

Proof. We just prove the bilinear case, the proof for the sesquilinear case being similar. The scalar product $\langle\cdot,\cdot\rangle_{\rm M}$ being unitary implies that αM is unitary for some $\alpha>0$. If x and y are normalized right and left eigenvectors associated with λ then $\alpha\overline{My}$ and $\alpha\overline{Mx}$ are right and left normalized eigenvectors belonging to the eigenvalue $1/\lambda$. The equality for $\kappa(A,\lambda)$ follows.

Let $E\in T_A\mathbb{G}$. Then $E=AH$ for some H in the Lie algebra $\mathbb{L}$ of $\langle\cdot,\cdot\rangle_{\rm M}$ and

$$|y^*Ex| = |\lambda|\,|y^*Hx|. \tag{5.5}$$

Also, $A\in\mathbb{G}\Rightarrow M^TA=A^{-T}M^T$, αM unitary $\Rightarrow M^{-T}=\alpha^2\overline{M}$ and $H\in\mathbb{L}\Rightarrow \alpha^2M^TH\overline{M}=-H^T$. Hence,

$$\begin{aligned}|(\alpha\overline{Mx})^*E(\alpha\overline{My})| &= |\alpha^2x^TM^TAH\overline{My}|\\ &= |\alpha^2(x^TA^{-T})(M^TH\overline{M})\overline{y}|\\ &= \frac{1}{|\lambda|}|x^TH^T\overline{y}|\\ &= \frac{1}{|\lambda|}|y^*Hx| = \frac{1}{|\lambda|^2}|y^*Ex|\end{aligned}$$

so that from (2.3), $\kappa(A,\lambda;\mathbb{G})=\kappa(A,1/\lambda;\mathbb{G})/|\lambda|^2$. □

Theorem 5.1 shows that the *relative structured* eigenvalue condition numbers for λ and $1/\lambda$ if the form is bilinear or λ and $1/\bar\lambda$ if the form is sesquilinear, are equal. On the other hand, the *relative unstructured* eigenvalue condition numbers for λ and $1/\lambda$ (or λ and $1/\bar\lambda$) is $1/|\lambda|^2$. Hence, if we use a non structure preserving algorithm, we should compute the larger of λ and $1/\lambda$ (or $1/\bar\lambda$). In other words, we should compute whichever member of the pair $(\lambda,1/\lambda)$ (or the pair $(\lambda,1/\bar\lambda)$) lies outside the unit circle and then obtain the other one by reciprocation.

5.3. Bounds for $\kappa_2(A,\lambda;\mathbb{G})$. Lower bounds for the eigenvalue structured condition number can be derived when $\langle\cdot,\cdot\rangle_{\rm M}$ is orthosymmetric and unitary.

THEOREM 5.2. *Let λ be a simple eigenvalue of $A\in\mathbb{G}$, where $\mathbb{G}$ is the automorphism group of an orthosymmetric and unitary scalar product $\langle\cdot,\cdot\rangle_M$ on $\mathbb{K}^n$. If $\mathbb{K}=\mathbb{C}$ or, if $\mathbb{K}=\mathbb{R}$ with λ real we have*

- *for symmetric bilinear forms,*

$$\frac{|\lambda|}{\|A\|_\nu}\le\kappa_\nu(A,\lambda;\mathbb{G})\le\kappa_\nu(A,\lambda)\ \textit{for}\ \lambda\ne\pm1,\qquad \nu=2,F,$$
$$\kappa(A,\lambda;\mathbb{G})=0\quad \textit{for}\ \lambda=\pm1\ \textit{and any norm,}$$

- *for skew-symmetric bilinear forms,*

$$\frac{|\lambda|}{\sqrt{2}\|A\|_\nu}\kappa_\nu(A,\lambda)\le\kappa_\nu(A,\lambda;\mathbb{G})\le\kappa_\nu(A,\lambda),\qquad \nu=2,F, \tag{5.6}$$

- *for sesquilinear forms ($\mathbb{K}=\mathbb{C}$),*

$$\frac{|\lambda|}{\|A\|_\nu}\kappa_\nu(A,\lambda)\le\kappa_\nu(A,\lambda;\mathbb{G})\le\kappa_\nu(A,\lambda),\qquad \nu=2,F.$$

For $\mathbb{K} = \mathbb{R}$ and λ complex, the lower bounds for the Frobenius norm need to be multiplied by $1/\sqrt{2}$.

Proof. Let $\mathbb{L}$ be the Lie algebra of $\langle \cdot, \cdot \rangle_{\mathrm{M}}$. From (2.3) and (5.1) we have

$$\kappa_2(A, \lambda; \mathbb{G}) = \frac{1}{|y^* x|} \sup\left\{|y^* AHx| : \ H \in \mathbb{L}, \ \|AH\|_2 = 1\right\}.$$

Since αM is unitarity for some $\alpha > 0$ and because the scalar product is orthosymmetric, using (4.1) and (4.2) we can rewrite $\kappa_2(A, \lambda; \mathbb{G})$ as

$$\kappa_2(A, \lambda; \mathbb{G}) = \frac{|\lambda|}{|y^* x|} \sup\left\{|(\alpha M y)^* \widetilde{H} x| : \ \widetilde{H} \in \widetilde{\mathbb{L}}, \ \|A(\alpha M)^* \widetilde{H}\|_2 = 1\right\}, \tag{5.7}$$

where for bilinear forms,

$$\widetilde{\mathbb{L}} = \begin{cases} \mathrm{Skew}(\mathbb{K}) \text{ if } M = M^T, \\ \mathrm{Sym}(\mathbb{K}) \text{ if } M = -M^T. \end{cases}$$

and $\widetilde{\mathbb{L}} = \mathrm{Herm}(\mathbb{C})$ for sesquilinear forms.

- $\widetilde{\mathbb{L}} = \mathrm{Skew}(\mathbb{C})$, i.e., $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is a skew-symmetric bilinear form on $\mathbb{C}^n$. We know that there exists $\widetilde{H} \in \mathrm{Skew}(\mathbb{C})$ such that $\widetilde{H} x = b$ for some $b \in \mathbb{C}^n$ if and only if $b^T x = 0$ [16]. So we need to find a vector b satisfying this orthogonality condition. We have that $A \in \mathbb{G}$ implies

  $$\lambda \langle x, x \rangle_{\mathrm{M}} = \langle Ax, x \rangle_{\mathrm{M}} = \langle x, A^{-1} x \rangle_{\mathrm{M}} = \frac{1}{\lambda} \langle x, x \rangle_{\mathrm{M}}. \tag{5.8}$$

  Hence if $\lambda \neq \pm 1$, we have $(Mx)^T x = 0$ and we can take $b = Mx$. As in the proof of Theorem 4.3, there exists $S \in \mathrm{Skew}(\mathbb{C})$ such that $Sx = \alpha M x$ and $\|S\|_2 = 1$. Let $\widetilde{H} = \xi S$ with $\xi > 0$ such that $\|A(\alpha M)^* \widetilde{H}\|_\nu = \|A(\alpha M)^* \xi S\|_\nu = 1$ which implies $\xi \geq 1/\|A\|_\nu$. Also, we have $|(\alpha M y)^* \widetilde{H} x| = \xi |y^* x|$. The lower bounds for $\lambda \neq \pm 1$ follows from (5.7). Note that if $\mathbb{K} = \mathbb{R}$ and λ is real, the worst case perturbation is real.
  If $\lambda = \pm 1$ then $\overline{M\bar{x}}$ is a left eigenvector and for any $H \in \mathbb{L}$, $y^* AHx = \lambda y^* Hx = x^T (MH) x = 0$ since $MH \in \mathrm{Skew}(\mathbb{C})$.
- $\widetilde{\mathbb{L}} = \mathrm{Sym}(\mathbb{C})$, i.e., $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is a skew-symmetric bilinear form on $\mathbb{C}^n$. As in the proof of Theorem 4.3 there exists $S \in \mathrm{Sym}(\mathbb{C})$ such that $Sx = \alpha M y$ and $\|S\|_2 \leq \|S\|_F \leq \sqrt{2}$. Let $\widetilde{H} = \xi S$ with $\xi > 0$ such that $\|A(\alpha M)^* \widetilde{H}\|_n u = \|A(\alpha M)^* \xi S\|_\nu = 1$. This implies $\xi \geq \frac{1}{\sqrt{2}\|A\|_n u}$ and the lower bound follow.
- $\widetilde{\mathbb{L}} = \mathrm{Herm}(\mathbb{C})$. Let $\widetilde{y} = \alpha M y$. As in the proof of Theorem 4.2 there exists $S \in \mathrm{Herm}(\mathbb{C})$ such that $Sx = \mu \widetilde{y}$, with $|\mu| = 1$ and $\|S\|_2 = 1$. Let $\widetilde{H} = \imath \xi S$ with $\xi > 0$ such that $\|A(\alpha M)^* (\imath \xi S)\|_\nu = 1$. This implies $\xi \geq \|A\|_\nu^{-1}$ and the lower bound follows.

For real bilinear forms ($\mathbb{K} = \mathbb{R}$), if λ is real we can take x and y real and the worst case perturbation is real. When λ is complex one need to use (2.9). ∎

When $M = I$ and $\langle \cdot, \cdot \rangle$ is a sesquilinear form, $\mathbb{G}$ is the set of unitary matrices (see Table 3.1). But unitary matrices are normal and therefore $\kappa_\nu(A, \lambda) = 1$ ($\nu = 2, F$). Thus we can expect $\kappa_\nu(A, \lambda; \mathbb{G}) \leq 1$. Theorem 5.2 implies that the structured condition number is exactly 1.

For $M = I$ and a real bilinear form, $\mathbb{G}$ is the set of orthogonal matrices. Theorem 5.2 says that $\kappa(A, \lambda; \mathbb{G}) = 0$ if $\lambda = \pm 1$ and $\frac{1}{\sqrt{2}} \leq \kappa_\nu(A, \lambda; \mathbb{G}) \leq 1$, otherwise. In fact

TABLE 5.1

Condition numbers for the eigenvalues of the symplectic matrix A in (5.9), ratio ρ between the structured and unstructured condition number, and lower bound γ for this ratio.

λ	10^4	10^2	2	$1/2$	10^{-2}	10^{-4}
$\kappa_F(A,\lambda;\mathbb{G})$	1.2	1.2	1.5	0.4	1.2×10^{-4}	1.2×10^{-8}
ρ	0.87	0.87	0.89	0.22	8.7×10^{-5}	8.7×10^{-9}
γ	0.5	5×10^{-3}	1×10^{-4}	2.5×10^{-5}	5×10^{-7}	5×10^{-9}

we can show that $\kappa_2(A,\lambda;\mathbb{G}) = 1$ for $\lambda \neq \pm1$. For that decompose $x = x_R + \imath x_I$ with real vectors x_R, x_I. If $\lambda \neq \pm1$ then the fact that $x^T x = 0$ (see (5.8)) yields $\|x_R\|_2 = \|x_I\|_2 = 1/\sqrt{2}$ and $x_R^T x_I = 0$. Consequently, the two nonzero singular values of the skew-symmetric matrix $H = 2(x_R x_I^T - x_I x_R^T)$ are both 1, and hence $\|H\|_2 = 1$. Moreover, $|x^* H x| = 4(\|x_R\|_2^2 \cdot \|x_I\|_2^2) = 1$, which shows from (5.5) and (2.3) that $\kappa_2(A,\lambda,\mathbb{G}) \geq 1$. A more general perturbation analysis of orthogonal and unitary eigenvalue problems, based on the Cayley transform, can be found in [3].

Suppose $\mathbb{G}$ is the automorphism group of a skew-symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ ($M = -M^T$). For an eigenvalue λ of A with $|\lambda| \approx \|A\|_2$, the bounds in Theorem 5.2 imply

$$\kappa_\nu(A,\lambda;\mathbb{G}) \approx \kappa_\nu(A,\lambda), \quad \nu = 2, F.$$

From Theorem 5.1 we then have

$$|\lambda|^2 \kappa_\nu(A,1/\lambda;\mathbb{G}) \approx \kappa_\nu(A,1/\lambda), \quad \nu = 2, F$$

showing that if $|\lambda|$ is large, the unstructured eigenvalue condition number for $1/\lambda$ is much larger than the structured one. The lower bound (5.6) may not be tight when $\max(|\lambda|, 1/|\lambda|) \ll \|A\|_\nu$ as shown by the following example. Suppose that $M = J$ and that $\langle\cdot,\cdot\rangle_{\mathrm{J}}$ is a real bilinear form ($\mathbb{K} = \mathbb{R}$). Then $\mathbb{G}$ is the set of real symplectic matrices (see table 3.1). Let us consider the symplectic matrix

$$A = \begin{bmatrix} D & D \\ 0 & D^{-1} \end{bmatrix}, \quad D = \mathrm{diag}(10^4, 10^2, 2). \tag{5.9}$$

Define the ratio

$$\rho = \kappa_F(A,\lambda;\mathbb{G})/\kappa_F(A,\lambda) \leq 1$$

between the structured and unstructured eigenvalue condition numbers. ρ is computed using (5.4) and its values are displayed in Table 5.1 together with the lower bound $\gamma = |\lambda|/(\sqrt{2}\|A\|_2)$ of Theorem 5.2. This example demonstrates the looseness of the bounds of Theorem 5.2 for eigenvalues in the interior of the spectrum. Hence for these eigenvalues the computable expressions in Section 5.1 are of interest.

6. Conclusions. We have derived directly computable expressions for structured eigenvalue condition numbers on a smooth manifold of structured matrices. Furthermore, we have obtained meaningful bounds on the ratios between the structured and unstructured eigenvalue condition numbers for a number of structures related to Jordan algebras, Lie algebras, and automorphism groups. We have identified classes of structured matrices for which this ratio is 1 or close to 1. Hence for these structures, the usual unstructured perturbation analysis is sufficient.

The important task of finding computable expressions for structured backward errors of nonlinearly structured eigenvalue problems is still largely open and remains to be addressed.

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