Asymptotic analysis of elastic curved rods*

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Abstract

We suppose a convergent sequence of curved rods made from an isotropic elastic material and clamped on the lower basis or on both bases, and the linearized elasticity system posed on the sequence of the curved rods. We study the asymptotic behaviour of the stress tensor and the solution to this system, when the radius of the domains tends to zero. The curved rods with a nonsmooth line of centroids are covered by the used asymptotic method as well.

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1 Introduction

The aim of this paper is to continue with the study of thin elastic curved rods which started in Tiba, Vodák [8]. We suppose the sequence of smooth curved rods, converging to a unit speed curve which has generally absolutely continuous regular parametrization, with the radius ϵ. In the general case of a nonsmooth curve, we introduce another small parameter ϵ^r associated to the approximating sequence of the smooth curved rods. We derive an asymptotic one dimensional model for the curved rod from the three dimensional linearized elasticity system posed on the sequence of the smooth curved rods, and we show that the used asymptotic method requires for the proof of the strong convergence of the stress tensors and the solutions to this three dimensional model in special cases the suitable choice of the body force $H = (H_{ij})_{i,j=1}^3$ or the approximating sequence of the curved rods, which can affect the form of the limit stress tensor as well.

The related results concerning with the asymptotic methods for isotropic or anisotropic straight rods can be found in Aganovič, Tutek [1] and Murat, Sili [6], respectively. The case of the smooth curved rods was studied in Jurak, Tambača [4], [5]. The construction of the approximating sequence of the smooth rods and the relaxation of the regularity assumptions was done in Tiba, Vodák [8], where the above mentioned one dimensional model was derived for the curved rods clamped on both bases and $H = 0$. We refer also the reader to [2] for the related theory for shells.

The paper is organized as follows: In Section 2, we establish the basic notation used throughout the paper. The Section 3 contains auxiliary lemmas. In Section 4, we introduce the linearized elasticity systems for the curved rods clamped on the lower basis and on both bases, and we transform the models on a cylinder, which does not depend on the parameter ϵ. Section 5 deals with the derivation of the asymptotic one dimensional model and with the analysis of the asymptotic behaviour of the displacements and the stress tensors. Section 6 contains a corrector result for the stress tensor.

Our main results can be summarized in the following theorems and corollary:

Theorem 1.1 *Assume that the function $\mathbf{\Phi} \in W^{1,\infty}(0,l)^3$ is the parametrization of a unit speed curve generating the local frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. Let the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ satisfying* (3.2)–(3.5) *form the smooth approximation of this local frame. Let, further, $\mathbf{F} \in L^2(\Omega)^3$, $\mathbf{G} \in L^2(0,l;L^2(\partial S)^3)$, $H \in L^2(\Omega)^9$, $\mathbf{K} \in L^2(S)^3$ and the functions $\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}$ and $\check{\mathbf{K}}$ be defined in Lemma* 5.5. *Then there exists a unique pair $\langle \mathbf{U}, \phi\rangle \in \mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, generating the unique solution $\mathbf{U}_*$ to the boundary value problem* (5.77), *such that*

$$\mathbf{U}_\epsilon \to \mathbf{U} \text{ in } H^1(\Omega)^3, \tag{1.1}$$

$$\frac{1}{2\epsilon}((\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - (\partial_3 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)) \rightharpoonup \phi \text{ in } L^2(\Omega), \tag{1.2}$$

where the functions $\mathbf{U}_\epsilon \in V_b(\Omega)$ are the unique solutions to the equation (4.6). *In addition,*

$$\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) \to \zeta \text{ in } L^2(\Omega)^9, \tag{1.3}$$

where the tensors $\omega^\epsilon(\mathbf{U}_\epsilon)$ *and* ζ *are defined by the relations* (4.8)–(4.11) *and* (5.71)–(5.76).

Theorem 1.2 *Let the assumptions of Theorem* 1.1 *be fulfilled and* $\mathbf{K} = 0$. *Then the convergences* (1.1)-(1.3) *remain valid for the functions* $\mathbf{U}_\epsilon \in V_{bb}(\Omega)$ *and* $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_{bb}^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, $\mathbf{U}_* \in H^1_{bb}(0,l)^3$ *solving the equations* (4.12) *and* (5.77) *(for all* $\langle \mathbf{V}, \psi \rangle \in \mathcal{V}_{bb}^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$*), respectively, if one of the following conditions holds:*

1. *there exist no constants* C_{10}, $C_{11} \in \mathbb{R}$ *such that* $t_2 = C_{10}t_1$ *and* $t_3 = C_{11}t_1$, *where* t_i, $i = 1,2,3$, *are the components of the tangent vector* $\mathbf{t}$;

2. *there exist the functions* $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ *satisfying* (3.2)–(3.5) *such that* $t_{1,\epsilon}(x_1) = n_{1,\epsilon}(x_1) = 0$, $b_{1,\epsilon}(x_1) = 1$ *for* $x_1 \in [\widehat{x}_1 - \epsilon^q, \widehat{x}_1 + \epsilon^q] \subset [0,l]$, *where* $q \in (0, \frac{2}{3})$.

3. *there exist constants* C_{10}, C_{11}, $C_{13} \in \mathbb{R}$ *and the functions* $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ *satisfying* (3.2)–(3.5) *such that* $t_2 = C_{10}t_1$, $t_3 = C_{11}t_1$ *and* $t_{j,\epsilon} = C_{13}t_{1,\epsilon} \neq 0$, $n_{j,\epsilon} = C_{13}n_{1,\epsilon}$, $b_{j,\epsilon} = C_{13}b_{1,\epsilon}$ *on an interval* I_ϵ, $|I_\epsilon| \to 0$ *for* $\epsilon \to 0$, *for all* $\epsilon \in (0,1)$ *and for* $j = 2$ *or* $j = 3$, *where* $C_{13} \neq C_{10}$ *or* $C_{13} \neq C_{11}$, *respectively;*

4. *there exist constants* C_{10}, $C_{11} \in \mathbb{R}$ *such that* $t_2 = C_{10}t_1$ *and* $t_3 = C_{11}t_1$, *and the identity*
$$\int_0^l t_1 \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\; dx_2dx_3dx_1 = 0. \tag{1.4}$$
holds;

5. *the vectors* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ *are constant vectors and the functions* $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ *are their smooth approximations in* $C^1([0,l])^3$ *satisfying* (3.2) *and such that* $\|\mathbf{t}_\epsilon - \mathbf{t}\|_{C([0,l])} \leq C\epsilon^p$, $p > 1$.

In the cases 1.–4., *the form of the tensor* ζ *is given by the relations* (5.71)–(5.76). *In the last case, we get the form of the tensor* ζ *adding the constant* $\frac{t_1}{l}\int_0^l \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\; dx_2dx_3dx_1$ *to the relations* (5.71), (5.74) *and* (5.75).

Corollary 1.3 *Let the function* $\mathbf{U}$ *be given by Theorem* 1.1 *or* 1.2 *and let* $\mathbf{U}_1^\epsilon$ *be its approximation introduced in Proposition* 3.4 *(see also Remark* 3.7*). Then there exist functions* $\mathbf{U}_2^\epsilon$ *and* $\mathbf{U}_\epsilon^3$ *bounded in* $L^2(\Omega)^3$ *such that*
$$\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\omega^\epsilon(\widehat{\mathbf{U}}_\epsilon) \to 0 \text{ in } L^2(\Omega)^9, \tag{1.5}$$
where $\widehat{\mathbf{U}}_\epsilon = \mathbf{U}_1^\epsilon + \epsilon\mathbf{U}_2^\epsilon + \epsilon^2\mathbf{U}_3^\epsilon$.

2 Preliminaries

Without risk of confusion, we denote by the symbol $|\cdot|$ the Lebesgue measure of some measurable set, absolute value of a scalar function and the norm in the three dimensional Euclidean space $\mathbb{R}^3$. This norm is generated by the usual scalar product $(\cdot,\cdot)$. We shall denote by $\langle\cdot,\cdot\rangle$ any ordered pair. The summation convention with respect to repeated indices will be also used, if not otherwise explicitly stated.

We denote by $S \subset \mathbb{R}^2$ a bounded simply connected domain of class C^1 satisfying the symmetry condition

$$\int_S x_2 \; dx_2dx_3 = \int_S x_3 \; dx_2dx_3 = \int_S x_2x_3 \; dx_2dx_3 = 0. \tag{2.1}$$

The symbols Ω and Ω_ϵ stand for the open cylinders $(0,l)\times S$ and $(0,l)\times\epsilon S$, respectively, where $l > 0$ and $\epsilon > 0$ small, are given.

We use for constants the symbols C or C_i, $i \in \mathbb{N}_0 = \{0,1,2,\ldots\}$. We adopt the usual notation for the function spaces and their norms, i.e. $C^m(\overline{O})$, with $m \in \mathbb{N}_0$, denotes the space of continuous functions, whose derivatives up to the order m are continuous in the domain $\overline{O}$, with the norm $\|\cdot\|_{C^m(\overline{O})}$, $H^1(O)$ and $L^p(O)$, $p\in[1,\infty]$, mean the standard Sobolev and Lebesgue spaces endowed with the norms $\|\cdot\|_{H^1(O)}$ and $\|\cdot\|_{L^p(O)}$, respectively, and the symbols $L^p(0,l;X)$ and $C([0,l];X)$, where X is a Banach space, stand for the Bochner spaces with the norms

$$\|v\|_{L^p(0,l;X)} = \left(\int_0^l \|v(x_1)\|_X^p \; dx_1\right)^{\frac{1}{p}} \text{ and } \|v\|_{C([0,l];X)} = \max_{x_1\in[0,l]} \|v(x_1)\|_X.$$

Further, we define the spaces:

$$H_b^1(0,l) = \{v \in H^1(0,l);\; v(0) = 0\},$$

$$H_{bb}^1(0,l) = \{v \in H^1(0,l);\; v(0) = v(l) = 0\},$$

$$rd_2(S) = \{\langle v_2,v_3\rangle;\; v_2 = C_1x_3 + C_2,\; v_3 = -C_1x_2 + C_3,\; C_i \in \mathbb{R},\; i = 1,2,3\},$$

$$rd_2^\perp(S) = \{\langle v_2,v_3\rangle \in L^2(S)^2;\; \int_S v_i \; dx_2dx_3 = 0,\; i = 2,3,$$

$$\int_S [-x_2v_3 + x_3v_2] \; dx_2dx_3 = 0\}.$$

Let $\mathcal{C}$ represent a unit speed curve in $\mathbb{R}^3$ defined by its parametrization $\mathbf{\Phi} : [0,l] \to \mathbb{R}^3$. The local frame of this curve is formed by its tangent, normal and binormal vectors denoted by $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, respectively. We use the analogous notation, i.e. $\mathbf{\Phi}_\epsilon$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$, for the smooth approximation of the curve $\mathcal{C}$ and its local frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, where the curves $\mathcal{C}_\epsilon$ defined by its parametrization $\mathbf{\Phi}_\epsilon$ remain unit speed curves for arbitrary $\epsilon > 0$. We refer the reader to Proposition 3.1 for other properties of the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon$.

Using the assumed orthonormality of the local basis $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$, we can derive the laws of motion of the local frame:

$$\begin{aligned} \mathbf{t}'_\epsilon &= \alpha_\epsilon \mathbf{b}_\epsilon + \beta_\epsilon \mathbf{n}_\epsilon, \\ \mathbf{n}'_\epsilon &= -\beta_\epsilon \mathbf{t}_\epsilon - \gamma_\epsilon \mathbf{b}_\epsilon, \\ \mathbf{b}'_\epsilon &= -\alpha_\epsilon \mathbf{t}_\epsilon + \gamma_\epsilon \mathbf{n}_\epsilon. \end{aligned} \tag{2.2}$$

The mappings $\mathbf{R}_\epsilon$ and $\bar{\mathbf{P}}_\epsilon$, defined by

$$\mathbf{R}_\epsilon : \Omega \to \Omega_\epsilon,\ \ \mathbf{R}_\epsilon(x_1, x_2, x_3) = (x_1, \epsilon x_2, \epsilon x_3), \tag{2.3}$$

$$\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \mathbb{R}^3,\ \ \bar{\mathbf{P}}_\epsilon(y) = \mathbf{\Phi}_\epsilon(y_1) + y_2 \mathbf{n}_\epsilon(y_1) + y_3 \mathbf{b}_\epsilon(y_1), \tag{2.4}$$

$(y_1, y_2, y_3) \in (0, l) \times \epsilon S$, represent the parametrization of the curved rod $\widetilde{\Omega}_\epsilon = (\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(\Omega)$. From Corollary 3.2 and (2.10), it follows that

$$\bar{d}_\epsilon(y) = \det\left(\bar{\nabla} \bar{\mathbf{P}}_\epsilon(y)\right) = 1 - \beta_\epsilon(y_1) y_2 - \alpha_\epsilon(y_1) y_3 > 0 \text{ for all } y \in \Omega_\epsilon \tag{2.5}$$

and thus the mapping $\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \widetilde{\Omega}_\epsilon$ is a C^1-diffeomorphism, Ciarlet [2], Theorem 3.1-1. We distinguish by the notation $\widetilde{\partial}_i \widetilde{V}(\widetilde{y}) = \frac{\partial}{\partial \widetilde{y}_i} \widetilde{V}(\widetilde{y})$, $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3) \in \widetilde{\Omega}_\epsilon$, $\bar{\partial}_i \bar{V}(y) = \frac{\partial}{\partial y_i} \bar{V}(y)$, $y = (y_1, y_2, y_3) \in \Omega_\epsilon$, $\partial_i V(x) = \frac{\partial}{\partial x_i} V(x)$, $x = (x_1, x_2, x_3) \in \Omega$, where a function and its derivatives are defined. We suppose throughout this subsection that all needed derivatives exist which follows from Proposition 3.1.

Using the relations $\bar{\mathbf{g}}_{i,\epsilon}(y) = \bar{\partial}_i \bar{\mathbf{P}}_\epsilon(y)$, $y \in \Omega_\epsilon$, and $(\bar{\mathbf{g}}_{i,\epsilon}, \bar{\mathbf{g}}^{j,\epsilon}) = \delta^{ij}$, $i,\ j = 1,2,3$, we can establish the covariant and contravariant basis by the vectors

$$\begin{aligned} \bar{\mathbf{g}}_{1,\epsilon}(y) &= (1 - y_2 \beta_\epsilon(y_1) - y_3 \alpha_\epsilon(y_1)) \mathbf{t}_\epsilon(y_1) + y_3 \gamma_\epsilon(y_1) \mathbf{n}_\epsilon(y_1) - y_2 \gamma_\epsilon(y_1) \mathbf{b}_\epsilon(y_1), \\ \bar{\mathbf{g}}_{2,\epsilon}(y) &= \mathbf{n}_\epsilon(y_1),\ \ \bar{\mathbf{g}}_{3,\epsilon}(y) = \mathbf{b}_\epsilon(y_1), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \bar{\mathbf{g}}^{1,\epsilon}(y) &= \frac{\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)},\ \ \bar{\mathbf{g}}^{2,\epsilon}(y) = \frac{-y_3 \gamma_\epsilon(y_1) \mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{n}_\epsilon(y_1), \\ \bar{\mathbf{g}}^{3,\epsilon}(y) &= \frac{y_2 \gamma_\epsilon(y_1) \mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{b}_\epsilon(y_1), \end{aligned} \tag{2.7}$$

respectively, and the covariant and contravariant metric tensors $(\bar{g}_{ij,\epsilon})_{i,j=1}^3$ and $(\bar{g}^{ij,\epsilon})_{i,j=1}^3$ by the matrices with the components

$$\bar{g}_{ij,\epsilon} = (\bar{\mathbf{g}}_{i,\epsilon}, \bar{\mathbf{g}}_{j,\epsilon}) \text{ and } \bar{g}^{ij,\epsilon} = (\bar{\mathbf{g}}^{i,\epsilon}, \bar{\mathbf{g}}^{j,\epsilon}), \tag{2.8}$$

respectively. After substitution $y = \mathbf{R}_\epsilon(x)$, we adopt the notation

$$g^{ij,\epsilon}(x) = \bar{g}^{ij,\epsilon}(\mathbf{R}_\epsilon(x)),\ \ g_{ij,\epsilon}(x) = \bar{g}_{ij,\epsilon}(\mathbf{R}_\epsilon(x)),\ \ \mathbf{g}_{i,\epsilon}(x) = \bar{\mathbf{g}}_{i,\epsilon}(\mathbf{R}_\epsilon(x)), \tag{2.9}$$

$$\mathbf{g}^{j,\epsilon}(x) = \bar{\mathbf{g}}^{j,\epsilon}(\mathbf{R}_\epsilon(x)),\ \ d_\epsilon(x) = \bar{d}_\epsilon(\mathbf{R}_\epsilon(x)), \tag{2.10}$$

where $x \in \Omega$. We can derive analogously the covariant and contravariant basis at the point $(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $x \in \Omega$, and the covariant and contravariant metric tensors $(o_{ij,\epsilon})_{i,j=1}^3$ and $(o^{ij,\epsilon})_{i,j=1}^3$, where the last one has the form

$$(o^{ij,\epsilon})_{i,j=1}^3 = \begin{pmatrix} \frac{1}{d_\epsilon^2} & \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{x_2\gamma_\epsilon}{d_\epsilon^2} \\ \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_3^2\gamma_\epsilon^2}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} \\ \frac{x_2\gamma_\epsilon}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_2^2\gamma_\epsilon^2}{d_\epsilon^2} \end{pmatrix}. \tag{2.11}$$

We refer the reader to [8] for the more detailed derivation.

The definitions of the domains $\widetilde{\Omega}_\epsilon$ and Ω enable us to introduce the function spaces

$$V_{bb}(\widetilde{\Omega}_\epsilon) = \{\widetilde{\mathbf{V}} \in H^1(\widetilde{\Omega}_\epsilon)^3 : \widetilde{\mathbf{V}}|_{\bar{\mathbf{P}}_\epsilon(\{0\}\times\epsilon S)} = \widetilde{\mathbf{V}}|_{\bar{\mathbf{P}}_\epsilon(\{l\}\times\epsilon S)} = 0\},$$

$$V_{bb}(\Omega) = \{\mathbf{V} \in H^1(\Omega)^3 : \mathbf{V}|_{(\{0\}\times S)} = \mathbf{V}|_{(\{l\}\times S)} = 0\}$$

and further we introduce the space

$$\mathcal{V}_{bb}^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l) = \{\langle \mathbf{V}, \psi\rangle \in H_{bb}^1(0,l)^3 \times L^2(0,l) : \ (\mathbf{V}', \mathbf{t}) = 0$$
$$\text{and } \mathbf{V}_* = -\psi\mathbf{t} + (\mathbf{V}', \mathbf{b})\mathbf{n} - (\mathbf{V}', \mathbf{n})\mathbf{b} \in H_{bb}^1(0,l)^3\}. \tag{2.12}$$

From the above definitions, we can deduce easily the definitions of the spaces $V_b(\widetilde{\Omega}_\epsilon)$, $V_b(\Omega)$ and $\mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (compare with the definition of the spaces $H_b^1(0,l)$ and $H_{bb}^1(0,l)$).

3 Auxiliary propositions

Proposition 3.1 [8] *Let $\mathbf{\Phi} \in W^{1,\infty}(0,l)^3$ be the parametrization of the unit speed curve $\mathcal{C}$. Then there exist vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, which belong to $L^\infty(0,l)^3$ and form the local frame corresponding to the curve $\mathcal{C}$, such that*

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1, \ \mathbf{t}\perp\mathbf{n}\perp\mathbf{b} \ \textit{a.e. in } (0,l). \tag{3.1}$$

In addition, there exist functions

$$\{\mathbf{\Phi}_\epsilon\}_{\epsilon\in(0,1)}, \ \{\mathbf{t}_\epsilon\}_{\epsilon\in(0,1)}, \ \{\mathbf{n}_\epsilon\}_{\epsilon\in(0,1)}, \ \{\mathbf{b}_\epsilon\}_{\epsilon\in(0,1)} \subset C^\infty([0,l])^3$$

such that

$$|\mathbf{t}_\epsilon| = |\mathbf{n}_\epsilon| = |\mathbf{b}_\epsilon| = 1, \ \mathbf{t}_\epsilon\perp\mathbf{n}_\epsilon\perp\mathbf{b}_\epsilon \ \textit{on } [0,l] \tag{3.2}$$

for all $\epsilon \in (0,1)$,

$$\mathbf{t}_\epsilon \to \mathbf{t}, \ \mathbf{n}_\epsilon \to \mathbf{n}, \ \mathbf{b}_\epsilon \to \mathbf{b} \ \textit{in measure in } (0,l) \tag{3.3}$$

for $\epsilon \to 0$,

$$\begin{aligned} \|\mathbf{t}_\epsilon'\|_{L^\infty(0,l)^3}, \ \|\mathbf{n}_\epsilon'\|_{L^\infty(0,l)^3}, \ \|\mathbf{b}_\epsilon'\|_{L^\infty(0,l)^3} \sim O(\tfrac{1}{\epsilon^r}), \\ \|\mathbf{t}_\epsilon''\|_{L^\infty(0,l)^3}, \ \|\mathbf{n}_\epsilon''\|_{L^\infty(0,l)^3}, \ \|\mathbf{b}_\epsilon''\|_{L^\infty(0,l)^3} \sim O(\tfrac{1}{\epsilon^{2r}}) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|\alpha_\epsilon\|_{L^\infty(0,l)}, \ \|\beta_\epsilon\|_{L^\infty(0,l)}, \ \|\gamma_\epsilon\|_{L^\infty(0,l)} \sim O(\tfrac{1}{\epsilon^r}), \\ \|\alpha_\epsilon'\|_{L^\infty(0,l)}, \ \|\beta_\epsilon'\|_{L^\infty(0,l)}, \ \|\gamma_\epsilon'\|_{L^\infty(0,l)} \sim O(\tfrac{1}{\epsilon^{2r}}), \ r \in (0, \tfrac{1}{3}), \end{aligned} \tag{3.5}$$

where the functions α_ϵ, β_ϵ, $\gamma_\epsilon \in C^\infty([0,l])$ are determined by (2.2).

Corollary 3.2 [8] *There exist constants C_j, $j = 4,5,6$, such that the function d_ϵ defined by (2.5) and (2.10) satisfies $0 < C_4 \le d_\epsilon(x) \le C_5$ for all $x \in \overline{\Omega}$, and the function $\epsilon d_\epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j}$ defined by (2.11), where ν_i, $i = 1,2,3$, are the components of the unit outward normal for $(0,l) \times \partial S$, satisfies $0 \le d_\epsilon(x)\epsilon\sqrt{\nu_i(x) o^{ij,\epsilon}(x)\nu_j(x)} \le C_6$ for all $x \in \overline{(0,l) \times \partial S}$ and $\epsilon \in (0,1)$. In addition,*

$$d_\epsilon \to 1 \text{ in } C(\overline{\Omega}), \tag{3.6}$$

$$\epsilon d_\epsilon(x)\sqrt{\nu_i(x) o^{ij,\epsilon}(x)\nu_j(x)} \to 1 \text{ in } C(\overline{(0,l) \times \partial S}), \tag{3.7}$$

for $\epsilon \to 0$.

Remark 3.3 After a simple modification of the proof of Proposition 3.1 in [8] and Theorem 3.1 in [3], we can construct the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$, which satisfy the condition 2. or 3. from Theorem 1.2.

Proposition 3.4 [8] *Let $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ be the functions from Proposition 3.1 and let the space $\mathcal{V}_{bb}^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ be defined by (2.12) using the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ instead of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. Let, further, $\langle \mathbf{V}, \psi \rangle \in \mathcal{V}_{bb}^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ be an arbitrary but fixed couple. Then there exist couples $\langle \mathbf{V}_\epsilon, \psi_\epsilon \rangle \in \mathcal{V}_{bb}^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ generating the functions $\mathbf{V}_{*,\epsilon}$ such that $\{\mathbf{V}_\epsilon\}_{\epsilon\in(0,1)}$, $\{\mathbf{V}_{*,\epsilon}\}_{\epsilon\in(0,1)} \subset C_{bb}^\infty(0,l)^3$, $\{\psi_\epsilon\}_{\epsilon\in(0,1)} \subset C_{bb}^\infty(0,l)$,*

$$\mathbf{V}_\epsilon \to \mathbf{V},\ \mathbf{V}_{*,\epsilon} \to \mathbf{V}_* \text{ in } H^1_{bb}(0,l)^3,\ \psi_\epsilon \to \psi \text{ in } L^p(0,l), \tag{3.8}$$

for $\epsilon \to 0$ and $p \in [1,\infty)$, and

$$\|\mathbf{V}''_\epsilon\|_{L^2(0,l)^3} \sim O(\frac{1}{\epsilon^r}),\ \|\psi'_\epsilon\|_{L^2(0,l)} \sim O(\frac{1}{\epsilon^r}),\ r \in (0,\frac{1}{3}). \tag{3.9}$$

Proposition 3.5 [8] *Let $\lambda \ge 0$, $\mu > 0$ and*

$$A_\epsilon^{ijkl} = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu(g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}). \tag{3.10}$$

Then there exists a constant $C > 0$ independent of ϵ such that the estimate

$$\|\mathbf{V}\|^2_{H^1(\Omega)^3} \le \frac{C}{\epsilon}\|\omega^\epsilon(\mathbf{V})\|^2_{L^2(\Omega)^9} \le C\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{V})\frac{1}{\epsilon}\omega^\epsilon_{ij}(\mathbf{V})\,dx \tag{3.11}$$

holds for all $\mathbf{V} \in V_{bb}(\Omega)$ and $\epsilon \in (0,1)$.

Proposition 3.6 [8] *Suppose that $\{\epsilon_n\}_{n=1}^\infty \subset (0,1)$ and $\epsilon_n \to 0$. Let, in addition, a sequence $\{\mathbf{U}_{\epsilon_n}\}_{n=1}^\infty \subset V_{bb}(\Omega)$ be such that*

$$\mathbf{U}_{\epsilon_n} \rightharpoonup \mathbf{U} \text{ in } H^1(\Omega)^3, \tag{3.12}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n}) \rightharpoonup \zeta \text{ in } L^2(\Omega)^9 \tag{3.13}$$

for $\epsilon_n \to 0$. Then the couple $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_{bb}^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (in the sense $\partial_j \mathbf{U} = 0$, $j = 2,3$), where the function ϕ is such that

$$\frac{1}{2\epsilon_n}\Big((\partial_2 \mathbf{U}_{\epsilon_n}, \mathbf{b}_{\epsilon_n}) - (\partial_3 \mathbf{U}_{\epsilon_n}, \mathbf{n}_{\epsilon_n})\Big) \rightharpoonup \phi$$

in $L^2(\Omega)$ for $\epsilon_n \to 0$. In addition, the couple $\langle \mathbf{U}, \phi \rangle$ generates the function $\mathbf{U}_* \in H^1_{bb}(0,l)^3$ which together with the function $\mathbf{U}$ satisfy the relations

$$(\mathbf{U}', \mathbf{t}) = 0 \ a.e. \ on \ [0,l], \tag{3.14}$$

$$(\mathbf{U}'_*, \mathbf{t}) = \partial_3 \zeta_{12} - \partial_2 \zeta_{13} \ in \ L^2(0,l; H^{-1}(S)), \tag{3.15}$$

$$(\mathbf{U}'_*, \mathbf{n}) = -\partial_3 \zeta_{11} \ a.e. \ on \ [0,l], \tag{3.16}$$

$$(\mathbf{U}'_*, \mathbf{b}) = \partial_2 \zeta_{11} \ a.e. \ on \ [0,l]. \tag{3.17}$$

If the sequence $\{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n})\}_{n=1}^{\infty}$ converges strongly in $L^2(\Omega)^9$, then the convergence in (3.12) is strong as well.

Remark 3.7 Proposition 3.4, 3.5 and 3.6 can be analogously checked on the spaces $C_b^\infty(0,l)$, $V_b(\Omega)$ and $\mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$.

4 Transformation of variational equations for the curved rods

Let $\widetilde{\Omega}_\epsilon$ be a three-dimensional homogeneous isotropic elastic body with the Lamé constants $\lambda \geq 0$ and $\mu > 0$ defined by the mapping $\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon$ (see (2.3)–(2.4)), for $\epsilon \in (0,1)$ arbitrary but fixed, and clamped on the basis $\bar{\mathbf{P}}_\epsilon(\{0\} \times \epsilon S)$. We consider the variational equation posed on $\widetilde{\Omega}_\epsilon$

$$\int_{\widetilde{\Omega}_\epsilon} \widetilde{A}^{ijkl} e_{kl}(\widetilde{\mathbf{U}}_\epsilon) e_{ij}(\widetilde{\mathbf{V}})\, d\widetilde{y} = \int_{\widetilde{\Omega}_\epsilon} (\widetilde{\mathbf{F}}_\epsilon, \widetilde{\mathbf{V}})\, d\widetilde{y} + \int_{\widetilde{\Omega}_\epsilon} \widetilde{H}_{ij,\epsilon} e_{ij}(\widetilde{\mathbf{V}})\, d\widetilde{y}$$

$$+ \int_{(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l)\times \partial S)} (\widetilde{\mathbf{G}}_\epsilon, \widetilde{\mathbf{V}})\, d\widetilde{S}_\epsilon d\widetilde{y}_1 + \int_{(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(\{l\}\times S)} (\widetilde{\mathbf{K}}_\epsilon, \widetilde{\mathbf{V}})\, d\widetilde{S}_{l,\epsilon}, \ \forall \widetilde{\mathbf{V}} \in V_b(\widetilde{\Omega}_\epsilon), \tag{4.1}$$

where $\widetilde{\mathbf{F}}_\epsilon$ and $(\widetilde{H}_{ij,\epsilon})_{i,j=1}^3$ are the body forces, $\widetilde{\mathbf{G}}_\epsilon$ and $\widetilde{\mathbf{K}}_\epsilon$ are the surface tractions acting on the curved rod $\widetilde{\Omega}_\epsilon$ such that $\widetilde{\mathbf{F}}_\epsilon \in L^2(\widetilde{\Omega}_\epsilon)^3$, $\widetilde{\mathbf{G}}_\epsilon \in L^2((\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S))^3$, $\widetilde{\mathbf{K}}_\epsilon \in L^2((\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(\{l\} \times S))^3$ and $(\widetilde{H}_{ij,\epsilon})_{i,j=1}^3 \in L^2(\widetilde{\Omega}_\epsilon)^9$ for $\epsilon \in (0,1)$. Further, $\widetilde{S}_\epsilon = (\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S)$, $\widetilde{S}_{l,\epsilon} = (\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(\{l\} \times S)$, $\widetilde{A}^{ijkl} = \lambda \delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ and $(e_{ij}(\widetilde{\mathbf{V}}))_{i,j=1}^3$ stands for the symmetric part of the gradient of the function $\widetilde{\mathbf{V}}$.

According to Theorem 1.2-1. (b) from [2], we can transform the last term in (4.1) as

$$\int_{(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(\{l\}\times S)} (\widetilde{\mathbf{K}}_\epsilon, \widetilde{\mathbf{V}})\, d\widetilde{S}_{l,\epsilon} = \int_{\{l\}\times \epsilon S} (\bar{\mathbf{K}}_\epsilon, \bar{\mathbf{V}}_\epsilon) d_\epsilon \sqrt{\eta_{i,\epsilon} \bar{g}^{ij,\epsilon} \eta_{j,\epsilon}}\, d\bar{S}_{l,\epsilon}$$

$$= \int_{\epsilon S} (\bar{\mathbf{K}}_\epsilon(l), \bar{\mathbf{V}}_\epsilon(l)) \; dy_2 y_3 = \epsilon^2 \int_S (\mathbf{K}_\epsilon, \mathbf{V}_\epsilon(l)) \; dx_2 x_3,$$

where $\eta_{i,\epsilon}$, $i = 1, 2, 3$, are the components of the unit outer normal vector to $\{l\} \times \epsilon S$ (i.e. $\eta_{1,\epsilon} = 1$, $\eta_{2,\epsilon} = 0$ and $\eta_{3,\epsilon} = 0$), and $\bar{\mathbf{K}}_\epsilon = \widetilde{\mathbf{K}}_\epsilon \circ \bar{\mathbf{P}}_\epsilon$, $\mathbf{K}_\epsilon = (\bar{\mathbf{K}}_\epsilon \circ \mathbf{R}_\epsilon)(l)$, $\bar{\mathbf{V}}_\epsilon = \widetilde{\mathbf{V}} \circ \bar{\mathbf{P}}_\epsilon$ and $\mathbf{V}_\epsilon = \bar{\mathbf{V}}_\epsilon \circ \mathbf{R}_\epsilon$. Now, we decompose the tensor $(\widetilde{H}_{ij,\epsilon})_{i,j=1}^3$ in the covariant basis

$$\widetilde{H}_{ij,\epsilon} \circ \bar{\mathbf{P}}_\epsilon = \widehat{H}_{i\widehat{j},\epsilon} [\bar{\mathbf{g}}_{\widehat{j},\epsilon}]_j \tag{4.2}$$

for arbitrary but fixed $i = 1, 2, 3$. We apply the same decomposition on $\widehat{H}_{i\widehat{j},\epsilon}[\bar{\mathbf{g}}_{\widehat{j},\epsilon}]_j$, for arbitrary but fixed $j = 1, 2, 3$, which yields

$$\widehat{H}_{i\widehat{j},\epsilon}[\bar{\mathbf{g}}_{\widehat{j},\epsilon}]_j = \bar{H}_{\widehat{i}\widehat{j},\epsilon}[\bar{\mathbf{g}}_{\widehat{i},\epsilon}]_i [\bar{\mathbf{g}}_{\widehat{j},\epsilon}]_j. \tag{4.3}$$

In [8] it was proved that

$$e_{ij}(\widetilde{\mathbf{V}}) \circ \bar{\mathbf{P}}_\epsilon = \bar{\omega}^\epsilon_{kl}(\bar{\mathbf{V}}_\epsilon)[\bar{\mathbf{g}}^{k,\epsilon}]_i [\bar{\mathbf{g}}^{l,\epsilon}]_j, \;\; i, j = 1, 2, 3. \tag{4.4}$$

Hence, together with the properties of the covariant and contravariant basis, we deduce after the substitution $\mathbf{R}_\epsilon$ that

$$\int_{\widetilde{\Omega}_\epsilon} \widetilde{H}_{ij,\epsilon} e_{ij}(\widetilde{\mathbf{V}}) \; d\widetilde{y} = \epsilon^2 \int_\Omega H_{ij,\epsilon} \omega^\epsilon_{ij}(\mathbf{V}_\epsilon) d_\epsilon \; dx \tag{4.5}$$

for $H_{ij,\epsilon} = \bar{H}_{ij,\epsilon} \circ \mathbf{R}_\epsilon$. We refer the reader to [8] for the detailed transformation of the other terms in (4.1). Using the scaling $\mathbf{F}_\epsilon = \epsilon^2 \mathbf{F}$, $\mathbf{G}_\epsilon = \epsilon^3 \mathbf{G}$, $(H_{ij,\epsilon})_{i,j=1}^3 = \epsilon (H_{ij})_{i,j=1}^3 = \epsilon H$, $\mathbf{K}_\epsilon = \epsilon^2 \mathbf{K}$, we can rewrite the equation (4.1) as

$$\begin{aligned} \int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon} \omega^\epsilon_{kl}(\mathbf{U}_\epsilon) \frac{1}{\epsilon} \omega^\epsilon_{ij}(\mathbf{V}) d_\epsilon \; dx &= \int_\Omega (\mathbf{F}, \mathbf{V}) d_\epsilon \; dx + \int_\Omega H_{ij} \frac{1}{\epsilon} \omega^\epsilon_{ij}(\mathbf{V}) d_\epsilon \; dx \\ &+ \int_{(0,l)} \int_{\partial S} (\mathbf{G}, \mathbf{V}) \epsilon d_\epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \; dS dx_1 + \int_S (\mathbf{K}, \mathbf{V}(l)) \; dx_2 dx_3 \end{aligned} \tag{4.6}$$

for all $\mathbf{V} \in V_b(\Omega)$, where ν_i, $i = 1, 2, 3$, are the components of the unit outward normal to $(0, l) \times \partial S$, and

$$A^{ijkl}_\epsilon = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu (g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}). \tag{4.7}$$

The symmetric tensor $\omega^\epsilon(\mathbf{V})$, obtained from (4.4) after the composition with $\mathbf{R}_\epsilon$, has the form

$$\omega^\epsilon(\mathbf{V}) = \frac{1}{\epsilon} \theta^\epsilon(\mathbf{V}) + \kappa^\epsilon(\mathbf{V}), \tag{4.8}$$

where the individual nonzero components of the symmetric tensors θ^ϵ and κ^ϵ are defined by

$$\theta^\epsilon_{12}(\mathbf{V}) = \frac{1}{2} (\partial_2 \mathbf{V}, \mathbf{g}_{1,\epsilon}), \;\; \theta^\epsilon_{22}(\mathbf{V}) = (\partial_2 \mathbf{V}, \mathbf{n}_\epsilon), \;\; \theta^\epsilon_{33}(\mathbf{V}) = (\partial_3 \mathbf{V}, \mathbf{b}_\epsilon), \tag{4.9}$$

$$\theta_{13}^\epsilon(\mathbf{V}) = \frac{1}{2}(\partial_3\mathbf{V}, \mathbf{g}_{1,\epsilon}),\ \theta_{23}^\epsilon(\mathbf{V}) = \frac{1}{2}\Big((\partial_2\mathbf{V}, \mathbf{b}_\epsilon) + (\partial_3\mathbf{V}, \mathbf{n}_\epsilon)\Big), \tag{4.10}$$

$$\kappa_{11}^\epsilon(\mathbf{V}) = (\partial_1\mathbf{V}, \mathbf{g}_{1,\epsilon}),\ \kappa_{12}^\epsilon(\mathbf{V}) = \frac{1}{2}(\partial_1\mathbf{V}, \mathbf{n}_\epsilon),\ \kappa_{13}^\epsilon(\mathbf{V}) = \frac{1}{2}(\partial_1\mathbf{V}, \mathbf{b}_\epsilon). \tag{4.11}$$

The other components of θ^ϵ and κ^ϵ are equal to zero.

Analogously we can derive the equation

$$\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon)\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{V})d_\epsilon\ dx = \int_\Omega(\mathbf{F},\mathbf{V})d_\epsilon\ dx + \int_\Omega H_{ij}\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{V})d_\epsilon\ dx$$

$$+\int_{(0,l)}\int_{\partial S}(\mathbf{G},\mathbf{V})\epsilon d_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\ dSdx_1,\ \forall\mathbf{V}\in V_{bb}(\Omega), \tag{4.12}$$

for the curved rods clamped on both bases.

5 Proofs of Theorem 1.1 and 1.2

The proofs of Theorem 1.1 and 1.2 will be decomposed in this section to several propositions, lemmas and corollaries.

Using (3.11), (4.6) and Corollary 3.2, we can derive easily the estimate

$$\frac{1}{\epsilon^2}\|\omega^\epsilon(\mathbf{U}_\epsilon)\|^2_{L^2(\Omega)^9} \le \frac{C^2}{C_4\epsilon^2}\int_\Omega A_\epsilon^{ijkl}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon)\omega_{ij}^\epsilon(\mathbf{U}_\epsilon)d_\epsilon\ dx = \frac{C^2}{C_4}\Bigg(\int_\Omega H_{ij}\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{U}_\epsilon)d_\epsilon\ dx$$

$$+\int_\Omega(\mathbf{F},\mathbf{U}_\epsilon)d_\epsilon\ dx + \int_0^l\int_{\partial S}(\mathbf{G},\mathbf{U}_\epsilon)d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j}\ dSdx_1 + \int_S(\mathbf{K},\mathbf{U}_\epsilon(l))\ dx_2dx_3\Bigg)$$

$$\le \frac{C^2C_5}{C_4}\Big(\|H\|_{L^2(\Omega)^9}\frac{1}{\epsilon}\|\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^2(\Omega)^9} + \|\mathbf{F}\|_{L^2(\Omega)^3}\|\mathbf{U}_\epsilon\|_{H^1(\Omega)^3}$$

$$+\|\mathbf{G}\|_{L^2(0,l;L^2(\partial S)^3)}\|\mathbf{U}_\epsilon\|_{L^2(0,l;L^2(\partial S)^3)} + \|\mathbf{K}\|_{L^2(S)^3}\|\mathbf{U}_\epsilon(l)\|_{L^2(S)^3}\Big)$$

$$\le C(\|\mathbf{U}_\epsilon\|_{H^1(\Omega)^3} + \frac{1}{\epsilon}\|\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^2(\Omega)^9}) \le C\frac{1}{\epsilon}\|\omega^\epsilon(\mathbf{U}_\epsilon)\|_{L^2(\Omega)^9} \tag{5.1}$$

for all $\epsilon\in(0,1)$, because $\mathbf{U}_\epsilon\in V_b(\Omega)$ which implies that $\mathbf{U}_\epsilon\in C([0,l];L^2(S)^3)$ and $\mathbf{U}_\epsilon\in L^2(0,l;L^2(\partial S)^3)$ in the sense of the trace. By the inequalities (3.11) and (5.1) (passing to a subsequence), we have that

$$\mathbf{U}_{\epsilon_n} \rightharpoonup \mathbf{U} \text{ in } H^1(\Omega)^3, \tag{5.2}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n}) \rightharpoonup \zeta \text{ in } L^2(\Omega)^9 \tag{5.3}$$

for $\epsilon_n\to 0$, where $\mathbf{U}\in H_b^1(0,l)^3$ according to Proposition 3.6 and Remark 3.7. To simplify the notation, we will use further ϵ instead of ϵ_n.

Now, we will study the properties of the tensor ζ.

Proposition 5.1 *Let the tensor ζ be the limit determined by* (5.3). *Then it satisfies the equation*

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\theta_{ij}^0(\mathbf{V})\ dx = \int_\Omega H_{ij}\theta_{ij}^0(\mathbf{V})\ dx,\ \forall \mathbf{V}\in L^2(0,l;H^1(S)^3), \tag{5.4}$$

where the tensor $\theta^0(\mathbf{V})$ is defined by

$$\theta^0(\mathbf{V}) = \begin{pmatrix} 0 & \frac{(\partial_2\mathbf{V},\mathbf{t})}{2} & \frac{(\partial_3\mathbf{V},\mathbf{t})}{2} \\ \frac{(\partial_2\mathbf{V},\mathbf{t})}{2} & (\partial_2\mathbf{V},\mathbf{n}) & \frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2} \\ \frac{(\partial_3\mathbf{V},\mathbf{t})}{2} & \frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2} & (\partial_3\mathbf{V},\mathbf{b}) \end{pmatrix}. \tag{5.5}$$

P r o o f: Analogously as in [8] we can prove that $\theta^\epsilon(\mathbf{V})+\epsilon\kappa^\epsilon(\mathbf{V})\to\theta^0(\mathbf{V})$ in $L^2(\Omega)^9$ for $\epsilon\to 0$ and that

$$A_\epsilon^{ijkl}\to A_0^{ijkl} \text{ in } C(\overline{\Omega}),\ \text{where } A_0^{ijkl}=\lambda\delta^{ij}\delta^{kl}+\mu(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}), \tag{5.6}$$

for $i,j,k,l=1,2,3$. The rest of the proof follows from density of the space $V_b(\Omega)$ in $L^2(0,l;H^1(S)^3)$ and from (5.5) and (5.6). □

Now, we introduce the notation:

$$\widehat{\zeta}_{22}=\zeta_{22}+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11},\ \widehat{\zeta}_{33}=\zeta_{33}+\frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11},\ \widehat{\zeta}_{23}=\zeta_{23}. \tag{5.7}$$

Corollary 5.2 *Let the equation* (5.4) *hold. Then* (*we do not use the summation convention here*)

$$\int_S\zeta_{1j}=\frac{1}{2\mu}\int_S H_{1j},\ \int_S\zeta_{1j}x_j=\frac{1}{2\mu}\int_S H_{1j}x_j,\ j=2,3,$$

$$\int_S\zeta_{12}x_3+\zeta_{13}x_2=\frac{1}{2\mu}\int_S H_{12}x_3+H_{13}x_2, \tag{5.8}$$

$$\int_S\widehat{\zeta}_{23}=\frac{1}{2\mu}\int_S H_{23},\ \int_S\widehat{\zeta}_{23}x_2=\frac{1}{2\mu}\int_S H_{23}x_2,\ \int_S\widehat{\zeta}_{23}x_3=\frac{1}{2\mu}\int_S H_{23}x_3, \tag{5.9}$$

$$\int_S\widehat{\zeta}_{22}+\widehat{\zeta}_{33}=\frac{1}{\lambda+2\mu}\int_S H_{22}+H_{33},\ \int_S(\widehat{\zeta}_{22}+\widehat{\zeta}_{33})x_2=\frac{1}{\lambda+2\mu}\int_S(H_{22}+H_{33})x_2,$$

$$\int_S(\widehat{\zeta}_{22}+\widehat{\zeta}_{33})x_3=\frac{1}{\lambda+2\mu}\int_S(H_{22}+H_{33})x_3. \tag{5.10}$$

P r o o f: Using in the equation (5.4) the test functions $\mathbf{V}x_2$, $\mathbf{V}x_3$, $\mathbf{V}x_2^2/2$, $\mathbf{V}x_3^2/2$ and $\mathbf{V}x_2x_3$, where $\mathbf{V}=v\mathbf{t}$, $\mathbf{V}=v\mathbf{n}$ and $\mathbf{V}=v\mathbf{b}$ for some function $v\in L^2(0,l)$, we deduce the relations (5.8)–(5.10) in the same way as in the proof of Corollary 8.2 in [8] for $H_{ij}=0$, $i,j=1,2,3$. □

Corollary 5.3 *We have*

$$\int_\Omega [\lambda(\widehat{\zeta}_{22}+\widehat{\zeta}_{33})^2 + 2\mu(\widehat{\zeta}_{22}^2+\widehat{\zeta}_{33}^2+2\widehat{\zeta}_{23}^2)]\ dx$$

$$= \int_\Omega [H_{22}\zeta_{22} + H_{33}\zeta_{33} + 2H_{23}\zeta_{23} + \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33})\zeta_{11}]\ dx. \tag{5.11}$$

P r o o f: If we take an arbitrary function $\mathbf{V} \in L^2(0,l;H^1(S)^3)$ such that $(\mathbf{V},\mathbf{t}) = 0$, we get from (5.4) that

$$\int_\Omega [\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})((\partial_2\mathbf{V},\mathbf{n}) + (\partial_3\mathbf{V},\mathbf{b})) + 2\mu(\zeta_{22}(\partial_2\mathbf{V},\mathbf{n}) + \zeta_{33}(\partial_3\mathbf{V},\mathbf{b})$$

$$+2\zeta_{23}\frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2})]\ dx = \int_\Omega [H_{22}(\partial_2\mathbf{V},\mathbf{n}) + H_{33}(\partial_3\mathbf{V},\mathbf{b})$$

$$+2H_{23}\frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2}]\ dx. \tag{5.12}$$

Now, we define the function

$$\mathbf{V}_{\mathbf{U}_\epsilon} = \frac{1}{\epsilon^2}((\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\mathbf{n} + (\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\mathbf{b}).$$

Since $\mathbf{U}_\epsilon \in V_b(\Omega)$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon \in C^\infty([0,l])^3$ and $\mathbf{n}$, $\mathbf{b} \in L^\infty(0,l)^3$, we can easily check that $\mathbf{V}_{\mathbf{U}_\epsilon} \in L^2(0,l;H^1(S)^3)$ for all $\epsilon \in (0,1)$. After the substitution of the function $\mathbf{V}_{\mathbf{U}_\epsilon}$ to the equality (5.12), we obtain, using the notation from (4.8)–(4.11), that

$$\int_\Omega [\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})(\frac{1}{\epsilon}\omega_{22}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\omega_{33}^\epsilon(\mathbf{U}_\epsilon)) + 2\mu(\zeta_{22}\frac{1}{\epsilon}\omega_{22}^\epsilon(\mathbf{U}_\epsilon) + \zeta_{33}\frac{1}{\epsilon}\omega_{33}^\epsilon(\mathbf{U}_\epsilon)$$

$$+2\zeta_{23}\frac{1}{\epsilon}\omega_{23}^\epsilon(\mathbf{U}_\epsilon))]\ dx = \int_\Omega [H_{22}\frac{1}{\epsilon}\omega_{22}^\epsilon(\mathbf{U}_\epsilon) + H_{33}\frac{1}{\epsilon}\omega_{33}^\epsilon(\mathbf{U}_\epsilon) + 2H_{23}\frac{1}{\epsilon}\omega_{23}^\epsilon(\mathbf{U}_\epsilon)]\ dx. \tag{5.13}$$

The functions ζ_{11}, ζ_{22}, ζ_{33} and ζ_{23} belong to $L^2(\Omega)$ and thus the convergence in (5.3) enables us to pass from the equality (5.13) to the equality

$$\int_\Omega [\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})(\zeta_{22}+\zeta_{33}) + 2\mu(\zeta_{22}^2+\zeta_{33}^2+2\zeta_{23}^2)]\ dx$$

$$= \int_\Omega [H_{22}\zeta_{22} + H_{33}\zeta_{33} + 2H_{23}\zeta_{23}]\ dx. \tag{5.14}$$

The term on the left-hand side can be rewritten as

$$\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})(\zeta_{22}+\zeta_{33}) + 2\mu(\zeta_{22}^2+\zeta_{33}^2+2\zeta_{23}^2)$$

$$\stackrel{(5.7)}{=} \lambda(\widehat{\zeta}_{22}+\widehat{\zeta}_{33}+\frac{\mu}{\lambda+\mu}\zeta_{11})(\widehat{\zeta}_{22}+\widehat{\zeta}_{33}-\frac{\lambda}{\lambda+\mu}\zeta_{11}) + 2\mu((\widehat{\zeta}_{22}-\frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11})^2$$

$$+(\widehat{\zeta}_{33}-\frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11})^2+2\widehat{\zeta}_{23}^2)=\lambda(\widehat{\zeta}_{22}+\widehat{\zeta}_{33})^2-\lambda\zeta_{11}(\widehat{\zeta}_{22}+\widehat{\zeta}_{33})$$
$$+2\mu(\widehat{\zeta}_{22}^2+\widehat{\zeta}_{33}^2+2\widehat{\zeta}_{23}^2). \tag{5.15}$$

From (3.16) and (3.17), it follows that

$$\zeta_{11}=Q_0+(\mathbf{U}'_*,\mathbf{b})x_2-(\mathbf{U}'_*,\mathbf{n})x_3 \tag{5.16}$$

for some function $Q_0\in L^2(0,l)$. After the substitution (5.15) to (5.14) and using (5.10) and (5.16), we get (5.11). □

Lemma 5.4 *Let S be a simply connected domain and let $\partial S\in C^1$. Then*

$$\langle\zeta_{12},\zeta_{13}\rangle=-\frac{1}{2}(\mathbf{U}'_*,\mathbf{t})\langle\partial_2 p-x_3,\partial_3 p+x_2\rangle+\langle\partial_2 p_H,\partial_3 p_H\rangle, \tag{5.17}$$

where the functions $p\in H^1(S)$ and $p_H\in L^2(0,l;H^1(S))$ are the unique solutions to the Neumann problems

$$\int_S[(\partial_2 p-x_3)\partial_2 r+(\partial_3 p+x_2)\partial_3 r]\ dx_2dx_3=0,\quad \int_S p\ dx_2dx_3=0, \tag{5.18}$$

for all $r\in H^1(S)$, and

$$\int_S[\partial_2 p_H\partial_2 r+\partial_3 p_H\partial_3 r]\ dx_2dx_3=\frac{1}{2\mu}\int_S[H_{12}\partial_2 r+H_{13}\partial_3 r]\ dx_2dx_3, \tag{5.19}$$

for all $r\in H^1(S)$,

$$\int_S p_H\ dx_2dx_3=0, \tag{5.20}$$

respectively, where (5.19)–(5.20) are fulfilled on the whole interval $(0,l)$.

P r o o f: After putting $\mathbf{V}=\varphi\mathbf{t}$, $\varphi\in L^2(0,l;H^1(S))$, as a test function in the equation (5.4) and taking the equality (3.15), we get the system of equations

$$\int_\Omega(\langle\zeta_{12},\zeta_{13}\rangle,\nabla_{23}\varphi)_2\ dx=\frac{1}{2\mu}\int_\Omega(\langle H_{12},H_{13}\rangle,\nabla_{23}\varphi)_2\ dx, \tag{5.21}$$

for all $\varphi\in L^2(0,l;H^1(S))$,

$$\int_\Omega(\langle\zeta_{12},\zeta_{13}\rangle,\mathrm{rot}_{23}\psi)_2\ dx=\int_\Omega(\mathbf{U}'_*,\mathbf{t})\psi\ dx,\ \forall\psi\in H^1_{bb}(\Omega), \tag{5.22}$$

where we have denoted $\nabla_{23}\varphi=\langle\partial_2\varphi,\partial_3\varphi\rangle$, $\mathrm{rot}_{23}\psi=\langle-\partial_3\psi,\partial_2\psi\rangle$, and $(\cdot,\cdot)_2$ means the scalar product in the usual two dimensional Euclidean space $\mathbb{R}^2$. Substituting (5.17) to (5.21)–(5.22), we can check that this couple is a solution to the system (5.21)–(5.22). We refer the reader to [4] or [8] for the proof of uniqueness. □

Now, we derive the asymptotic one-dimensional model. First, we introduce the notation

$$I_{x_2^2} = \int_S x_2^2 \; dx_2 dx_3, \;\; I_{x_3^2} = \int_S x_3^2 \; dx_2 dx_3, \tag{5.23}$$

$$E = \mu \frac{3\lambda + 2\mu}{\lambda + \mu}, \;\; K = \int_S [(\partial_2 p - x_3)^2 + (\partial_3 p + x_2)^2] \; dx_2 dx_3, \tag{5.24}$$

where $p \in H^1(S)$ is the unique solution of the Neumann problem (5.18).

Lemma 5.5 *Let the functions* $\mathbf{U}_\epsilon$ *be the solutions of the problem* (4.6) *satisfying* (5.2) *and* (5.3). *Then the limit couple* $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ *obtained in Proposition* 3.6 (*see Remark* 3.7) *generates the function* $\mathbf{U}_*$, *which satisfies the equation*

$$\int_0^l [E(Q_0|S|(\mathbf{W}_P', \mathbf{t}) + I_{x_2^2}(\mathbf{U}_*', \mathbf{b})(\mathbf{V}_*', \mathbf{b}) + I_{x_3^2}(\mathbf{U}_*', \mathbf{n})(\mathbf{V}_*', \mathbf{n})) + \mu K(\mathbf{U}_*', \mathbf{t})(\mathbf{V}_*', \mathbf{t})] \; dx_1$$

$$= -\frac{\lambda}{\lambda + 2\mu} \int_\Omega (H_{22} + H_{33})(x_2(\mathbf{V}_*', \mathbf{b}) - x_3(\mathbf{V}_*', \mathbf{n}) + (\mathbf{W}_P', \mathbf{t})) \; dx$$

$$+ \int_\Omega [H_{12}(\mathbf{V}_*', \mathbf{t})(-\partial_2 p + x_3) - H_{13}(\mathbf{V}_*', \mathbf{t})(\partial_3 p + x_2)] \; dx$$

$$+ \int_\Omega H_{11}((\mathbf{V}_*', \mathbf{b})x_2 - (\mathbf{V}_*', \mathbf{n})x_3 + (\mathbf{W}_P', \mathbf{t})) \; dx + \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{V}) \; dx_1 + (\check{\mathbf{K}}, \mathbf{V}(l)) \tag{5.25}$$

for all functions $\mathbf{W}_P \in H_b^1(0,l)^3$ *and* $\mathbf{V}_* \in H_b^1(0,l)^3$ *generated by any arbitrary couple* $\langle \mathbf{V}, \psi \rangle \in \mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (*see* (2.12)), *where* $\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}(x_1) = \int_S \mathbf{F} \; dx_2 dx_3 + \int_{\partial S} \mathbf{G} \; dS$, $x_1 \in (0,l)$, *and* $\check{\mathbf{K}} = \int_S \mathbf{K} \; dx_2 dx_3$.

P r o o f: Let $\langle \mathbf{V}, \psi \rangle$ be an arbitrary couple from the space $\mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. Proposition 3.4 and Remark 3.7 enable us to approximate the couple $\langle \mathbf{V}, \psi \rangle$ with couples $\langle \mathbf{V}_\epsilon, \psi_\epsilon \rangle \in \mathcal{V}_b^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ satisfying (3.8), (3.9). Further, we define the functions $\mathbf{W}_\epsilon \in C^\infty(\overline{\Omega})^3$ by

$$\mathbf{W}_\epsilon(x_1, x_2, x_3) = -\Big((\mathbf{V}_\epsilon'(x_1), \mathbf{n}_\epsilon(x_1))x_2 + (\mathbf{V}_\epsilon'(x_1), \mathbf{b}_\epsilon(x_1))x_3\Big)\mathbf{t}_\epsilon(x_1)$$

$$-x_3\psi_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) + x_2\psi_\epsilon(x_1)\mathbf{b}_\epsilon(x_1) \tag{5.26}$$

for $(x_1, x_2, x_3) \in \Omega$. Let $\mathbf{W}_P$ be an arbitrary function from $H_b^1(0,l)^3$. Using the functions $\mathbf{V}_\epsilon$, $\mathbf{W}_\epsilon$ and $\mathbf{W}_P$, we establish the function $\widehat{\mathbf{V}}_\epsilon$ by

$$\widehat{\mathbf{V}}_\epsilon = \mathbf{V}_\epsilon + \epsilon \mathbf{W}_\epsilon + \epsilon \mathbf{W}_P \in C^\infty(\overline{\Omega})^3 \cap V_b(\Omega). \tag{5.27}$$

Analogously as in [8] Lemma 8.4, we can derive that the tensor $B_\epsilon = (B_\epsilon^{ij})_{i,j=1}^3$ is such that $B_\epsilon^{ij} = 0$ except for $i = j = 1$ and

$$B_\epsilon^{11} = \epsilon^2 \Big((\beta_\epsilon x_2 + \alpha_\epsilon x_3)(x_2(\mathbf{V}_\epsilon', \mathbf{n}_\epsilon)' + x_3(\mathbf{V}_\epsilon', \mathbf{b}_\epsilon)' - \beta_\epsilon x_3 \psi_\epsilon + \alpha_\epsilon x_2 \psi_\epsilon - (\mathbf{W}_P', \mathbf{t}_\epsilon))$$

$$+\gamma_\epsilon x_3\big((\partial_1\mathbf{W}_\epsilon,\mathbf{n}_\epsilon)+(\mathbf{W}'_P,\mathbf{n}_\epsilon)\big)-\gamma_\epsilon x_2\big((\partial_1\mathbf{W}_\epsilon,\mathbf{b}_\epsilon)+(\mathbf{W}'_P,\mathbf{b}_\epsilon)\big)\Big). \tag{5.28}$$

Hence and from (4.8)–(4.11), it follows that

$$\omega^\epsilon(\widehat{\mathbf{V}}_\epsilon)=\epsilon\Upsilon^\epsilon(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)+B_\epsilon, \tag{5.29}$$

where

$$\Upsilon^\epsilon_{11}(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)=-(\mathbf{V}'_{*,\epsilon},\mathbf{n}_\epsilon)x_3+(\mathbf{V}'_{*,\epsilon},\mathbf{b}_\epsilon)x_2+(\mathbf{W}'_P,\mathbf{t}_\epsilon), \tag{5.30}$$

$$\Upsilon^\epsilon_{12}(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)=\Upsilon^\epsilon_{21}(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)=\frac{x_3}{2}(\mathbf{V}'_{*,\epsilon},\mathbf{t}_\epsilon)+\frac{1}{2}(\mathbf{W}'_P,\mathbf{n}_\epsilon), \tag{5.31}$$

$$\Upsilon^\epsilon_{13}(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)=\Upsilon^\epsilon_{31}(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)=-\frac{x_2}{2}(\mathbf{V}'_{*,\epsilon},\mathbf{t}_\epsilon)+\frac{1}{2}(\mathbf{W}'_P,\mathbf{b}_\epsilon) \tag{5.32}$$

and

$$\Upsilon^\epsilon_{ij}(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)=0,\ \ i,j=2,3. \tag{5.33}$$

Since we know that $\mathbf{t}_\epsilon\to\mathbf{t}$, $\mathbf{n}_\epsilon\to\mathbf{n}$, $\mathbf{b}_\epsilon\to\mathbf{b}$ in measure in $(0,l)$, we can prove that

$$\Upsilon^\epsilon_{ij}(\mathbf{V}_{*,\epsilon},\mathbf{W}'_P)\to\Upsilon_{ij}(\mathbf{V}_*,\mathbf{W}'_P)\ \text{in}\ L^2(\Omega),\ \ i,j=1,2,3. \tag{5.34}$$

Moreover, using (3.4), (3.5), (3.9) and (5.28), we can easily check that

$$\|B_\epsilon\|_2=\|B^{11}_\epsilon\|_2\le C\epsilon^{2(1-r)},\ \ r\in(0,\frac{1}{3}). \tag{5.35}$$

These convergences and estimates together with (3.6), (3.7), (5.2), (5.3), (5.6) enable us to pass to the limit in the equation (since $\widehat{\mathbf{V}}_\epsilon\in C^\infty(\overline{\Omega})^3\cap V_b(\Omega)$)

$$\int_\Omega A^{ijkl}_\epsilon\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)\frac{1}{\epsilon}\omega^\epsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon\,dx=\int_\Omega(\mathbf{F},\widehat{\mathbf{V}}_\epsilon)d_\epsilon\,dx+\int_\Omega H_{ij}\frac{1}{\epsilon}\omega^\epsilon_{ij}(\widehat{\mathbf{V}}_\epsilon)d_\epsilon\,dx$$

$$+\int_0^l\int_{\partial S}(\mathbf{G},\widehat{\mathbf{V}}_\epsilon)\epsilon d_\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j}\,dSdx_1+\int_S(\mathbf{K},\widehat{\mathbf{V}}_\epsilon(l))\,dx_2dx_3$$

and to establish

$$\int_\Omega A^{ijkl}_0\zeta_{kl}\Upsilon_{ij}(\mathbf{V}_*,\mathbf{W}'_P)\,dx=\int_\Omega(\mathbf{F},\mathbf{V})\,dx+\int_\Omega H_{ij}\Upsilon_{ij}(\mathbf{V}_*,\mathbf{W}'_P)\,dx$$

$$+\int_0^l\int_{\partial S}(\mathbf{G},\mathbf{V})\,dSdx_1+\int_S(\mathbf{K},\mathbf{V}(l))\,dx_2dx_3 \tag{5.36}$$

for all $\mathbf{W}_P\in H^1_b(0,l)^3$ and $\langle\mathbf{V},\psi\rangle\in\mathcal{V}^{\mathbf{t},\mathbf{n},\mathbf{b}}_b(0,l)$, where the last couples generate the functions $\mathbf{V}_*$ (see (2.12)).

By the form of the tensor $(A^{ijkl}_0)^3_{i,j,k,l=1}$ (see (5.6)), we have after the substitution (5.30)–(5.33) for "$\epsilon=0$" (see (5.34)) to (5.36) that

$$\int_\Omega A^{ijkl}_0\zeta_{kl}\Upsilon_{ij}(\mathbf{V}_*,\mathbf{W}'_P)\,dx=\int_\Omega[\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})+2\mu\zeta_{11}]\Upsilon_{11}(\mathbf{V}_*,\mathbf{W}'_P)\,dx$$

$$+\int_\Omega [4\mu(\zeta_{12}\Upsilon_{12}(\mathbf{V}_*,\mathbf{W}'_P)+\zeta_{13}\Upsilon_{13}(\mathbf{V}_*,\mathbf{W}'_P))]\ dx.$$

Hence, using (5.7), (5.24) and (5.30)–(5.32), we can rewrite (5.36) as

$$\mathcal{I}_1+\mathcal{I}_2=\int_\Omega(\mathbf{F},\mathbf{V})\ dx+\int_\Omega H_{ij}\Upsilon_{ij}(\mathbf{V}_*,\mathbf{W}'_P)\ dx$$

$$+\int_0^l\int_{\partial S}(\mathbf{G},\mathbf{V})\ dSdx_1+\int_S(\mathbf{K},\mathbf{V}(l))\ dx_2dx_3, \tag{5.37}$$

where

$$\mathcal{I}_1=\int_\Omega[E\zeta_{11}+\lambda(\widehat{\zeta}_{22}+\widehat{\zeta}_{33})][(\mathbf{V}'_*,\mathbf{b})x_2-(\mathbf{V}'_*,\mathbf{n})x_3+(\mathbf{W}'_P,\mathbf{t})]\ dx, \tag{5.38}$$

$$\mathcal{I}_2=2\mu\int_\Omega[\zeta_{12}(\mathbf{V}'_*,\mathbf{t})x_3+\zeta_{12}(\mathbf{W}'_P,\mathbf{n})-\zeta_{13}(\mathbf{V}'_*,\mathbf{t})x_2+\zeta_{13}(\mathbf{W}'_P,\mathbf{b})]\ dx. \tag{5.39}$$

After the substitution of (5.10) and (5.16) to (5.38), we can conclude using (2.1) and (5.23)–(5.24) that

$$\mathcal{I}_1=\int_0^l E[Q_0|S|(\mathbf{W}'_P,\mathbf{t})+I_{x_2^2}(\mathbf{U}'_*,\mathbf{b})(\mathbf{V}'_*,\mathbf{b})+I_{x_3^2}(\mathbf{U}'_*,\mathbf{n})(\mathbf{V}'_*,\mathbf{n})]\ dx_1$$

$$+\frac{\lambda}{\lambda+2\mu}\int_\Omega[(H_{22}+H_{33})(x_2(\mathbf{V}'_*,\mathbf{b})-x_3(\mathbf{V}'_*,\mathbf{n})+(\mathbf{W}'_P,\mathbf{t}))]\ dx. \tag{5.40}$$

After the substitution of (5.17) to (5.39), we obtain

$$\mathcal{I}_2=\mu\int_\Omega[(-(\partial_2 p-x_3)x_3+(\partial_3 p+x_2)x_2)(\mathbf{U}'_*,\mathbf{t})$$

$$+2\partial_2 p_H x_3-2\partial_3 p_H x_2](\mathbf{V}'_*,\mathbf{t})\ dx+2\mu\int_\Omega\zeta_{12}(\mathbf{W}'_P,\mathbf{n})-\zeta_{13}(\mathbf{W}'_P,\mathbf{b})\ dx, \tag{5.41}$$

where the functions p and p_H are the unique solutions to the Neumann problems (5.18) and (5.19)–(5.20), respectively. Analogously as in [8], we can verify that

$$\mu\int_\Omega\left(-(\partial_2 p-x_3)x_3+(\partial_3 p+x_2)x_2\right)(\mathbf{U}'_*,\mathbf{t})(\mathbf{V}'_*,\mathbf{t})\ dx_1$$

$$=\int_0^l\mu K(\mathbf{U}'_*,\mathbf{t})(\mathbf{V}'_*,\mathbf{t})\ dx_1. \tag{5.42}$$

In addition, from (5.18)–(5.20), it follows that

$$2\mu\int_\Omega\partial_2 p_H x_3(\mathbf{V}'_*,\mathbf{t})-\partial_3 p_H x_2(\mathbf{V}'_*,\mathbf{t})\ dx=2\mu\int_\Omega[\partial_2 p_H(-\partial_2 p+x_3)(\mathbf{V}'_*,\mathbf{t})$$

$$-\partial_3 p_H(\partial_3 p + x_2)(\mathbf{V}'_*, \mathbf{t})]\ dx + 2\mu \int_\Omega \partial_2 p_H \partial_2 p(\mathbf{V}'_*, \mathbf{t}) + \partial_3 p_H \partial_3 p(\mathbf{V}'_*, \mathbf{t})\ dx$$

$$= \int_\Omega (H_{12}\partial_2 p + H_{13}\partial_3 p)(\mathbf{V}'_*, \mathbf{t})\ dx, \tag{5.43}$$

and we deduce from (5.8) that

$$2\mu \int_\Omega \zeta_{12}(\mathbf{W}'_P, \mathbf{n}) + \zeta_{13}(\mathbf{W}'_P, \mathbf{b})\ dx = \int_\Omega H_{12}(\mathbf{W}'_P, \mathbf{n}) + H_{13}(\mathbf{W}'_P, \mathbf{b})\ dx. \tag{5.44}$$

The relations (5.29)–(5.34) enable us to express the second term from the right-hand side of the equation (5.37) as a sum of the integrals

$$\int_\Omega H_{11}\Upsilon_{11}(\mathbf{V}_*, \mathbf{W}'_P)\ dx = \int_\Omega -H_{11}(\mathbf{V}'_*, \mathbf{n})x_3 + H_{11}(\mathbf{V}'_*, \mathbf{b})x_2 + H_{11}(\mathbf{W}'_P, \mathbf{t})\ dx, \tag{5.45}$$

$$2\int_\Omega H_{12}\Upsilon_{12}(\mathbf{V}_*, \mathbf{W}'_P)\ dx = \int_\Omega H_{12}(\mathbf{V}'_*, \mathbf{t})x_3 + H_{12}(\mathbf{W}'_P, \mathbf{n})\ dx, \tag{5.46}$$

$$2\int_\Omega H_{13}\Upsilon_{13}(\mathbf{V}_*, \mathbf{W}'_P)\ dx = \int_\Omega -H_{13}(\mathbf{V}'_*, \mathbf{t})x_2 + H_{13}(\mathbf{W}'_P, \mathbf{b})\ dx. \tag{5.47}$$

Substituting (5.38)–(5.47) to (5.37) we obtain (5.25). □

Corollary 5.6 *Let the assumptions of Lemma* 5.5 *hold. Then*

$$EQ_0|S| + \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2 dx_3 = 0 \ \textit{in}\ (0, l). \tag{5.48}$$

P r o o f: If we put $\mathbf{V}_* = \mathbf{0}$ (i.e. $\mathbf{V} = \mathbf{0}$ and $\psi = 0$) as a test function in (5.25), we get that

$$\int_0^l [EQ_0|S| + (\int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2 dx_3)](\mathbf{W}', \mathbf{t})\ dx_1 = 0 \tag{5.49}$$

for all $\mathbf{W} \in H^1_b(0, l)^3$. If we put $\mathbf{W} = (W_1, 0, 0)$, $\mathbf{W} = (0, W_2, 0)$ and $\mathbf{W} = (0, 0, W_3)$, where $W_j \in H^1_{bb}(0, l)$, $j = 1, 2, 3$, we conclude that

$$[EQ_0|S| + (\int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2 dx_3)]t_i = C_{i+6} \text{ in } (0, l),\ i = 1, 2, 3. \tag{5.50}$$

But we can take also the functions W_j such that $W_j(l) = 1$, $j = 1, 2, 3$, because $\mathbf{W} \in H^1_b(0, l)^3$. Then the relations (5.49) and (5.50) give (5.48). □

Lemma 5.7 *The sequence* $\{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n})\}_{n=1}^\infty$ *from* (5.3) *converges strongly to* ζ *in* $L^2(\Omega)^9$ *for* $\epsilon_n \to 0$.

P r o o f: In the proof, we will write ϵ instead of ϵ_n to simplify the notation. Let us define

$$\Lambda_\epsilon = \int_\Omega A_\epsilon^{ijkl} \left(\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon) - \zeta_{kl}\right) \left(\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{U}_\epsilon) - \zeta_{ij}\right) d_\epsilon \; dx. \tag{5.51}$$

According to Proposition 3.5 and Remark 3.7, there exists a constant $C > 0$ independent of ϵ such that

$$\left\|\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) - \zeta\right\|^2_{L^2(\Omega)^9} \le C\Lambda_\epsilon. \tag{5.52}$$

Equation (4.6) implies that

$$\begin{aligned}\Lambda_\epsilon = &\int_\Omega (\mathbf{F}, \mathbf{U}_\epsilon) d_\epsilon \; dx + \int_0^l \int_{\partial S} (\mathbf{G}, \mathbf{U}_\epsilon) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \; dS dx_1 \\ &+ \int_S (\mathbf{K}, \mathbf{U}_\epsilon(l)) \; dx_2 dx_3 + \int_\Omega H_{ij} \frac{1}{\epsilon} \omega^\epsilon(\mathbf{U}_\epsilon) d_\epsilon \; dx \\ &+ \int_\Omega A_\epsilon^{ijkl} \left(\left(\zeta_{kl} - \frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon)\right)\zeta_{ij} - \zeta_{kl}\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{U}_\epsilon)\right) d_\epsilon \; dx.\end{aligned} \tag{5.53}$$

As a result of (3.6), (3.7), (5.2), (5.3) and (5.6), we obtain the convergence of the sequence Λ_ϵ, i.e.

$$\Lambda = \lim_{\epsilon \to 0} \Lambda_\epsilon = \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{U}) \; dx_1 + (\check{\mathbf{K}}, \mathbf{U}(l)) + \int_\Omega H_{ij}\zeta_{ij} \; dx - \int_\Omega A_0^{ijkl} \zeta_{kl}\zeta_{ij} \; dx. \tag{5.54}$$

In the same way as in [8] we can derive the identity

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\zeta_{ij} \; dx =$$

$$\int_\Omega [E\zeta_{11}^2 + 4\mu(\zeta_{12}^2 + \zeta_{13}^2) + \lambda(\widehat{\zeta}_{22} + \widehat{\zeta}_{33})^2 + 2\mu((\widehat{\zeta}_{22})^2 + (\widehat{\zeta}_{33})^2 + 2(\widehat{\zeta}_{23})^2)] \; dx. \tag{5.55}$$

The expressions for ζ_{11}, ζ_{12} and ζ_{13}, i.e (5.16) and (5.17), imply after their substitution to (5.55) that

$$\begin{aligned}\int_\Omega A_0^{ijkl}\zeta_{kl}\zeta_{ij} \; dx = &\int_\Omega [E\zeta_{11}^2 + 4\mu(\zeta_{12}^2 + \zeta_{13}^2) + \lambda(\widehat{\zeta}_{22} + \widehat{\zeta}_{33})^2 \\ +2\mu((\widehat{\zeta}_{22})^2 + (\widehat{\zeta}_{33})^2 + 2(\widehat{\zeta}_{23})^2)] \; dx &\overset{(5.16),(5.17)}{=} \int_\Omega [E(Q_0 + (\mathbf{U}_*', \mathbf{b})x_2 - (\mathbf{U}_*', \mathbf{n})x_3)^2 \\ +4\mu\big(-\frac{1}{2}(\mathbf{U}_*', \mathbf{t})(\partial_2 p - x_3) + \partial_2 p_H\big)^2 &+ 4\mu\big(-\frac{1}{2}(\mathbf{U}_*', \mathbf{t})(\partial_3 p + x_2) + \partial_3 p_H\big)^2 + \lambda(\widehat{\zeta}_{22} + \widehat{\zeta}_{33})^2 \\ +2\mu((\widehat{\zeta}_{22})^2 + (\widehat{\zeta}_{33})^2 + 2(\widehat{\zeta}_{23})^2)] \; dx &\overset{(2.1),(5.18)}{=}\end{aligned}$$

$$
\begin{aligned}
&\int_0^l [E(I_{x_2^2}(\mathbf{U}'_*, \mathbf{b})^2 + I_{x_3^3}(\mathbf{U}'_*, \mathbf{n})^2) + \mu K(\mathbf{U}'_*, \mathbf{t})^2]\ dx_1 \\
&+ \int_\Omega [EQ_0^2 + 4\mu(\partial_2 p_H)^2 + 4\mu(\partial_3 p_H)^2]\ dx + \int_\Omega [\lambda(\widehat{\zeta}_{22} + \widehat{\zeta}_{33})^2 \\
&+2\mu((\widehat{\zeta}_{22})^2 + (\widehat{\zeta}_{33})^2 + 2(\widehat{\zeta}_{23})^2)]\ dx \overset{(5.19),(5.25)\ \text{for}\ \mathbf{w}_P=\mathbf{0}}{=} \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{U})\ dx_1 + (\check{\mathbf{K}}, \mathbf{U}(l)) \\
&- \frac{\lambda}{\lambda + 2\mu} \int_\Omega (H_{22} + H_{33})(x_2(\mathbf{U}'_*, \mathbf{b}) - x_3(\mathbf{U}'_*, \mathbf{n}))\ dx \\
&+ \int_\Omega [H_{12}(\mathbf{U}'_*, \mathbf{t})(-\partial_2 p + x_3) - H_{13}(\mathbf{U}'_*, \mathbf{t})(\partial_3 p + x_2)]\ dx \\
&+ \int_\Omega H_{11}((\mathbf{U}'_*, \mathbf{b})x_2 - (\mathbf{U}'_*, \mathbf{n})x_3)\ dx + \int_0^l EQ_0^2|S|\ dx_1 \\
&+2 \int_\Omega H_{12}\partial_2 p_H + H_{13}\partial_3 p_H\ dx + \int_\Omega \lambda(\widehat{\zeta}_{22} + \widehat{\zeta}_{33})^2 + 2\mu((\widehat{\zeta}_{22})^2 + (\widehat{\zeta}_{33})^2 + 2(\widehat{\zeta}_{23})^2)\ dx \\
&\overset{(5.11),(5.16),(5.17)}{=} \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{U})\ dx_1 + (\check{\mathbf{K}}(l), \mathbf{U}(l)) \\
&+ \int_0^l Q_0(EQ_0|S| + \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2 dx_3)\ dx_1 + \int_\Omega H_{ij}\zeta_{ij}\ dx. \qquad (5.56)
\end{aligned}
$$

Using (5.48) we get, after the substitution of (5.56) to (5.54), that $\Lambda = 0$. $\square$

It remains to express the components ζ_{22}, ζ_{33}, ζ_{23} of the tensor ζ. To find their forms, we use the decomposition of the space $H^1(S)^2$ given by

$$H^1(S)^2 = rd_2(S) \oplus rd_2^\perp(S) \qquad (5.57)$$

([7]), where $rd_2(S)$ can be also defined by

$$rd_2(S) = \{\mathbf{v} = (v_2, v_3) \in H^1(S)^2;\ e_{ij}(\mathbf{v}) = 0;\ i, j = 2, 3\}, \qquad (5.58)$$

where $(e_{ij}(\mathbf{v}))_{i,j=2,3}$ means the symmetric part of the gradient of the function $\mathbf{v}$. It is easy to verify that $rd_2^\perp(S)$ is a nontrivial Hilbert space with the scalar product

$$((\mathbf{v}, \mathbf{w})) = \int_S e_{ij}(\mathbf{v})e_{ij}(\mathbf{w})\ dx_2 dx_3 \qquad (5.59)$$

and that the Korn inequality

$$\|\mathbf{v}\|_{1,2} \le C \sum_{i,j=2}^{3} \|e_{ij}(\mathbf{v})\|_{L^2(S)} \qquad (5.60)$$

holds for all $\mathbf{v} \in rd_2^{\perp}(S)$. Then the problem

$$\lambda \int_S (\partial_2 \widehat{p}_2^H + \partial_3 \widehat{p}_3^H)(\partial_2 v_2 + \partial_3 v_3)\ dx_2 dx_3 + 2\mu \int_S e_{ij}(\widehat{\mathbf{p}}^H) e_{ij}(\mathbf{v})\ dx_2 dx_3$$

$$= \int_S H_{ij} e_{ij}(\mathbf{v})\ dx_2 dx_3 \tag{5.61}$$

has a unique solution $\widehat{\mathbf{p}}^H \in L^2(0,l; rd_2^{\perp}(S))$ satisfying the estimate (see (5.60) and (5.61))

$$\|\widehat{\mathbf{p}}^H\|_{L^2(0,l;H^1(S)^2)} \leq C \sum_{i,j=2}^{3} \|e_{ij}(\widehat{\mathbf{p}}^H)\|_{L^2(0,l;L^2(S))} \leq C \sum_{i,j=2}^{3} \|H_{ij}\|_{L^2(0,l;L^2(S))}. \tag{5.62}$$

Analogously as we have derived the relations (5.9) and (5.10), we can check that

$$\int_S \frac{\partial_2 \widehat{p}_3^H + \partial_3 \widehat{p}_2^H}{2} = \frac{1}{2\mu} \int_S H_{23}, \ \int_S \frac{\partial_2 \widehat{p}_3^H + \partial_3 \widehat{p}_2^H}{2} x_2 = \frac{1}{2\mu} \int_S H_{23} x_2,$$

$$\int_S \frac{\partial_2 \widehat{p}_3^H + \partial_3 \widehat{p}_2^H}{2} x_3 = \frac{1}{2\mu} \int_S H_{23} x_3, \tag{5.63}$$

$$\int_S \partial_2 \widehat{p}_2^H + \partial_3 \widehat{p}_3^H = \frac{1}{\lambda + 2\mu} \int_S H_{22} + H_{33},$$

$$\int_S (\partial_2 \widehat{p}_2^H + \partial_3 \widehat{p}_3^H) x_2 = \frac{1}{\lambda + 2\mu} \int_S (H_{22} + H_{33}) x_2,$$

$$\int_S (\partial_2 \widehat{p}_2^H + \partial_3 \widehat{p}_3^H) x_3 = \frac{1}{\lambda + 2\mu} \int_S (H_{22} + H_{33}) x_3. \tag{5.64}$$

Lemma 5.8 *We have*

$$\widehat{\zeta}_{22} = \partial_2 \widehat{p}_2^H,\ \widehat{\zeta}_{33} = \partial_3 \widehat{p}_3^H,\ \widehat{\zeta}_{23} = \frac{\partial_2 \widehat{p}_3^H + \partial_3 \widehat{p}_2^H}{2}. \tag{5.65}$$

P r o o f: If we use the function $\mathbf{v} = (v_2, v_3)$, where $v_2 = (\mathbf{V}, \mathbf{n})$ and $v_3 = (\mathbf{V}, \mathbf{b})$ for $\mathbf{V} \in L^2(0,l; H^1(S)^3)$, as a test function in the equation (5.61), we get that the right-hand side in the equation (5.61) is nothing but the right-hand side in the equation (5.12). Subtracting (5.61), after the above substitution, from (5.12), we obtain that

$$\int_\Omega [\lambda(\zeta_{11} + \zeta_{22} - \partial_2 \widehat{p}_2^H + \zeta_{33} - \partial_3 \widehat{p}_3)((\partial_2 \mathbf{V}, \mathbf{n}) + (\partial_3 \mathbf{V}, \mathbf{b})) + 2\mu((\zeta_{22} - \partial_2 \widehat{p}_2^H)(\partial_2 \mathbf{V}, \mathbf{n})$$

$$+(\zeta_{33} - \partial_3 \widehat{p}_3^H)(\partial_3 \mathbf{V}, \mathbf{b})) + 2(\zeta_{23} - \frac{\partial_2 \widehat{p}_3^H + \partial_3 \widehat{p}_2^H}{2}) \frac{(\partial_2 \mathbf{V}, \mathbf{b}) + (\partial_3 \mathbf{V}, \mathbf{n})}{2}]\ dx = 0. \tag{5.66}$$

Further, we define the function

$$\mathbf{V}_{\mathbf{U}_\epsilon, \widehat{\mathbf{p}}^H} = [(\frac{\mathbf{U}_\epsilon}{\epsilon^2}, \mathbf{n}_\epsilon) - (\widehat{\mathbf{P}}^H, \mathbf{n})]\mathbf{n} + [(\frac{\mathbf{U}_\epsilon}{\epsilon^2}, \mathbf{b}_\epsilon) - (\widehat{\mathbf{P}}^H, \mathbf{b})]\mathbf{b}, \tag{5.67}$$

where

$$\mathbf{P}^H = \widehat{p}_2^H \mathbf{n} + \widehat{p}_3^H \mathbf{b}. \tag{5.68}$$

Using the estimate (5.62) and the fact that $\mathbf{U}_\epsilon \in V_b(\Omega)$, we can easily check that $\mathbf{V}_{\mathbf{U}_\epsilon, \widehat{\mathbf{p}}^H} \in L^2(0,l;H^1(S)^3)$ for all $\epsilon \in (0,1)$. After the substitution of the function $\mathbf{V}_{\mathbf{U}_\epsilon, \widehat{\mathbf{p}}^H}$ to the equality (5.66), we obtain analogously as in (5.13) and (5.14) that

$$\int_\Omega [\lambda(\zeta_{11} + \zeta_{22} - \partial_2 \widehat{p}_2^H + \zeta_{33} - \partial_3 \widehat{p}_3^H)(\zeta_{22} - \partial_2 \widehat{p}_2^H + \zeta_{33} - \partial_3 \widehat{p}_3^H)$$

$$+2\mu((\zeta_{22} - \partial_2 \widehat{p}_2^H)^2 + (\zeta_{33} - \partial_3 \widehat{p}_3^H)^2 + 2(\zeta_{23} - \frac{\partial_2 \widehat{p}_3^H + \partial_3 \widehat{p}_2^H}{2})^2)]\, dx = 0. \tag{5.69}$$

Using the analogous computation as in (5.15), we conclude that

$$\int_\Omega [\lambda(\widehat{\zeta}_{22} - \partial_2 \widehat{p}_2^H + \widehat{\zeta}_{33} - \partial_3 \widehat{p}_3^H)^2$$

$$+2\mu((\widehat{\zeta}_{22} - \partial_2 \widehat{p}_2^H)^2 + (\widehat{\zeta}_{33} - \partial_3 \widehat{p}_3^H)^2 + 2(\widehat{\zeta}_{23} - \frac{\partial_2 \widehat{p}_3^H + \partial_3 \widehat{p}_2^H}{2})^2)]\, dx$$

$$= \lambda \int_\Omega \zeta_{11}(\widehat{\zeta}_{22} + \widehat{\zeta}_{33} - \partial_2 \widehat{p}_2^H - \partial_3 \widehat{p}_3^H)\, dx \overset{(5.10),(5.16),(5.64)}{=} 0. \tag{5.70}$$

$\square$

Corollary 5.9 *The tensor ζ has the following form:*

$$\zeta_{11} \overset{(5.16),(5.48)}{=} (\mathbf{U}_*', \mathbf{b})x_2 - (\mathbf{U}_*', \mathbf{n})x_3$$

$$-\frac{1}{E|S|}\left(\int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\, dx_2 dx_3 \right), \tag{5.71}$$

$$\zeta_{12} \overset{(5.17)}{=} -\frac{1}{2}(\mathbf{U}_*', \mathbf{t})(\partial_2 p - x_3) + \partial_2 p_H, \tag{5.72}$$

$$\zeta_{13} \overset{(5.17)}{=} -\frac{1}{2}(\mathbf{U}_*', \mathbf{t})(\partial_3 p + x_2) + \partial_3 p_H, \tag{5.73}$$

$$\zeta_{22} \overset{(5.7),(5.16),(5.48),(5.65)}{=} \partial_2 \widehat{p}_2^H - \frac{1}{2}\frac{\lambda}{\lambda + \mu}\Bigg[(\mathbf{U}_*', \mathbf{b})x_2 - (\mathbf{U}_*', \mathbf{n})x_3$$

$$-\frac{1}{E|S|}\left(\int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\, dx_2 dx_3 \right)\Bigg], \tag{5.74}$$

$$\zeta_{33} \overset{(5.7),(5.16),(5.48),(5.65)}{=} \partial_3 \widehat{p}_3^H - \frac{1}{2}\frac{\lambda}{\lambda + \mu}\Bigg[(\mathbf{U}_*', \mathbf{b})x_2 - (\mathbf{U}_*', \mathbf{n})x_3$$

$$
-\frac{1}{E|S|}\left(\int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33})-H_{11}\ dx_2dx_3\right)\Bigg], \tag{5.75}
$$

$$
\zeta_{23} \overset{(5.7),(5.65)}{=} \frac{\partial_2\widehat{p}_3^H+\partial_3\widehat{p}_2^H}{2}, \tag{5.76}
$$

where the functions p, p_H and $\widehat{\mathbf{p}}^H$ are the unique solutions to the problems (5.18), (5.19)–(5.20) *and* (5.61), *respectively.*

Remark 5.10 In Lemma 5.5 for $\mathbf{W}_P=\mathbf{0}$, we have proved that the asymptotic one-dimensional model for the curved rods has the form

$$
\begin{aligned}
&\int_0^l [E(I_{x_2^2}(\mathbf{U}_*',\mathbf{b})(\mathbf{V}_*',\mathbf{b})+I_{x_3^2}(\mathbf{U}_*',\mathbf{n})(\mathbf{V}_*',\mathbf{n}))+\mu K(\mathbf{U}_*',\mathbf{t})(\mathbf{V}_*',\mathbf{t})]\ dx_1\\
&\quad=-\frac{\lambda}{\lambda+2\mu}\int_\Omega [(H_{22}+H_{33})(x_2(\mathbf{V}_*',\mathbf{b})-x_3(\mathbf{V}_*',\mathbf{n}))]\ dx\\
&\quad+\int_\Omega [H_{12}(\mathbf{V}_*',\mathbf{t})(-\partial_2 p+x_3)-H_{13}(\mathbf{V}_*',\mathbf{t})(\partial_3 p+x_2)]\ dx\\
&+\int_\Omega [H_{11}((\mathbf{V}_*',\mathbf{b})x_2-(\mathbf{V}_*',\mathbf{n})x_3)]\ dx+\int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}},\mathbf{V})\ dx_1+(\check{\mathbf{K}},\mathbf{V}(l)),
\end{aligned} \tag{5.77}
$$

for all functions $\mathbf{V}_*\in H_b^1(0,l)^3$ generated by any arbitrary couple $\langle\mathbf{V},\psi\rangle\in\mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (see (2.12)). We refer the reader to Proposition 8.7 in [8] for the proof of the uniqueness. Thus it is not necessary to pass to weak convergent subsequences in (5.2) and (5.3), which are actually strong convergent according to Proposition 3.6, Remark 3.7 and Lemma 5.7.

Now, we will concentrate our attention on the curved rods clamped on both bases.

Remark 5.11 In the case of the curved rods clamped on both bases, we can derive in the same way the assertions of Proposition 5.1, Corollary 5.2, 5.3 and Lemma 5.4, 5.5, 5.8, and thus the asymptotic one-dimensional model has the form (5.77) for $\check{\mathbf{K}}=0$, $\mathbf{V}_*\in H_{bb}^1(0,l)^3$ and $\langle\mathbf{V},\psi\rangle\in\mathcal{V}_{bb}^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. In what follows, we want to express the function Q_0 from the relation (5.16) and thus to find the form of the tensor ζ. We saw in the proof of Lemma 5.7 that this problem is connected via the identity (5.48) with the problem about the strong convergence of the tensors $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)$.

Lemma 5.12 *Let there exist no constants C_{10}, $C_{11}\in\mathbb{R}$ such that $t_2=C_{10}t_1$ and $t_3=C_{11}t_1$, where t_i, $i=1,2,3$, are the components of the tangent vector $\mathbf{t}$. Then*

$$
EQ_0|S|+\int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33})-H_{11}\ dx_2dx_3=0 \ \textit{in}\ (0,l) \tag{5.78}
$$

and

$$
\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)\to\zeta \ \textit{in}\ L^2(\Omega)^9\ \textit{for}\ \epsilon\to 0.
$$

Otherwise,

$$EQ_0|S|t_1 + t_1 \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3 = C_{12} \in [-I_l(H), I_l(H)] \quad (5.79)$$

on $(0, l)$, *where*

$$I_l(H) = \left| \frac{1}{l} \int_0^l t_1 \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3dx_1 \right|. \quad (5.80)$$

P r o o f: We start with the proof of the first part of the lemma. Analogously as in the proof of Corollary 5.6, we can check the relation (5.50). Assuming the contrary, we suppose that the function $EQ_0|S| + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3$ is not equal to zero on $(0, l)$ and without loss of generality we can suppose that $t_1 \neq 0$ on $(0, l)$ and thus the constant C_7 from (5.50) is not equal to zero. Thus the relation (5.50) enables us to express the components of the tangent vector $\mathbf{t}$ as

$$t_j = \frac{C_{j+6}}{EQ_0|S| + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3} \text{ on } (0, l),\ j = 1, 2, 3, \quad (5.81)$$

and thus

$$t_j = \frac{C_{j+6}C_7}{C_7EQ_0|S| + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3)} = \frac{C_{j+6}}{C_7}t_1,\ j = 2, 3, \quad (5.82)$$

which is a contradiction.

If there exist constants C_{10}, C_{11} such that $t_2 = C_{10}t_1$, $t_3 = C_{11}t_1$, we have only one identity, namely,

$$EQ_0|S|t_1 + t_1 \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3 = C_{12} \text{ on } (0, l). \quad (5.83)$$

We assume again the contrary, i.e. $|C_{12}| > I_l(H)$. Then

$$\begin{aligned} &\int_0^l Q_0(EQ_0|S| + \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3)dx_1 \\ &= \frac{1}{E|S|} \int_0^l \left(EQ_0|S| + \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3 \right)^2\ dx_1 \\ &\quad - \frac{1}{E|S|} \int_0^l (\int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3)(EQ_0|S| \\ &\quad + \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3) = (Z) \end{aligned}$$

(new notation). Using the identity $uv = (u\mathbf{t}, v\mathbf{t})$, the dependence of the functions t_2, t_3 on t_1, (5.83) and the assumption $|C_{12}| > I_l(H)$, we deduce that

$$(Z) = \frac{1}{E|S|}\big((1 + C_{10}^2 + C_{11}^2)C_{12}^2 l$$

$$-(1 + C_{10}^2 + C_{11}^2)C_{12}\int_0^l t_1 \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3dx_1\big) \geq 0. \tag{5.84}$$

Now, we can repeat the proof of Lemma 5.7 and from (5.52), (5.54), (5.56) and (5.84), it follows that

$$\int_0^l Q_0(EQ_0|S| + \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3)dx_1 = 0. \tag{5.85}$$

On the other hand, we get from (5.83) that

$$\int_0^l Q_0(EQ_0|S| + \int_S \frac{\lambda}{\lambda + 2\mu}(H_{22} + H_{33}) - H_{11}\ dx_2dx_3)dx_1$$

$$= (1 + C_{10}^2 + C_{11}^2)C_{12}\int_0^l Q_0 t_1\ dx_1, \tag{5.86}$$

which together with (5.85) imply that either $C_{12} = 0$ or $\int_0^l Q_0 t_1\ dx_1 = 0$. Then the identity (5.83) gives a contradiction. □

Remark 5.13 In the proof of Lemma 5.12, we could see that the straightforward way, which was possible in the proof of Corollary 5.6 for the curved rods clamped on the lower basis, does not provide in general an analogous expression for the function Q_0 as (5.48). Thus we infer that the form of the constant C_{12} in (5.83) depends on

1. the properties of the function Q_0, which can be also defined by the weak convergence
$$\int_S \frac{(\partial_1 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})}{\epsilon}\ dx_2dx_3 \rightharpoonup Q_0|S| \text{ in } L^2(0,l) \tag{5.87}$$
(see (2.1), (5.3) and (5.16));

2. the properties of the functions H_{ii}, $i = 1, 2, 3$;

3. the properties of the approximating local frames given by the vector functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$.

Hereafter, we concentrate our attention on two last cases, because it has no sense to suppose some properties of the functions Q_0 or $\int_S \frac{(\partial_1 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})}{\epsilon}\ dx_2dx_3$ in a general case.

Lemma 5.14 *1. Let the components t_i, n_i, b_i, $i = 1,2,3$, of the vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ be constant vectors and let the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ be their smooth approximations in $C^1([0,l])^3$ satisfying (3.2) and such that $\|\mathbf{t}_\epsilon - \mathbf{t}\|_{C([0,l])} \le C\epsilon^p$, $p > 1$. Then*

$$\int_0^l Q_0(E|S|Q_0 + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3)dx_1 = 0, \tag{5.88}$$

$$Q_0 = \frac{1}{E|S|}\left(\frac{t_1}{l}\int_0^l \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3dx_1 \right.$$
$$\left. - \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3\right) \text{ on } (0,l); \tag{5.89}$$

2. *Let there exist constants C_{10}, $C_{11} \in \mathbb{R}$ such that $t_2 = C_{10}t_1$ and $t_3 = C_{11}t_1$, and let*
$$\int_0^l t_1 \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3dx_1 = 0. \tag{5.90}$$
Then
$$EQ_0|S| + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3 = 0 \text{ on } (0,l). \tag{5.91}$$

In both cases
$$\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) \to \zeta \text{ in } L^2(\Omega)^9 \text{ for } \epsilon \to 0. \tag{5.92}$$

P r o o f:

1. The convergence in (5.87) together with (2.6) and the assumptions of this lemma imply that
$$\int_0^l Q_0|S|\ dx_1 = \int_0^l \int_S \zeta_{11}\ dx = \lim_{\epsilon\to 0}\int_\Omega \frac{(\partial_1 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})}{\epsilon}\ dx \overset{(2.2),(2.6)}{\sim}$$
$$\lim_{\epsilon\to 0}\int_\Omega \frac{(\partial_1 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)}{\epsilon}\ dx \sim \lim_{\epsilon\to 0}\int_0^l\int_S \frac{(\partial_1 \mathbf{U}_\epsilon, \mathbf{t})}{\epsilon}\ dx$$
$$= \lim_{\epsilon\to 0}\int_0^l \partial_1 \int_S \frac{(\mathbf{U}_\epsilon, \mathbf{t})}{\epsilon}\ dx = 0, \tag{5.93}$$
because $\mathbf{U}_\epsilon \in V_{bb}(\Omega)$ and thus $\int_S (\mathbf{U}_\epsilon, \mathbf{t})\ dx_2dx_3 \in H^1_{bb}(0,l)$ for all $\epsilon \in (0,1)$. The rest of the proof follows from (5.56) and (5.83).

2. Analogously as in (5.84) we can derive that
$$\int_0^l Q_0(EQ_0|S| + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3)dx_1 \ge 0, \tag{5.94}$$

which together with (5.56) imply the strong convergence of the functions $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)$ and provide the identity

$$\int_0^l Q_0(EQ_0|S| + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3)dx_1 = 0. \tag{5.95}$$

Hence we can conclude using (5.86) that either the constant C_{12} in (5.83) is equal to zero or $\int_0^l Q_0 t_1\ dx_1 = 0$. The rest follows from (5.83).

□

Lemma 5.15 *Let the functions* $\mathbf{t}_\epsilon$*,* $\mathbf{n}_\epsilon$*,* $\mathbf{b}_\epsilon$ *satisfy* (3.2)–(3.5) *and the condition* 2. *from Theorem* 1.2. *Then*

$$EQ_0|S| + \int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3 = 0 \ \textit{on}\ (0,l) \tag{5.96}$$

and the functions $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)$ *converge strongly to* ζ *in* $L^2(\Omega)^9$*.*

P r o o f: Let us take the function $\mathbf{V} = \epsilon\mathbf{W}$, $\mathbf{W} \in H^1_{bb}(0,l)^3$, as a test function in the equation (4.12). Since

$$\omega^\epsilon_{11}(\epsilon\mathbf{W}) = \epsilon(\mathbf{W}', \mathbf{g}_{1,\epsilon}),\ \omega^\epsilon_{12}(\epsilon\mathbf{W}) = \frac{\epsilon}{2}(\mathbf{W}', \mathbf{n}_\epsilon),\ \omega^\epsilon_{13}(\epsilon\mathbf{W}) = \frac{\epsilon}{2}(\mathbf{W}', \mathbf{b}_\epsilon), \tag{5.97}$$

$$\omega^\epsilon_{22}(\epsilon\mathbf{W}) = \omega^\epsilon_{33}(\epsilon\mathbf{W}) = \omega^\epsilon_{23}(\epsilon\mathbf{W}) = 0 \tag{5.98}$$

according to (4.8)–(4.11) (compare with (5.29)–(5.33)), we can rewrite the left-hand side of the equation (4.12) as

$$\int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)\frac{1}{\epsilon}\omega^\epsilon_{ij}(\epsilon\mathbf{W})d_\epsilon\ dx$$

$$= \int_0^l (I_1^{\mathbf{U}_\epsilon}(\mathbf{g}_{1,\epsilon}) + I_2^{\mathbf{U}_\epsilon}(\mathbf{n}_\epsilon) + I_3^{\mathbf{U}_\epsilon}(\mathbf{b}_\epsilon), \mathbf{W}')\ dx_1, \tag{5.99}$$

where

$$I_1^{\mathbf{U}_\epsilon}(\mathbf{g}_{1,\epsilon}) = \int_S (\lambda g^{11,\epsilon}g^{kl,\epsilon} + 2\mu g^{1k,\epsilon}g^{1l,\epsilon})\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)d_\epsilon\mathbf{g}_{1,\epsilon}\ dx_2dx_3, \tag{5.100}$$

$$I_2^{\mathbf{U}_\epsilon}(\mathbf{n}_\epsilon) = \int_S (\lambda g^{12,\epsilon}g^{kl,\epsilon} + \mu(g^{1k,\epsilon}g^{2l,\epsilon} + g^{2k,\epsilon}g^{1l,\epsilon}))\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)d_\epsilon\mathbf{n}_\epsilon\ dx_2dx_3, \tag{5.101}$$

$$I_3^{\mathbf{U}_\epsilon}(\mathbf{b}_\epsilon) = \int_S (\lambda g^{13,\epsilon}g^{kl,\epsilon} + \mu(g^{1k,\epsilon}g^{3l,\epsilon} + g^{3k,\epsilon}g^{1l,\epsilon}))\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)d_\epsilon\mathbf{b}_\epsilon\ dx_2dx_3. \tag{5.102}$$

We get analogously for the right-hand side of the equation (4.12) that

$$\epsilon\int_\Omega(\mathbf{F},\mathbf{W})d_\epsilon\ dx + \int_\Omega H_{ij}\frac{1}{\epsilon}\omega^\epsilon_{ij}(\epsilon\mathbf{W})d_\epsilon\ dx + \epsilon\int_{(0,l)}\int_{\partial S}(\mathbf{G},\mathbf{W})\epsilon d_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\ dSdx_1$$

$$= \epsilon \int_0^l (I_4^{\mathbf{F}+\mathbf{G}}, \mathbf{W})\ dx_1 + \int_0^l (I_1^{H_{11}}(\mathbf{g}_{1,\epsilon}) + I_2^{H_{12}}(\mathbf{n}_\epsilon) + I_3^{H_{13}}(\mathbf{b}_\epsilon), \mathbf{W}')\ dx_1, \tag{5.103}$$

where

$$I_4^{\mathbf{F}+\mathbf{G}} = \int_S \mathbf{F} d_\epsilon\ dx_2 dx_3 + \int_{\partial S} \mathbf{G} \epsilon d_\epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j}\ dS, \tag{5.104}$$

$$I_1^{H_{11}}(\mathbf{g}_{1,\epsilon}) = \int_S H_{11} \mathbf{g}_{1,\epsilon} d_\epsilon\ dx_2 dx_3,\ \ I_2^{H_{12}}(\mathbf{n}_\epsilon) = \int_S H_{12} \mathbf{n}_\epsilon d_\epsilon\ dx_2 dx_3, \tag{5.105}$$

$$I_3^{H_{13}}(\mathbf{b}_\epsilon) = \int_S H_{13} \mathbf{b}_\epsilon d_\epsilon\ dx_2 dx_3. \tag{5.106}$$

Further, we will use the notation $I_j^{\mathbf{U}_\epsilon}(w)$, $I_j^{H_{1i}}(w)$ and I_4^w, $i,j = 1,2,3$, if we have a function w instead of the functions $\mathbf{g}_{1,\epsilon}$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon$. Using (5.99) and (5.103), we can rewrite the equation (4.12) as

$$\int_0^l (I_1^{\mathbf{U}_\epsilon}(\mathbf{g}_{1,\epsilon}) - I_1^{H_{11}}(\mathbf{g}_{1,\epsilon}) + I_2^{\mathbf{U}_\epsilon}(\mathbf{n}_\epsilon) - I_2^{H_{12}}(\mathbf{n}_\epsilon) + I_3^{\mathbf{U}_\epsilon}(\mathbf{b}_\epsilon) - I_3^{H_{13}}(\mathbf{b}_\epsilon), \mathbf{W}')\ dx_1$$

$$= \epsilon \int_0^l (I_4^{\mathbf{F}+\mathbf{G}}, \mathbf{W})\ dx_1. \tag{5.107}$$

Hence we get that

$$\frac{d}{dx_1}\big(I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_i) - I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_i) + I_2^{\mathbf{U}_\epsilon}(n_{i,\epsilon}) - I_2^{H_{12}}(n_{i,\epsilon}) + I_3^{\mathbf{U}_\epsilon}(b_{i,\epsilon}) - I_3^{H_{13}}(b_{i,\epsilon})\big)$$

$$= \epsilon I_4^{F_i+G_i} \text{ in } (0,l) \tag{5.108}$$

for $i = 1,2,3$, and both terms belong to $L^2(0,l)$. After integration of (5.108) over an interval $[z_1, z_2] \subset (0,l)$, we obtain the equality

$$\begin{aligned}
&(I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_i) - I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_i) + I_2^{\mathbf{U}_\epsilon}(n_{i,\epsilon}) - I_2^{H_{12}}(n_{i,\epsilon}) + I_3^{\mathbf{U}_\epsilon}(b_{i,\epsilon}) - I_3^{H_{13}}(b_{i,\epsilon}))(z_1)\\
&-(I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_i) - I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_i) + I_2^{\mathbf{U}_\epsilon}(n_{i,\epsilon}) - I_2^{H_{12}}(n_{i,\epsilon}) + I_3^{\mathbf{U}_\epsilon}(b_{i,\epsilon}) - I_3^{H_{13}}(b_{i,\epsilon}))(z_2)\\
&= \epsilon \int_{z_2}^{z_1} I_4^{F_i+G_i}(x_1)\ dx_1,\ \ i = 1,2,3.
\end{aligned} \tag{5.109}$$

We can take $z_2 \in [\widehat{x}_1 - \epsilon^q, \widehat{x}_1 + \epsilon^q] = \bigcup_{\widehat{x}_2 \in [\widehat{x}_1 - \frac{\epsilon^q}{2}, \widehat{x}_1 + \frac{\epsilon^q}{2}]} [\widehat{x}_2 - \frac{\epsilon^q}{2}, \widehat{x}_2 + \frac{\epsilon^q}{2}]$ (see the condition 2. from Theorem 1.2) and integrate the equality (5.109) over the intervals $[\widehat{x}_j - \frac{\epsilon^q}{2}, \widehat{x}_j + \frac{\epsilon^q}{2}]$, $j = 1,2$. Then using the properties of the functions $t_{1,\epsilon}$, $n_{1,\epsilon}$, $b_{1,\epsilon}$ and (2.6) lead to the estimate

$$|(I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_1) - I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_1) + I_2^{\mathbf{U}_\epsilon}(n_{1,\epsilon}) - I_2^{H_{12}}(n_{1,\epsilon}) + I_3^{\mathbf{U}_\epsilon}(b_{1,\epsilon}) - I_3^{H_{13}}(b_{1,\epsilon}))(z_1)|$$

$$\leq |\frac{1}{\epsilon^q} \int_{\widehat{x}_1 - \frac{\epsilon^q}{2}}^{\widehat{x}_1 + \frac{\epsilon^q}{2}} \frac{1}{\epsilon^q} \int_{\widehat{x}_2 - \frac{\epsilon^q}{2}}^{\widehat{x}_2 + \frac{\epsilon^q}{2}} (I_1^{\mathbf{U}_\epsilon}(-\epsilon\gamma_\epsilon z_2) - I_1^{H_{11}}(-\epsilon\gamma_\epsilon z_2) + I_3^{\mathbf{U}_\epsilon}(1) - I_3^{H_{13}}(1))(z_2)\ dz_2 d\widehat{x}_2|$$

$$+|\frac{1}{\epsilon^q}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\frac{1}{\epsilon^q}\int_{\widehat{x}_2-\frac{\epsilon^q}{2}}^{\widehat{x}_2+\frac{\epsilon^q}{2}}\epsilon\int_{z_2}^{z_1} I_4^{F_1+G_1}(x_1)\ dx_1dz_2d\widehat{x}_2|, \tag{5.110}$$

for all $z_1\in(0,l)$, which implies that

$$\|I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_1)-I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_1)+I_2^{\mathbf{U}_\epsilon}(n_{1,\epsilon})-I_2^{H_{12}}(n_{1,\epsilon})+I_3^{\mathbf{U}_\epsilon}(b_{1,\epsilon})-I_3^{H_{13}}(b_{1,\epsilon})\|_{L^2(0,l)}\leq$$

$$C\Big(|\frac{1}{\epsilon^q}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\frac{1}{\epsilon^q}\int_{\widehat{x}_2-\frac{\epsilon^q}{2}}^{\widehat{x}_2+\frac{\epsilon^q}{2}}(I_1^{\mathbf{U}_\epsilon}(-\epsilon\gamma_\epsilon z_2)-I_1^{H_{11}}(-\epsilon\gamma_\epsilon z_2)+I_3^{\mathbf{U}_\epsilon}(1)-I_3^{H_{13}}(1))(z_2)\ dz_2d\widehat{x}_2|$$

$$+\epsilon(\|\mathbf{F}\|_{L^2(\Omega)^3}+\|\mathbf{G}\|_{L^2(0,l;L^2(\partial S)^3)})\Big). \tag{5.111}$$

Let us suppose for a moment that the convergence

$$|\frac{1}{\epsilon^{2q}}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\int_{\widehat{x}_2-\frac{\epsilon^q}{2}}^{\widehat{x}_2+\frac{\epsilon^q}{2}}[(I_1^{\mathbf{U}_\epsilon}(-\epsilon\gamma_\epsilon z_2)-I_1^{H_{11}}(-\epsilon\gamma_\epsilon z_2)$$

$$+I_3^{\mathbf{U}_\epsilon}(1)-I_3^{H_{13}}(1))(z_2)]\ dz_2d\widehat{x}_2\to 0 \tag{5.112}$$

holds for $\epsilon\to 0$. Then the estimate (5.111) yields

$$\|I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_1)-I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_1)+I_2^{\mathbf{U}_\epsilon}(n_{1,\epsilon})-I_2^{H_{12}}(n_{1,\epsilon})$$

$$+I_3^{\mathbf{U}_\epsilon}(b_{1,\epsilon})-I_3^{H_{13}}(b_{1,\epsilon})\|_{L^2(0,l)}\to 0 \tag{5.113}$$

for $\epsilon\to 0$. Further, we can derive from (2.7)–(2.8) and (3.4)–(3.5) that

$$\|g^{ii,\epsilon}\|_{L^\infty(0,l)}\sim 1+O(\epsilon^{2(1-r)}),\ i=1,2,3,\ \|g^{12,\epsilon}\|_{L^\infty(0,l)}\sim O(\epsilon^{1-r}), \tag{5.114}$$

$$\|g^{13,\epsilon}\|_{L^\infty(0,l)}\sim O(\epsilon^{1-r}),\ \|g^{23,\epsilon}\|_{L^\infty(0,l)}\sim O(\epsilon^{2(1-r)}). \tag{5.115}$$

Using the boundedness of the tensors $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)$ in $L^2(\Omega)^9$, (5.100)–(5.102), (5.104)–(5.106), (5.111) and (5.113)–(5.115) lead to the convergence

$$\|\int_S[\lambda g^{11,\epsilon}g^{kk,\epsilon}\frac{1}{\epsilon}\omega_{kk}^\epsilon(\mathbf{U}_\epsilon)[\mathbf{g}_{1,\epsilon}]_1+2\mu(g^{11,\epsilon}g^{11,\epsilon}\frac{1}{\epsilon}\omega_{11,\epsilon}(\mathbf{U}_\epsilon)[\mathbf{g}_{1,\epsilon}]_1$$

$$+g^{22,\epsilon}g^{11,\epsilon}\frac{1}{\epsilon}\omega_{12}^\epsilon(\mathbf{U}_\epsilon)n_{1,\epsilon}+g^{33,\epsilon}g^{11,\epsilon}\frac{1}{\epsilon}\omega_{13}^\epsilon(\mathbf{U}_\epsilon)b_{1,\epsilon})$$

$$-H_{11}[\mathbf{g}_{1,\epsilon}]_1-H_{12}n_{1,\epsilon}-H_{13}b_{1,\epsilon}]d_\epsilon\ dx_2dx_3\|_{L^2(0,l)}\to 0 \text{ for } \epsilon\to 0. \tag{5.116}$$

Furthermore, we know that

$$\int_S[2\mu g^{22,\epsilon}g^{11,\epsilon}\frac{1}{\epsilon}\omega_{12}^\epsilon(\mathbf{U}_\epsilon)-H_{12}]n_{1,\epsilon}d_\epsilon\ dx_2dx_3\rightharpoonup 0, \tag{5.117}$$

$$\int_S[2\mu g^{33,\epsilon}g^{11,\epsilon}\frac{1}{\epsilon}\omega_{13}^\epsilon(\mathbf{U}_\epsilon)-H_{13}]b_{1,\epsilon}d_\epsilon\ dx_2dx_3\rightharpoonup 0 \text{ in } L^2(0,l) \tag{5.118}$$

according to (3.6), (5.3) and (5.8), and

$$\int_S [\lambda g^{11,\epsilon} g^{kk,\epsilon} \frac{1}{\epsilon} \omega^\epsilon_{kk}(\mathbf{U}_\epsilon) + 2\mu g^{11,\epsilon} g^{11,\epsilon} \frac{1}{\epsilon} \omega_{11,\epsilon}(\mathbf{U}_\epsilon) - H_{11}][\mathbf{g}_{1,\epsilon}]_1 d_\epsilon \ dx_2 dx_3 \ \text{ in } \xrightarrow{L^2(0,l)}$$

$$\int_S [\lambda(\zeta_{11} + \zeta_{22} + \zeta_{33}) + 2\mu\zeta_{11} - H_{11}] t_1 \ dx_2 dx_3$$

$$\overset{(5.7),(5.24)}{=} \int_S [E\zeta_{11} + \lambda(\widehat{\zeta}_{22} + \widehat{\zeta}_{33}) - H_{11}] t_1 \ dx_2 dx_3 \overset{(2.1),(5.10),(5.16),(5.83)}{=} C_{12}. \tag{5.119}$$

Hence and from (5.116), we conclude that $C_{12} = 0$ and we can prove analogously as in the proof of Lemma 5.7 that the tensors $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)$ converge strongly to ζ in $L^2(\Omega)^9$.

It remains to prove (5.112). First, we detect the terms in the integral

$$|\frac{1}{\epsilon^{2q}} \int_{\widehat{x}_1 - \frac{\epsilon^q}{2}}^{\widehat{x}_1 + \frac{\epsilon^q}{2}} \int_{\widehat{x}_2 - \frac{\epsilon^q}{2}}^{\widehat{x}_2 + \frac{\epsilon^q}{2}} (I_1^{\mathbf{U}_\epsilon}(-\epsilon\gamma_\epsilon z_2) - I_1^{H_{11}}(-\epsilon\gamma_\epsilon z_2) + I_3^{\mathbf{U}_\epsilon}(1) - I_3^{H_{13}}(1))(z_2) \ dz_2 d\widehat{x}_2|,$$

which need not converge to zero. Using (5.100), (5.102), (5.114)–(5.115) and the boundedness of the tensors $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)$ in $L^2(\Omega)^9$, we can deduce, for instance, the estimate

$$|\frac{\epsilon^{1-r}}{\epsilon^{2q}} \int_{\widehat{x}_1 - \frac{\epsilon^q}{2}}^{\widehat{x}_1 + \frac{\epsilon^q}{2}} \int_{\widehat{x}_2 - \frac{\epsilon^q}{2}}^{\widehat{x}_2 + \frac{\epsilon^q}{2}} 2\mu \int_S g^{11,\epsilon} g^{33,\epsilon} \frac{1}{\epsilon} \omega^\epsilon_{13}(\mathbf{U}_\epsilon) d_\epsilon \ dx_2 dx_3 dz_2 d\widehat{x}_2|$$

$$\leq C| \sup_{\widehat{x}_2 \in [\widehat{x}_1 - \frac{\epsilon^q}{2}, \widehat{x}_1 + \frac{\epsilon^q}{2}]} \frac{\epsilon^{1-r}}{\epsilon^q} \int_{\widehat{x}_2 - \frac{\epsilon^q}{2}}^{\widehat{x}_2 + \frac{\epsilon^q}{2}} \|\frac{1}{\epsilon} \omega^\epsilon_{13}(\mathbf{U}_\epsilon)(z_2, \cdot, \cdot)\|_{L^2(S)^9} \ dz_2|$$

$$\leq C\epsilon^{1-r-\frac{q}{2}} \to 0 \tag{5.120}$$

for $\epsilon \to 0$, because $q \in (0, \frac{2}{3})$ and $r \in (0, \frac{1}{3})$. We can estimate analogously the other terms using (5.114)–(5.115) and we find that the only terms, which need not converge to zero, are contained in the integral

$$\frac{1}{\epsilon^{2q}} \int_{\widehat{x}_1 - \frac{\epsilon^q}{2}}^{\widehat{x}_1 + \frac{\epsilon^q}{2}} \int_{\widehat{x}_2 - \frac{\epsilon^q}{2}}^{\widehat{x}_2 + \frac{\epsilon^q}{2}} 2\mu \int_S [g^{11,\epsilon} g^{33,\epsilon} \frac{1}{\epsilon} \omega^\epsilon_{13}(\mathbf{U}_\epsilon) - H_{13}] d_\epsilon \ dx_2 dx_3 dz_2 d\widehat{x}_2. \tag{5.121}$$

Now, we show that this integral converges to zero as well. Multiplying the equation (4.12) by ϵ^2, we obtain the equation

$$\int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon} \omega^\epsilon_{kl}(\mathbf{U}_\epsilon) \epsilon \omega^\epsilon_{ij}(\mathbf{V}) d_\epsilon \ dx = \epsilon^2 \int_\Omega (\mathbf{F}, \mathbf{V}) d_\epsilon \ dx + \int_\Omega H_{ij} \epsilon \omega^\epsilon_{ij}(\mathbf{V}) d_\epsilon \ dx$$

$$+\epsilon^2 \int_{(0,l)} \int_{\partial S} (\mathbf{G}, \mathbf{V}) \epsilon d_\epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j} \ dS dx_1. \tag{5.122}$$

We put $\mathbf{V} = Wx_3\mathbf{t}_\epsilon$, where $W \in H^1_{bb}(0,l)$. From (2.2), (2.6), (4.8)–(4.11), it follows that

$$\epsilon\omega^\epsilon_{11}(\mathbf{V}) = \epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)x_3W' + \epsilon^2x_3^2\gamma_\epsilon\beta_\epsilon W - \epsilon^2 x_2x_3\alpha_\epsilon\gamma_\epsilon W, \tag{5.123}$$

$$\epsilon\omega^\epsilon_{12}(\mathbf{V}) = \frac{\epsilon}{2}\beta_\epsilon x_3 W,\ \ \epsilon\omega^\epsilon_{13}(\mathbf{V}) = \frac{1}{2}(1-\epsilon x_2\beta_\epsilon)W, \tag{5.124}$$

$$\epsilon\omega^\epsilon_{22}(\mathbf{V}) = \epsilon\omega^\epsilon_{23}(\mathbf{V}) = \epsilon\omega^\epsilon_{33}(\mathbf{V}) = 0. \tag{5.125}$$

After the substitution of (5.123)–(5.125) to (5.122), we get that

$$\int_\Omega A^{ijkl}_\epsilon \frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)\epsilon\omega^\epsilon_{ij}(\mathbf{V})d_\epsilon\ dx = \int_0^l I_5W' + I_6W\ dx_1, \tag{5.126}$$

where

$$I_5 = \int_S \epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)x_3(\lambda g^{11,\epsilon}g^{kl,\epsilon} + 2\mu g^{1k,\epsilon}g^{1l,\epsilon})\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)d_\epsilon\ dx_2dx_3, \tag{5.127}$$

$$\begin{aligned} I_6 = \int_S &(\epsilon^2x_3^2\gamma_\epsilon\beta_\epsilon - \epsilon^2x_2x_3\alpha_\epsilon\gamma_\epsilon)(\lambda g^{11,\epsilon}g^{kl,\epsilon} + 2\mu g^{1k,\epsilon}g^{1l,\epsilon})\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon) \\ &+\epsilon\beta_\epsilon x_3(\lambda g^{12,\epsilon}g^{kl,\epsilon} + \mu(g^{1k,\epsilon}g^{2l,\epsilon} + g^{2k,\epsilon}g^{1l,\epsilon}))\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon) \\ &+(1-\epsilon x_2\beta_\epsilon)(\lambda g^{13,\epsilon}g^{kl,\epsilon} + \mu(g^{1k,\epsilon}g^{3l,\epsilon} + g^{3k,\epsilon}g^{1l,\epsilon}))\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)d_\epsilon\ dx_2dx_3. \end{aligned} \tag{5.128}$$

The right-hand side of the equation (5.122) can be rewritten as

$$\begin{aligned} \epsilon^2\int_\Omega(\mathbf{F},\mathbf{V})d_\epsilon\ dx + \epsilon^2\int_{(0,l)}\int_{\partial S}(\mathbf{G},\mathbf{V})\epsilon d_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\ dSdx_1 + \int_\Omega H_{ij}\epsilon\omega^\epsilon_{ij}(\mathbf{V})d_\epsilon\ dx \\ = \int_0^l \epsilon^2 I_4^{x_3((\mathbf{F},\mathbf{t}_\epsilon)+(\mathbf{G},\mathbf{t}_\epsilon))}W + I_5^HW' + I_6^HW\ dx_1, \end{aligned} \tag{5.129}$$

where

$$I_5^H = \int_S H_{11}\epsilon(1-\epsilon x_2\beta_\epsilon - x_3\alpha_\epsilon)x_3d_\epsilon\ dx_2dx_3, \tag{5.130}$$

$$I_6^H = \int_S H_{11}(\epsilon^2x_3^2\gamma_\epsilon\beta_\epsilon - \epsilon^2x_2x_3\alpha_\epsilon\gamma_\epsilon) + H_{12}\epsilon\beta_\epsilon x_3 + H_{13}(1-\epsilon x_2\beta_\epsilon)d_\epsilon\ dx_2dx_3. \tag{5.131}$$

After the substitution of (5.126) and (5.129) to (5.122), we find the equation

$$\int_0^l (I_6 - I_6^H)W\ dx_1 = \epsilon^2\int_0^l I_4^{x_3((\mathbf{F},\mathbf{t}_\epsilon)+(\mathbf{G},\mathbf{t}_\epsilon))}W\ dx_1 - \int_0^l (I_5 - I_5^H)W'\ dx_1 \tag{5.132}$$

and thus

$$I_6 - I_6^H = \epsilon^2 I_4^{x_3((\mathbf{F},\mathbf{t}_\epsilon)+(\mathbf{G},\mathbf{t}_\epsilon))} - \frac{d}{dx_1}(I_5 - I_5^H)\ \text{on}\ (0,l), \tag{5.133}$$

where all terms belong obviously to $L^2(0,l)$. Applying the integrals $\frac{1}{\epsilon^{2q}}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\int_{\widehat{x}_2-\frac{\epsilon^q}{2}}^{\widehat{x}_2+\frac{\epsilon^q}{2}}$ to (5.133), we obtain the estimate

$$|\frac{1}{\epsilon^{2q}}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\int_{\widehat{x}_2-\frac{\epsilon^q}{2}}^{\widehat{x}_2+\frac{\epsilon^q}{2}}[I_6(\widehat{x}_2)-I_6^H(\widehat{x}_2)]\ d\widehat{x}_2|\le C(\epsilon^2(\|\mathbf{F}\|_{L^2(\Omega)^3}+\|\mathbf{G}\|_{L^2(0,l;L^2(\partial S)^3)})$$

$$+|\frac{1}{\epsilon^{2q}}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}[I_5(\widehat{x}_2+\frac{\epsilon^q}{2})-I_5^H(\widehat{x}_2+\frac{\epsilon^q}{2})-I_5(\widehat{x}_2-\frac{\epsilon^q}{2})+I_5^H(\widehat{x}_2-\frac{\epsilon^q}{2})]\ d\widehat{x}_2).\quad(5.134)$$

Using (5.128) and (5.131), we can deduce analogously as in the estimate (5.120) that the only terms, which need not converge to zero from the integral on the left-hand side of the estimate (5.134), are contained in the integral

$$\frac{1}{\epsilon^{2q}}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\int_{\widehat{x}_2-\frac{\epsilon^q}{2}}^{\widehat{x}_2+\frac{\epsilon^q}{2}}2\mu\int_S[g^{11,\epsilon}g^{33,\epsilon}\frac{1}{\epsilon}\omega_{13}^\epsilon(\mathbf{U}_\epsilon)-H_{13}]d_\epsilon\ dx_2dx_3dz_2d\widehat{x}_2.\quad(5.135)$$

Since the estimate

$$|\frac{1}{\epsilon^{2q}}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\int_S\epsilon g^{11,\epsilon}g^{11,\epsilon}x_3\frac{1}{\epsilon}\omega_{11}^\epsilon(\mathbf{U}_\epsilon(\widehat{x}_2+\frac{\epsilon^q}{2},x_2,x_3))d_\epsilon\ dx_2dx_3d\widehat{x}_2|$$

$$\le C\epsilon^{1-2q}\int_{\widehat{x}_1}^{\widehat{x}_1+\epsilon^q}\|\frac{1}{\epsilon}\omega_{11}^\epsilon(\mathbf{U}_\epsilon(z_1,\cdot,\cdot))\|_{L^2(S)^9}\ dz_1|\le C\epsilon^{1-\frac{3q}{2}}\to 0\quad(5.136)$$

holds because of $q\in(0,\frac{2}{3})$ and the other terms from the last integral from (5.134) satisfy analogous estimates, then from (5.134)–(5.136), it follows that

$$|\frac{1}{\epsilon^{2q}}\int_{\widehat{x}_1-\frac{\epsilon^q}{2}}^{\widehat{x}_1+\frac{\epsilon^q}{2}}\int_{\widehat{x}_2-\frac{\epsilon^q}{2}}^{\widehat{x}_2+\frac{\epsilon^q}{2}}2\mu\int_S[g^{11,\epsilon}g^{33,\epsilon}\frac{1}{\epsilon}\omega_{13}^\epsilon(\mathbf{U}_\epsilon)-H_{13}]d_\epsilon\ dx_2dx_3dz_2d\widehat{x}_2|\to 0\quad(5.137)$$

for $\epsilon\to 0$. □

Corollary 5.16 *From Lemma* 5.14-1 *and* 5.15., *it follows that the form of the function* Q_0 *depends on the choice of approximating local frames if the components of the tangent vector* $\mathbf{t}$ *are constant functions.*

Remark 5.17 The situation is simpler if we construct such approximating local frame that, for instance, $t_{2,\epsilon}=C_{13}t_{1,\epsilon}\ne 0$, $n_{2,\epsilon}=C_{13}n_{1,\epsilon}$, $b_{2,\epsilon}=C_{13}b_{1,\epsilon}$ on an interval I_ϵ for all $\epsilon\in(0,1)$, $|I_\epsilon|\to 0$ for $\epsilon\to 0$, where $C_{13}\ne C_{10}$ (see the condition 3. from Theorem 1.2). Subtracting (5.109) with $i=2$ from (5.109) with $i=1$ leads to the estimate

$$\|[I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_1)-I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_1)+I_2^{\mathbf{U}_\epsilon}(n_{1,\epsilon})-I_2^{H_{12}}(n_{1,\epsilon})+I_3^{\mathbf{U}_\epsilon}(b_{1,\epsilon})-I_3^{H_{13}}(b_{1,\epsilon})]$$
$$-C_{13}[I_1^{\mathbf{U}_\epsilon}([\mathbf{g}_{1,\epsilon}]_1)-I_1^{H_{11}}([\mathbf{g}_{1,\epsilon}]_1)+I_2^{\mathbf{U}_\epsilon}(n_{1,\epsilon})-I_2^{H_{12}}(n_{1,\epsilon})+I_3^{\mathbf{U}_\epsilon}(b_{1,\epsilon})-I_3^{H_{13}}(b_{1,\epsilon})]\|_{L^2(0,l)}$$
$$\le C\epsilon(\|\mathbf{F}\|_{L^2(\Omega)}+\|\mathbf{G}\|_{L^2(0,l;L^2(\partial S)^3)})\to 0\text{ for }\epsilon\to 0$$

which together with (5.50), (5.117)–(5.119) and the assumption $C_{13}\ne C_{10}$ imply (5.96).

6 Approximation of the stress tensor

In this section, we prove Corollary 1.3. We know from the previous section that the function $\mathbf{U}_*$ generated by the couple $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_b^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is the unique solution to the equation (5.77). Now, we seek for a suitable approximation $\widehat{\mathbf{U}}_\epsilon$ of the function $\mathbf{U}$ in the form

$$\widehat{\mathbf{U}}_\epsilon = \mathbf{U}_1^\epsilon + \epsilon \mathbf{U}_2^\epsilon + \epsilon^2 \mathbf{U}_3^\epsilon, \tag{6.1}$$

which satisfies

$$\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\omega^\epsilon(\widehat{\mathbf{U}}_\epsilon) \to 0 \text{ in } L^2(\Omega)^9. \tag{6.2}$$

Let the function $\mathbf{U}_1^\epsilon \in \mathcal{V}_b^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ be the approximation of the function $\mathbf{U}$ from Proposition 3.4 and Remark 3.7, which, in addition, satisfies

$$\|(\mathbf{U}_{1,*}^\epsilon)''\|_{L^2(0,l)^3} \sim O(\frac{1}{\epsilon^{2r}}),\ \|\phi_\epsilon''\|_{L^2(0,l)} \sim O(\frac{1}{\epsilon^{2r}}),\ r \in (0, \frac{1}{3}). \tag{6.3}$$

The verification of (6.3) is left to the reader, because it follows from Proposition 4.2 in [8] after a simple modification of the proof. Let, further,

$$\begin{aligned}\mathbf{U}_2^\epsilon(x_1,x_2,x_3) = -\Big(&((\mathbf{U}_1^\epsilon)'(x_1), \mathbf{n}_\epsilon(x_1))x_2 + ((\mathbf{U}_1^\epsilon)'(x_1), \mathbf{b}_\epsilon(x_1))x_3 \Big)\mathbf{t}_\epsilon(x_1) \\ &-x_3\phi_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) + x_2\phi_\epsilon(x_1)\mathbf{b}_\epsilon(x_1)\end{aligned} \tag{6.4}$$

for $(x_1,x_2,x_3) \in \Omega$. Analogously as in the proof of Lemma 8.4 in [8], we can derive that

$$\omega^\epsilon(\mathbf{U}_1^\epsilon + \epsilon \mathbf{U}_2^\epsilon) = \epsilon \Upsilon^\epsilon(\mathbf{U}_{1,*}^\epsilon) + B_\epsilon, \tag{6.5}$$

where

$$\Upsilon_{11}^\epsilon(\mathbf{U}_{1,*}^\epsilon) = -((\mathbf{U}_{1,*}^\epsilon)', \mathbf{n}_\epsilon)x_3 + ((\mathbf{U}_{1,*}^\epsilon)', \mathbf{b}_\epsilon)x_2, \tag{6.6}$$

$$\Upsilon_{12}^\epsilon(\mathbf{U}_{1,*}^\epsilon) = \Upsilon_{21}^\epsilon(\mathbf{U}_{1,*}^\epsilon) = \frac{x_3}{2}((\mathbf{U}_{1,*}^\epsilon)', \mathbf{t}_\epsilon), \tag{6.7}$$

$$\Upsilon_{13}^\epsilon(\mathbf{U}_{1,*}^\epsilon) = \Upsilon_{31}^\epsilon(\mathbf{U}_{1,*}^\epsilon) = -\frac{x_2}{2}((\mathbf{U}_{1,*}^\epsilon)', \mathbf{t}_\epsilon), \tag{6.8}$$

$$\Upsilon_{ij}^\epsilon(\mathbf{U}_{1,*}^\epsilon) = 0,\ i,j = 2,3, \tag{6.9}$$

and

$$\|B_\epsilon\|_2 = \|B_\epsilon^{11}\|_2 \le C\epsilon^{2(1-r)},\ r \in (0, \frac{1}{3}). \tag{6.10}$$

At the end we define the function $\mathbf{U}_2^\epsilon$ by

$$\mathbf{U}_2^\epsilon = (\mathbf{U}_2^\epsilon, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon + (\mathbf{U}_2^\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon + (\mathbf{U}_2^\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon, \tag{6.11}$$

where

$$(\mathbf{U}_2^\epsilon, \mathbf{t}_\epsilon) = -((\mathbf{U}_{1,*}^\epsilon)', \mathbf{t}_\epsilon)p + p_H, \tag{6.12}$$

$$(\mathbf{U}_2^\epsilon, \mathbf{n}_\epsilon) = \widehat{p}_2^H + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\Bigg[((\mathbf{U}_{1,*}^\epsilon)', \mathbf{b}_\epsilon)(\frac{x_3^2 - x_2^2}{2}) + ((\mathbf{U}_{1,*}^\epsilon)', \mathbf{n}_\epsilon)x_2x_3$$

$$-\frac{x_2}{E|S|}\left(\int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3\right)\Bigg], \tag{6.13}$$

$$(\mathbf{U}_2^\epsilon, \mathbf{b}_\epsilon) = \widehat{p}_3^H + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\Bigg[((\mathbf{U}_{1,*}^\epsilon)', \mathbf{n}_\epsilon)(\frac{x_3^2 - x_2^2}{2}) - ((\mathbf{U}_{1,*}^\epsilon)', \mathbf{b}_\epsilon)x_2x_3$$

$$-\frac{x_3}{E|S|}\left(\int_S \frac{\lambda}{\lambda+2\mu}(H_{22}+H_{33}) - H_{11}\ dx_2dx_3\right)\Bigg]. \tag{6.14}$$

After the substitution of $\widehat{\mathbf{U}}_\epsilon$ to $\frac{1}{\epsilon}\omega^\epsilon(\widehat{\mathbf{U}}_\epsilon)$ we can check (6.2) using Lemma 5.7, (5.71)-(5.76), (6.1) and (6.3)–(6.14). The same result is valid for the curved rods clamped on both basis (see Lemma 5.12, 5.14, 5.15).

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