

Equivalence Between Generalized Symplectic Pencil and Skew-Hermitian/Hermitian Pencil

Hongguo Xu*

Abstract

We introduce a transformation between the generalized symplectic pencils and the skew-Hermitian/Hermitian pencils. Under the transformation the regularity of the matrix pencils is preserved, and the equivalence relations about their eigenvalues and deflating subspaces are established. The eigenvalue problems of the generalized symplectic pencils and skew-Hermitian/Hermitian pencils are strongly related to the discrete-time and continuous-time robust control problems, respectively. With the transformation a simple connection between these two types of robust control problems is made. The connection may help to develop unified methods for solving the robust control problems.

Keywords Generalized symplectic pencil, skew-Hermitian/Hermitian pencil, eigen-structure, eigenvalue, deflating subspace, regularity, robust control.
AMS subject classification. 15A18, 15A22, 93B40, 93B36, 65F15.

1 Introduction

We investigate the relationship between the *generalized symplectic pencils* of the form

$$\lambda \mathcal{E}_D - \mathcal{A}_D = \lambda \begin{bmatrix} 0 & F \\ -G^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^* & D \end{bmatrix} \quad (1)$$

and the *skew-Hermitian/Hermitian pencils* of the form

$$\mu \mathcal{E}_C - \mathcal{A}_C = \lambda \begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}, \quad (2)$$

*Department of Mathematics, University of Kansas, Lawrence Kansas 66045, USA.
xu@math.ku.edu. Partially supported by *National Science Foundation grants 0314427*
and the *University of Kansas General Research Fund 2301717*

where $F, G, \tilde{F}, \tilde{G} \in \mathbb{C}^{n,m}$, $D, \tilde{D} \in \mathbb{C}^{m,m}$, and $D^* = D$, $\tilde{D}^* = \tilde{D}$. In this paper A^* stands for the conjugate transpose of A . Both pencils play the central role in optimal and robust control, see, e.g., [7, 10, 3, 14]. The generalized symplectic pencils are related to the discrete-time control problems and the skew-Hermitian/Hermitian pencils are related to the continuous-time control problems. For instance, the discrete-time linear quadratic optimal control problem

$$\begin{aligned} \min_{u_k} \frac{1}{2} \sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ \text{subject to} \quad Ex_{k+1} = Ax_k + Bu_k \quad x_0 = x^0, \end{aligned}$$

with $Q^* = Q$, $R^* = R$, is related to the eigenvalue problem of the generalized symplectic pencil

$$\lambda \left[\begin{array}{c|cc} 0 & E & 0 \\ \hline -A^* & 0 & 0 \\ -B^* & 0 & 0 \end{array} \right] - \left[\begin{array}{c|cc} 0 & A & B \\ \hline -E^* & Q & S \\ 0 & S^* & R \end{array} \right], \quad (3)$$

see, e.g., [9, 11, 7, 3]. The continuous-time linear quadratic optimal problem

$$\begin{aligned} \min_u \frac{1}{2} \int_0^{\infty} \begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \\ \text{subject to} \quad \tilde{E}\dot{x} = \tilde{A}x + \tilde{B}u \quad x(0) = x^0, \end{aligned}$$

with $\tilde{Q}^* = \tilde{Q}$, $\tilde{R}^* = \tilde{R}$, is related to the eigenvalue problem of the skew-Hermitian/Hermitian pencil

$$\mu \left[\begin{array}{c|cc} 0 & \tilde{E} & 0 \\ \hline -\tilde{E}^* & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] - \left[\begin{array}{c|cc} 0 & \tilde{A} & \tilde{B} \\ \hline \tilde{A}^* & \tilde{Q} & \tilde{S} \\ \tilde{B}^* & \tilde{S}^* & \tilde{R} \end{array} \right], \quad (4)$$

see, e.g., [4, 11, 7, 3].

It is well-known that the discrete-time and continuous-time robust control problems are closely related. For instance, a discrete-time control problem is usually obtained from its continuous-time counterpart by discretization. The eigen-structures of the generalized symplectic pencils and skew-Hermitian/Hermitian pencils are also closely related. Both of them have certain symmetry. For instance, the eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ are symmetric with respect to the unit circle, and the eigenvalues of $\mu\mathcal{E}_C - \mathcal{A}_C$ are symmetric with respect to the imaginary axis. These two sets of eigenvalues can

be simply connected with the Cayley transformation ([5, 7, 3]). For this reason many efforts have been made in order to find a simple relation between these two types of pencils so that the related robust control problems can be treated in a unified way. The Cayley transformation seems to be a natural choice. Unfortunately, if the Cayley transformation or its inverse transformation is applied directly to the pencils, the transformed pencils may lose the matrix structures as in (1) or (2). One way to remedy this is as follows. First, reduce the pencils to matrices by deflating the blocks associated with the eigenvalue infinity. Then use the Cayley transformation to connect the resulting matrices ([5, 7, 3, 8]). However, new problems arise from this approach. Firstly, it is not always possible to do the deflation. Secondly, even if it is possible, matrix inversion is usually involved. This makes the transformed matrices complicated. More seriously, this may cause stability problem for numerical computations.

In this paper we introduce a simple one-to-one transformation

$$\mu\mathcal{E}_C - \mathcal{A}_C = \mathcal{T}(\lambda\mathcal{E}_D - \mathcal{A}_D)$$

with

$$\tilde{F} = G + F, \quad \tilde{G} = G - F, \quad \tilde{D} = D.$$

The transformation makes a direct connection between the generalized symplectic pencils and the skew-Hermitian/Hermitian pencils. More importantly, we will show that under this transformation the eigenvalues of the pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\mu\mathcal{E}_C - \mathcal{A}_C$ are connected basically in the same way as under the Cayley transformation, and the corresponding deflating subspaces are also simply connected. Consequently, under the transformation \mathcal{T} , the generalized symplectic pencils and the skew-Hermitian/Hermitian pencils and their eigen-structures can be treated equally. Therefore, an equivalence relation between the underlying robust control problems can be established.

The paper is organized as follows. In Section 2 we provide some basic definitions and properties about the eigen-structure of a general pencil. In Section 3 we give some properties about the eigen-structures of the pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\mu\mathcal{E}_C - \mathcal{A}_C$. We also give some well-known properties about the Cayley transformation in this section. In Section 4 we give our main results. We provide the precise relations about the eigenvalues and the deflating subspaces of the pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\mu\mathcal{E}_C - \mathcal{A}_C$ under the transformation \mathcal{T} . Conclusions are given in Section 5.

2 Preliminaries

In this section we introduce some basic concepts and properties about the eigen-structure of a general matrix pencil.

Definition 1 Consider a pencil $\lambda E - A \in \mathbb{C}^{n,m}$. A scalar $\lambda_0 \in \mathbb{C}$ is a finite eigenvalue of $\lambda E - A$ if there exists a nonzero vector $x \in \mathbb{C}^m$ such that

$$\lambda_0 E x = A x, \quad E x \neq 0.$$

$\lambda E - A$ has the eigenvalue infinity if there exists a nonzero vector $x \in \mathbb{C}^m$ such that

$$E x = 0, \quad A x \neq 0.$$

Definition 2 A pencil $\lambda E - A \in \mathbb{C}^{n,m}$ is regular if E, A are square ($n = m$), and $\det(cE - A) \neq 0$ for some $c \in \mathbb{C}$.

Proposition 3 Suppose that $\lambda E - A$ is regular. Then λ_0 is a finite eigenvalue of $\lambda E - A$ if and only if $\det(\lambda_0 E - A) = 0$. ∞ is an eigenvalue of $\lambda E - A$ if and only if $\det E = 0$.

Proof. Omitted. \square

The set of the eigenvalues of pencil $\lambda E - A$ (including the eigenvalue infinity) is denoted by $\Lambda(E, A)$. When $E = I$, it is simply denoted by $\Lambda(A)$.

Definition 4 Suppose $\lambda E - A \in \mathbb{C}^{n,n}$ is regular.

1. A matrix $U \in \mathbb{C}^{n,r}$ spans the right deflating subspace of $\lambda E - A$ corresponding to a finite eigenvalue λ_0 , if r is the algebraic multiplicity of λ_0 and there exists a matrix $T \in \mathbb{C}^{r,r}$ such that $\Lambda(T) = \{\lambda_0\}$ and

$$E U T = A U, \quad \text{rank } E U = r.$$

A matrix $V \in \mathbb{C}^{n,r}$ spans the left deflating subspace of $\lambda E - A$ corresponding to a finite eigenvalue λ_0 , if r is the algebraic multiplicity of λ_0 and there exists a matrix $S \in \mathbb{C}^{r,r}$ such that $\Lambda(S^*) = \{\lambda_0\}$ and

$$E^* V S = A^* V, \quad \text{rank } E^* V = r.$$

2. A matrix $U \in \mathbb{C}^{n,r}$ spans the right deflating subspace of $\lambda E - A$ corresponding to the eigenvalue ∞ , if r is the algebraic multiplicity of ∞ and there exists a nilpotent matrix $T \in \mathbb{C}^{r,r}$ such that

$$E U = A U T, \quad \text{rank } A U = r.$$

A matrix $V \in \mathbb{C}^{n,r}$ spans the left deflating subspace of $\lambda E - A$ corresponding to the eigenvalue ∞ , if r is the algebraic multiplicity of ∞ and there exists a nilpotent matrix $S \in \mathbb{C}^{r,r}$ such that

$$E^*V = A^*VS, \quad \text{rank } A^*V = r.$$

When $\lambda E - A$ is regular, it is well-known that the right and left deflating subspaces corresponding to a single eigenvalue λ_0 (including ∞) are unique, we denote them by \mathcal{R}_{λ_0} and \mathcal{L}_{λ_0} , respectively.

Proposition 5 *Suppose $\lambda E - A \in \mathbb{C}^{n,n}$ is regular.*

- (a) *The matrices $U, V \in \mathbb{C}^{n,r}$ span the right and left deflating subspaces of $\lambda E - A$, respectively, corresponding to a finite eigenvalue λ_0 , if and only if r is the algebraic multiplicity of λ_0 and*

$$EUT = AU, \quad E^*VS = A^*V, \quad \det V^*EU \neq 0$$

for some matrices $T, S \in \mathbb{C}^{r,r}$ with $\Lambda(T) = \Lambda(S^) = \{\lambda_0\}$.*

- (b) *The matrices $U, V \in \mathbb{C}^{n,r}$ span the right and left deflating subspaces of $\lambda E - A$, respectively, corresponding to the eigenvalue ∞ , if and only if r is the algebraic multiplicity of ∞ and*

$$EU = AUT, \quad E^*V = A^*VS, \quad \det V^*AU \neq 0$$

for some nilpotent matrices $T, S \in \mathbb{C}^{r,r}$.

Proof. The results are simply from the Weierstraß form of the regular pencil $\lambda E - A$ ([13]). \square

3 Eigen-structures of generalized symplectic pencil and skew-Hermitian/Hermitian pencil, and Cayley transformation

In this paper we always assume that the pencils (1) and (2) are regular. So we first give some necessary conditions about the regularity.

Proposition 6 *If the generalized symplectic pencil $\lambda \mathcal{E}_D - \mathcal{A}_D$ defined in (1) is regular, then*

$$m - \text{rank } D \leq n \leq m.$$

If the skew-Hermitian/Hermitian pencil $\mu \mathcal{E}_C - \mathcal{A}_C$ defined in (2) is regular, then

$$m - \text{rank } \tilde{D} \leq n \leq m.$$

Proof. If $\lambda\mathcal{E}_D - \mathcal{A}_D$ is regular, then there is a complex number c such that

$$\det(c\mathcal{E}_D - \mathcal{A}_D) = \det \begin{bmatrix} 0 & cF - G \\ F^* - cG^* & -D \end{bmatrix} \neq 0.$$

The blocks $cF - G, F - \bar{c}G \in \mathbb{C}^{n,m}$ must have full row rank, since otherwise $\det(c\mathcal{E}_D - \mathcal{A}_D) = 0$. So we have $n \leq m$.

Suppose $p = \text{rank } D$. Let $D = Q \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} Q^*$ be the Schur form, where $\Sigma \in \mathbb{C}^{p,p}$ is diagonal and nonsingular. Denote

$$FQ = n \begin{bmatrix} p & m-p \\ F_1 & F_2 \end{bmatrix}, \quad GQ = n \begin{bmatrix} p & m-p \\ G_1 & G_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix}^* (c\mathcal{E}_D - \mathcal{A}_D) \begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & cF_1 - G_1 & cF_2 - G_2 \\ F_1^* - cG_1^* & -\Sigma & 0 \\ F_2^* - cG_2^* & 0 & 0 \end{bmatrix}$$

Similarly, $cF_2 - G_2$ and $F_2 - \bar{c}G_2$ must have full column rank, since otherwise $\det(c\mathcal{E}_D - \mathcal{A}_D) = 0$. So $m - p \leq n$.

The second part can be proved in the same way. \square

Next, we will give some properties about the eigenvalues and deflating subspaces of a general skew-Hermitian/Hermitian pencil. The results are well-known.

Proposition 7 *Consider the regular skew-Hermitian/Hermitian pencil $\mu\mathcal{E} - \mathcal{A} \in \mathbb{C}^{n,n}$ with $\mathcal{E} = -\mathcal{E}^*$ and $\mathcal{A} = \mathcal{A}^*$. We have the following properties.*

- (a) μ_0 is an eigenvalue of $\mu\mathcal{E} - \mathcal{A}$ if and only if $-\bar{\mu}_0$ is also an eigenvalue of $\mu\mathcal{E} - \mathcal{A}$. Moreover, μ_0 and $-\bar{\mu}_0$ has the same algebraic and geometric multiplicities.
- (b) The matrices U and V span \mathcal{R}_{μ_0} and \mathcal{L}_{μ_0} , respectively, corresponding the eigenvalue μ_0 (including ∞), if and only if U and V also span $\mathcal{L}_{-\bar{\mu}_0}$ and $\mathcal{R}_{-\bar{\mu}_0}$, respectively.

Proof. The results are simply from the structured Kronecker canonical form, see, e.g., [2, 1, 6]. \square

We will also show the properties about the eigen-structure of the generalized symplectic pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$. But first let us give the following lemma that will be used later.

Lemma 8 *Suppose that the pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ defined in (1) is regular, and λ_0 is a finite eigenvalue of $\lambda\mathcal{E}_D - \mathcal{A}_D$ with algebraic multiplicity r .*

If λ_0 is also an eigenvalue of the pencil $\lambda G^ - F^*$ with algebraic multiplicity r_1 , let the matrix $U_{11} \in \mathbb{C}^{n,r_1}$ satisfy*

$$G^*U_{11}T_{11} = F^*U_{11}, \quad \text{rank } G^*U_{11} = r_1$$

for some $T_{11} \in \mathbb{C}^{r_1,r_1}$ with $\Lambda(T_{11}) = \{\lambda_0\}$. Then there exists a matrix

$$U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \in \mathbb{C}^{n+m,r}$$

such that

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad \mathcal{E}_D U = r,$$

where $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$ and $\Lambda(T) = \{\lambda_0\}$, i.e., U spans \mathcal{R}_{λ_0} of $\lambda\mathcal{E}_D - \mathcal{A}_D$.

If λ_0 is also an eigenvalue of the pencil $\lambda F - G$ with algebraic multiplicity r_2 , let the matrix $V_{11} \in \mathbb{C}^{n,r_2}$ satisfy

$$F^*V_{11}S_{11} = G^*V_{11}, \quad \text{rank } F^*V_{11} = r_2$$

for some $S_{11} \in \mathbb{C}^{r_2,r_2}$ with $\Lambda(S_{11}^) = \{\lambda_0\}$. Then there exists a matrix*

$$V = \begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix} \in \mathbb{C}^{n+m,r}$$

such that

$$\mathcal{E}_D^* V S = \mathcal{A}_D^* V, \quad \text{rank } \mathcal{E}_D^* V = r,$$

where $S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$ and $\Lambda(S^) = \{\lambda_0\}$, i.e., V spans \mathcal{L}_{λ_0} of $\lambda\mathcal{E}_D - \mathcal{A}_D$.*

Proof. Since $\lambda\mathcal{E}_D - \mathcal{A}_D$ is regular, $\det(c\mathcal{E}_D - \mathcal{A}_D) \neq 0$ for some scalar c . From the block structure $cG^* - F^*$ must have full column rank. That implies that the sub-pencil $\lambda G^* - F^*$ doesn't have any right singular blocks. If λ_0 is an eigenvalue of $\lambda G^* - F^*$, its right deflating subspace is unique ([12]). So the matrix U_{11} is well-defined. Clearly the matrix U_{11} also satisfies

$$\mathcal{E}_D \begin{bmatrix} U_{11} \\ 0 \end{bmatrix} T_{11} = \mathcal{A}_D \begin{bmatrix} U_{11} \\ 0 \end{bmatrix}, \quad \text{rank } \mathcal{E}_D \begin{bmatrix} U_{11} \\ 0 \end{bmatrix} = \text{rank } G^*U_{11} = r_1,$$

i.e., $\begin{bmatrix} U_{11} \\ 0 \end{bmatrix}$ spans a subspace of \mathcal{R}_{λ_0} of $\lambda\mathcal{E}_D - \mathcal{A}_D$. Therefore, $\begin{bmatrix} U_{11} \\ 0 \end{bmatrix}$ can be extended to the matrix U as required.

Note that the pencil $\lambda\mathcal{E}_D^* - \mathcal{A}_D^*$ has the same block structure as $\lambda\mathcal{E}_D - \mathcal{A}_D$. So the second part can be proved in the same way. \square

The eigen-structure of $\lambda\mathcal{E}_D - \mathcal{A}_D$ is relatively complicated. But as one can see from the following theorem, it has also some interesting symmetry. The results given below are essentially the same as those in [7, Proposition 4.18]. The only improvement is the relation about the eigenvalues 0 and ∞ .

Theorem 9 *Suppose that the pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ defined in (1) is regular.*

(a) *Let λ_0 be a nonzero finite eigenvalue of $\lambda\mathcal{E}_D - \mathcal{A}_D$ with algebraic multiplicity r . Then the matrices*

$$U = \begin{matrix} & r \\ n & \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ m & \end{matrix}, \quad V = \begin{matrix} & r \\ n & \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ m & \end{matrix} \in \mathbb{C}^{n+m,r},$$

satisfy

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad \mathcal{E}_D^* V S = \mathcal{A}_D^* V, \quad \det V^* \mathcal{E}_D U \neq 0$$

for some $T, S \in \mathbb{C}^{r,r}$ and $\Lambda(T) = \Lambda(S^) = \{\lambda_0\}$, if and only if*

$$\hat{U} = \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix}, \quad \hat{V} = \begin{bmatrix} V_1 S \\ V_2 \end{bmatrix}$$

satisfies

$$\mathcal{E}_D^* \hat{U} T^{-1} = \mathcal{A}_D^* \hat{U}, \quad \mathcal{E}_D \hat{V} S^{-1} = \mathcal{A}_D \hat{V}, \quad \det \hat{U}^* \mathcal{E}_D \hat{V} \neq 0.$$

Consequently, both $\lambda_0, \bar{\lambda}_0^{-1}$ are the eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ with the same geometric and algebraic multiplicities. Moreover, U, V span \mathcal{R}_{λ_0} and \mathcal{L}_{λ_0} , respectively, if and only if \hat{U}, \hat{V} span $\mathcal{L}_{\bar{\lambda}_0^{-1}}$ and $\mathcal{R}_{\bar{\lambda}_0^{-1}}$, respectively.

(b) *Suppose $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. Let*

$$U = \begin{matrix} & r_1 & r - r_1 \\ n & \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \\ m & \end{matrix}, \quad V = \begin{matrix} & r_2 & r - r_2 \\ n & \begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix} \\ m & \end{matrix} \in \mathbb{C}^{n+m,r}$$

be the matrices that span the right and left deflating subspaces \mathcal{R}_0 and \mathcal{L}_0 , respectively, generated as in Lemma 8 with $\lambda_0 = 0$, and satisfy

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad \mathcal{E}_D^* V S = \mathcal{A}_D^* V, \quad \text{rank } \mathcal{E}_D U = \text{rank } \mathcal{E}_D^* V = r$$

with the nilpotent matrices

$$T = \begin{matrix} & r_1 & r - r_1 \\ r_1 & \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \\ r - r_1 & \end{matrix}, \quad S = \begin{matrix} & r_2 & r - r_2 \\ r_2 & \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \\ r - r_2 & \end{matrix}.$$

Let

$$\hat{U} = \begin{bmatrix} U_{11} & U_{11}T_{12} + U_{12}T_{22} \\ 0 & U_{22} \end{bmatrix}, \quad \hat{V} = \begin{bmatrix} V_{11} & V_{11}S_{12} + V_{12}S_{22} \\ 0 & V_{22} \end{bmatrix}$$

and

$$\hat{T} = \begin{bmatrix} T_{11} & T_{11}T_{12} \\ 0 & T_{22} \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} S_{11} & S_{11}S_{12} \\ 0 & S_{22} \end{bmatrix}.$$

Then

$$\mathcal{E}_D^* \hat{U} = \mathcal{A}_D^* \hat{U} \hat{T}, \quad \mathcal{E}_D \hat{V} = \mathcal{A}_D \hat{V} \hat{S}, \quad \text{rank } \mathcal{A}_D^* \hat{U} = \text{rank } \mathcal{A}_D \hat{V} = r.$$

Consequently, if $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and has algebraic multiplicity r , then $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and has algebraic multiplicity at least r . Moreover, if U, V defined above span \mathcal{R}_0 and \mathcal{L}_0 , respectively, then \hat{U}, \hat{V} span a subspace of \mathcal{L}_∞ and \mathcal{R}_∞ , respectively.

Proof. The proof is given in Appendix A. \square

The above theorem shows that the nonzero finite eigenvalues of the generalized symplectic pencil $\lambda \mathcal{E}_D - \mathcal{A}_D$ are paired as $(\lambda_0, \bar{\lambda}_0^{-1})$. Their deflating subspaces are also simply related, though it is not as simple as in the skew-Hermitian/Hermitian pencil case. When λ_0 is on the unit circle, i.e., $\lambda_0 = \bar{\lambda}_0^{-1}$, the matrix V can be chosen as $\begin{bmatrix} U_1^T \\ U_2 \end{bmatrix}$, and the matrix U can be chosen as $\begin{bmatrix} V_1 S \\ V_2 \end{bmatrix}$. So in this case one basis matrix U or V can determine both the deflating subspaces \mathcal{R}_{λ_0} and \mathcal{L}_{λ_0} . The eigenvalue 0 corresponds to the eigenvalue ∞ . But usually $\lambda \mathcal{E}_D - \mathcal{A}_D$ may have more eigenvalues ∞ than 0.

Example 1 Consider the generalized symplectic pencil

$$\lambda \mathcal{E}_D - \mathcal{A}_D = \lambda \left[\begin{array}{c|cc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right] - \left[\begin{array}{c|cc} 0 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right].$$

Simple calculations yield

$$\mathcal{E}_D U_1 \cdot 0 = \mathcal{A}_D U_1 \quad \mathcal{E}_D^* V_1 \cdot 0 = \mathcal{A}_D^* V_1, \quad U_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};$$

and

$$\begin{aligned}\mathcal{E}_D U_2 &= \mathcal{A}_D U_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & U_2 &= \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \mathcal{E}_D^* V_2 &= \mathcal{A}_D^* V_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & V_2 &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.\end{aligned}$$

Clearly $\lambda \mathcal{E}_D - \mathcal{A}_D$ has the eigenvalues 0 and ∞ with algebraic multiplicities 1 and 2, respectively. Based on Theorem 9 (b), from U_1, V_1 we get

$$\hat{U} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \hat{V} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which are just the first column of V_2 and U_2 , respectively. Note that in this example, $r_1 = r_2 = 0$.

In the end of this section we give some well-known properties about the Cayley transformation. The Cayley transformation considered in this paper is the one-to-one transformation (if we consider ∞ as a "number") in the complex plane defined by

$$\mu = \rho(\lambda) = (\lambda - 1)(\lambda + 1)^{-1}.$$

Its inverse transformation is

$$\lambda = \rho^{-1}(\mu) = (1 + \mu)(1 - \mu)^{-1}.$$

The correspondence between λ and μ under the Cayley transformation is summarized in Table 1.

λ	∞	-1	0	1	$ \lambda < 1$	$ \lambda = 1$	$ \lambda > 1$
μ	1	∞	-1	0	$\operatorname{Re} \mu < 0$	$\operatorname{Re} \mu = 0$	$\operatorname{Re} \mu > 0$

Table 1: Correspondence between λ and $\mu = \rho(\lambda)$

The following property is obvious.

Proposition 10 *If $\mu = \rho(\lambda)$ then $-\bar{\mu} = \rho(\bar{\lambda}^{-1})$. Conversely, if $\lambda = \rho^{-1}(\mu)$ then $\bar{\lambda}^{-1} = \rho^{-1}(-\bar{\mu})$.*

Proof. the proof is trivial. \square

The matrix version of the Cayley transformation is

$$\rho(A) = (A - I)(A + I)^{-1}.$$

The matrix pencil version is

$$\mu F - B := \rho(\lambda E - A) = \mu(A + E) - (A - E).$$

The Cayley transformation preserves the regularity of the matrices pencils. Also, when $\lambda E - A$ is transformed to $\mu F - B$, the eigenvalues are transformed according to the rule given in Table 1, while the deflating subspaces remain unchanged.

Proposition 11 *Let $\lambda E - A \in \mathbb{C}^{n,n}$ and $\mu F - B = \mu(A + E) - (A - E)$.*

- (a) *$\lambda E - A$ is regular if and only if $\mu F - B$ is regular.*
- (b) *Suppose $\lambda E - A$ is regular. λ_0 is an eigenvalue (including ∞) of $\lambda E - A$ if and only if $\mu_0 = \rho(\lambda_0)$ is an eigenvalue of $\mu F - B$. Let \mathcal{R}_{λ_0} , \mathcal{L}_{λ_0} be the right and left deflating subspaces of $\lambda E - A$ corresponding to λ_0 , and let \mathcal{R}_{μ_0} , \mathcal{L}_{μ_0} be the right and left deflating subspaces of $\mu F - B$ corresponding to μ_0 . Then $\mathcal{R}_{\lambda_0} = \mathcal{R}_{\mu_0}$ and $\mathcal{L}_{\lambda_0} = \mathcal{L}_{\mu_0}$.*

Proof. The results can be found in [5, 7, 8]. \square

Let $\lambda \mathcal{E}_D - \mathcal{A}_D$ be defined in (1). Define

$$\begin{aligned} \mu \mathcal{F} - \mathcal{B} &= \rho(\lambda \mathcal{E}_D - \mathcal{A}_D) := \mu(\mathcal{A}_D + \mathcal{E}_D) - (\mathcal{A}_D - \mathcal{E}_D) \\ &= \mu \begin{bmatrix} 0 & G + F \\ -(G + F)^* & D \end{bmatrix} - \begin{bmatrix} 0 & G - F \\ (G - F)^* & D \end{bmatrix}. \end{aligned} \quad (5)$$

The pencil $\mu \mathcal{F} - \mathcal{B}$ has the following extra property.

Proposition 12 *The eigenvalues of the pencil $\mu \mathcal{F} - \mathcal{B}$ are paired as $(\mu_0, -\bar{\mu}_0)$.*

Proof. The result is simply from Theorem 9 (a) and Proposition 10. \square

4 Relation between $\lambda \mathcal{E}_D - \mathcal{A}_D$ and $\mu \mathcal{E}_C - \mathcal{A}_C$

Let

$$\lambda \mathcal{E}_D - \mathcal{A}_D = \begin{bmatrix} 0 & F \\ -G^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^* & D \end{bmatrix},$$

and

$$\mu \mathcal{E}_C - \mathcal{A}_C = \mu \begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}.$$

We define the transformation

$$\mu\mathcal{E}_C - \mathcal{A}_C = \mathcal{T}(\lambda\mathcal{E}_D - \mathcal{A}_D)$$

with

$$\tilde{F} = G + F, \quad \tilde{G} = G - F, \quad \tilde{D} = D.$$

The inverse transformation of \mathcal{T} exists and is

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \mathcal{T}^{-1}(\mu\mathcal{E}_C - \mathcal{A}_C)$$

with

$$F = \frac{1}{2}(\tilde{F} - \tilde{G}), \quad G = \frac{1}{2}(\tilde{F} + \tilde{G}), \quad D = \tilde{D}.$$

The transformation \mathcal{T} can be interpreted as the composition of the Cayley transformation and an annihilation: apply the Cayley transformation to $\lambda\mathcal{E}_D - \mathcal{A}_D$ to obtain the pencil $\mu\mathcal{F} - \mathcal{B}$ in (5), which now can be written as

$$\mu\mathcal{F} - \mathcal{B} = \begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & \tilde{D} \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}, \quad (6)$$

then simply drop off the block \tilde{D} in \mathcal{F} to get $\mu\mathcal{E}_C - \mathcal{A}_C$.

We begin to show the relation about the eigen-structures of the pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\mu\mathcal{E}_C - \mathcal{A}_C$ under the transformation \mathcal{T} . Our first result is about the regularity.

Theorem 13 *Under the transformation \mathcal{T} the pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ is regular if and only if $\mu\mathcal{E}_C - \mathcal{A}_C$ is regular.*

Proof. By Proposition 11 (a), it sufficient to show that $\mu\mathcal{F} - \mathcal{B}$ is regular if and only if $\mu\mathcal{E}_C - \mathcal{A}_C$ is regular.

If $\mu\mathcal{F} - \mathcal{B}$ regular, we have $\det(\mu\mathcal{F} - \mathcal{B}) \neq 0$. This implies that $\det(\mu\mathcal{F} - \mathcal{B})$ is a nonzero polynomial of μ . So one can always choose a scalar $c \neq -1$ such that $\det(c\mathcal{F} - \mathcal{B}) \neq 0$. The same argument applies to $\mu\mathcal{E}_C - \mathcal{A}_C$.

For a scalar $c \neq -1$, by using the block forms, we have

$$\begin{aligned} \det(c\mathcal{F} - \mathcal{B}) &= \det \begin{bmatrix} 0 & c\tilde{F} - \tilde{G} \\ -c\tilde{F} - \tilde{G}^* & (c-1)\tilde{D} \end{bmatrix} \\ &= \det \left(\begin{bmatrix} (1-c)^{-1}I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & c\tilde{F} - \tilde{G} \\ -c\tilde{F}^* - \tilde{G}^* & -\tilde{D} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (1-c)I \end{bmatrix} \right) \\ &= \det \left(\begin{bmatrix} (1-c)^{-1}I & 0 \\ 0 & I \end{bmatrix} (c\mathcal{E}_C - \mathcal{A}_C) \begin{bmatrix} I & 0 \\ 0 & (1-c)I \end{bmatrix} \right). \end{aligned}$$

Clearly, $\det(c\mathcal{E}_C - \mathcal{A}_C) \neq 0$ if and only if $\det(c\mathcal{F} - \mathcal{B}) \neq 0$. \square

The relations about the eigenvalues and the deflating subspaces of the pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\mu\mathcal{E}_C - \mathcal{A}_C$ are presented in the following theorem.

Theorem 14 Suppose $\lambda\mathcal{E}_D - \mathcal{A}_D$ is regular and $\mu\mathcal{E}_C - \mathcal{A}_C = \mathcal{T}(\lambda\mathcal{E}_D - \mathcal{A}_D)$.

(a) Let $\lambda_0 \neq -1$ be a finite eigenvalue of $\lambda\mathcal{E}_D - \mathcal{A}_D$ with algebraic multiplicity r . Then the matrices

$$U = \begin{matrix} & r \\ n & \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ m & \end{matrix}, \quad V = \begin{matrix} & r \\ n & \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ m & \end{matrix} \in \mathbb{C}^{n+m,r}$$

satisfy

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad \mathcal{E}_D^* V S = \mathcal{A}_D^* V, \quad \det V^* \mathcal{E}_D U \neq 0,$$

for some matrices $T, S \in \mathbb{C}^{r,r}$ with $\Lambda(T) = \Lambda(S^*) = \{\lambda_0\}$, if and only if the matrices

$$\tilde{U} = \begin{bmatrix} U_1(I+T) \\ 2U_2 \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} V_1(I+S) \\ 2V_2 \end{bmatrix}$$

satisfy

$$\mathcal{E}_C \tilde{U} \tilde{T} = \mathcal{A}_C \tilde{U}, \quad \mathcal{E}_C^* \tilde{V} \tilde{S} = \mathcal{A}_C^* \tilde{V}, \quad \det \tilde{V}^* \mathcal{E}_C \tilde{U} \neq 0,$$

where $\tilde{T} = \rho(T)$, $\tilde{S} = \rho(S)$, and $\Lambda(\tilde{T}) = \Lambda(\tilde{S}^*) = \{\mu_0\}$ with $\mu_0 = \rho(\lambda_0)$.

Consequently, $-1 \neq \lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ if and only if $\mu_0 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. λ_0 and μ_0 have the same algebraic and geometric multiplicities. Moreover, U, V span \mathcal{R}_{λ_0} and \mathcal{L}_{λ_0} of $\lambda\mathcal{E}_D - \mathcal{A}_D$, respectively, if and only if \tilde{U}, \tilde{V} span \mathcal{R}_{μ_0} and \mathcal{L}_{μ_0} of $\mu\mathcal{E}_C - \mathcal{A}_C$, respectively.

(b) Suppose $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ with algebraic multiplicity r . Let

$$U = \begin{matrix} & r_1 & r - r_1 \\ n & \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \\ m & \end{matrix} \in \mathbb{C}^{n+m,r},$$

be the matrix that spans \mathcal{R}_{-1} of $\lambda\mathcal{E}_D - \mathcal{A}_D$, generated as in Lemma 8 with $\lambda_0 = -1$, and satisfies

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad \text{rank } \mathcal{E}_D U = r$$

where

$$T = \begin{matrix} & r_1 & r - r_1 \\ r_1 & \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \\ r - r_1 & \end{matrix} \in \mathbb{C}^{r,r}, \quad \Lambda(T) = \{-1\}.$$

Then

$$\tilde{U} = \begin{bmatrix} 2U_{11} & U_{12}(T_{22} + I) \\ 0 & 2U_{22} \end{bmatrix}$$

and the nilpotent matrix

$$\tilde{T} = \begin{bmatrix} (T_{11} + I)(T_{11} - I)^{-1} & -(T_{11} - I)^{-1}T_{12}(T_{22} + I)(T_{22} - I)^{-1} \\ 0 & (T_{22} + I)(T_{22} - I)^{-1} \end{bmatrix}$$

satisfy

$$\mathcal{E}_C \tilde{U} = \mathcal{A}_C \tilde{U} \tilde{T}, \quad \text{rank } \mathcal{A}_C \tilde{U} = r.$$

Consequently, if $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and has algebraic multiplicity r , then $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ and has algebraic multiplicity at least r . Moreover, if U defined above spans \mathcal{R}_{-1} of $\lambda\mathcal{E}_D - \mathcal{A}_D$, then \tilde{U} spans a subspace of \mathcal{R}_∞ (and \mathcal{L}_∞) of $\mu\mathcal{E}_C - \mathcal{A}_C$.

- (c) Let r_{-1} , r_0 , and r_∞ be the algebraic multiplicities of the eigenvalues $-1, 0, \infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$, respectively, and \tilde{r}_1 , \tilde{r}_{-1} , and \tilde{r}_∞ be the algebraic multiplicities of the eigenvalues $1, -1, \infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$, respectively. Then

$$\tilde{r}_1 = \tilde{r}_{-1} = r_0, \quad \tilde{r}_\infty + \tilde{r}_1 = r_{-1} + r_\infty$$

Consequently, under the transformation \mathcal{T} , r_0 eigenvalues $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ become $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. the rest eigenvalue $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ correspond to $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. Under the inverse transformation \mathcal{T}^{-1} , r_{-1} eigenvalues $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ become $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and the rest eigenvalue ∞ correspond to $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$.

Proof. The proof is given in Appendix B. \square

The correspondence between the eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\mu\mathcal{E}_C - \mathcal{A}_C$ is summarized in Table 2.

$\lambda\mathcal{E}_D - \mathcal{A}_D$	∞	-1	0	1	$ \lambda < 0$	$ \lambda = 1$	$ \lambda > 1$
$\mu\mathcal{E}_C - \mathcal{A}_C$	1	∞	-1	0	$\text{Re } \mu < 0$	$\text{Re } \mu = 0$	$\text{Re } \mu > 0$

Table 2: Correspondence between the eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\mu\mathcal{E}_C - \mathcal{A}_C$

Recall that by Theorem 9, if $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ spans \mathcal{R}_{λ_0} of $\lambda\mathcal{E}_D - \mathcal{A}_D$ corresponding to the finite eigenvalue λ_0 , then $\hat{U} = \begin{bmatrix} U_1^T \\ U_2 \end{bmatrix}$ spans $\mathcal{L}_{\lambda_0^{-1}}$ of $\lambda\mathcal{E}_D - \mathcal{A}_D$. Theorem 14 shows that $\tilde{U} = U + \hat{U}$ spans \mathcal{R}_{μ_0} of $\mu\mathcal{E}_C - \mathcal{A}_C$. Note that by

Proposition 7, \tilde{U} also spans $\mathcal{L}_{-\bar{\mu}_0}$ of $\mu\mathcal{E}_C - \mathcal{A}_C$. This shows how the deflating subspaces of $\lambda_0, \bar{\lambda}_0^{-1}$ and $\mu_0, -\bar{\mu}_0$ are precisely related under the transformation \mathcal{T} . The same argument applies to V, \hat{V} , and \tilde{V} .

Theorem 14 shows that if we know the eigen-structure of $\lambda\mathcal{E}_D - \mathcal{A}_D$, we also know the eigen-structure of $\mu\mathcal{E}_C - \mathcal{A}_C$. The only missing part is the deflating subspaces corresponding to $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. But by Proposition 7, they can be obtained from the deflating subspaces corresponding to $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. Conversely, if we know the eigen-structure of $\mu\mathcal{E}_C - \mathcal{A}_C$, we essentially know the eigen-structure of $\mathcal{E}_D - \mathcal{A}_D$. However, Theorem 14 doesn't tell how to extract the information about the eigenvalue $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ from $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. It seems that extra work needs to be done in order to check if $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. One way is to check the singularity of $\mathcal{E}_D + \mathcal{A}_D$. The following result provides an alternative way.

Proposition 15 *Consider the regular pencils $\mu\mathcal{E}_C - \mathcal{A}_C$ and $\lambda\mathcal{E}_D - \mathcal{A}_D = \mathcal{T}^{-1}(\mu\mathcal{E}_C - \mathcal{A}_C)$ as in (2) and (1), respectively. Suppose that the matrix Q has full column rank and spans the null space of \tilde{F} . Let $\hat{D} = Q^* \tilde{D} Q$. Then $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ if and only if at least one of the following conditions holds.*

(i) $\text{rank } \tilde{F} < n$.

(ii) \hat{D} is singular.

Moreover, let $p = \text{rank } \tilde{F}$ and $\text{rank } \hat{D} = q$. Then the dimension of the right eigenvector space of $\lambda\mathcal{E}_D - \mathcal{A}_D$ corresponding to -1 is $n + m - 2p - q$.

Proof. Let

$$\tilde{F} = W \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tilde{Q}^*$$

be the singular value decomposition of \tilde{F} , where $\Sigma \in \mathbb{C}^{p,p}$ is positive diagonal, $W \in \mathbb{C}^{n,n}$ and $\tilde{Q} \in \mathbb{C}^{m,m}$ are unitary. Since the last $m - p$ columns of \tilde{Q} also span the null space of \tilde{F} , without loss of generality we express

$$\tilde{Q} = [\tilde{Q}_1 \quad Q].$$

Let

$$\hat{D} = V \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix} V^*$$

be the Schur form of \hat{D} , where $\Delta \in \mathbb{C}^{q,q}$ is diagonal and nonsingular. Define

$$\mathcal{P} = \text{diag} (W, [\tilde{Q}_1 \quad QV]).$$

Then by (6),

$$\mathcal{P}^*(\mu\mathcal{F}-\mathcal{B})\mathcal{P} = \mu \left[\begin{array}{cc|ccc} 0 & 0 & \Sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -\Sigma & 0 & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ 0 & 0 & \tilde{D}_{12}^* & 0 & 0 \\ 0 & 0 & \tilde{D}_{13}^* & 0 & \Delta \end{array} \right] - \left[\begin{array}{cc|ccc} 0 & 0 & \tilde{G}_{11} & \tilde{G}_{12} & \tilde{G}_{13} \\ 0 & 0 & \tilde{G}_{21} & \tilde{G}_{22} & \tilde{G}_{23} \\ \hline \tilde{G}_{11}^* & \tilde{G}_{12}^* & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ \tilde{G}_{12}^* & \tilde{G}_{22}^* & \tilde{D}_{12} & 0 & 0 \\ \tilde{G}_{13}^* & \tilde{G}_{23}^* & \tilde{D}_{13} & 0 & \Delta \end{array} \right].$$

Clearly, $\text{rank } \mathcal{F} = 2p + q$. So \mathcal{F} is nonsingular if and only if $2p + q = n + m$. Since $p \leq n$ and $q \leq m - p = \text{rank } Q$, $2p + q = n + m$ if and only if $p = n$ and $q = m - p$. This implies that \mathcal{F} is singular if and only if $p < n$ or $q < m - p$, or both. Since $\mathcal{F} = \mathcal{A}_D + \mathcal{E}_D$, when \mathcal{F} is singular, the null space of \mathcal{F} is just the right eigenvector space of $\lambda\mathcal{E}_D - \mathcal{A}_D$ corresponding to the eigenvalue -1 . So the dimension of the eigenvector space equals $\dim \mathcal{F} = n + m - 2p - q$. \square

Note that the result given above is only related to blocks in $\mu\mathcal{E}_C - \mathcal{A}_C$. However, this result still can't provide the complete information about the deflating subspaces of $\lambda\mathcal{E}_D - \mathcal{A}_D$ corresponding to -1 . For this it seems that one has to work on the pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ directly. Another way to remove this difficulty is to use the Cayley transformation

$$\begin{aligned} \mu\mathcal{E}_C - \mathcal{A}_C &= \mathcal{T}_\alpha(\lambda\mathcal{E}_D - \mathcal{A}_D) \\ &= \mu \begin{bmatrix} 0 & G + \alpha F \\ -(G + \alpha F)^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G - \alpha F \\ (G - \alpha F)^* & D \end{bmatrix}, \end{aligned}$$

where α is a unit complex number and $-\alpha \notin \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. Under this transformation, similar results as in Theorem 14 can be derived. The advantage is that no finite eigenvalue in $\Lambda(\mathcal{E}_D, \mathcal{A}_D)$ will correspond to $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$, and vice versa. So the above problem never occurs. However, such a transformation is not a good choice for real pencils, since it can not keep them in real form.

We give two examples to illustrate the behavior about the eigenvalue ∞ and the deflating subspaces under the transformation \mathcal{T} .

Example 2 The generalized symplectic pencil

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \lambda \left[\begin{array}{c|cc} 0 & -1 & 0 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right] - \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & a \\ 0 & a & b \end{array} \right]$$

($a \neq b$) has the eigenvalues -1 and ∞ with the algebraic multiplicities 2 and 1, respectively. With simple calculations we have

$$\mathcal{E}_D U_1 \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \mathcal{A}_D U_1, \quad \mathcal{E}_D^* V_1 \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \mathcal{A}_D^* V_1,$$

where

$$U_1 = \begin{bmatrix} a & b-a \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} a & -b \\ -1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\mathcal{E}_D U_2 = \mathcal{A}_D U_2 \cdot 0, \quad \mathcal{E}_D^* V_2 = \mathcal{A}_D^* V_2 \cdot 0, \quad U_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Under the transformation \mathcal{T} the corresponding skew-Hermitian/Hermitian pencil is

$$\mu \mathcal{E}_C - \mathcal{A}_C = \mu \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right] - \left[\begin{array}{c|cc} 0 & 2 & 1 \\ \hline 2 & 0 & a \\ 1 & a & b \end{array} \right].$$

Simple calculations yield

$$\mathcal{E}_C \tilde{U} = \mathcal{A}_C \tilde{U} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} 0 & -a & a-2b \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

This shows that under the transformation, all the eigenvalues of $\lambda \mathcal{E}_D - \mathcal{A}_D$ become $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. We can also get the relations about the deflating subspaces. By Theorem 9 (a), from U_1 and $T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, we get

$$\hat{U} = \begin{bmatrix} [a & b-a] T \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -a & -b \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = V_1 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $\text{range } \hat{U} = \text{range } V_1 = \mathcal{L}_{-1}$.

Based the fact that $r_1 = 0$, by Theorem 14 (b) we have

$$\tilde{U} = U_1 + \hat{U} = \begin{bmatrix} 0 & -a \\ 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which is just the submatrix formed by first two columns of \tilde{U} .

Finally let us check the result in Proposition 15. Note that in our example $n = 1$, $m = 2$. In the pencil $\mu \mathcal{E}_C - \mathcal{A}_C$,

$$\tilde{F} = [0 \ 1], \quad \tilde{D} = \begin{bmatrix} 0 & a \\ a & b \end{bmatrix}.$$

So $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ spans the null space of \tilde{F} . Since $p = \text{rank } \tilde{F} = 1$, and $q = e_1^* \tilde{D} e_1 = 0$, we conclude that $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and the dimension of the right eigenvector space is $n + m - 2p - q = 1$, which is obviously true.

Example 3 Consider the pencil in Example 1,

$$\lambda \mathcal{E}_D - \mathcal{A}_D = \lambda \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right] - \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline -1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right].$$

We had

$$\mathcal{E}_D U_1 \cdot 0 = \mathcal{A}_D U_1, \quad \mathcal{E}_D^* V_1 \cdot 0 = \mathcal{A}_D^* V_1, \quad U_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\mathcal{E}_D U_2 = \mathcal{A}_D U_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{E}_D^* V_2 = \mathcal{A}_D^* V_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where

$$U_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

The corresponding skew-Hermitian/Hermitian pencil is

$$\mu \mathcal{E}_C - \mathcal{A}_C = \mu \left[\begin{array}{c|cc} 0 & 2 & 1 \\ \hline -2 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right] - \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right].$$

It is easily verified that for

$$\tilde{U}_1 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \quad \tilde{V}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \tilde{U}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},$$

we have

$$\begin{aligned} -\mathcal{E}_C \tilde{U}_1 &= \mathcal{A}_C \tilde{U}_1, & -\mathcal{E}_C^* \tilde{V}_1 &= \mathcal{A}_C^* \tilde{V}_1 \\ \mathcal{E}_C \tilde{V}_1 &= \mathcal{A}_C \tilde{V}_1, & \mathcal{E}_C^* \tilde{U}_1 &= \mathcal{A}_C^* \tilde{U}_1 \\ \mathcal{E}_C \tilde{U}_2 &= \mathcal{A}_C \tilde{U}_2 \cdot 0, & \mathcal{E}_C^* \tilde{U}_2 &= \mathcal{A}_C^* \tilde{U}_2 \cdot 0. \end{aligned}$$

This shows that under the transformation \mathcal{T} , $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ corresponds to $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$, one eigenvalue ∞ of $\lambda \mathcal{E}_D - \mathcal{A}_D$ corresponds to $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$, and another one corresponds to $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. Moreover, it is easily verified that \tilde{U}_1 and \tilde{V}_1 are just the vectors derived from U_1 and V_1 , respectively, by using the formulas in Theorem 14 (a).

Finally, we just point it out that for real pencils a real version of Theorem 14 can be developed basically in the same way.

5 Conclusion

We have shown that the transformation \mathcal{T} sets up the one-to-one relation between the generalized symplectic pencils and the skew-Hermitian/Hermitian pencils. More importantly, under this transformation the regularity of the pencils is preserved, and the eigenvalues and the deflating subspaces are also simply connected. We should mention that the pencils derived from the robust control problems usually have the block forms similar to (3) and (4), which look more special than (1) and (2). However, any pencil like (1) can be transformed to (3) by simply compressing the block F . Similarly, any pencil as (2) can be transformed to (4) by compressing the block \tilde{F} . Therefore, with \mathcal{T} and \mathcal{T}^{-1} the pencils (3) and (4) can also be transformed back and forth. This provides a simple connection between the discrete-time and continuous-time robust control problems.

Acknowledgement. The author thanks Ralph Byers from the University of Kansas and Volker Mehrmann from TU Berlin for their valuable suggestions and comments.

References

- [1] D.Ž. Djoković, J. Patera, P. Winternitz, and H. Zassenhaus. Normal forms of elements of classical real and complex lie and jordan algebras. *J. Math. Phys.*, 24:1363–1373, 1983.
- [2] F.R. Gantmacher. *Theory of Matrices*, volume 2. Chelsea, New York, 1959.
- [3] P. Lancaster and L. Rodman. *The Algebraic Riccati Equation*. Oxford University Press, Oxford, 1995.
- [4] A.J. Laub. A Schur method for solving algebraic Riccati equations. *IEEE Trans. Automat. Control*, AC-24:913–921, 1979.
- [5] A.J. Laub. Invariant subspace methods for the numerical solution of Riccati equations. In S. Bittanti, A.J. Laub, and J.C. Willems, editors, *The Riccati Equation*, pages 163–196. Springer-Verlag, Berlin, 1991.
- [6] W.-W. Lin, V. Mehrmann, and H. Xu. Canonical forms for Hamiltonian and symplectic matrices and pencils. *Linear Algebra Appl.*, 301–303:469–533, 1999.

- [7] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution*. Number 163 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, July 1991.
- [8] V. Mehrmann. A step towards a unified treatment of continuous and discrete time control problems. *Linear Algebra Appl.*, 241-243(749-779), 1996.
- [9] T. Pappas, A.J. Laub, and N.R. Sandell. On the numerical solution of the discrete-time algebraic Riccati equation. *IEEE Trans. Automat. Control*, AC-25:631–641, 1980.
- [10] A. Stoerovogel. *The H_∞ Control Problem: A State-Space Approach*. Prentice-Hall, Englewood Cliffs, NJ, 1992.
- [11] P. Van Dooren. A generalized eigenvalue approach for solving Riccati equations. *SIAM J. Sci. Statist. Comput.*, 2:121–135, 1981.
- [12] P. Van Dooren. *Reducing Subspaces: Definitions, Properties and Algorithms*, volume 973 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, FRG, 1983.
- [13] K. Weiestraß. Zur Theorie der bilinearen und quadratischen Formen. *Monatsh. Akad. D. Wiss. Berlin*, 310–338, 1967.
- [14] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ, 1995.

Appendix A

Proof of Theorem 9

(a) By using the block structures, the matrix equation $\mathcal{E}_D U T = \mathcal{A}_D U$ yields

$$F U_2 T = G U_2, \quad -G^* U_1 T = -F^* U_1 + D U_2.$$

Multiplying the first equation by -1 and post-multiplying the second equation by T we have

$$-G U_2 = -F U_2 T, \quad F^*(U_1 T) = (G^*(U_1 T) + D U_2) T,$$

which can be written as

$$\begin{bmatrix} 0 & -G \\ F^* & 0 \end{bmatrix} \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 & -F \\ G^* & D \end{bmatrix} \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix} T,$$

or equivalently, $\mathcal{E}_D^* \hat{U} = \mathcal{A}_D^* \hat{U} T$.

Conversely, because T is nonsingular, one can express

$$\hat{U} = \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix}.$$

Then by reversing the above steps we can get $\mathcal{E}_D U T = \mathcal{A}_D U$ from the equation $\mathcal{E}_D^* \hat{U} = \mathcal{A}_D^* \hat{U} T$.

Because the pencil $\lambda \mathcal{E}_D^* - \mathcal{A}_D^*$ has the same block structure as $\lambda \mathcal{E}_D - \mathcal{A}_D$, the equivalence relation between $\mathcal{E}_D^* V S = \mathcal{A}_D^* V$ and $\mathcal{E}_D \hat{V} = \mathcal{A}_D \hat{V} S$ can be established in the same way.

We still need to prove the equivalence between $\det V^* \mathcal{E}_D U \neq 0$ and $\det \hat{U}^* \mathcal{E}_D \hat{V} \neq 0$. Using the relations

$$F U_2 T = G U_2, \quad G V_2 S = F V_2,$$

which are from the equations $\mathcal{E}_D U T = \mathcal{A}_D U$ and $\mathcal{E}_D^* V S = \mathcal{A}_D^* V$, respectively, we have

$$\begin{aligned} \hat{U}^* \mathcal{E}_D \hat{V} &= T^* U_1^* F V_2 - U_2^* G^* V_1 S = T^* U_1^* G V_2 S - T^* U_2^* F^* V_1 S \\ &= T^* (U_1^* G V_2 - U_2^* F^* V_1) S = -T^* (V^* \mathcal{E}_D U)^* S. \end{aligned}$$

Since both T and S are nonsingular, it is clear that $\det \hat{U}^* \mathcal{E}_D \hat{V} \neq 0$ if and only if $\det V^* \mathcal{E}_D U \neq 0$.

The above relations show that if $\lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and has algebraic multiplicity r , then $\bar{\lambda}_0^{-1} \in \Lambda(\mathcal{E}_D^*, \mathcal{A}_D^*)$ and has algebraic multiplicity at least r . But the algebraic multiplicity can't be bigger, since otherwise by the duality the algebraic multiplicity of λ_0 will be larger than r . The rest follow simply from the above relations.

(b) The equation $\mathcal{E}_D U T = \mathcal{A}_D U$ implies

$$\begin{aligned} F U_{22} T_{22} &= G U_{22}, \\ G^* U_{11} T_{11} &= F^* U_{11}, \\ -G^* (U_{11} T_{12} + U_{12} T_{22}) &= -F^* U_{12} + D U_{22}. \end{aligned} \tag{7}$$

By using the above equations,

$$\begin{aligned} &\mathcal{A}_D^* \hat{U} \begin{bmatrix} T_{11} & T_{11} T_{12} \\ 0 & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -F \\ G^* & D \end{bmatrix} \begin{bmatrix} U_{11} & U_{11} T_{12} + U_{12} T_{22} \\ 0 & U_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{11} T_{12} \\ 0 & T_{22} \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 0 & -FU_{22}T_{22} \\ G^*U_{11}T_{11} & G^*U_{11}T_{11}T_{12} + [G^*(U_{11}T_{12} + U_{12}T_{22}) + DU_{22}]T_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -GU_{22} \\ F^*U_{11} & F^*U_{11}T_{12} + F^*U_{12}T_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -G \\ F^* & 0 \end{bmatrix} \begin{bmatrix} U_{11} & U_{11}T_{12} + U_{12}T_{22} \\ 0 & U_{22} \end{bmatrix} = \mathcal{E}_D^* \hat{U}.
\end{aligned}$$

Next we show that $\text{rank } \mathcal{E}_D U = r$ implies $\text{rank } \mathcal{A}_D^* \hat{U} = r$. By using the last equation in (7) we have

$$\mathcal{A}_D^* \hat{U} = \begin{bmatrix} 0 & -FU_{22} \\ G^*U_{11} & G^*(U_{11}T_{12} + U_{12}T_{22}) + DU_{22} \end{bmatrix} = \begin{bmatrix} 0 & -FU_{22} \\ G^*U_{11} & F^*U_{12} \end{bmatrix}.$$

If $\mathcal{A}_D^* \hat{U}$ is rank deficient, then there exists a nonzero vector

$$x = \begin{matrix} r_1 \\ r - r_1 \end{matrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

such that $\mathcal{A}_D^* \hat{U} x = 0$. This implies

$$-FU_{22}x_2 = 0, \quad G^*U_{11}x_1 + F^*U_{12}x_2 = 0.$$

We first show $x_2 \neq 0$. Otherwise $x_1 \neq 0$ and $G^*U_{11}x_1 = 0$. But then

$$\mathcal{E}_D U \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & FU_{22} \\ -G^*U_{11} & -G^*U_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = 0,$$

which contradicts to $\text{rank } \mathcal{E}_D U = r$.

Now combining $G^*U_{11}x_1 + F^*U_{12}x_2 = 0$ with $G^*U_{11}T_{11} = F^*U_{11}$, which is the second equation in (7), we have

$$G^* \begin{bmatrix} U_{11} & U_{12}x_2 \end{bmatrix} \begin{bmatrix} T_{11} & -x_1 \\ 0 & 0 \end{bmatrix} = F^* \begin{bmatrix} U_{11} & U_{12}x_2 \end{bmatrix}.$$

Since it is assumed that $\lambda G^* - F^*$ has only r_1 eigenvalues 0, the above relation implies $\text{rank} \begin{bmatrix} U_{11} & U_{12}x_2 \end{bmatrix} = r_1$. Because $\text{rank } U_{11} = r_1$, we have

$$U_{12}x_2 = U_{11}x_3$$

for some vector $x_3 \in \mathbb{C}^{r_1}$.

Now $\begin{bmatrix} -x_3 \\ x_2 \end{bmatrix} \neq 0$, but it is easily verified that

$$\mathcal{E}_D U \begin{bmatrix} -x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} FU_{22}x_2 \\ G^*(U_{11}x_3 - U_{12}x_2) \end{bmatrix} = 0.$$

Then $\text{rank } \mathcal{E}_D U < r$, which again is a contradiction. Hence it must be $\text{rank } \mathcal{A}_D^* \hat{U} = r$. Obviously the pencil $\lambda \mathcal{E}_D - \mathcal{A}_D$ has at least r eigenvalues ∞ and \hat{U} span a subspace of \mathcal{L}_∞ .

By the duality, the relation about V and \hat{V} can be proved in the same way.

Appendix B

Proof of Theorem 14. Let $\mu \mathcal{F} - \mathcal{B} = \rho(\lambda \mathcal{E}_D - \mathcal{A}_D)$ as in (6).

(a) It is easily verified that the equations

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad \mathcal{E}_D^* V S = \mathcal{A}_D^* V$$

are equivalent to

$$\mathcal{F} U \rho(T) = \mathcal{B} U, \quad \mathcal{F}^* V \rho(S) = \mathcal{B}^* V.$$

Clearly, $\Lambda(\rho(T)) = \Lambda(\rho(S)^*) = \{\mu_0\}$, where $\mu_0 = \rho(\lambda_0)$. Since $\lambda_0 \neq -1, \infty$, we have $\mu_0 \neq \infty, 1$.

From $\mathcal{F} U \rho(T) = \mathcal{B} U$, we have

$$\tilde{F} U_2 \rho(T) = \tilde{G} U_2, \quad -\tilde{F}^* U_1 \rho(T) = \tilde{G}^* U_1 + \tilde{D} U_2 (I - \rho(T)).$$

Because

$$I - \rho(T) = I - (T - I)(T + I)^{-1} = 2(T + I)^{-1},$$

and $\rho(T)$ and $T + I$ commute, multiplying the first equation by 2 and post-multiplying the second equation by $T + I$, we get

$$\tilde{F}(2U_2)\rho(T) = \tilde{G}(2U_2), \quad -\tilde{F}^* U_1 (T + I) \rho(T) = \tilde{G}^* U_1 (T + I) + \tilde{D}(2U_2).$$

So we have

$$\begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix} \begin{bmatrix} U_1(I + T) \\ 2U_2 \end{bmatrix} \rho(T) = \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix} \begin{bmatrix} U_1(I + T) \\ 2U_2 \end{bmatrix},$$

which is just

$$\mathcal{E}_C \tilde{U} \tilde{T} = \mathcal{A}_C \tilde{U}.$$

Conversely, suppose that \tilde{U} satisfies

$$\mathcal{E}_C \tilde{U} \tilde{T} = \mathcal{A}_C \tilde{U},$$

with $\Lambda(\tilde{T}) = \{\mu_0\}$ and $\mu_0 \neq \infty, 1$. Let

$$T = \rho^{-1}(\tilde{T}) = (I + \tilde{T})(I - \tilde{T})^{-1}.$$

Then $\Lambda(T) = \{\lambda_0\}$, where $\lambda_0 = \rho^{-1}(\mu_0) \neq -1, \infty$. Because $I+T$ is invertible, we can express \tilde{U} as

$$\tilde{U} = \begin{bmatrix} U_1(I+T) \\ 2U_2 \end{bmatrix}.$$

Define $U := \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$. By reversing the above steps we can show that U and T satisfy

$$\mathcal{E}_D U T = \mathcal{A}_D U.$$

Similarly, we can show that $\mathcal{E}_D^* V S = \mathcal{A}_D^* V$ if and only if $\mathcal{E}_C^* \tilde{V} \tilde{S} = \mathcal{A}_C^* \tilde{V}$.

We now prove that $\det V^* \mathcal{E}_D U \neq 0$ and $\det \tilde{V}^* \mathcal{E}_C \tilde{U} \neq 0$ are equivalent. From the equations $\mathcal{E}_D U T = \mathcal{A}_D U$ and $\mathcal{E}_D^* V S = \mathcal{A}_D^* V$, we have

$$F U_2 T = G U_2, \quad G V_2 S = F V_2.$$

So

$$\begin{aligned} \tilde{V}^* \mathcal{E}_C \tilde{U} &= 2(I+S)^* V_1^* (G+F) U_2 - 2V_2^* (G+F)^* U_1 (I+T) \\ &= 2(I+S)^* V_1^* (G U_2 + F U_2) - 2(V_2^* G^* + V_2^* F^*) U_1 (I+T) \\ &= 2(I+S)^* V_1^* (F U_2 T + F U_2) - 2(V_2^* G^* + S^* V_2^* G^*) U_1 (I+T) \\ &= 2(I+S)^* V_1^* F U_2 (I+T) - 2(I+S)^* V_2^* G^* U_1 (I+T) \\ &= 2(I+S)^* (V_1^* F U_2 - V_2^* G^* U_1) (I+T) \\ &= 2(I+S)^* (V^* \mathcal{E}_D U) (I+T). \end{aligned}$$

Since $(I+S)^*$ and $(I+T)$ are nonsingular, it is clear that $\det \tilde{V}^* \mathcal{E}_C \tilde{U} \neq 0$ if and only if $\det V^* \mathcal{E}_D U \neq 0$. The rest are obvious.

(b) When $\Lambda(T) = \{-1\}$, it is easily verified that $\mathcal{E}_D U T = \mathcal{A}_D U$ implies $\mathcal{F} U = \mathcal{B} U \hat{T}$, where

$$\begin{aligned} \hat{T} &= (T+I)(T-I)^{-1} \\ &= \begin{bmatrix} (T_{11}+I)(T_{11}-I)^{-1} & -2(T_{11}-I)^{-1} T_{12} (T_{22}-I)^{-1} \\ 0 & (T_{22}+I)(T_{22}-I)^{-1} \end{bmatrix} \\ &=: \begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} \\ 0 & \hat{T}_{22} \end{bmatrix}. \end{aligned}$$

The equation $\mathcal{F} U = \mathcal{B} U \hat{T}$ gives

$$\begin{aligned} \tilde{F} U_{22} &= \tilde{G} U_{22} \hat{T}_{22} \\ -\tilde{F}^* U_{11} &= -\tilde{G}^* U_{11} \hat{T}_{11} \\ -\tilde{F}^* U_{12} &= -\tilde{G}^* (U_{11} \hat{T}_{12} + U_{12} \hat{T}_{22}) + \tilde{D} U_{22} (\hat{T}_{22} - I). \end{aligned}$$

Since $\hat{T}_{22} - I = 2(T_{22} - I)^{-1}$, multiplying the first and the second equations by 2 and post-multiplying the 3rd equation by $(T_{22} + I)$, we have

$$\begin{aligned}\tilde{F}(2U_{22}) &= \tilde{G}(2U_{22})\hat{T}_{22} \\ -\tilde{F}^*(2U_{11}) &= -\tilde{G}^*(2U_{11})\hat{T}_{11} \\ -\tilde{F}^*U_{12}(T_{22} + I) &= -\tilde{G}^*(-(2U_{11})(T_{11} - I)^{-1}T_{12}\hat{T}_{22} + U_{12}(T_{22} + I)\hat{T}_{22}) \\ &\quad + \tilde{D}(2U_{22})\hat{T}_{22}.\end{aligned}$$

With these equations we have

$$\mathcal{E}_C \tilde{U} = \mathcal{A}_C \tilde{U} \tilde{T},$$

where \tilde{U}, \tilde{T} are as given in the theorem.

We now show that $\text{rank } \mathcal{E}_D U = r$ implies $\text{rank } \mathcal{A}_C \tilde{U} = r$. From the equation $\mathcal{E}_D U T = \mathcal{A}_D U$, we have

$$\begin{aligned}FU_{22}T_{22} &= GU_{22} \\ G^*U_{11}T_{11} &= F^*U_{11} \\ -G^*(U_{11}T_{12} + U_{12}T_{22}) &= -F^*U_{12} + DU_{22}.\end{aligned}$$

By using these relations,

$$\begin{aligned}\mathcal{A}_C \tilde{U} &= \begin{bmatrix} 0 & 2(G - F)U_{22} \\ 2(G - F)^*U_{11} & (G - F)^*U_{12}(T_{22} + I) + 2DU_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2FU_{22}(T_{22} - I) \\ 2G^*U_{11}(I - T_{11}) & (G + F)^*U_{12}(I - T_{22}) - 2G^*U_{11}T_{12} \end{bmatrix}.\end{aligned}$$

If $\mathcal{A}_C \tilde{U}$ is rank deficient, then there is a nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $\mathcal{A}_C \tilde{U} x = 0$. This implies

$$FU_{22}(T_{22} - I)x_2 = 0 \tag{8}$$

and

$$2G^*U_{11}(I - T_{11})x_1 + ((G + F)^*U_{12}(I - T_{22}) - 2G^*U_{11}T_{12})x_2 = 0. \tag{9}$$

We have $x_2 \neq 0$. Otherwise $x_1 \neq 0$ but $G^*U_{11}(I - T_{11})x_1 = 0$. Because $\Lambda(T_{11}) = \{-1\}$, we have $(I - T_{11})x_1 \neq 0$. Then

$$\mathcal{E}_D U \begin{bmatrix} (I - T_{11})x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -G^*U_{11}(I - T_{11})x_1 \\ 0 \end{bmatrix} = 0$$

implies $\text{rank } \mathcal{E}_D U < r$. This contradicts to $\text{rank } \mathcal{E}_D U = r$. Now rewrite (9) as

$$G^*[2U_{11}((I - T_{11})x_1 - T_{12}x_2) + U_{12}(I - T_{22})x_2] = -F^*U_{12}(I - T_{22})x_2.$$

Combining it with $G^*U_{11}T_{11} = F^*U_{11}$, which is from $\mathcal{E}_D UT = \mathcal{A}_D U$, we have

$$\begin{aligned} G^* \begin{bmatrix} U_{11} & -U_{12}(I - T_{22})x_2 \end{bmatrix} \begin{bmatrix} T_{11} & 2((I - T_{11})x_1 - T_{12}x_2) \\ 0 & -1 \end{bmatrix} \\ = F^* \begin{bmatrix} U_{11} & -U_{12}(I - T_{22})x_2 \end{bmatrix}. \end{aligned}$$

Since U_{11} spans the right deflating subspace of $\lambda G^* - F^*$ corresponding to the eigenvalue -1 , we have

$$U_{12}(I - T_{22})x_2 = U_{11}x_3, \quad (10)$$

for some vector x_3 . Let $z = \begin{bmatrix} -x_3 \\ (I - T_{22})x_2 \end{bmatrix}$. Because $\Lambda(T_{22}) = \{-1\}$ and $x_2 \neq 0$, we have $z \neq 0$. But (8) and (10) imply

$$\mathcal{E}_D U z = \begin{bmatrix} FU_{22}(I - T_{22})x_2 \\ -G^*(-U_{11}x_3 + U_{12}(I - T_{22})x_2) \end{bmatrix} = 0,$$

which contradicts to $\text{rank } \mathcal{E}_D U = r$ again. So we have $\text{rank } \mathcal{A}_C \tilde{U} = r$.

The rest are obvious.

(c) By Theorem 9, if $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and has algebraic multiplicity r_0 , then there are at least r_0 eigenvalues $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. From part (a) these r_0 eigenvalues 0 correspond to r_0 eigenvalues $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. So $\tilde{r}_{-1} = r_0$. By Proposition 7 (a), $\mu \mathcal{E}_C - \mathcal{A}_C$ has also r_0 eigenvalues 1. So $\tilde{r}_1 = r_0$. Under the transformation \mathcal{T} , no finite eigenvalue in $\Lambda(\mathcal{E}_D, \mathcal{A}_D)$ corresponds to $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. So these r_0 eigenvalues 1 must be from r_0 eigenvalues $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. Now, if $r_\infty > r_0$, then there are additional $r_\infty - r_0$ eigenvalues $\infty \in (\mathcal{E}_D, \mathcal{A}_D)$. Clearly, these eigenvalues have to correspond to $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. Since other eigenvalues $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ are from $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$, we have

$$\tilde{r}_\infty = r_\infty - r_0 + r_{-1} = r_\infty - \tilde{r}_1 + r_{-1},$$

or equivalently,

$$\tilde{r}_\infty + \tilde{r}_1 = r_\infty + r_{-1}.$$