

REVLEX-INITIAL 0/1-POLYTOPES

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ABSTRACT. We introduce *revlex-initial* 0/1-polytopes as the convex hulls of reverse-lexicographically initial sets of 0/1-vectors. Thus for each $n \leq 2^d$, we consider the 0/1-knapsack polytope given by all 0/1-vectors $x = (x_1, \dots, x_d)$ such that $\sum_{i=0}^{d-1} x_i 2^i$ is at most n .

The revlex-initial 0/1-polytopes have remarkable extremal properties. In particular, they have surprisingly low numbers of facets and small average degrees. Thus we establish the existence of 0/1-polytopes of given dimension d and prescribed number n of vertices, $d < n \leq 2^d$, with no more than $3d$ facets and with average degree bounded by $d + 8$.

Despite these “lower bound” type extremal properties, we prove that the revlex-initial 0/1-polytopes satisfy the Mihail–Vazirani conjecture, according to which all graphs of 0/1-polytopes should have edge-expansion at least one.

1. INTRODUCTION

Let us call a subset X of $\{0, 1\}^d$ *revlex-initial* if, for every $x \in X$, all points in $\{0, 1\}^d$ that are reverse-lexicographically smaller than x are contained in X . The convex hulls of revlex-initial subsets of $\{0, 1\}^d$ are the *revlex-initial 0/1-polytopes*. Phrased differently, the revlex-initial 0/1-polytopes are the convex hulls of those sets of 0/1-vectors of length d that correspond to the binary representations of all numbers $0, 1, \dots, n - 1$ for some n . In particular, for every $1 \leq n \leq 2^d$ there is precisely one revlex-initial 0/1-polytope with n vertices in \mathbb{R}^d .

Why should one be interested in such special polytopes? The general interest in 0/1-polytopes stems from their importance in combinatorial optimization. Investigations of 0/1-polytopes like traveling salesman polytopes, cut polytopes, stable set polytopes, and matching polytopes have not only led to beautiful insights into the interplay of combinatorics and geometry, but also to great algorithmic progress with respect to the corresponding optimization problems. From that work on such special 0/1-polytopes quite a few general questions on 0/1-polytopes have emerged, such as, e.g., the question for the maximal number of facets a d -dimensional 0/1-polytope may have (see Ziegler [14]).

With respect to this extremal question, Bárány and Pór [2] obtained a remarkable result. They showed that a random d -dimensional 0/1-polytope with

Date: December 14, 2004.

This work has been supported by the DFG Research Group *Algorithms, Structure, Randomness* and by the DFG Research Center MATHEON.

roughly $2^{d/\log_2 d}$ vertices in expectation has at least (roughly) $d^{d/4}$ facets. Recently, this bound was even improved to $d^{d/2}$ by Gatzouras, Giannopoulos, and Markoulakis [5]. The best known upper bound currently is $O((d-2)!)$ (due to Fleiner, Kaibel, and Rote [4]). It turns out that the revlex-initial 0/1-polytopes studied in this paper give some answers to two opposite extremal questions: How *few* facets or edges can a d -dimensional 0/1-polytope with a specified number of vertices have?

In contrast to the situation of general polytopes, the number of vertices of a 0/1-polytope may impose severe restrictions on the combinatorial type. For instance, a 0/1-polytope is simple if and only if it is a product of simplices (Kaibel and Wolff [8]). Thus, d -dimensional simple 0/1-polytopes with n vertices do only exist if there is a factorization $n = \prod n_i$ of n with $d = \sum (n_i - 1)$. Therefore, within the realm of 0/1-polytopes, it seems interesting to investigate extremal questions for *all* (reasonable) pairs (d, n) .

Our paper contains three main results.

(1) Revlex-initial 0/1-polytopes in \mathbb{R}^d have no more than $3d$ facets (Theorem 2); from this we deduce that the smallest number of facets $g_{\text{nfac}}(d, n)$ of a d -dimensional 0/1-polytope with exactly n vertices satisfies $g_{\text{nfac}}(d, n) \leq 3d$ for all d and n and $g_{\text{nfac}}(d, n(d)) \leq d + o(d)$ if $n(d)$ grows subexponentially with d (Theorem 6).

(2) The average degree of every revlex-initial 0/1-polytope in \mathbb{R}^d is at most $d+8$ (Theorem 4); from this we deduce that the smallest average degree $g_{\text{avdeg}}(d, n)$ of a d -dimensional 0/1-polytope with exactly n vertices satisfies $g_{\text{avdeg}}(d, n) \leq d+8$ (Theorem 7).

Since revlex-initial 0/1-polytopes have extremely sparse graphs, at first sight they look like candidates for counter-examples to an important conjecture due to Mihail and Vazirani (cited, e.g., in [3, 9]) that the graph of every 0/1-polytope has edge-expansion at least one. However, supporting that conjecture, we prove:

(3) Revlex-initial 0/1-polytopes have edge-expansion at least one (Theorem 5); from this we deduce that, for every (reasonable) pair (d, n) , there are d -dimensional 0/1-polytopes with n vertices, sparse graphs, and edge-expansion at least one (Theorem 8).

The context in which we came to study the special class of revlex-initial 0/1-polytopes is described in Section 3.4. They appeared from investigating an apparently strange behaviour of certain convex hull algorithms on random 0/1-polytopes.

The notion of revlex-initial subsets of $\{0, 1\}^d$, or, equivalently, of a system of subsets of $\{1, \dots, d\}$, is not new. It is related to the notion of *compression* of a set system, which plays an important role in the Kruskal-Katona theorem (see, e.g., [13, Thm. 8.32]) characterizing the f -vectors of simplicial complexes. Here, a system \mathcal{S} of subsets of $\{1, \dots, d\}$ (corresponding to a subset $X \subseteq \{0, 1\}^d$) is called *compressed* if, for every i , the subsystem of \mathcal{S} containing all sets from \mathcal{S} of

cardinality i is reverse-lexicographically initial within the i -subsets of $\{1, \dots, d\}$. Clearly, every revlex-initial subset of $\{0, 1\}^d$ corresponds to a compressed system of subsets of $\{1, \dots, d\}$, but the converse is not true.

In the context of the Kruskal-Katona theorem only compressed set systems that are closed under taking subsets are considered. Of course, all revlex-initial 0/1-polytopes correspond to compressed set systems with that property (i.e., revlex-initial 0/1-polytopes are *monotone*). But even more: Exploiting the interpretation in terms of binary representations of numbers, one finds that revlex-initial 0/1-polytopes are a special kind of knapsack polytopes (see Section 2).

Note that the term 'compressed polytope' has already been coined with a different meaning (see, e.g., [12]).

Acknowledgements. We are grateful to Jens Hillmann for computer implementations and for performing several computer experiments, to Michael Joswig for stimulating discussions, and to Günter M. Ziegler for useful comments on an earlier version of the paper.

2. DEFINITIONS

Throughout the paper, we assume that d is a positive integer number. We start with fixing some notions and notation.

Definition 1 (Index ranges). *For a positive integer number k , let*

$$[k] := \{1, 2, \dots, k\} \quad \text{and} \quad [k]_0 := \{0, 1, \dots, k-1\} .$$

We will identify \mathbb{R}^d with $\mathbb{R}^{[d]_0}$, i.e. vectors $x \in \mathbb{R}^d$ have components x_0, x_1, \dots, x_{d-1} , similarly for \mathbb{N}^d .

Definition 2 (Reverse-lexicographical order). *A point $x \in \{0, 1\}^d$ is reverse-lexicographically smaller than another point $y \in \{0, 1\}^d \setminus \{x\}$ ($x \prec_{rlex} y$) if $x_{i_{\max}} < y_{i_{\max}}$ holds for $i_{\max} := \max\{i : x_i \neq y_i\}$. For $x, y \in \{0, 1\}^d$, we denote $x \preceq_{rlex} y$ if $x = y$ or $x \prec_{rlex} y$ hold.*

For $x \in \{0, 1\}^d$ denote $S(x) := \{i \in [d]_0 : x_i = 1\}$. Then we have

$$x \prec_{rlex} y \Leftrightarrow \max(S(x) \Delta S(y)) \in S(y)$$

for all $x, y \in \{0, 1\}^d$ ($x \neq y$), where Δ denotes the symmetric difference of two sets.

Definition 3 (Revlex-Initial 0/1-polytope). *A subset $X \subseteq \{0, 1\}^d$ is revlex-initial if, for every $x \in X$, it contains all $y \in \{0, 1\}^d$ with $y \prec_{rlex} x$.*

$$X^{\prec v} := \{x \in \{0, 1\}^d : x \prec_{rlex} v\} .$$

A revlex-initial 0/1-polytope is the convex hull of any revlex-initial 0/1-set. We denote

$$P^{\prec v} := \text{conv } X^{\prec v} .$$

Since \prec_{rlex} defines a total ordering of $\{0, 1\}^d$, every revlex-initial 0/1-set X with $|X| < 2^d$ is of the form $X^{\prec v}$ for some $v \in \{0, 1\}^d \setminus \{0\}$. Note that $v \notin P^{\prec v}$.

Definition 4 (Signature of a 0/1-point). *Let $v \in \{0, 1\}^d \setminus \{0\}$. Its weight $w(v) := \mathbb{1}^T v$ is the number of ones of v . Its signature is the vector*

$$(\sigma_1(v), \dots, \sigma_{w(v)}(v))$$

with

$$S(v) = \{\sigma_1(v), \dots, \sigma_{w(v)}(v)\} \quad \text{and} \quad \sigma_1(v) > \sigma_2(v) > \dots > \sigma_{w(v)}(v).$$

Further we define the set index-set of all zero-components

$$\bar{S}(v) := [d]_0 \setminus S(v).$$

Definition 5 (Block decomposition). *For a 0/1-point $v \in \{0, 1\}^d \setminus \{0\}$ with signature $(\sigma_1(v), \dots, \sigma_{w(v)})$, we call*

$$X_q^{\prec v} := \{x \in \{0, 1\}^d : x_{\sigma_q(v)} = 0, x_{\sigma_q(v)+1} = v_{\sigma_q(v)+1}, \dots, x_{d-1} = v_{d-1}\}$$

(for $q \in [w(v)]$) the blocks of $P^{\prec v}$. Clearly, $X^{\prec v}$ is the disjoint union

$$X^{\prec v} = X_1^{\prec v} \uplus \dots \uplus X_{w(v)}^{\prec v}$$

of its blocks. The faces $P_q^{\prec v} := \text{conv } X_q^{\prec v}$ are the block faces of $P^{\prec v}$. The vector

$$(\dim P_1^{\prec v}, \dots, \dim P_{w(v)}^{\prec v}) = (\sigma_1(v), \dots, \sigma_{w(v)})$$

is the signature of the revlex-initial 0/1-polytope $P^{\prec v}$.

TABLE 1. Example illustrating some of the definitions: We have $d = 10$, $w(v) = 5$, $S(v) = \{0, 2, 3, 6, 9\}$, and $\bar{S}(v) = \{1, 4, 5, 7, 8\}$.

v	1	0	1	1	0	0	1	0	0	1
indices	0	1	2	3	4	5	6	7	8	9
signature	$\sigma_5(v)$		$\sigma_4(v)$	$\sigma_3(v)$			$\sigma_2(v)$			$\sigma_1(v)$
$P_1^{\prec v}$	*	*	*	*	*	*	*	*	*	0
$P_2^{\prec v}$	*	*	*	*	*	*	0	0	0	1
$P_3^{\prec v}$	*	*	*	0	0	0	1	0	0	1
$P_4^{\prec v}$	*	*	0	1	0	0	1	0	0	1
$P_5^{\prec v}$	0	0	1	1	0	0	1	0	0	1

As mentioned in the introduction, revlex-initial 0/1-polytopes are a special kind of knapsack polytopes. Indeed, for $d \in \mathbb{N}$ we define $a \in \mathbb{N}^d$ as $a_i := 2^i$. Then for two 0/1-vectors $v, w \in \{0, 1\}^d$ we have $v \prec_{\text{rlex}} w$ if and only if $a^\top v < a^\top w$ holds. Thus we can identify all 0/1-vectors $v \in \{0, 1\}^d$ with $v_{d-1} = 1$ uniquely

with the set of natural numbers \mathbb{N} via $n = a^\top v$. And we can write $P^{<n}$ with $n \in \mathbb{N}$ instead of $P^{<v}$ with $v \in \{0, 1\}^d$ with $v_{d-1} = 1$. Also note that $P^{<n}$ has exactly the n vertices $0, 1, \dots, n-1$. In other words, $P^{<v}$ with $v \in \{0, 1\}^d$ is the knapsack polytope $\text{conv}\{x \in \{0, 1\}^d : a^\top x \leq a^\top v - 1\}$.

3. THE FACETS OF REVLEX-INITIAL 0/1-POLYTOPES

3.1. Optimizing Linear Functions. For $c \in \mathbb{R}^d$ and $I \subseteq [d]_0$, define

$$c^+(I) := \sum_{i \in I} \max\{c_i, 0\}.$$

The following statement follows immediately from the block decomposition of revlex-initial 0/1-polytopes.

Proposition 1. *For every $v \in \{0, 1\}^d \setminus \{0\}$ and $c \in \mathbb{R}^d$, we have*

$$\max\{c^T x : x \in P^{<v}\} = \max \left\{ \sum_{p=1}^{q-1} c_{\sigma_p(v)} + c^+([\sigma_q(v)]_0) : q \in S(v) \right\}.$$

In particular, the optimization problem $\max\{c^T x : x \in P^{<v}\}$ (for given $v \in \{0, 1\}^d$ and $c \in \mathbb{Q}^d$) can be solved in polynomial time.

3.2. A Linear Description. If $i \in \bar{S}(v)$ and $x \in X^{<v}$ with $x_i = 1$, then $x_j = 0$ must hold for some $j \in S(v)$ with $j > i$. Let us denote

$$S^{>i}(v) := \{j \in S(v) : j > i\} \quad \text{and} \quad \bar{S}^{>i}(v) := \{j \in \bar{S}(v) : j > i\}.$$

We will use similar notations with respect to $<$, \leq and \geq . Thus, the inequalities

$$(1) \quad x_i + \sum_{j \in S^{>i}(v)} x_j \leq |S^{>i}(v)| \quad \text{for all } i \in \bar{S}(v)$$

and (since $v \notin P^{<v}$)

$$(2) \quad \sum_{j \in S(v)} x_j \leq |S(v)| - 1$$

are valid for $P^{<v}$.

Theorem 1 (Linear descriptions of revlex-initial 0/1-polytopes). *For every $v \in \{0, 1\}^d \setminus \{0\}$ the revlex-initial 0/1-polytope $P^{<v}$ has the following linear description:*

$$(3) \quad P^{<v} = \{x \in \mathbb{R}^d : 0 \leq x \leq \mathbb{1}, x \text{ satisfies (1) and (2)}\}$$

Proof. Denote the polytope defined by the right-hand side of (3) by $Q(v)$. Thus, $Q(v)$ is the set of all $x \in \mathbb{R}^d$ satisfying the following system of inequalities:

$$\begin{aligned}
(4) \quad & -x_i \leq 0 \quad \text{for all } i \in [d]_0 \\
(5) \quad & x_i \leq 1 \quad \text{for all } i \in [d]_0 \\
(6) \quad & \sum_{j \in S(v)} x_j \leq w(v) - 1 \\
(7) \quad & x_i + \sum_{j \in S^{>i}(v)} x_j \leq |S^{>i}(v)| \quad \text{for all } i \in \bar{S}(v)
\end{aligned}$$

Since all these inequalities are valid for $P^{<v}$, we have $P^{<v} \subseteq Q(v)$. In order to prove the theorem, it therefore suffices to show that every vertex of $Q(v)$ is contained in $P^{<v}$.

Let z^* be any vertex of $Q(v)$. Thus, there is some $c \in \mathbb{R}^d$ such that the linear functional $c^T x$ is uniquely maximized by z^* over $Q(v)$. We may assume that there is also a unique $x^* \in X^{<v}$ that maximizes $c^T x$ over $P^{<v}$. It suffices to show that c can be obtained as a *conic combination* (i.e., a linear combination with nonnegative coefficients) of the coefficient vectors of (the left-hand sides of) those inequalities (4), (5), (6), and (7) that are satisfied with equality by x^* (since this proves that x^* even maximizes $c^T x$ over $Q(v)$ – thus showing $z^* = x^* \in P^{<v}$).

Let

$$q^* := \min\{q \in [w(x^*)] : x_{\sigma_q(v)}^* = 0\}$$

be the number of the block face of $P^{<v}$ containing x^* and

$$\alpha := \sigma_{q^*}(v).$$

The following table illustrates the most important parameters used in the proof (some of which will be defined only later). In the example displayed, we have $q^* = 4$:

	β	L	L	L	L	α										
v	0	0	1	0	0	1	0	0	1	1	0	0	1	0	0	1
x^*	?	?	0	1	1	1	1	0	0	1	0	0	1	0	0	1

Let us denote, for $i \in \bar{S}(v)$, the coefficient vector of (the left-hand-side) of (7) by $a^{(i)} \in \mathbb{R}^d$, and the coefficient vector of (6) by $a \in \mathbb{R}^d$. Clearly, the coefficient vectors of (4) and (5) are $-\mathbf{e}_i$ and \mathbf{e}_i , respectively. Let C^* be the set of all coefficient vectors of those among the inequalities (4), (5), (6), and (7) that are satisfied at equality by x^* . By cone C^* we denote the set of all conic combinations of vectors from C^* . Thus, we have to show $c \in \text{cone } C^*$.

For each $i \in [d]_0$, let us define $\text{type}(i) := (v_i, x_i^*)$. Thus, we have $\text{type}(i) \in \{(0, 0), (1, 1)\}$ for all $i > \alpha$. In particular, $\bar{S}^{>\alpha}(v) = \{i > \alpha : x_i^* = 0\}$ holds, and

thus we have

$$c' := \sum_{\substack{i > \alpha \\ x_i^* = 0, c_i \leq 0}} (-c_i)(-\mathbf{e}_i) + \sum_{\substack{i > \alpha \\ x_i^* = 0, c_i > 0}} c_i a^{(i)} \in \text{cone } C^*$$

with $c'_i = c_i$ for all $i > \alpha$ with $x_i^* = 0$.

If $c_\alpha \leq 0$, then define

$$c'' := c' + (-c_\alpha)(-\mathbf{e}_\alpha) \in \text{cone } C^* .$$

Let us consider the other case $c_\alpha > 0$. If $x^* = v - \mathbf{e}_\alpha$, then $a \in \text{cone } C^*$, and thus we have

$$c'' := c' + c_\alpha a \in \text{cone } C^* .$$

Otherwise, we set

$$\beta := \max(\{-1\} \cup \{k \in [\alpha]_0 : \text{type}(k) = (1, 0)\}) .$$

We conclude

$$L := \{l : \beta < l < \alpha, \text{type}(l) = (0, 1)\} \neq \emptyset ,$$

since if this was not the case, then switching the α -th component of x^* from 0 to 1 would yield a point in $P^{\prec v}$ (note $x^* \neq v - \mathbf{e}_\alpha$), contradicting the optimality of x^* due to $c_\alpha > 0$. Because $\text{type}(k) \neq (1, 0)$ holds for all $\beta < k < \alpha$, we have

$$a^{(l)} \in C^* \quad \text{for all } l \in L .$$

Moreover, we derive $\sum_{l \in L} c_l \geq c_\alpha$ from the optimality of x^* . Thus, there are coefficients $0 \leq \lambda_l \leq c_l$ ($l \in L$) with

$$c_\alpha = \sum_{l \in L} \lambda_l$$

(recall $c_\alpha > 0$). We define (for the case $c_\alpha > 0$)

$$c'' := c' + \sum_{l \in L} \lambda_l a^{(l)} .$$

Clearly, we have

$$(8) \quad c''_i = \lambda_l \leq c_l \quad \text{for all } l \in L .$$

In fact, it will be important for our purposes to choose the coefficients λ_l carefully. One way to do this is to choose them such that $\sum_{l' \in L, l' \geq l} \lambda_{l'}$ is maximized for all $l \in L$. This implies

$$c''_i \leq c_\alpha - \sum_{l \in L, l > i} c_l \quad \text{for all } \beta < i < \alpha, \text{type}(i) = (1, 1) .$$

On the other hand, we can conclude for each $\beta < i < \alpha$ with $\text{type}(i) = (1, 1)$ that

$$c_\alpha \leq c_i + \sum_{l \in L, l > i} c_l$$

must hold (since otherwise switching x_α^* from 0 to 1 as well as x_i^* and x_l^* ($l \in L$, $l > i$) from 1 to 0 would produce some $\tilde{x} \in P^{<v}$ from x^* with $c^T \tilde{x} > c^T x^*$, contradicting the optimality of x^*). From these two inequalities we derive

$$(9) \quad c_i'' \leq c_i \quad \text{for all } \beta < i < \alpha, \text{type}(i) = (1, 1) .$$

In both cases ($c_\alpha \leq 0$ and $c_\alpha > 0$), we have $c'' \in \text{cone } C^*$ with $c_i'' = c_i$ for all $i \geq \alpha$ with $x_i^* = 0$. For all $i > \alpha$ with $x_i^* = 1$, we have

$$c_i \geq \sum_{\substack{\alpha \leq j < i \\ x_j^* = 0}} c_j \geq c_i''$$

since, if the first inequality was not true, switching x_i^* from 1 to 0 and all x_j^* ($\alpha \leq j < i$) from 0 to 1 would change x^* into some $\tilde{x} \in P^{<v}$ with $c^T \tilde{x} > c^T x^*$ (contradicting the optimality of x^*). Thus

$$c''' := c'' + \sum_{i > \alpha, x_i^* = 1} (c_i - c_i'') \mathbf{e}_i \in \text{cone } C^*$$

with $c_i''' = c_i$ for all $i \geq \alpha$.

For $i < \alpha$, we have (where, in the first case $c_\alpha \leq 0$, we set $L := \emptyset$ and $\beta := \alpha$)

$$\begin{aligned} c_i''' &= 0 & \text{for all } i \leq \beta \\ c_i''' &= 0 & \text{for all } \beta < i < \alpha, i \notin L, \text{type}(i) \neq (1, 1) \\ c_l''' &\leq c_l & \text{for all } l \in L \text{ by (8)} \\ c_i''' &\leq c_i & \text{for all } \beta < i < \alpha, \text{type}(i) = (1, 1) \text{ by (9)} . \end{aligned}$$

Clearly, due to the (unique) optimality of x^* , we have $c_i > 0$ for all $i < \alpha$ with $x_i^* = 1$ and $c_i < 0$ for all $i < \alpha$ with $x_i^* = 0$. Therefore, adding to c'''

$$\begin{aligned} (-c_i)(-\mathbf{e}_i) & \text{ for all } i \leq \alpha, x_i^* = 0 \text{ and} \\ (c_i - c_i''')\mathbf{e}_i & \text{ for all } i \leq \alpha, x_i^* = 1 \end{aligned}$$

yields the desired nonnegative linear combination of c from vectors in C^* . □

3.3. The Facet Defining Inequalities. Let us first describe the dimension of a revlex-initial 0/1-polytope.

Proposition 2. *For each $v \in \{0, 1\}^d \setminus \{0\}$ the dimension of the revlex-initial 0/1-polytope $P^{<v}$ is*

$$\dim P^{<v} = 1 + \max(\{i \in [d]_0 : \mathbf{e}_i \prec_{\text{revlex}} v\} \cup \{-1\}) ,$$

if $v \neq \mathbf{e}_0$. For the latter case we have $\dim P^{<\mathbf{e}_0} = 0$.

In our knapsack notation we have for $n \in \mathbb{N}$

$$\dim P^{<n} = 1 + \max\{i \in \mathbb{N} \cup \{-1\} : 2^i < n\} = \min\{j \in \mathbb{N} : n \leq 2^j\} .$$

Proof. This follows from the block decomposition of $P^{<v}$. □

In particular, $P^{\prec v}$ is full-dimensional if and only if $\mathbf{e}_{d-1} \prec_{\text{rlex}} v$. The following three propositions describe the facets of full-dimensional revlex-initial 0/1-polytopes.

Proposition 3. *For each $v \in \{0, 1\}^d$ with $\mathbf{e}_{d-1} \prec_{\text{rlex}} v$, and for every $i \in [d]_0$, the inequality $x_i \geq 0$ defines a facet of $P^{\prec v}$.*

Proof. By Theorem 1, the inequalities (4)–(6) provide a linear description of $P^{\prec v}$. Since the trivial inequalities (4) are the only ones in this description which have negative coefficients, none of them can be conically combined from others. Hence, they all define facets of $P^{\prec v}$ (since $P^{\prec v}$ is full-dimensional). \square

Proposition 4. *For each $v \in \{0, 1\}^d$ with $\mathbf{e}_{d-1} \prec_{\text{rlex}} v$, the inequality $\sum_{j \in \mathbb{S}(v)} x_j \leq \mathbf{w}(v) - 1$ defines a facet of $P^{\prec v}$.*

Proof. The inequality $\sum_{j \in \mathbb{S}(v)} x_j \leq \mathbf{w}(v) - 1$ is the only inequality in the linear description (4)–(6) of $P^{\prec v}$ provided by Theorem 1 that is violated by the point v , which is not contained in $P^{\prec v}$. Thus, that inequality must define a facet of $P^{\prec v}$. \square

Proposition 5. *For each $v \in \{0, 1\}^d$ with $\mathbf{e}_{d-1} \prec_{\text{rlex}} v$, and for every $i \in [d]_0$, the inequality $x_i \leq 1$ defines a facet of $P^{\prec v}$ unless*

$$\mathbf{w}(v) = 2 \text{ and } i \in \mathbb{S}(v)$$

or

$$\sigma_2(v) < d - 2 \text{ and } \sigma_2(v) < i \leq d - 1$$

(in which cases they do not define facets).

Proof. Unless one of the exceptions listed in the proposition holds, all inequalities from the linear description (4)–(6) of $P^{\prec v}$ provided by Theorem 1 that have a positive i -th coefficient have right-hand-side at least two. Since the only ones with negative i -th coefficient have right-hand-side zero, the inequality $x_i \leq 1$ cannot be conically combined from the others in that linear description. Hence it defines a facet of $P^{\prec v}$ (since $P^{\prec v}$ is full-dimensional).

In case of $\mathbf{w}(v) = 2$ and $i \in \mathbb{S}(v)$, let j be such that $\mathbb{S}(v) = \{i, j\}$. Thus, $x_i \leq 1$ is the sum of inequality (6) and $-x_j \leq 0$. Hence, it does not define a facet of $P^{\prec v}$.

Finally, consider the case $\sigma_2(v) < d - 2$. If $\sigma_2(v) < i < d - 1$, then the type-(7) inequality $x_i + x_{d-1} \leq 1$ implies $x_i \leq 1$ by adding $-x_{d-1} \leq 0$. If $i = d - 1$ then the type (7) inequality $x_j + x_{d-1} \leq 1$ for any $\sigma_2(v) < j < d - 1$ implies $x_{d-1} \leq 1$ by adding $-x_j \leq 0$. Thus, in both cases, $x_i \leq 1$ does not define a facet of $P^{\prec v}$. \square

Proposition 6. *For each $v \in \{0, 1\}^d$ with $\mathbf{e}_{d-1} \prec_{\text{rlex}} v$, and for every $i \in \overline{\mathbb{S}}(v)$, the inequality $x_i + \sum_{j \in \mathbb{S}^{>i}(v)} x_j \leq |\mathbb{S}^{>i}(v)|$ defines a facet of $P^{\prec v}$ unless $i < \sigma_{\mathbf{w}(v)}(v)$ (in which case they do not define facets).*

Proof. For each $i \in \overline{\mathbb{S}}(v)$ with $i > \sigma_{\mathbf{w}(v)}(v)$, the inequality $x_i + \sum_{j \in \mathbb{S}^{>i}(v)} x_j \leq |\mathbb{S}^{>i}(v)|$ is the only inequality in the linear description (4)–(6) of $P^{\prec v}$ provided

by Theorem 1 that is violated by the point $v + \mathbf{e}_i - \mathbf{e}_{w(v)}$, which is not contained in $P^{\prec v}$. Thus, that inequality must define a facet of $P^{\prec v}$.

If $i < \sigma_{w(v)}(v)$, then $x_i + \sum_{j \in S^{>i}(v)} x_j \leq |S^{>i}(v)|$ does not define a facet since it equals the sum of the two inequalities $\sum_{j \in S(v)} x_j \leq w(v) - 1$ and $x_i \leq 1$. \square

Combining Theorem 1 and the five preceding propositions, we obtain the following result.

Theorem 2 (Facets of revlex-initial 0/1-polytopes). *Let $v \in \{0, 1\}^d$ with $\mathbf{e}_{d-1} \prec_{rlx} v$, i.e., $P^{\prec v}$ is a full-dimensional revlex-initial 0/1-polytope. Let*

$$D(v) := D_1(v) \cup D_2(v)$$

with

$$D_1(v) := \begin{cases} S(v) & \text{if } w(v) = 2 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$D_2(v) := \begin{cases} \{\sigma_2(v) + 1, \dots, d - 1\} & \text{if } \sigma_2(v) < d - 2 \\ \emptyset & \text{otherwise} \end{cases}.$$

- (1) *The following system is a linear description of $P^{\prec v}$ by inequalities defining pairwise disjoint facets of $P^{\prec v}$:*

$$\begin{aligned} x_i &\geq 0 && \text{for all } i \in [d]_0 \\ x_i &\leq 1 && \text{for all } i \in [d]_0 \setminus D(v) \\ x_i + \sum_{j \in S^{>i}(v)} x_j &\leq |S^{>i}(v)| && \text{for all } i \in \bar{S}(v), i > \sigma_{w(v)}(v) \\ \sum_{j \in S(v)} x_j &\leq w(v) - 1 \end{aligned}.$$

- (2) *The number of facets of $P^{\prec v}$ is*

$$f_{d-1}(P^{\prec v}) = 2d + |\{\sigma_{w(v)}(v) < i < \sigma_2(v) : v_i = 0\}| + \epsilon,$$

where

$$\epsilon := \begin{cases} -1 & \text{if } w(v) = 2 \\ 0 & \text{if } w(v) > 2, v_{d-2} = 0 \\ 1 & \text{otherwise (i.e., } w(v) > 2, v_{d-2} = 1) \end{cases}.$$

We have

$$2d - 1 \leq f_{d-1}(P^{\prec v}) \leq 3d - 2.$$

The minimum number $2d - 1$ of facets is attained if and only if $w(v) = 2$, and the maximum $f_{d-1}(P^{\prec v}) = 3d - 2$ is achieved only by $v = \mathbf{e}_0 + \mathbf{e}_{d-2} + \mathbf{e}_{d-1}$ (for $d \geq 3$).

See Figure 2 for an illustration of the facet numbers of revlex-initial 0/1-polytopes.

3.4. Incremental Convex-Hull Algorithms. The origin of our investigations on revlex-initial 0/1-polytopes lies in some experiments on computing the convex hulls of random 0/1-polytopes that we performed with the `polymake` system. Some of the results of the experiments are illustrated in Figure 1, showing the running times for computing the convex hulls of (uniformly) random 0/1-polytopes in \mathbb{R}^d depending on the number n of vertices. The picture shows two curves, one for the *beneath-beyond* and one for the *double-description* method (where `polymake` uses Komei Fukuda’s implementation `cdd` for the latter method).

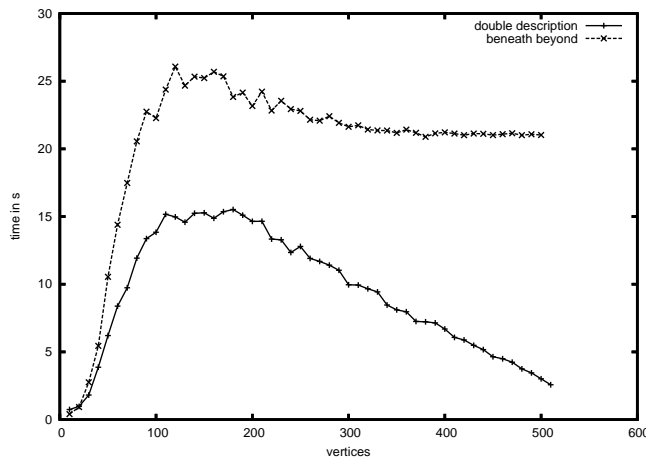


FIGURE 1. Incremental convex hull algorithms: running times on 9-dimensional random 0/1-polytopes.

These two methods are *incremental* in the sense that they iteratedly compute the convex hull of the first $i + 1$ vertices from the convex hull of the first i vertices. Since $n - 1$ vertices of a random 0/1-polytope with n vertices should make a random 0/1-polytope with $n - 1$ vertices, we had expected the curves to be monotonically increasing. However, the *first* $n - 1$ vertices do only make a (uniform) random 0/1-polytope with $n - 1$ vertices if the order of the n vertices is (uniformly) random.

As it turned out, this is not the case for random 0/1-polytopes produced by the `polymake` system. Instead, the `rand01` client of `polymake` is implemented in such a way that the vertices of the random 0/1-polytope that is produced appear in lexicographic order. This led us to studying revlex-initial 0/1-polytopes.

And in fact, our results on the facet numbers of revlex-initial 0/1-polytopes make the curves in Figure 1 plausible: For 0/1-polytopes with large numbers of vertices, which furthermore are lexicographically ordered, the intermediate polytopes appearing during the runs of incremental convex hull algorithms are quite close to revlex-initial 0/1-polytopes. Therefore, it is plausible that these intermediate polytopes have extremely few facets compared to random 0/1-polytopes with the same numbers of vertices.

In particular, if the 2^d vertices of the entire cube are ordered lexicographically, then the total number of facets of all intermediate polytopes produced by an incremental convex hull algorithm is bounded from above by $3d \cdot 2^d$, while for an arbitrary (even for a random ordering) there might be intermediate polytopes with super-exponentially many vertices (due to the results of Bárány and Pór [2] and Gatzouras, Giannopoulos, and Markoulakis [5]).

These results indicate that it might be a good strategy to sort the vertices lexicographically before applying an incremental convex hull algorithm to a 0/1-polytope. However, we do not yet have any thorough computational study to support this.

4. THE GRAPHS OF REVLEX-INITIAL 0/1-POLYTOPES

4.1. Characterization of Adjacency. The one-dimensional faces of a polytope (forming its 1-*skeleton* or *graph*) are particularly important, for instance, since the simplex algorithm for linear programming proceeds along them. Moreover, in the special case of 0/1-polytopes, the graphs are important also for different reasons (see Section 4.3).

Here, we describe the graphs of revlex-initial 0/1-polytopes.

Definition 6. For $v \in \{0, 1\}^d \setminus \{0\}$ and $1 \leq p < q \leq w(v)$ and $x \in \{0, 1\}^{[d]_0}$ we define the sets

$$A_{p,q}^{\prec v}(x) := \{z \in P_p^{\prec v} : z_i = x_i \text{ for all } 0 \leq i < \sigma_q(v), \\ z_{\sigma_q(v)} = 0, z_{\sigma_r(v)} = 1 \text{ for all } p < r < q\}$$

and

$$B_{p,q}^{\prec v}(x) := \{z \in P_p^{\prec v} : z z_i = x_i \text{ for all } 0 \leq i < \sigma_q(v), \\ z_{\sigma_q(v)} = 1, z_{\sigma_r(v)} = 1 \text{ for all } p < r < q\}.$$

Theorem 3 (Graphs of revlex-initial 0/1-polytopes). For $v \in \{0, 1\}^d \setminus \{0\}$, the graph of the corresponding revlex-initial 0/1-polytope $P^{\prec v}$ has the following structure.

- (1) Let $x \in X^{\prec v}$ be a vertex of $P^{\prec v}$ contained in the block $P_q^{\prec v}$. Let p be some block number with $1 \leq p < q$.
 - (a) The pyramid over $A_{p,q}^{\prec v}(x)$ with apex x is a face of $P^{\prec v}$; in particular, x is adjacent to all vertices of $A_{p,q}^{\prec v}(x)$.
 - (b) If $\max(\{i \in [\sigma_q(v)]_0 : x_i \neq v_i\} \cup \{-1\}) \notin S(v)$, then the pyramid over $B_{p,q}^{\prec v}(x)$ with apex x is a face of $P^{\prec v}$; in particular, x then is also adjacent to all vertices of $B_{p,q}^{\prec v}(x)$.
- (2) The graph of $P^{\prec v}$ does not contain any other edges than the (cube-)edges of the blocks $P_1^{\prec v}, \dots, P_{w(v)}^{\prec v}$ and the edges described in part (i) of this theorem.

TABLE 2. Illustration of the definitions (with $p = 1$, $q = 4$, and $x = (1, 0, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \{0, 1\}^{10} \setminus \{0\}$) at the example from the introduction.

v	1	0	1	1	0	0	1	0	0	1
indices	0	1	2	3	4	5	6	7	8	9
signature	$\sigma_5(v)$		$\sigma_4(v)$	$\sigma_3(v)$			$\sigma_2(v)$			$\sigma_1(v)$
$P_1^{\prec v}$	*	*	*	*	*	*	*	*	*	0
$A_{1,4}^{\prec v}(x)$	1	0	0	1	*	*	1	*	*	0
$B_{1,4}^{\prec v}(x)$	1	0	1	1	*	*	1	*	*	0
$P_4^{\prec v}$	*	*	0	1	0	0	1	0	0	1

Proof. For the proof of part (1), let us denote by F the face of $P^{\prec v}$ that is defined by the following equations:

$$(10) \quad z_i = x_i \quad (0 \leq i < \sigma_q(v))$$

$$(11) \quad z_{\sigma_r(v)} = 1 \quad (p < r < q)$$

$$(12) \quad z_i = v_i \quad (\sigma_p(v) < i)$$

The claim in (a) follows from the fact that the only vertices of the face $\{z \in F : z_{\sigma_q(v)} = 0\}$ of $P^{\prec v}$ are the vertices of $A_{p,q}^{\prec v}(x)$ and x itself. Since $A_{p,q}^{\prec v}(x)$ is contained in the hyperplane defined by $z_{\sigma_p(v)} = 0$, while x is not, that face must be the claimed pyramid.

In order to prove part (b), assume $\max(\{i \in [\sigma_q(v)]_0 : x_i \neq v_i\} \cup \{-1\}) \notin S(v)$. Thus, there is no block $P_r^{\prec v}$ with $r > q$ that has a common vertex with the face $\{z \in F : z_{\sigma_q(v)} = 1\}$. Hence, the only vertices of that face are the vertices of $B_{p,q}^{\prec v}(x)$ and x itself. Again, since $B_{p,q}^{\prec v}(x)$ is contained in the hyperplane defined by $z_{\sigma_p(v)} = 0$, while x is not, that face must be the claimed pyramid.

For the proof of part (2), suppose that x and y are adjacent vertices of $P^{\prec v}$ not contained in the same block. We may assume $x \in P_q^{\prec v}$ and $y \in P_p^{\prec v}$ with $1 \leq p < q \leq w(v)$.

We will first show that y is contained in the face F of $P^{\prec v}$ defined in the proof of part (1). Therefore, we have to prove that (10)–(12) is satisfied by $z = y$.

Let us assume (10) was not satisfied by $z = y$, i.e., there is some $0 \leq i < \sigma_q(v)$ with $x_i \neq y_i$. If we denote, for $a, b \in \{0, 1\}^d$, by $a \oplus b$ the componentwise addition modulo two, then we have $x \oplus \mathbf{e}_i \in P^{\prec v}$ (since $i < \sigma_q(v)$) and $y \oplus \mathbf{e}_i \in P^{\prec v}$ (since $i < \sigma_q(v) < \sigma_p(v)$) with

$$\{x \oplus \mathbf{e}_i, y \oplus \mathbf{e}_i\} \neq \{x, y\}$$

(since $x_{\sigma_p(v)} = 1 \neq 0 = y_{\sigma_p(v)}$). But then

$$\frac{1}{2}(x + y) = \frac{1}{2}(x \oplus \mathbf{e}_i + y \oplus \mathbf{e}_i)$$

contradicts the adjacency of x and y . Thus, $z = y$ satisfies (10).

If (11) would not be satisfied by $z = y$, then there was some $p < r < q$ with $y_{\sigma_r(v)} = 0$. Due to $x \in P_q^{\prec v}$, $x_{\sigma_r(v)} = 1$ holds. Thus, we have $x - \mathbf{e}_{\sigma_r(v)} \in P^{\prec v}$ and $y + \mathbf{e}_{\sigma_r(v)} \in P^{\prec v}$ (since $y \in P_p^{\prec v}$ with $r < p$). Again,

$$\{x - \mathbf{e}_{\sigma_r(v)}, y + \mathbf{e}_{\sigma_r(v)}\} \neq \{x, y\}$$

holds, and therefore,

$$\frac{1}{2}(x + y) = \frac{1}{2}((x - \mathbf{e}_{\sigma_r(v)}) + (y + \mathbf{e}_{\sigma_r(v)}))$$

contradicts the adjacency of x and y . Hence, (11) is satisfied by $z = y$.

Since $q > p$ and $x \in P_q^{\prec v}$, $y \in P_p^{\prec v}$, we clearly have $x_i = y_i = v_i$ for all $i > \sigma_p(v)$. Therefore, also (12) is satisfied by $z = y$, and thus, the claim $y \in F$ is proved.

We obtain $y \in A_{p,q}^{\prec v}(x) \cup B_{p,q}^{\prec v}(x)$. It hence suffices to show that, in case of $y \in B_{p,q}^{\prec v}(x)$, we have

$$\max(\{i \in [\sigma_q(v)]_0 : x_i \neq v_i\} \cup \{-1\}) \notin S(v).$$

Therefore, suppose we have $y \in B_{p,q}^{\prec v}(x)$ and there is some $q < s \leq w(v)$ with $x_{\sigma_s(v)} = 0$ and $x_i = v_i$ for all $\sigma_s(v) < i < \sigma_q(v)$. Then we have $y - \mathbf{e}_{\sigma_q(v)} \in P^{\prec v}$ (due to $y \in P_p^{\prec v}$, $p < q$, and $y_{\sigma_q(v)} = 1$) and $x + \mathbf{e}_{\sigma_q(v)} \in P^{\prec v}$ (in fact: $x + \mathbf{e}_{\sigma_q(v)} \in P_s^{\prec v}$). Also here, we have

$$\{x + \mathbf{e}_{\sigma_q(v)}, y - \mathbf{e}_{\sigma_q(v)}\} \neq \{x, y\},$$

and thus,

$$\frac{1}{2}(x + y) = \frac{1}{2}((x + \mathbf{e}_{\sigma_q(v)}) + (y - \mathbf{e}_{\sigma_q(v)}))$$

contradicts the adjacency of x and y . □

4.2. The Number of Edges. Having the structural description given in Theorem 3 at hands, we can now derive a formula for the number of edges of a revlex-initial 0/1-polytope.

Theorem 4 (Edge numbers of revlex-initial 0/1-polytopes). *For $v \in \{0, 1\}^d \setminus \{0\}$, the graph of the corresponding revlex-initial 0/1-polytope $P^{\prec v}$ has*

$$\sum_{p=1}^{w(v)} 2^{\sigma_p(v)} \left(\frac{\sigma_p(v)}{2} + \sum_{q=p+1}^{w(v)} 2^{p-q} (2 - (w(v) - q) 2^{\sigma_q(v)}) \right)$$

edges. In particular, its average node degree is bounded from above by $d + 8$.

Proof. The statement on the average degree follows from the exact expression for the number of edges: Inside the (outmost) brackets, the fraction $\frac{\sigma_p(v)}{2}$ is bounded from above by $\frac{d}{2}$ while the remaining sum clearly is at most 4. Thus the number of edges is at most $(\frac{d}{2} + 4)$ times the number $\sum 2^{\sigma_p(v)}$ of vertices of $P^{\prec v}$.

In order to determine the total number of edges, let $1 \leq p < q \leq w(v)$. We have

$$\dim A_{p,y}^{\prec v}(x) = \dim B_{p,q}^{\prec v}(x) = (p + \sigma_p(v)) - (q + \sigma_q(v)) =: \delta_{p,q}$$

for each $x \in X_q^{\prec v}$.

Clearly, the number of edges between $P_q^{\prec v}$ and $P_p^{\prec v}$ described in part (1a) of Theorem 3 thus is

$$2^{\sigma_q(v)} \cdot 2^{\delta_{p,q}} = 2^{p+\sigma_p(v)-q}.$$

The number of $x \in P_q^{\prec v}$ that do not satisfy the condition of part (1b) of Theorem 3 is $w(v) - q$. Thus, the number of edges between $P_q^{\prec v}$ and $P_p^{\prec v}$ described in part (1b) is

$$(2^{\sigma_q(v)} - (w(v) - q)) \cdot 2^{\delta_{p,q}} = 2^{p+\sigma_p(v)-q}.$$

Therefore, the total number of edges is

$$\sum_{p=1}^{w(v)} \sigma_p(v) 2^{\sigma_p(v)-1} + \sum_{1 \leq p < q \leq w(v)} (2 \cdot 2^{p+\sigma_p(v)-q} - (w(v) - q) 2^{(p+\sigma_p(v))-(q+\sigma_q(v))}),$$

where the first sum accounts for the edges inside the blocks and the second one (the double-sum) counts the edges running across different blocks. That expression can easily be simplified to the one stated in the theorem. \square

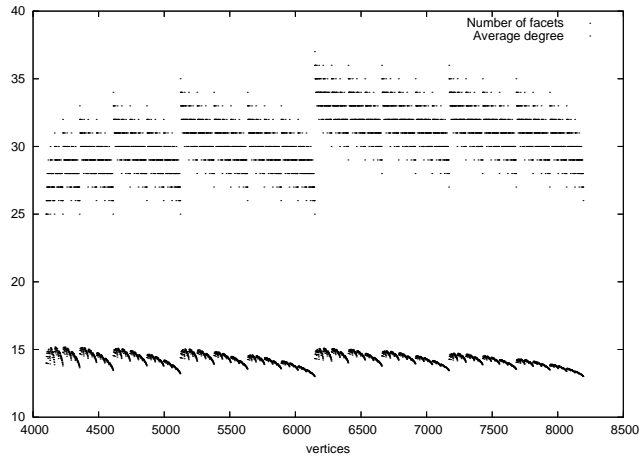


FIGURE 2. The numbers of facets and the average degrees of all full-dimensional revlex-initial 0/1-polytopes for $d = 13$.

4.3. The Edge-Expansion. The geometry of a 0/1-polytope P (more precisely: its 1-skeleton, i.e., its graph) defines a natural neighborhood structure on the set system \mathcal{S} corresponding to the vertices of P . Such a neighborhood structure can be used in order to design random walk algorithms for generating elements from \mathcal{S} at random (according to a certain prespecified probability distribution). Random walk algorithms are of great importance, for instance with respect to randomized approximative counting algorithms (see, e.g., [6]).

In many cases, the neighborhood structure defined geometrically via the associated 0/1-polytope has turned out to be quite appropriate for designing such random walk algorithms. A crucial parameter with respect to the time complexity of these methods is the *edge expansion* of the neighborhood structure. The rule of thumb here is that the expansion should be bounded from below polynomially in $1/d$ (where d is the dimension of the polytope) in order to achieve an efficient time algorithm.

Definition 7. *The edge expansion $\mathcal{X}(G)$ of a graph $G = (V, E)$ is defined as*

$$\mathcal{X}(G) := \min \left\{ \frac{|\delta(S)|}{|S|} : S \subset V, 0 < |S| \leq \frac{|V|}{2} \right\}$$

(with $\delta(S)$ denoting the set of all edges with one end node in S and the other one in $V \setminus S$).

It has been conjectured by Mihail and Vazirani (cited, e.g., in [3, 9]) that the graph of every 0/1-polytope has edge expansion at least one. In fact, this conjecture is known to be true for several classes of 0/1-polytopes, including stable set polytopes, (perfect) matching polytopes, and polytopes associated with the bases of *balanced* (in particular: regular) matroids (see [9, 3, 7]). For more details and references, we refer to [7].

Here, further supporting the Mihail-Vazirani conjecture, we prove that despite the sparsity of their graphs, revlex-initial 0/1-polytopes have edge expansion at least one.

Theorem 5 (Edge expansion of revlex-initial 0/1-polytopes). *For $v \in \{0, 1\}^d \setminus \{0\}$, the graph of the corresponding revlex-initial 0/1-polytope $P^{\prec v}$ has edge expansion at least one.*

In order to bound the edge expansion of a graph $G = (V, E)$ from below we will construct certain flows in the (uncapacitated) network $\mathcal{N}(G) = (V, A)$, where A contains for each edge $\{u, v\} \in E$ both arcs (u, v) and (v, u) . This strategy dates back to the method of “canonical paths” developed by Sinclair (see [11]). The extension to flows was explicitly exploited by Morris and Sinclair [10]. Feder and Mihail [3] use random canonical paths, which can equivalently be formulated in terms of flows.

The crucial idea is to construct for each ordered pair $(x, y) \in V \times V$ a flow $\phi_{(x,y)} : A \rightarrow \mathbb{Q}^{\geq 0}$ in the network $\mathcal{N}(G)$ sending one unit of some commodity

from x to y . Define the multi commodity flow (MCF) $\phi := \sum_{(x,y) \in V \times V} \phi_{(x,y)}$ as the sum of all the flows $\phi_{(x,y)}$. By

$$\phi_{\max} := \max\{\phi(a) : a \in A\}$$

we denote the maximal amount of ϕ -flow on any arc. By construction of ϕ , the total amount $\phi(S : V \setminus S)$ of ϕ -flow leaving S is at least $|S| \cdot (n - |S|)$, where $n = |V|$. On the other hand, we have $\phi(S : V \setminus S) \leq \phi_{\max} \cdot |\delta(S)|$. This implies $|S| \cdot (n - |S|) \leq \phi_{\max} \cdot |\delta(S)|$, and hence, if $|S| \leq \frac{n}{2}$ holds,

$$\frac{|\delta(S)|}{|S|} \geq \frac{n}{2 \cdot \phi_{\max}} .$$

Thus, we have proven

$$(13) \quad \mathcal{X}(G) \geq \frac{n}{2 \cdot \phi_{\max}} .$$

In the light of inequality (13) it is clear that the task is to construct a flow ϕ as above with $\phi_{\max} \leq \frac{n}{2}$ (where $n = |V|$).

Proof of Theorem 5. We will use the notations $P^{<n} := P^{<v} \subset \mathbb{R}^d$ and $X^{<n} := X^{<v}$, where $n \in \mathbb{N}$ is the number having binary representation v (i.e., n is the number of vertices of $P^{<n} = P^{<v}$). Clearly, we may assume $v_{d-1} = 1$, i.e., $n > 2^{d-1}$ and $\dim P^{<n} = d$. Thus, in particular, the dimension d and the 0/1-vector $v \in \{0, 1\}^d$ are uniquely determined by the vertex number n .

We will prove the theorem by showing via induction on n that, for every $n \in \mathbb{N}$, there is an MCF $\Phi^n = \sum_{(x,y) \in X^{<n} \times X^{<n}} \Phi_{(x,y)}^n$ on $\mathcal{N}(G(P^{<n}))$ such that $\Phi_{\max}^n \leq \frac{n}{2}$.

The statement obviously holds for $n = 2$, since in that case, the polytope $P^{<n}$ consists of two vertices joint by an edge.

Thus let us suppose that for all $2 \leq n' < n$ there is such an MCF $\Phi^{n'}$ on $\mathcal{N}(G(P^{<n'}))$ with $\Phi_{\max}^{n'} \leq \frac{n'}{2}$. The induction step, i.e., the construction of an appropriate MCF Φ^n , will be subdivided into two cases.

Let $G := G(P^{<n})$. For a subset A of the nodes of G , we denote by $G[A]$ the subgraph of G induced by A (similarly, we use $\mathcal{N}(G)[A]$). Two 0/1-polytopes P and Q are called *0/1-equivalent* if they can be transformed into each other by (potentially) lifting one of them into the space of the other and applying a symmetry of the cube (i.e., by flipping and permuting coordinates). Of course, such a transformation induces an isomorphism between the graphs of P and Q . Note that for $w \in \{0, 1\}^d$ the vertex $w \oplus \mathbf{e}_0$ is the one obtained by flipping the first coordinate of w .

Case 1 ($v_0 = 0$). Define the following faces of $P^{<n}$ and the corresponding vertex sets:

$$\begin{aligned} F_A &:= \{w \in P^{<n} : w_0 = 0\} & F_B &:= \{w \in P^{<n} : w_0 = 1\} \\ X_A &:= \{w \in X^{<n} : w_0 = 0\} & X_B &:= \{w \in X^{<n} : w_0 = 1\} \end{aligned}$$

Then, for every $x \in X_A$, we have $x \oplus \mathbf{e}_0 \in X_B$ (and vice versa). Thus $P^{<n}$ is a prism over F_A . In particular, F_A and F_B are 0/1-equivalent. Furthermore, they both are 0/1-equivalent to $P^{<n'}$ with $n' = \frac{n}{2}$. Thus, $G[X_A]$ and $G[X_B]$ both are isomorphic to $G(P^{<n'})$.

Let Φ^A and Φ^B be the MCFs induced by $\Phi^{n'}$ on $\mathcal{N}(G)[X_A]$ and $\mathcal{N}(G)[X_B]$, respectively. Thus $\Phi_{\max}^A = \Phi_{\max}^B = \Phi_{\max}^{n'} \leq \frac{n}{4}$ by the induction hypothesis. Now we construct the MCF Φ^n on $\mathcal{N}(G)$ by defining each $\Phi_{(x,y)}^n$ in the following way (note that $G[X_A]$ and $G[X_B]$ are edge-disjoint):

$$\begin{aligned} x, y \in X_A : \quad & \Phi_{(x,y)}^n := \Phi_{(x,y)}^A \\ x, y \in X_B : \quad & \Phi_{(x,y)}^n := \Phi_{(x,y)}^B \\ x \in X_A, y \in X_B : \quad & \Phi_{(x,y)}^n := \Psi_{(x, x \oplus \mathbf{e}_0)} + \Phi_{(x \oplus \mathbf{e}_0, y)}^B \\ x \in X_B, y \in X_A : \quad & \Phi_{(x,y)}^n := \Psi_{(x, x \oplus \mathbf{e}_0)} + \Phi_{(x \oplus \mathbf{e}_0, y)}^A. \end{aligned}$$

Here, $\Psi_{(x, x \oplus \mathbf{e}_0)}$ denotes the flow just sending one unit along the arc $(x, x \oplus \mathbf{e}_0) \in A$ (and nothing along any other arc). In the resulting MCF Φ^n , every arc $(x, x \oplus \mathbf{e}_0)$ with $x \in X_A$ carries one unit of flow for each of the $|X_B| = \frac{n}{2}$ pairs (x, y) , $y \in X_B$. The same holds for the reverse arcs $(x, x \oplus \mathbf{e}_0)$, $x \in X_B$. Thus we have $\Phi(x, x \oplus \mathbf{e}_0) = \frac{n}{2}$ for every such arc, and we conclude

$$\Phi_{\max}^n \leq \max \left\{ \frac{n}{2}, 2 \cdot \Phi_{\max}^{n'} \right\} = \frac{n}{2}.$$

Case 2 ($v_0 = 1$). Let $\hat{x} \in \{0, 1\}^d$ be the revlex-predecessor of v , i.e., the 0/1-vector corresponding to the number $(n-1)$. Then \hat{x} is the ‘‘last’’ vertex of $P^{<n}$ and $\{\hat{x}\} = P_{w(n)}^{<n}$ is the ‘‘last’’ block of $P^{<n}$. Define the following faces of $P^{<n}$ and the corresponding vertex sets (see Fig. 3):

$$\begin{aligned} F_A &:= \{w \in P^{<n} : w_0 = 0\} & F_B &:= \{w \in P^{<n} : w_0 = 1\} \\ X_A &:= \{w \in X^{<n} : w_0 = 0\} & X_B &:= \{w \in X^{<n} : w_0 = 1\} \cup \{\hat{x}\} \\ X'_A &:= X_A \setminus \{\hat{x}\} & X'_B &:= X_B \setminus \{\hat{x}\} \end{aligned}$$

Thus $P^{<n}$ is a *partial prism* over F_A , i.e., $P^{<n}$ arises from a true prism over F_A via removing the vertex $(\hat{x} + \mathbf{e}_0)$ corresponding to \hat{x} (and taking the convex hull).

We will first prove that there is a subgraph of $G[X_B]$ that is isomorphic to $G[X_A]$. Indeed, this is a simple consequence of the fact that $P^{<n}$ is a partial prism over F_A (with \hat{x} being the ‘‘not duplicated vertex’’): Every edge $\{a, a'\}$ of F_A with $a, a' \neq \hat{x}$ gives rise to a quadrangular 2-face $\{a, a', a \oplus \mathbf{e}_0, a' \oplus \mathbf{e}_0\}$ of the partial prism (showing that $\{a \oplus \mathbf{e}_0, a' \oplus \mathbf{e}_0\}$ is an edge of G), and every edge $\{\hat{x}, a\}$ of F_A yields a triangular 2-face $\{\hat{x}, a, a \oplus \mathbf{e}_0\}$ of the partial prism (showing that $\{\hat{x}, a \oplus \mathbf{e}_0\}$ is an edge of G).

Hence, there is a subgraph of $G[X_B]$ that is isomorphic to $G[X_A]$. Furthermore, the face F_A of $P^{<n}$ is 0/1-equivalent to $P^{<n'}$ with $n' = (n+1)/2$. Therefore, the MCF $\Phi^{n'}$ induces MCFs Φ^A and Φ^B on $\mathcal{N}(G)[X_A]$ and $\mathcal{N}(G)[X_B]$, respectively,

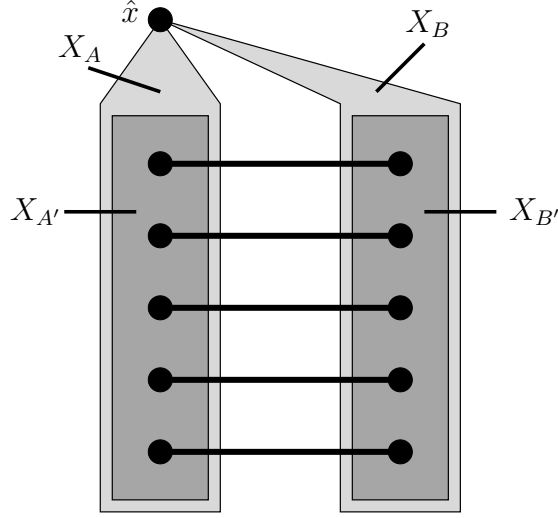


FIGURE 3. Illustration of the sets used in case 2 of the proof of Theorem 5.

with

$$\Phi_{\max}^A = \Phi_{\max}^B = \Phi_{\max}^{n'} \leq \frac{n+1}{4}$$

by the induction hypothesis.

With $\alpha := \frac{n-1}{n+1} < 1$ we have $(1+\alpha)\Phi_{\max}^{n'} \leq \frac{n}{2}$. Thus we can increase each of the flows Φ^A and Φ^B by an α -fraction without making the flow exceed the desired limit of $n/2$ at any arc. We construct the MCF Φ^n on $\mathcal{N}(G)$ by defining each $\Phi_{(x,y)}^n$ in the following way (note that $G[X_A]$ and $G[X_B]$ are edge-disjoint):

$$\begin{aligned} x, y \in X_A : & \quad \Phi_{(x,y)}^n := \Phi_{(x,y)}^A \\ x, y \in X_B : & \quad \Phi_{(x,y)}^n := \Phi_{(x,y)}^B \\ x \in X_A', y \in X_B' : & \quad \Phi_{(x,y)}^n := \alpha \left(\Psi_{(x, x \oplus \mathbf{e}_0)} + \Phi_{(x \oplus \mathbf{e}_0, y)}^B \right) + (1-\alpha) \left(\Phi_{(x, \hat{x})}^A + \Phi_{(\hat{x}, y)}^B \right) \\ x \in X_B', y \in X_A' : & \quad \Phi_{(x,y)}^n := \alpha \left(\Psi_{(x, x \oplus \mathbf{e}_0)} + \Phi_{(x \oplus \mathbf{e}_0, y)}^A \right) + (1-\alpha) \left(\Phi_{(x, \hat{x})}^B + \Phi_{(\hat{x}, y)}^A \right) \end{aligned}$$

Here, as in the first case, $\Psi_{(x, x \oplus \mathbf{e}_0)}$ is the flow sending one unit along the arc $(x, x \oplus \mathbf{e}_0) \in A$ and nothing along any other arc.

It is easy to see that this is a valid MCF (i.e. for each pair (x, y) the flow $\Phi_{(x,y)}^n$ really sends one unit of flow). Thus let us check Φ_{\max}^n . Firstly, in order to estimate the flow on the arcs inside $G[X_A]$ and $G[X_B]$, we determine the multiplier by which each flow $\Phi_{(s,t)}^A$ respectively $\Phi_{(s,t)}^B$ appears in the definition of Φ^n . By symmetry, it suffices to do this for all pairs $s, t \in X_A$.

Each pair $s, t \in X_A'$ is used once with multiplier one (for $(x, y) = (s, t)$) and once with multiplier α (for $(x, y) = (s \oplus \mathbf{e}_0, t)$). Thus, each $\Phi_{(s,t)}^A$ appears with multiplier $(1+\alpha)$ for $s, t \neq \hat{x}$.

Each pair $s = \hat{x}$ and $t \in X_A'$ is used once with multiplier one (for $(x, y) = (s, t)$) and, for each of the $(n-1)/2$ pairs $x \in X_B$ and $y = t$, with multiplier $(1-\alpha)$.

Each pair $s \in X'_A$ and $t = \hat{x}$ is used once with multiplier one (for $(x, y) = (s, t)$) and, for each of the $(n-1)/2$ pairs $x = s$ and $y \in X_B$, with multiplier $(1-\alpha)$.

Thus, due to

$$(1-\alpha)\frac{n-1}{2} = \frac{n+1-(n-1)}{n+1} \frac{n-1}{2} = \frac{n-1}{n+1} = \alpha,$$

each $\Phi_{(s,t)}^A$ with $s = \hat{x}$ or $t = \hat{x}$ appears with multiplier $(1+\alpha)$.

Secondly, we estimate the flow along the arcs $(x, x \oplus \mathbf{e}_0)$ with $x \neq \hat{x}$. By symmetry we restrict our attention to the case $x \in X'_A$ and $y \in X'_B$, and we find that each arc $(x, x \oplus \mathbf{e}_0)$ is used $(n-1)/2$ times with flow-value α .

Alltogether, this yields

$$\Phi_{\max}^n \leq \max \left\{ (1+\alpha) \cdot \Phi_{\max}^{n'}, \frac{n-1}{2} \cdot \alpha \right\} \leq \frac{n}{2},$$

which concludes the inductive step, and thus, the proof. \square

5. TOWARDS A LOWER-BOUND-THEOREM FOR 0/1-POLYTOPES

In the sequel, we will exploit the following construction (using revlex-initial 0/1-polytopes) several times.

Proposition 7. *For $d, n \in \mathbb{N}$ with $d+1 \leq n \leq 2^d$ there exist $\tilde{d} \in \mathbb{N}$ such that for $\tilde{n} := n - (d - \tilde{d})$ the following inequalities hold.*

$$(14) \quad 0 \leq \tilde{d} \leq d$$

$$(15) \quad 2^{\tilde{d}-1} \leq \tilde{n} \leq 2^{\tilde{d}}$$

$$(16) \quad \tilde{d} \leq 1 + \log_2 n$$

Furthermore $P^{<\tilde{n}}$ is a \tilde{d} -dimensional revlex-initial 0/1-polytope with \tilde{n} vertices.

Proof. To see that such a \tilde{d} and \tilde{n} exist, observe that with $\tilde{n}(k) := n - (d - k)$ we have $\tilde{n}(k) \geq 2^{k-1}$ for $k = 0$ and $\tilde{n}(k) \leq 2^k$ for $k = d$; note that for these estimates we need $d+1 \leq n \leq 2^d$. Then, we have that

$$\tilde{d} := \min\{k \in \mathbb{N} : \tilde{n}(k) \leq 2^k\} \leq d$$

satisfies (14).

By definition, we have $\tilde{n}(\tilde{d}) \leq 2^{\tilde{d}}$. If $\tilde{d} = 0$, then (as stated above) also $\tilde{n}(\tilde{d}) \geq 2^{\tilde{d}-1}$ is true, and otherwise, from the minimality of \tilde{d} we conclude $\tilde{n}(\tilde{d}-1) > 2^{\tilde{d}-1}$, which, of course, implies $\tilde{n}(\tilde{d}) \geq 2^{\tilde{d}-1}$. Hence, \tilde{d} also satisfies (15).

Finally, (16) trivially follows from (15).

Thus, with $\tilde{n} := \tilde{n}(\tilde{d})$, by (15) and Proposition 2 the revlex-initial 0/1-polytope $P^{<\tilde{n}}$ has dimension \tilde{d} . \square

Definition 8. For arbitrary $d, n \in \mathbb{N}$ with $d + 1 \leq n \leq 2^d$ and $\tilde{d}, \tilde{n} \in \mathbb{N}$ as in Proposition 7 we define $P(d, n)$ to be the d -dimensional 0/1-polytope with n vertices obtained by building the $(d - \tilde{d})$ -fold pyramid over $P^{<\tilde{n}}$.

We denote the parameters \tilde{d} and \tilde{n} by $\tilde{d}(d, n)$ and $\tilde{n}(d, n)$.

Proposition 7 guarantees that this construction works always as claimed in the definition of $P(d, n)$.

5.1. An Upper Bound on the Minimal Number of Facets.

Definition 9. For $d, n \in \mathbb{N}$ with $d + 1 \leq n \leq 2^d$, denote by $g_{\text{nfac}}(d, n)$ the minimal number of facets of a d -dimensional 0/1-polytope with n vertices.

Note that a k -dimensional 0/1-polytope in \mathbb{R}^d (with $k < d$) can isometrically be projected to a k -dimensional 0/1-polytope in \mathbb{R}^k . Thus, the definition is independent of the ambient spaces of the polytopes.

Proposition 8. For every $d + 1 \leq n \leq 2^d$ we have $g_{\text{nfac}}(d, n) \leq d + 2 \log_2 n$.

Proof. By Theorem 2(2), the revlex-initial 0/1-polytope $P^{<\tilde{n}(d, n)}$ has at most $3\tilde{d} - 2$ facets. Thus $P(d, n)$ has at most $3\tilde{d} - 2 + n - \tilde{n} = 2\tilde{d} + d - 2$ facets. The claim of the proposition follows by (16). \square

The proposition immediately implies the following results.

Theorem 6.

- (1) For every $d + 1 \leq n \leq 2^d$ we have $g_{\text{nfac}}(d, n) \leq 3d$.
- (2) For $d + 1 \leq n(d) \leq 2^{o(d)}$ we have $g_{\text{nfac}}(d, n(d)) = d + o(d)$.
- (3) For $1 < \alpha < 2$ and $n(d) := \lfloor \alpha^d \rfloor$ we have $g_{\text{nfac}}(d, n(d)) \leq (1 + 2 \log_2 \alpha)d + o(d)$.

The upper bounds on $g_{\text{nfac}}(d, n)$ provided by the polytopes $P(d, n)$ in Proposition 8 are not sharp, at least not for all parameters d and n . This follows, for instance, from the examples of Cartesian products of r 0/1-simplices of dimension d_1, \dots, d_r (which are precisely the simple 0/1-polytopes, see Kaibel and Wolff [8]). Such a product is a 0/1-polytope of dimension $d = \sum d_i$ with $\prod (d_i + 1)$ vertices and $d + r$ facets. In particular, this yields

$$g(d, (\lfloor \frac{d}{2} \rfloor + 1)(\lceil \frac{d}{2} \rceil + 1)) = d + 2,$$

while the polytopes $P(d, n)$ have $d + \Omega(\log_2 d)$ facets.

The right part of Figure 4 shows that for $d = 5$ the polytopes $P(5, n)$ achieve the respective minimum number of facets in all but 10 cases (i.e., in 17 out of 27 cases). Figure 5 depicts the numbers of facets (and the average degrees) of the polytopes $P(13, n)$.

For subexponential numbers of vertices, Part (2) of Theorem 6 shows that the minimum number of facets is asymptotically as small as the number of facets of any d -dimensional polytope can be (up to an additive $o(1)$ -term). The range of

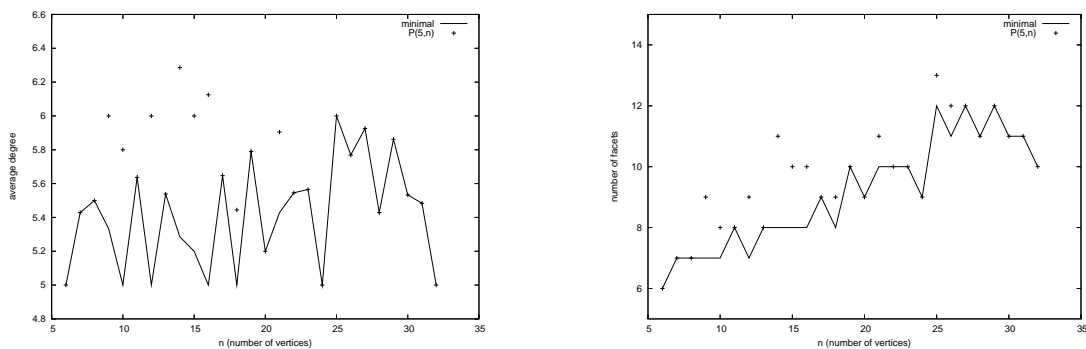


FIGURE 4. Comparison of the lower bounds on $g_{\text{nfac}}(5, n)$ and $g_{\text{avdeg}}(5, n)$ obtained from the polytopes $P(5, n)$ in the proofs of Propositions 7 and 8 with the true values of $g_{\text{nfac}}(5, n)$ and $g_{\text{avdeg}}(5, n)$ obtained from Aichholzer’s enumeration [1].

subexponential vertex numbers is particularly interesting for two reasons: Firstly, many 0/1-polytopes that are relevant in Combinatorial Optimization have subexponentially many vertices (e.g., cut polytopes of complete graphs and travelling salesman polytopes). Secondly, the papers by Bárány and Pór [2] and Gatzouras, Giannopoulos, and Markoulakis [5] show that within subexponential ranges of vertex numbers a random 0/1-polytope has very many facets. In fact, it may well be that the maximum numbers of facets of 0/1-polytopes is (roughly) attained by these polytopes.

The examples of products of simplices (i.e., simple 0/1-polytopes) seem to indicate that it might be hopeless to derive an explicit formula for $g_{\text{nfac}}(d, n)$, i.e., a sharp lower bound theorem for the facet numbers of 0/1-polytopes. Nevertheless, the question for the (asymptotic) best upper bound on $g_{\text{nfac}}(d, n)$ that does only depend on d (and not on n) might be within reach. In particular, we do not know whether there is some constant $\alpha < 3$ such that $g_{\text{nfac}}(d, n) \leq \alpha d + o(d)$ holds for all d and n . This might even be true for $\alpha = 2$.

5.2. An Upper Bound on the Minimal Number of Edges.

Definition 10. For $d, n \in \mathbb{N}$ with $d + 1 \leq n \leq 2^d$, denote by $g_{\text{avdeg}}(d, n)$ the minimal average degree among all graphs of d -dimensional 0/1-polytopes with n vertices.

Revlex-initial 0/1-polytopes and the pyramidal construction yield the following bound of the minimum average degrees.

Theorem 7. For $d + 1 \leq n \leq 2^d$, we have $g_{\text{avdeg}}(d, n) \leq d + 8$.

Proof. Set $\tilde{d} := \tilde{d}(d, n)$ and $\tilde{n} := \tilde{n}(d, n)$. By Theorem 3, the revlex-initial 0/1-polytope $P^{<\tilde{n}}$ has at most $(\tilde{d} + 8)\tilde{n}$ edges. Thus, $P(d, n)$ (the $(d - \tilde{d})$ -fold pyramid

over $P^{<\tilde{n}}$ has at most

$$(\tilde{d} + 8)\tilde{n} + (d - \tilde{d})n \leq (d + 8)n$$

edges. □

The left part of Figure 4 shows that for $d = 5$ the polytopes $P(5, n)$ achieve the respective minimum average degree in all but 8 cases (i.e., in 19 out of 27 cases).

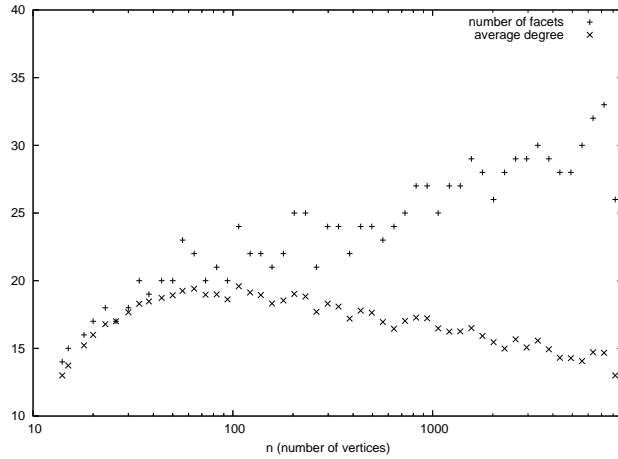


FIGURE 5. Numbers of facets and average degrees of the polytopes $P(13, n)$ providing the upper bounds on $g_{\text{fac}}(13, n)$ and $g_{\text{avdeg}}(13, n)$.

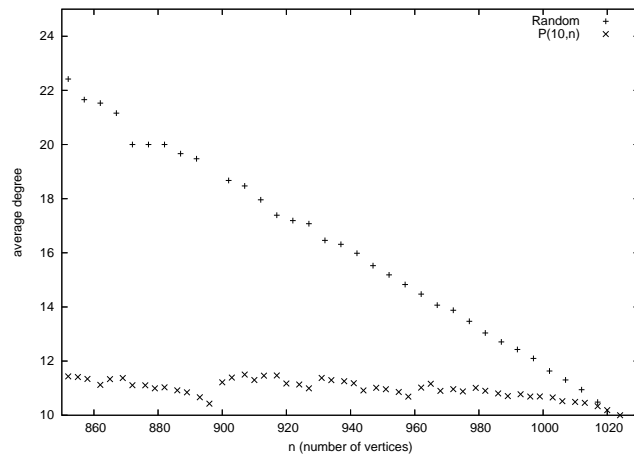


FIGURE 6. Average degrees of the polytopes $P(10, n)$ and uniformly random 10-dimensional 0/1-polytopes (by sampling).

Finally, the polytopes $P(d, n)$ yield examples of 0/1-polytopes with remarkably sparse graphs, satisfying, nevertheless, the Mihail-Vazirani conjecture.

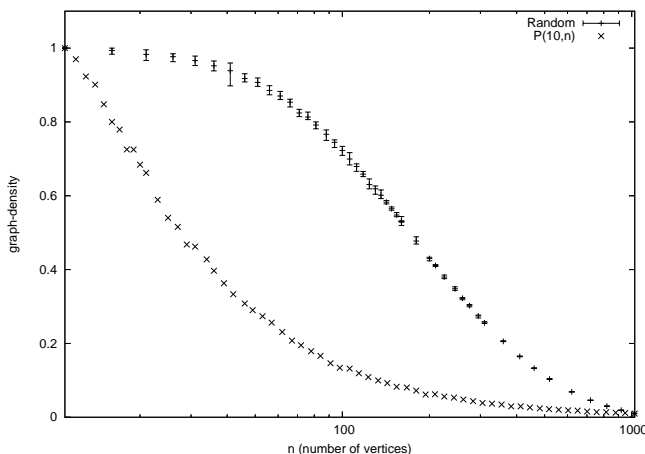


FIGURE 7. The graph densities of the polytopes $P(10, n)$ used in the proof of Theorem 7 versus the graph densities of random 0/1-polytopes with the same parameters (by sampling).

Theorem 8. *For every $d + 1 \leq n \leq 2^d$, there is a d -dimensional 0/1-polytope with n vertices, at most $(d + 8)n$ edges, and edge expansion at least one.*

Proof. By Theorem 7 the polytope $P(d, n)$ has at most $(d + 8)n$ edges. Since $P(d, n)$ is a k -fold pyramid over the revlex-initial 0/1-polytope $P^{<\tilde{n}(d,n)}$ the multi commodity flow constructed in the proof of 5 can be easily extended to a multi commodity flow of $P(d, n)$ sending one unit of flow from every vertex to every other vertex. \square

6. CONCLUDING REMARKS

The contributions of this paper concern three topics: (1) Investigations of a 'natural' class of 0/1-polytopes, (2) lower bound theorem(s) for 0/1-polytopes, and (3) support of the Mihail-Vazirani conjecture on the edge expansion of the graphs of 0/1-polytopes.

With respect to the first topic, one may be interested also in studying the convex hulls of sets of 0/1-vectors that are only *gradually* revlex-initial (the 0/1-polytopes corresponding to compressed set systems), i.e., convex hulls of sets X of 0/1-vectors which, with every $x \in X$, contain all 0/1-vectors y which have the same number of ones as x and are revlex-smaller than x . Due to the important role played by the *monotone* ones among them (more precisely: by the corresponding set systems) in the theory of simplicial complexes, it might be that these objects bear some connections between 0/1-polytopes and combinatorial topology. This would be quite interesting.

It seems that precise lower bound theorems on the number of facets (edges, or even other-dimensional faces) are hard to obtain. Nevertheless, with respect to

topic (2) some questions remain open that may be tractable, e.g., the question whether there is some $\alpha < 3$ (maybe $\alpha = 2$?) with $g_{\text{nfac}}(d, n) \leq \alpha d + o(d)$.

Another observation is that the 0/1-polytopes with small average degrees that we constructed have, in general, quite large maximum degrees. We have no idea how small the maximum degree of the graph of a d -dimensional 0/1-polytope with n vertices can be.

Perhaps the most interesting and promising line to follow up this research concerns topic (3). Extending our techniques for construction of the multi-commodity flows showing that revlex-initial 0/1-polytopes (as special knapsack polytopes) have edge expansion at least one to all knapsack polytopes (or even to all monotone polytopes) would be a big support for the Mihail-Vazirani conjecture (which itself is of great importance in the theory of random generation and approximate counting, as mentioned in Section 4.3). It follows from work of Morris and Sinclair [10] that the edge-expansion of the graphs of d -dimensional 0/1-knapsack polytopes is bounded from below by a polynomial in $1/d$. Their proof in fact shows that this is true even for the subgraph that is formed by those edges which are also edges of the cube. Since our flows extensively use non-cube edges, the techniques used in the proof of Theorem 5 seem to have good potential to improve the current lower bound, maybe even to 'one' as conjectured by Mihail and Vazirani.

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